Global attractors and bifurcations

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Abstract

We present some recent developments in the study of attractors of smooth dynamical systems, specially attractors whose basin has a global character. A key point in our approach is to explore the relations between this study and that of main bifurcation mechanisms.

Contents

1 Introduction

We consider both continuous time dynamical systems (flows) and discrete time dynamical systems (smooth transformations, diffeomorphisms) on manifolds. In the first setting we use $\varphi^t\colon M \longrightarrow M$, $t \in \mathbb{R}$, to denote the flow. In the second one we let $\varphi\colon M \longrightarrow M$ be the transformation and denote its t-iterate $\varphi^t = \varphi \circ \cdots \circ \varphi$, for each integer $t \geq 1$; if φ is invertible we also write $\varphi^{-t} = (\varphi^t)^{-1}$.

A main problem in Dynamics, which we want to address here, is to describe the (typical) asymptotic behaviour of trajectories $\varphi^t(z)$, $z \in M$, as time t goes to $+\infty$. Let an *attractor* be a (compact) subset A of the ambient manifold M such that

- A is *invariant* under time evolution: $\varphi^t(A) = A$ for every $t > 0$;
- A is *dinamically indivisible*: it contains some dense orbit (alternatively, one may ask that A support an ergodic invariant measure);
- *the basin of* A, defined by $B(A) = \{z \in M : \varphi^t(z) \to A \text{ as } t \to +\infty\}$, *is a large set*: it contains a neighbourhood of A (weaker definitions are obtained by requiring $B(A)$ to have nonempty interior or even just positive Lebesgue measure).

Then this problem can be rephrased in terms of describing the properties of attractors, namely

- geometric and topological properties (fractional dimensions, topological invariants);
- dynamical properties (symbolic dynamics, (non)hyperbolicity, Lyapunov exponents);
- ergodic properties (asymptotic measures, statistical parameters).

Of particular interest is to investigate the robustness (or *persistence*) of these features of the dynamics when the system is perturbed (either deterministically or randomly).

Besides the beautiful theory developed throughout the sixties and the seventies for the case of Axiom A systems, see e.g. [Sm], [Bo], a great deal of interest has been devoted in recent years to trying to provide a satisfactory answer to these questions for more general classes of attractors, lacking uniform hyperbolicity. Motivation comes both from the applications (models of natural phenomena are seldom uniformly hyperbolic) and from the intrinsic richness of such systems, which combine (structural) unstability with some remarkable forms of persistence.

A fruitful approach, strongly advocated by J. Palis, has been to try and relate the study of (nonhyperbolic) attractors with that of the generic processes through which the dynamics varies as the initial system is modified (bifurcation processes). More precisely, one considers parametrized families of dynamical systems unfolding a given type of bifurcation (such as nontransverse homoclinic trajectories or nonstable cycles envolving periodic trajectories, for instance) and one tries to describe the presence and the properties of attractors in those families. Results such as [MV], [DRV], [Mo] or [MP], for instance, may be thought of from this perspective. A second, kind of converse, step has also been proposed by Palis: to show that generic dynamical systems with nonhyperbolic attractors (or other relevant unstable phenomena) can be approximated by others exhibiting one of a small number of bifurcation types. Results of this kind include e.g. [Ur], [Ca].

Here we discuss a number of recent progresses in this general program. In Section 2 we analyse *the basin of Hénon-like attractors*, to prove that it *contains a neighbourhood of the attractor*, at least for a large set of parameters. This is well-known in the orientation-reversing case, but the, possibly even more relevant, orientation-preserving case seems to be new. We also announce a more quantitative result, of ergodic flavour, recently established by M. Benedicks, and myself: *almost every point in the basin of attraction is generic with respect to the Sinai-Ruelle-Bowen measure of the attractor.*

Section 3 corresponds to joint work with V. Baladi concerning the *ergodic properties of certain nonuniformly hyperbolic unimodal maps* of the interval. The main result asserts that those properties, including the fact that such maps are exponentially mixing (exponential decay of correlations), *are robust under random perturbations of the map* (stochastic stability).

Section 4 was written jointly with S. Luzzatto and contains a discussion of an extended geometric model for the behaviour of Lorenz equations. The goal of this model is to provide insight into the way the strange attractor is destroyed through the introduction of "folds", as the parameters are varied. The main statement is that *the attractor persists after the appearance of the folds, but only for a positive measure set of parameter values.*

In Section 5, we report on joint work with M. J. Pacifico and A. Rovella. We consider smooth flows in 3-dimensional manifolds exhibiting homoclinic connections associated to equilibrium points of saddle-focus type. Then we prove that *a new type of global attractor, with spiraling geometry, occurs (and is even a persistent phenomenon) in such families.*

2 The basin of Hénon-like attractors

In Section 2.1 we prove that *the basin of Hénon-like attractors contains a full neighbourhood of the attractor,* for a large set of parameter values, and we also state two related conjectures. Then, in Section 2.2, we discuss a substantial refinement of this result: *Lebesgue almost every orbit in the basin is (exponentially) asymptotic to some orbit in the attractor*. As a consequence, *almost every point in the basin is generic (in the sense of the ergodic theorem) with respect to the SRB-measure of the attractor.*

2.1 The topological basin

Let $(\Psi_a)_{1<a<2}$ denote the family of quadratic real maps $\Psi_a(x) = 1 - ax^2$ (this may replaced by much more general families of unimodal or multimodal maps of the interval, see e.g. [DRV, Section 5]). We also consider the corresponding family of endomorphisms ψ_a of the plane, given by $\psi_a(x, y) = (\Psi_a(x), 0)$. By a *Hénon-like* *map* we mean here any map φ on the plane which is close enough to some ψ_a in the C^r-sense

$$\|\varphi - \psi_a\|_{C^r} < b, \qquad b > 0 \quad \text{small}$$

(usually one assumes $r \geq 3$; just how small b should be depends on the context). In all that follows we suppose that φ is an orientation-preserving diffeomorphism but similar arguments apply in the orientation-reversing case.

It is straightforward to check that for every $a \in (1, 2)$ the map Ψ_a has exactly two fixed points $Q_a < 0 < P_a$ and that these are both hyperbolic (repelling). Moreover, the unstable set of P_a is a compact interval contained in $(Q_a, -Q_a)$. Then a corresponding statement holds for ψ_a: it has exactly two fixed points $q_a = (Q_a, 0)$ and $p_a = (P_a, 0)$, which are hyperbolic saddles, and the unstable set of p_a lies inside $(Q_a, -Q_a) \times \{0\}$. Now let $p = p(\varphi)$, $q = q(\varphi)$ be the continuation of these fixed points for a nearby diffeomorphism φ. Then p and q are still hyperbolic saddles and the unstable manifold $W^u(p)$ is contained in a bounded region $(Q_a, -Q_a) \times (-b, b)$. It is well known (see e.g. [BC2]) that the compact set $A = A(\varphi) = \text{closure}(W^u(p))$ has a basin $B(A)$ with nonempty interior. Moreover, [BC2], [MV], proved that very often (positive measure set of parameters) A contains a dense orbit with expanding behaviour (positive Lyapunov exponent).

Here we want to prove

Theorem 2.1 *There exist sequences $(I_j)_j$ of compact intervals converging to $a = 2$ and $(b_j)_j$ of positive numbers, such that, for any diffeomorphism φ satisfying $\|\varphi - \psi_a\|_{C^1} < b_j$ for some $a \in \cup I_j$, the basin $B(A)$ contains a neighbourhood of A.*

Proof: In order to exhibit the intervals I_j we go back to the quadratic family $\Psi_a(x) = 1 - ax^2$. An explicit calculation shows that if a is close to $a = 2$ then P_a is close to $x = 1/2$. Moreover, $\Psi_a^{-1}(P_a)$ consists of two points $P_{a,1} < 0 < P_{a,0} = P_a$ and $\Psi_a^{-2}(P_a)$ consists of four points $P_{a,2} < P_{a,1} < 0 < P_{a,0} < P'_{a,2}$. We denote $J_{a,0} = [P_{a,0}, P'_{a,2}]$ and let $J_{a,1} = [P_{a,1}, P'_{a,3}]$ be the connected component of $\Psi_a^{-1}(J_{a,0})$ situated to the left of zero (assuming once more that a is close enough to 2). More generally, for $j \geq 1$ we let $J_{a,j} = [P_{a,j}, P'_{a,j+2}]$ be the connected component of $\Psi_a^{-1}(J_{a,j-1})$ contained in $\{x < 0\}$. Observe that the $J_{a,j}$ converge to the repelling fixed point Q_a as $j \to +\infty$. Now we fix a slightly smaller compact interval $\tilde{J}_{a,j} \subset \text{int}\,(J_{a,j})$ and define the parameter interval I_j by

$$a \in I_j \iff \Psi_a(1) \in \tilde{J}_{a,j}$$

(1 is the critical value of Ψ_a). Remarking that $\Psi_a^2(0) = Q_a$ when $a = 2$, one concludes without difficulty that the sets I_j defined in this way are indeed compact intervals accumulating on $a = 2$.

Now, for $a \in I_j$ we consider the endomorphism ψ_a. Note that the stable set $W^s(p_a) = \{z: \psi_a^n(z) \to p_a \text{ as } n \to +\infty\}$ consists of all the vertical lines of the form $\{(x, y): \Psi_a^n(x) = P_a \text{ for some } n \geq 0\}$. In particular, it contains the vertical lines $P_{a,i} \times \mathrm{R}$, $P'_{a,i+2} \times \mathrm{R}$, for each $i \geq 0$.

Then we let φ be any orientation-preserving diffeomorphism sufficiently C^1-close to ψ_a. More precisely, we take φ to be defined in some large square $S = [-l, l]^2$, with $\|\varphi - \psi_a\|_{C^1(S)} < b_j$ for some small b_j. It is convenient to begin by extending φ to a (proper) diffeomorphism of the whole plane and from now on φ will denote such an extension (note that neither A nor the fact that $B(A)$ is a neighbourhood of it depend on the choice of the extension). Since local invariant manifolds of periodic points vary continuously with the the dynamics, see e.g. [Shu] or [PT, Appendix 1], we get that, provided $b_j > 0$ is small enough,

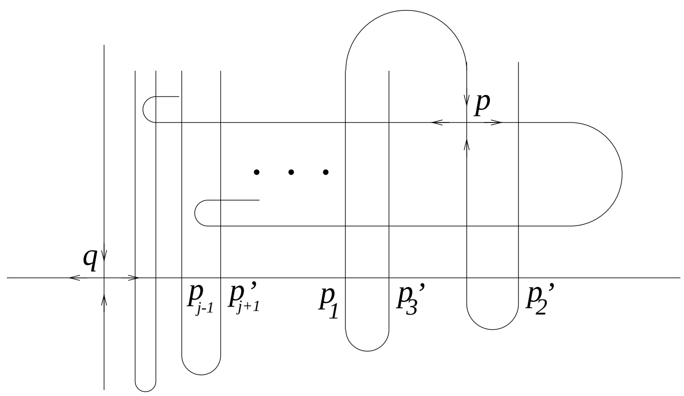

Figure 1: Invariant manifolds of p

1. $W^s(p)$ contains segments C^1-near each $P_{a,i} \times [-2l, 2l]$ and $P'_{a,i+2} \times [-l, l]$ with $0 \leq i \leq j$.

2. $W^u(p(\varphi))$ folds near $x = 1$; the first (resp. the second) image of this fold is near $x = \Psi_a(1)$ (resp. $x = \Psi_a^2(1)$) and hence it is contained in the interior of $J_{a,j} \times [-b, b]$ (resp. $J_{a,j-1} \times [-b, b]$).

In particular, the segments of $W^s(p)$ mentioned in 1 intersect $W^u(p)$ and $W^u(q)$; we denote by p_i, p'_i the points of intersection with the local unstable manifold of q, see Figure 1. Now, $W^s(p)$ is an immersed submanifold of the plane and so these segments must be connected in some way. Using the assumption that φ preserves orientation (hence both eigenvalues of $D\varphi(p)$ are negative) one checks easily that the segments passing through p, p'_2, p_1, must be connected as described in Figure 1. Moreover, iterating backwards we conclude that the segments passing through p_i and p'_{i+2}, $1 \leq i \leq j$, connect to each other as depicted. Observe here that each of these segments intersects the boundary of the horseshoe-shaped region $\varphi(S)$ in exactly four points, and so the corresponding preimage intersects ∂S also in four points.

At this point we fix some $\delta_j > 0$ and denote $K_j = [-2\delta_j, 2\delta_j] \times [-l, l]$. As long as δ_j and b_j are small enough, $\varphi(K_j)$, $\varphi^2(K_j)$, $\varphi^3(K_j)$ are small regions near

$x = 1$, $x = \Psi_a(1)$, $x = \Psi_a^2(1)$, respectively. In particular, $\varphi^3(K_j)$ is contained in the interior of $J_{a,j-1} \times [-b, b]$ and, in fact, in the region Ω bounded by $W^u(p)$ and the piece of $W^s(p)$ connecting the nearly straight segments passing through p_{j-1} and p_{j+1}. It follows from well-known arguments (see e.g. Theorem 4 in [BC2]) that $\varphi^3(K_j) \subset \Omega$ is contained in the basin of the attractor $A = \text{closure}\,(W^u(p))$ and so the same holds for K_j.

We are left to consider those points in a neighbourhood of A whose forward orbit does not intersect K_j and we do this as follows. A result in [Ma] asserts that the set E_j of points in the interval whose forward trajectory is disjoint from $(-\delta_j, \delta_j)$ is a compact hyperbolic set for Ψ_a. Clearly, E_j contains the fixed point P_a and it is not difficult to deduce that $\Psi_a | E_j$ is transitive. Since hyperbolic sets are persistent under perturbations of the system, see e.g. [Shu] or [PT, Appendix 1], it follows that the set H_j of points whose full orbit remains outside $\tilde{K}_j = [-\delta_j, \delta_j] \times [-1, 1]$ is hyperbolic and transitive for φ. Moreover, the set of points whose forward orbit never enters K_j is contained in the stable set of H_j. In other words, all the points under consideration at this stage are attracted to H_j. On the other hand, H_j contains the fixed point p and so it is contained in $\text{closure}\,(W^u(p)) = A$. This completes our argument. □

It is not difficult to check that the arguments in [MV] can be carried out within the parameter intervals I_j constructed above and so the present conclusions apply to the Hénon-like strange attractors found in there. Moreover, combining the present ideas with those in [Vi, Section 3] one obtains a similar conclusion for the quadratic-like attractors in higher-dimensional manifolds constructed there.

On the other hand, the previous result is somewhat unsatisfactory, in that one may expect the conclusion to hold for *all* values of a close to 2 (and small b), as happens in the orientation-reversing case. In this direction we state the following two conjectures.

1: There is a positive function $b(a)$, defined for a in a whole interval $(a_0, 2)$, such that the conclusion of the theorem holds if $\|\varphi - \psi_a\|_{C^1} < b(a)$ for some $a \in (a_0, 2)$.

2: Moreover, $b(a)$ may be taken so that if $\varphi(x, y) = (1 - ax^2 + by, \pm bx)$ (a Hénon map) with $b < b(a)$ then its nonwandering set $\Omega(\varphi)$ coincides with $Q(\varphi) \cup A(\varphi)$.

2.2 Exponential convergence and the ergodic basin

Another important problem is to characterize the ergodic basin of attraction of A. Let us formulate this more precisely. It was shown in [BY2] that for the parameter values such as in [BC2], [MV] the attractor A supports a measure of Sinai-Ruelle-Bowen. By such they mean an invariant probability measure μ supported on A, which is ergodic, has a positive Lyapunov exponent and, most important, induces absolutely continuous conditional measures along unstable manifolds (absolute continuity is with respect to the riemannian measure on the unstable manifold). Then standard arguments yield the following key property: μ *determines the asymptotic behaviour of time averages of all continuous functions* $g: B(A) \longrightarrow \mathrm{R}$

$$\frac{1}{t}\sum_{j=0}^{t-1} g(\varphi^j(x)) \to \int g\, d\mu \quad \text{as } t \to +\infty$$

for a positive (two-dimensional) Lebesgue measure set of points $x \in B(A)$. Of course, one would like to know whether this property holds for a *full measure* subset of the basin and this is the problem we are considering here.

Now, it is very easy to see that any (forward) asymptotic trajectories have the same (forward) limit time averages, for all continuous functions. Therefore, the previous problem is somewhat related to the question whether all (or almost all) the orbits in the basin are asymptotic to some orbit inside the attractor (as in the Axiom A case). Unfortunately, the elementary arguments we used in the proof of Theorem 2.1 seem of little help for solving this question. However, it turns out that the answer to both questions above is indeed positive, as shown recently by M. Benedicks and myself. More precisely, we prove that for the parameter values in [BC2] or in [MV], the attractor $A = A(\varphi)$ satisfies

Theorem 2.2 *Through Lebesgue almost every point in* $B(A)$ *there is a local stable manifold which intersects* A. *Moreover, we have*

$$\frac{1}{t}\sum_{j=0}^{t-1} g(\varphi^j(x)) \to \int g\, d\mu \quad \textit{as } t \to +\infty,$$

for every continuous function g *and for Lebesgue almost every* $x \in B(A)$.

Recall that Lebesgue refers to the two-dimensional Lebesgue measure. Also, by a stable manifold we mean a curve which is exponentially contracted under all positive iterates of φ. The proof of Theorem 2.2 is to appear elsewhere.

3 Stochastic stability and exponential mixing

In this section we deal with nonuniformly hyperbolic unimodal maps of the interval $\varphi: I \longrightarrow I$, with $\varphi(I) \subset \operatorname{int}(I)$. Our goal is to describe the main results and techniques in our paper with V. Baladi [BaV]. In an ongoing joint work with M. Benedicks we are extending part of these results (stochastic stability) to attractors of dissipative diffeomorphisms in higher-dimensional manifolds.

For simplicity we take φ to be quadratic, $\varphi(x) = a - x^2$, but our arguments hold for general unimodal maps with negative schwarzian derivative and nondegenerate critical point. We formulate the nonuniform hyperbolicity property in terms of the orbit of the critical point $c = 0$: let us assume that

1. $\left|(\varphi^k)'(\varphi(c))\right| \geq \lambda_c^k$ (positive Lyapunov exponent);

2. $\left|\varphi^k(c) - c\right| \geq e^{-\alpha k}$ (exponential recurrence bound)

for every $k \geq 1$ and for some constants $0 < \alpha \ll 1 < \lambda_c$. We also suppose that φ is topologically mixing (on the interval $\varphi^2(I)$).

This formulation is motivated by [BC1], [BC2], where it is proved that 1 and 2 above are satisfied by quadratic maps for a positive measure set of values of the parameter a. It follows from condition 1 and [Si] that φ can not have attracting periodic orbits. In contrast, as observed already by [BC1], [BC2], maps $\varphi_s(x) = \varphi(x) + s$ with small s may exhibit such periodic attractors. This means, in particular, that the dynamics of φ is very unstable under perturbations of the map. However, here we want to prove that *from a different, statistical, perspective the dynamics of such maps is, in fact, quite robust*. In order to state this in a precise way let us comment a bit more on conditions 1 and 2.

It is now well understood, [No], that condition 1 implies the conclusion of [Ja]: φ admits an invariant probability measure μ_0 which is absolutely continuous with respect to the Lebesgue measure m on I (even equivalent to m restricted to $\varphi^2(I)$). This measure μ_0 is unique, ergodic, and determines the asymptotics of typical orbits of φ:

$$\frac{1}{t}\sum_{j=0}^{t-1} g(\varphi^j(x)) \to \int g\, d\mu_0 \quad \text{as } t \to +\infty$$

for every continuous function g and m-almost all $x \in I$.

Now we want to consider the effect of adding random noise to the iteration of φ. More precisely, we want to compare the asymptotic behaviour of φ^t with that of $\varphi_{s_t} \circ \cdots \circ \varphi_{s_1}$, where $s_1, \ldots, s_t$ are chosen randomly and independently in some small interval $[-\varepsilon, \varepsilon]$ (we shall denote by θ_ε the corresponding probability distribution). Under general conditions, satisfied in our context, such a random scheme admits a stationary measure μ_ε, with

$$\frac{1}{t}\sum_{j=0}^{t-1} g(\varphi_{s_j} \circ \cdots \varphi_{s_1}(x)) \to \int g\, d\mu_\varepsilon \quad \text{as } t \to +\infty$$

for every continuous function g, m-almost all $x \in I$, and almost all choices of $(s_j)_{j\geq 1}$ (in the present context μ_ε is unique and absolutely continuous with respect to Lebesgue measure).

Then we say that φ is *(weakly) stochastically stable* if μ_ε is close to μ_0 (in the weak*-sense) when the noise level ε is close to zero.

Theorem 3.1 *[BaV] φ is strongly stochastically stable (hence stochastically stable), that is*

$$\frac{d\mu_\varepsilon}{dm} \to \frac{d\mu_0}{dm} \quad \textit{in the } L^1\textit{-sense}, \quad \textit{as } \varepsilon \to 0.$$

That is, *small random noise has a neglectable effect on the asymptotic behaviour of the map*. Results such as this may be thought to provide some conceptual legitimacy to information concerning "chaotic" systems extracted from finite-precision numerical experiments (although round-off errors are not really random noise).

Before sketching the main points underlying Theorem 3.1, let us introduce another important, somewhat related, notion. We say that (φ, μ_0) is *exponentially mixing* (equivalently, has *exponential decay of correlations*), if there exists $\tau < 1$ such that, given test functions f and g,

$$\left| \int U_0^t(f) \cdot g \, d\mu_0 - \int f \, d\mu_0 \int g \, d\mu_0 \right| \leq C(f,g)\tau^t \quad \text{for all } t \geq 1$$

(U_0 denotes the spectral operator $U_0(f) = f \circ \varphi$). In other words, *$f \circ \varphi^t$ and g, viewed as random variables, become independent exponentially fast as $t \to +\infty$.* Formally speaking, f and g should be taken in some convenient Banach space, in our case this will be the space $BV(I)$ of functions of bounded variation on I. One can also define a notion of exponential mixing for the random scheme φ_s, $|s| \leq \varepsilon$, just by replacing above μ_0 by μ_ε, and U_0 by the perturbed spectral operator $U_\varepsilon(f)(x) = \int f(\varphi_s(x))\theta_\varepsilon(s)\, ds$.

Theorem 3.2 *[BaV] Both φ and its random perturbation schemes $(\varphi_s)_{|s| \leq \varepsilon}$, are exponentially mixing, with mixing rates τ, τ_ε, uniformly bounded away from 1.*

Not all the content of Theorems 3.1, 3.2 is new in [BaV]. Weak stochastic stability for quadratic maps was first proved by [KK], for uncountably many parameters, and by [BY1], for a positive measure set of parameters (but see also [Co], where strong stability was already considered). Exponential decay of correlations for (unperturbed) quadratic maps was proved independently by [KN] and [Yo]. See also [Ki] for many other references and general background.

3.1 Towers, co-cycles, and transfer operators

Now we outline the main ingredients in the proof of Theorems 3.1 and 3.2, referring the reader to [BaV] for details. The global strategy is inspired by [BaY], where similar results were obtained for certain uniformly hyperbolic systems.

Here we have to circumvent the lack of hyperbolicity and a first step in this direction is to construct a tower extension $\hat{\varphi}: \hat{I} \longrightarrow \hat{I}$ of $\varphi: I \longrightarrow I$. We fix positive constants $\beta \approx 2\alpha$ and $\delta \ll \alpha$ and then define

- $\hat{I} = \cup_{k \geq 0}(B_k \times \{k\})$, with $B_0 = I$ and B_k being the $e^{-\beta k}$-neighbourhood of $\varphi^k(c)$ for each $k \geq 1$.
- $\hat{\varphi}(x,k) = (\varphi(x), k+1)$ whenever $\varphi(x) \in B_{k+1}$ and either $k \geq 1$ or $k = 0$ with $|x| < \delta$; in all other cases $\hat{\varphi}(x,k) = (\varphi(x), 0)$.

A main point in this construction is that return maps to the "ground floor" $E_0 = B_0 \times \{0\}$ are uniformly expanding:

(a) there is a constant $\lambda > 1$ such that $|(\varphi^k)'(x)| \geq \lambda^{2k}$ whenever $(x,0) \in E_0$, $\hat{\varphi}^k(x,0) \in E_0$, and $\hat{\varphi}^i(x,0) \notin E_0$ for $0 < i < k$.

Note also that extensions $\hat{\varphi}_s \colon \hat{I} \longrightarrow \hat{I}$ of the perturbed maps $\varphi_s \colon \hat{I} \longrightarrow I$ can be defined in just the same way.

Next, we introduce a co-cycle $w_0 \colon \hat{I} \longrightarrow [0,\infty)$, given by

- if $(y,k) = \hat{\varphi}^k(x,0)$ for some $x \in B_0$ then $w_0(y,k) = \lambda^k / |(\varphi^k)'(x)|$ (in particular, $w_0 \equiv 1$ on E_0); otherwise $w_0(y,k) = 0$.

This definition ensures that the map $\hat{\varphi}$ is λ-expanding with respect to the metric $w_0\, dx$ on the tower $\hat{I}$: this is automatic at points (x,k) with $\hat{\varphi}(x,k) = (\varphi(x), k+1)$ and (a) implies that it remains true when $\hat{\varphi}(x,k) = (\varphi(x), 0)$. Then we also need a perturbed version w_ε of w_0, which we define by

$$w_\varepsilon(y,k) = \frac{1}{2} \int \frac{\lambda^k}{|(\varphi_{s_k} \circ \cdots \circ \varphi_{s_1})'(x_{s_1 \cdots s_k})|} \theta_\varepsilon(s_1)\, ds_1 \cdots \theta_\varepsilon(s_k)\, ds_k,$$

where the integral is taken over $(\hat{\varphi}_{s_k} \circ \cdots \circ \hat{\varphi}_{s_1})(x_{s_1 \cdots s_k}, 0) = (y,k)$ (the factor $1/2$ is introduced to compensate for the noninjectiveness of $\hat{\varphi}_{s_1}$ on E_0).

Now we define transfer operators $\mathcal{L}_0$ and $\mathcal{L}_\varepsilon$, associated to $\hat{\varphi}$ and its random perturbations $\hat{\varphi}_s$, $|s| \leq \varepsilon$,

$$\mathcal{L}_0(\hat{f})(y,k) = \sum_{\hat{\varphi}(x,l)=(y,k)} \frac{w_0(x,l)}{w_0(y,k)} \frac{\hat{f}(x,l)}{|\varphi'(x)|}$$

$$\mathcal{L}_\varepsilon(\hat{f})(y,k) = \int \sum_{\hat{\varphi}_s(x,l)=(y,k)} \frac{w_\varepsilon(x,l)}{w_\varepsilon(y,k)} \frac{\hat{f}(x,l)}{|\varphi_s'(x)|} \theta_\varepsilon(s)\, ds$$

acting on the Banach space $BV(\hat{I})$ of functions $\hat{f} \colon \hat{I} \longrightarrow \mathrm{R}$ such that

$$\left\| \hat{f} \right\|_{BV} = \operatorname{var} \hat{f} + \sup |\hat{f}| + \int |\hat{f}| w_0\, dx < \infty$$

($\operatorname{var} \hat{f}$ denotes the total variation of $\hat{f}$ on $\hat{I}$, that is, the sum of the variations on each $B_k \times \{k\}$). Note that (both for $\varepsilon = 0$ or for $\varepsilon > 0$) we have the duality relation

(b) $\int \hat{U}_\varepsilon(\hat{f}) \cdot g\, w_\varepsilon\, dx = \int \hat{f} \cdot \mathcal{L}_\varepsilon(\hat{g})\, w_\varepsilon\, dx$

($\hat{U}_\varepsilon$ is defined in the same way as U_ε, with φ replaced by $\hat{\varphi}$). The main analytic step is to prove that

(c) $\mathcal{L}_0$ bounded and quasi-compact on $BV(\hat{I})$;

(d) $\mathcal{L}_\varepsilon$ is "close" to $\mathcal{L}_0$ (in a convenient sense, which we borrow from [BaY]) if ε is small.

Here quasi-compactness means that the spectrum of $\mathcal{L}_0$ splits as $\sigma(\mathcal{L}_0) = \{1\} \cup S_0$ with S_0 contained in a disk of radius $\tau < 1$. Then the closeness in (d) implies that, for all small ε, the operator $\mathcal{L}_\varepsilon$ is also quasi-compact: $\sigma(\mathcal{L}_\varepsilon) = \{1\} \cup S_\varepsilon$ and S_ε contained in a disk of radius τ_ε bounded away from zero (uniformly in ε).

The proof of (c) and (d) relies on a delicate analysis of the action of the transfer operators on the L^1-norm, the supremum, and the variation (inequalities of Lasota-Yorke type), which falls outside the scope of this sketch. On the other hand, once these spectral properties have been derived the conclusions of our theorems follow through fairly standard arguments.

Indeed, if ρ_0 is an eigenfunction of $\mathcal{L}_0$ associated to the eigenvalue 1 then $\hat{\mu}_0 = \rho_0 w_0 \, dx$ is an invariant measure for $\hat{\varphi}$ (use (b)). We normalize ρ_0 so that $\hat{\mu}_0$ be a probability and this defines ρ_0 uniquely. Then the absolutely continuous invariant probability measure of φ is given by $\mu_0 = p_*(\hat{\mu}_0)$, where $p: \hat{I} \longrightarrow I$ is the canonical projection. Similar statements hold for $\rho_\varepsilon, \mathcal{L}_\varepsilon, w_\varepsilon, \hat{\mu}_\varepsilon, \mu_\varepsilon$. Moreover, the fact that $\mathcal{L}_\varepsilon$ is close to $\mathcal{L}_0$ in the sense of [BaY] ensures that ρ_ε is close to ρ_0 in L^1-norm. From this one deduces that μ_ε is close to μ_0, as claimed in Theorem 3.1.

In order to prove Theorem 3.2 one uses the fact that (both for $\varepsilon = 0$ or $\varepsilon > 0$) the spectral projection π_ε associated to S_ε is given $\pi_\varepsilon(f) = f - \rho_\varepsilon \int f w_0 \, dx$. As a consequence of this and the duality (b),

$$\int \hat{U}_\varepsilon^t(\hat{f}) \cdot \hat{g} \, d\hat{\mu}_\varepsilon - \int \hat{f} \, d\hat{\mu}_\varepsilon \int \hat{g} \, d\hat{\mu}_\varepsilon = \int \hat{f} \mathcal{L}_\varepsilon^t(\pi_\varepsilon(\hat{g}\rho_\varepsilon)) w_\varepsilon \, dx$$

and, since $\mathcal{L}_\varepsilon$ acts as a τ_ε-contraction on $\pi_\varepsilon(BV(\hat{I}))$, this proves exponential mixing (with mixing rate bounded by τ_ε) at the tower level, for test functions in $BV(\hat{I})$. Finally, exponential mixing in $BV(I)$, as claimed in the theorem, is easily deduced by lifting bounded variation functions $f, g: I \longrightarrow \mathbb{R}$ to $BV(\hat{I})$ via the canonical projection $p: \hat{I} \longrightarrow I$.

4 Destruction of Lorenz attractors (joint with S. Luzzatto)

In this section we describe an extended geometric model for the dynamics of the system of differential equations

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}$$

introduced by Lorenz [Lo]. Numerical analysis of this system for parameter values $\sigma \approx 10$, $b \approx 8/3$, and $r \approx 28$ led Lorenz to identify sensitive dependence to initial points as a main source of unpredictability in deterministic dynamical systems.

A rigorous description of the dynamics for these parameter values remains a challenging open problem to the present day, although some limited facts can be proved by classical methods. For instance, it is easy to see that, for all parameter values, there exists a singularity (equilibrium point) at the origin. Also, using the theory of Lyapunov functions one can find a (large) neighbourhood of this singularity which all trajectories enter and never leave. Since Lorenz equations are dissipative, this implies that there exists a compact invariant set Λ of zero Lebesgue measure containing the omega-limit sets of all trajectories. However it seems hard to prove any specific properties of this attractor (see [Sp] for a thorough discussion of numerical studies and classical approaches).

Results in this direction include [Ro], [Ry], where the existence of a "strange" attractor was proved for systems of (cubic) differential equations similar to that of Lorenz. Also, rigorous computer assisted proofs have been announced recently, see e.g. [HT], concerning the existence of "chaotic" sets of trajectories in Lorenz equations. Such sets do not seem, in general, to be attracting and thus, even though their presence is very significant both from a mathematical and an experimental viewpoint, they only concern a set of trajectories of zero Lebesgue measure.

In fact, a large share of what we believe to know about chaotic behaviour in Lorenz equations comes from the study of geometric models. These were first introduced in [ABS], [GW], to try to describe the dynamics for the parameter values considered by Lorenz himself. Numerical studies of those equations indicate the presence of a nontrivial attractor containing the singularity at the origin and with strongly hyperbolic behaviour (exponentially contracting and exponentially expanding directions transverse to the flow). The papers mentioned above describe flows exhibiting attractors which do have these properties, and prove rigorous results on the corresponding topological, geometrical, and dynamical features. More recent results [BS], [Bu], [Pe1], [AP], [Pe2], [Sa] have built an extensive theory of such *generalized hyperbolic attractors*. Even more recently, [ACL], [Mo], [MP], provided a fairly detailed picture of the way these attractors can be formed already at the boundary of Morse-Smale flows.

Let us note that the methods used to study generalized hyperbolic systems are nontrivial generalizations of those developed for uniformly hyperbolic systems without singularities (e.g. geodesic flows on manifolds of negative curvature). Indeed the presence of a singularity constitutes an intrinsic obstruction to the existence of a uniform hyperbolic structure: since the hyperbolic decomposition $E^s \oplus E^u \oplus E^0$ of the tangent space at regular points must include a neutral direction E^0 tangent to the flow, which has no analog at singularities, such a decomposition can never be continuous on invariant sets containing regular trajectories accumulating at a singularity. This lack of a hyperbolic structure has more serious consequences than one might expect: the dynamics of flows is very often analysed in terms of Poincaré return maps to convenient cross-sections, however several features of smooth uniformly hyperbolic systems, like local product structure or continuous foliations by stable or unstable leaves, do not exist in general for such maps. This is related with the fact that the presence of singularities in the vector field naturally translates in the form of discontinuities for the return maps (e.g. at the intersection of the cross-section with the local stable manifold of some singularity). As a consequence, global invariant (stable or unstable) sets are, in general, not connected and local invariant manifolds may have arbitrarily small size. An additional complication, which affects the ergodic properties of the attractor, is that the contraction and expansion rates are unbounded as the discontinuity is approached.

In the geometric models, and in most of the papers we mentioned above, these problems were partly overcome by assuming the existence of a smooth invariant stable foliation transverse to the flow. This is also the case in [Rv], which exhibited the first examples of attractors containing sungularities and with measure-theoretical (but not full) persistence. This hypothesis permits to reduce the analysis of the flow to that of a one-dimensional map and, in this way, to deduce several strong results (e.g. ergodicity of the attractor) from their one-dimensional analogs. However, this strategy breaks down in many other important situations, such as the one we want to consider here and which we describe in detail in the next section: the geometry of the problem (more precisely, the presence of criticalities) obstructs the existence of invariant foliations with any reasonable degree of regularity.

4.1 Critical and singular dynamics in Lorenz equations

The first part of Figure 2 is well-known: it describes the image of certain Poincaré return maps associated to the geometric models of [ABS], [GW]; the features of these return maps are coherent with the numerical data concerning Lorenz equations, for the original parameters of Lorenz. Subsequent numerical analysis of

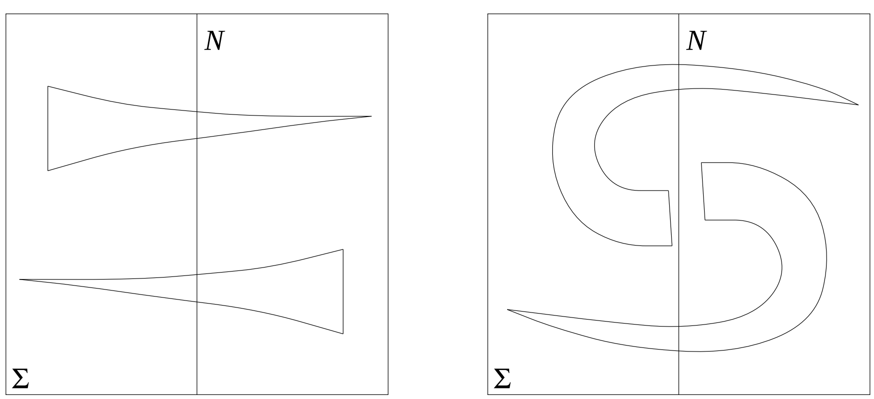

Figure 2: Formation of criticalities in the return map

these equations, [Sp], [GS], revealed that as the parameter r is increased to values of around 30 the flow begins to twist and fold in such a way that the image of the return map becomes as shown in the righthand half of Figure 2: it consists of two "hooks". See [GS] for an interpretation of this folding effect.

The new picture strongly suggests that the dynamics now contains *criticalities* (that is, nontransverse intersections between stable and unstable leaves) which, as we already mentioned, constitute a definite obstruction to the existence of regular invariant foliations (or of any uniform hyperbolic structure). The consequences of the loss of hyperbolicity due to the creation of such criticalities have been and continue to be the object of intense study, see [PT] and references therein for a presentation of the rich theory of homoclinic tangencies and a detailed study of the various dynamical phenomena occurring in their unfolding.

Flows with the characteristics above were first considered by [HP], [He], who introduced a family of smooth plane diffeomorphisms (the famous Hénon family) as a simplified model for the first return maps of the flow. These diffeomorphisms exhibit dynamical features arising from the presence of criticalities, without the additional complexity coming from the presence of a singularity. Notwithstanding this simplification, Hénon maps have been remarkably difficult to study rigorously. A major breakthrough occurred with the work of [BC2] in which new ideas and techniques were introduced to prove the existence and (measure-theoretical) persistence in the Hénon family of nontrivial attractors containing a dense orbit with a positive Lyapunov exponent. This result was generalized to strongly dissipative quadratic-like diffeomorphisms in [MV], [Vi] and the existence of SRB-measures for these attractors was proved in [BY2].

Our objective here is to recover the original project of Hénon-Pomeau and to develop a model for the dynamics exhibited by the Lorenz equation in the region of parameter values in which both criticalities and singularities are present. We define a class of one-parameter families of vector fields which exhibit, for a certain range of parameter values, generalized hyperbolic attractors as discussed above. As the parameter is varied a sequence of bifurcations takes place through which criticalities are formed. Beyond this sequence of bifurcations we encounter attractors in which features deriving from the presence of a singularity coexist with features related to the presence of criticalities. We study the way in which these two dynamical phenomena interact and show that a form of hyperbolicity remains, in a measure-theoretically persistent way.

Theorem 4.1 *There exists an open set of families $\{\mathcal{X}_a\}$ of smooth vector fields in $\mathbb{R}^3$ with the following properties. Let $(\varphi_a^t)_t$ denote the flow generated by the vector field $\mathcal{X}_a$. Then there exists a set $\mathcal{A}$ of positive Lebesgue measure in parameter space such that for each $a \in \mathcal{A}$ the flow $(\varphi_a^t)_t$ exhibits an attractor Λ_a and*

1. *Λ_a contains an equilibrium point with real eigenvalues and an infinite number of critical trajectories (consisting of criticalities).*

2. *Λ_a is transitive and (nonuniformly) hyperbolic in the following sense: there*

exists a point $z \in \Lambda_a$ *and a vector* $v \in T_z\mathbb{R}^3$ *such that*

$$\mathit{closure}\,(\bigcup_{t\geq 0} \varphi_a^t(z)) = \Lambda_a \quad \mathit{and} \quad \limsup_{T\to\infty} \frac{1}{T} \log \|D\varphi_a^T(z)v\| > 0.$$

A final remark is in order, concerning an important difference between this theorem and the kind of results discussed above for generalized (or even uniformly) hyperbolic attractors. There, some form of hyperbolicity is assumed *a priori* and the effort then goes into proving that various dynamical properties follow from this hyperbolicity. In the presence of criticalities, however, one can expect a wide variety of dynamical behaviour, including periodic and quasi-periodic attractors, which occur intermittently alongside each other. For the theorem presented above we make some assumptions on the geometry of the flow and from this we draw the conclusion that there are indeed many parameter values for which a certain form of hyperbolicity exists. This is clearly a first fundamental step towards a more detailed analysis of the dynamical properties of the attractor.

4.2 Recurrence control yields positive Lyapunov exponent

The proof of Theorem 4.1 consists of two main parts. First we give a condition which implies the existence of an attractor with a positive Lyapunov exponent along certain "critical" orbits. Then we show that this condition is satisfied for a positive measure set of parameters and that for most of these parameters some critical orbit is dense. In this brief outline we shall concentrate on the first part, which already contains several interesting aspects from the point of view of the dynamics. A detailed proof is to appear in [LV1], [LV2].

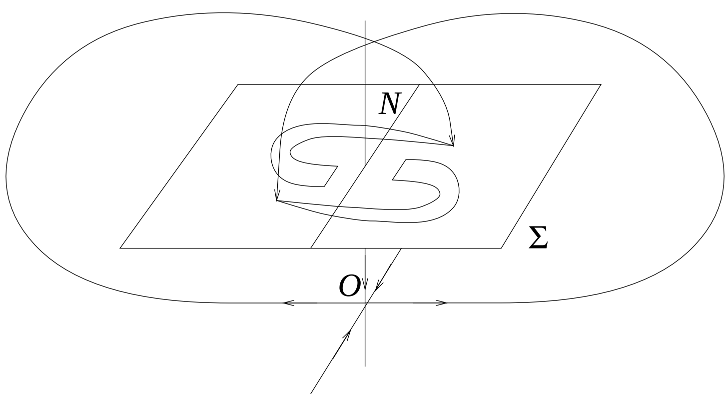

Figure 3: Defining the first-return map

Let $\{\mathcal{X}_a\}$ be a smooth family of vector fields in $\mathbb{R}^3$ with a singularity at the origin having real eigenvalues $\lambda_{ss} < \lambda_s < 0 < \lambda_u$ such that $|\lambda_s/\lambda_u| < 1/2$

and $|\lambda_{ss}/\lambda_u| > 1$. We define a first return map Φ_a to a horizontal cross-section $\Sigma \subset \{z = \epsilon\} \subset \mathbb{R}^3$ associating to each point $(x, y) \in \Sigma$ the point $\Phi_a(x, y) \in \Sigma$ given by the first intersection with Σ of the positive semi-trajectory $\{\varphi_a^t(x, y, \epsilon)\}_{t>0}$, see Figure 3. Notice that Φ_a is not defined on $N = \Sigma \cap W^s_{loc}(0)$, since points belonging to this set (which we assume to be given by $\{x = 0\} \subset \Sigma$) never leave a neighbourhood of the singularity and thus never return to intersect Σ. On the other hand, we always assume that $\Phi_a(\Sigma \setminus N) \subset \Sigma$, to guarantee the existence of some attracting set to which most orbits converge. Then we can write $\Phi_a = \Psi_a \circ P$ where Ψ_a is a diffeomorphism, corresponding to the holonomy of the flow far from the singularity, and P describes the holonomy of the flow near the origin. Ψ_a is responsable for the "folding" which we discussed above and which gives rise to the formation of criticalities. If one takes the flow to be locally linearizable then P has a simple explicit expression

$$P(x, y) = (|x|^{|\lambda_s/\lambda_u|} sgn(x), y|x|^{|\lambda_{ss}/\lambda_u|})$$

Note that the presence of the singularity yields some strong (local) hyperbolicity. Indeed, $|\lambda_s/\lambda_u| < 1$ gives $|\partial_x P_1| \approx |x|^{(|\lambda_s/\lambda_u|-1)} \to \infty$ as $x \to 0$, corresponding to a powerful horizontal expansion. Similarly, $|\partial_y P_2| \approx |x|^{|\lambda_{ss}/\lambda_u|-1} \to 0$ as $x \to 0$, corresponding to a strong contraction in the vertical direction.

Our strategy to obtain measure-theoretical persistence of positive Lyapunov exponents is very much inspired by [BC2], but we also have to deal with a difficulty which has no analog in the smooth case: controlling the recurrence of trajectories to the vicinity of the discontinuity N of Φ_a. Let us explain how this is done.

For the time being, the parameter a is fixed. Suppose that a certain set $\mathcal{C}$ of critical points is well defined for Φ_a (these lie roughly in the preimages of the folds and correspond to special points of nontransverse intersection between stable and unstable leaves). We fix some small neighbourhood $\Delta = \Delta_c \cup \Delta_0$ of the critical set $\mathcal{C}$ and of the line of discontinuity $\{x = 0\}$. Let $\{c_i\}_{i=0}^{\infty}$ be the orbit of some critical value $c_0 \in \Phi_a(\mathcal{C})$. According to a procedure which will be briefly recalled below, every iterate i is either *free* or *bound*. We say that ν is a *return* if $c_\nu \in \Delta$. A return is either free or bound according as to whether ν is a free iterate or a bound iterate. If ν is a return and $c_\nu \in \Delta_0$ we let $|||c_\nu|||$ denote the distance between c_ν and the line of discontinuity, if $c_\nu \in \Delta_c$ then we let $|||c_\nu|||$ denote the distance between c_ν and some critical point $\tilde{c} \in \mathcal{C}$ chosen so that the straight line joining c_ν and $\tilde{c}$ is essentially horizontal ("tangential position" [BC2]).

The key condition for controlling the recurrence to both the critical region and the singular region is the following. We fix some small $\alpha > 0$ and then we say that a critical point c satisfies condition $(*)$ if

$$\sum_{\nu \le n} -\log |||c_\nu||| \le \alpha n \quad \text{for all } n \ge 1,$$

where $c_\nu = \Phi_a^{\nu+1}(c)$ and the sum is taken over all *free* returns $\nu \le n$. We also denote $w_j(c_0) = D\Phi_a^j(c_0) \cdot (1, 0)$. Then we have, if $\alpha > 0$ is small enough,

Proposition 4.2 *If all the critical points $c \in \mathcal{C}$ satisfy condition $(*)$, then there exists a constant $\lambda > 0$ such that*

$$\|w_n(c_0)\| \geq e^{\lambda n} \quad \text{for all } n \geq 1 \text{ and all } c_0 \in \Phi_a(\mathcal{C}).$$

The proof of this fact relies on the decomposition of the orbit of each critical point into free and bound iterates mentioned above. Let a critical point c be fixed and ν be the first time that $c_\nu \in \Delta_c$. Then the first ν iterates are free, by definition. A crucial lemma says that during these iterates the vectors $w_j(c_0)$ are growing exponentially fast: there is $\lambda_0 > 0$ such that $\|w_j(c_0)\| \geq e^{\lambda_0 j}$ for all $j \leq \nu$. This expresses the fact that Φ_a has some hyperbolicity outside Δ_c: vectors which are roughly horizontal remain roughly horizontal and are expanded exponentially. The problems begin precisely when points fall into Δ_c, since there Φ_a is strongly contracting in all directions and, possibly even worse, rotates tangent vectors. This last fact means that $w_{\nu+1}(c_0)$ is likely to have large slope, making it difficult to control its growth during following iterates. Indeed, the hyperbolicity of Φ_a in the complement of Δ_c means that nearly horizontal vectors are expanded, but vectors which are almost vertical tend to be sharply contracted. The key to bypassing this effect is condition $(*)$, which implies $|||c_\nu||| \geq e^{-\alpha\nu}$, thence prevents the return from occurring too close to the critical set $\mathcal{C}$. This allows us to control the norm and the slope of $w_{\nu+1}(c_0)$ and to prove the following estimate:

Proposition 4.3 *There exist $\beta > 0$ and $p \approx \log |||c_\nu|||$ such that*

$$\|w_{\nu+p}(c_0)\| = \|D\Phi_a^p(c_\nu)w_\nu(c_0)\| \geq e^{\beta p}\|w_\nu(c_0)\|.$$

Moreover, the vector $w_{\nu+p}$ is roughly horizontal.

All iterates contained in the interval $[\nu+1, \nu+p)$ are called bound iterates. Starting at $\nu + p$ we then have a sequence of free iterates which continues until the next return to Δ_c. After this return another interval of bound iterates starts, followed by more free iterates, and so on. The above estimates are actually valid for each interval of free or bound iterates, respectively, and we get $\|w_n(c_0)\| \geq e^{\lambda_0 Q + \beta P}$ for $n \geq 1$, where P is the total number of bound iterates less than n and $Q = n - P$ is the number of free iterates. If the bound iterates correspond to returns $\nu_1, \ldots, \nu_s$ then by Proposition 4.3 and $(*)$ we have $P = \sum_{i=1}^s p_i \approx \sum_{i=1}^s -\log |||c_{\nu_i}||| \leq \alpha n$, which implies that $Q = n - P \geq (1 - \text{const}\,\alpha)n$. From this we easily get

$$\|w_n(c_0)\| \geq e^{\lambda_0 Q + \beta P} \geq e^{\lambda_0(1-\text{const}\,\alpha)n} \geq e^{\lambda n}$$

for some $0 < \lambda < (1 - \text{const}\,\alpha)$, if α is small enough. This completes a heuristic outline of the proof of Proposition 4.2.

The second part of the proof of the theorem consists of an algorithm for excluding parameters for which condition $(*)$ fails. Though we shall not go into any detail concerning this algorithm, let us make a couple of brief remarks. Our estimates to guarantee that a positive measure set of parameters survives all the exclusions depend heavily on a global uniform bound on the distortion. This can be obtained only by controlling the recurrence of critical points near the discontinuity as well as in the critical region and that is one of the reasons why returns to Δ_0 must also be taken into account in $(*)$. A second remark is that, just as in the Hénon case, the set of critical points which we have implicitly assumed in the proposition is not given a priori but rather needs to be constructed by successive approximations. This construction proceeds alongside the algorithm for excluding parameters and even depends on it in the sense that it can be carried out at each iteration n only for those parameters which are not excluded up to that time. So, eventually, a set $\mathcal{C}$ of critical points is well defined only for those parameters for which all critical points satisfy $(*)$ at all times $n \geq 1$.

Finally, to obtain the statement in the theorem we need to show that the exponential growth of the vectors $w_j(c_0)$ for the map Φ_a implies exponential growth of the $w_j(c_0)$ also with respect to the (continuous) flow $(\varphi_a^t)_t$. The problem here is that return times to Σ are unbounded as points approach the discontinuity and so a given expansion may be distributed along longer and longer time intervals. In principle, this could give rise to a strictly subexponential growth for the flow but the control over the recurrence of critical points near the discontinuity provided by condition $(*)$ allows us to show that this is not the case. Some difficulty arises from the fact that we have to worry about *all* (both free and bound) returns to Δ_0, since all of them give rise to large return times, whereas $(*)$ only commits explicitly the free returns. However, using the fact that every bound return to Δ_0 occurs during bound periods associated to free returns to Δ_c, one can show that

$$\sum_{i:c_i \in \Delta_0} -\log |||c_i||| \leq \text{ const} \sum -\log |||c_i||| \leq \text{ const}\, \alpha n$$

where the second sum is taken over all free returns. Now this allows us to deduce that the total contribution to the return times corresponding to bound returns is dominated by that of the free returns. Indeed, let $t(z)$ denote the return time for the point z and let t_0 be the supremum of return times for points outside Δ_0. If $|z|$ denotes the distance of z to the discontinuity, then $t(z) \approx \log |z|$, by straightforward computation. Thus, for each $c_0 \in \Phi_a(\mathcal{C})$, and corresponding to each iterate $n \geq 1$ of the return map, we have a "continuous flow time"

$$T_n = \sum_{i=0}^{n-1} t(\Phi_a^i(c_0)) \leq \sum_{i:c_i \notin \Delta_0} t_0 + \sum_{i:c_i \in \Delta_0} t(c_i) \leq n t_0 + \text{ const}\, \alpha n \leq \gamma n$$

for some constant $\gamma > 0$. Then $T_n^{-1} \log \|w_n\| \geq (\lambda n / \gamma n) = (\lambda/\gamma) > 0$, which implies the desired result.

5 Global spiral attractors

Our goal in this section is to prove that "chaotic" attractors with spiraling geometry occur, even in a measure-theoretically persistent way, in certain families of vector fields. This corresponds to very recent joint work with M. J. Pacifico and A. Rovella. The possible existence of spiral attractors seems to have been first mentioned by Ya. Sinai. Our results are motivated by the observations in [ACT] for which, in particular, they provide rigorous confirmation.

5.1 Saddle-focus connections

We consider smooth flows $(\varphi^t)_{t\in\mathbf{R}}$ in 3-dimensional space exhibiting a double saddle-focus homoclinic connection. By this we mean the following, see Figure 4. The flow has an equilibrium point O, at the origin say, which is of saddle-focus type: one expanding eigenvalue $\theta > 0$ and two complex contracting eigenvalues $\lambda \pm \omega i$, where $\lambda < 0$ and $\omega \neq 0$. Moreover, both unstable separatrices of O are contained in the stable manifold of O, that is, they are homoclinic trajectories.

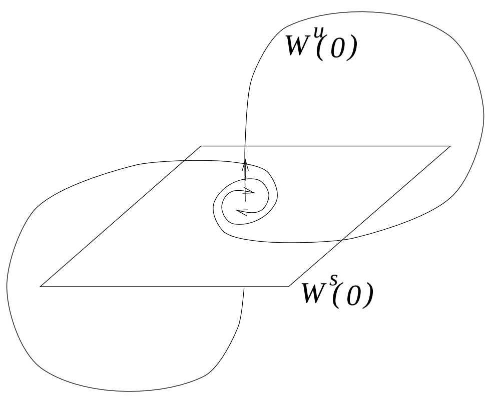

Figure 4: Double saddle-focus connections

For simplicity we assume that the flow is symmetric with respect to the origin, i.e. invariant under $(x, y, x) \longrightarrow (-x, -y, -z)$, but this is not strictly necessary for what follows. Furthermore, a convenient reformulation of our results holds when there is a single homoclinic connection, cf. comments below. These results also extend in straightforward way to general 3-dimensional manifolds. Generalization to higher dimensions was not yet carried out but seems a realistic task and is, certainly, an interesting one.

We want to describe the typical asymptotic behaviour of points in a neighbourhood of the homoclinic connections, not only for the initial flow $(\varphi^t)_t$ but also for "generic" nearby flows. More precisely, we consider smooth parametrized families of flows $(\varphi^t_\mu)_{t\in\mathbf{R}}$, $\mu \in (-\delta, \delta)$, generically unfolding the homoclinic connections: $(\varphi_0^t)_t = (\varphi^t)_t$ and as the parameter μ varies the unstable separatrices move with nonzero speed with respect to the (local) stable manifold. Then we want to describe the attractors of such flows, close to the unstable separatrices, for a sizable portion of parameter values.

The answer to this problem depends crucially on the relative strength of the contracting and the expanding eigenvalues, that is on the value of $\alpha = -\lambda/\theta$. The *contracting case* $\alpha > 1$ is comparatively simple. The union A_0 of the two homoclinic connections is an attracting set for the unperturbed flow $(\varphi_0^t)_t$, with basin containing a neighbourhood of A_0. Moreover, this attracting set is, in some sense, persistent: varying μ leads to the formation of attracting periodic orbits close to (and with basins containing) the unstable separatrices. Therefore, periodic asymptotic behaviour is typical for small parameter values.

The dynamics is much richer in the *expanding case* $\alpha < 1$, as was already attested by the pioneer work of Shil'nikov [Shi]: he proved that infinitely many periodic orbits of sadlle type (contained in suspended horseshoes) coexist in this situation. The main result in the present section states that, under convenient assumptions to be described below, *for a large (positive Lebesgue measure) set of values of* μ *the flow* $(\varphi_\mu^t)_t$ *admits a unique* (global) *attractor* A_μ, *close to* A_0. Moreover, this attractor is *chaotic* (sensitive dependence on initial conditions) and *singular* (contains the equilibrium point as well as regular trajectories) and has an intrincate *spiraling geometry*.

Before going into discussing this result, let us point out that some complex dynamical phenomena are also present in the case $\alpha = 1$. Recently, Pumariño [Pu] used this context to give examples of coexistence of suspended Hénon-like attractors: he even finds parameter values for which infinitely many such attractors occur simultaneously.

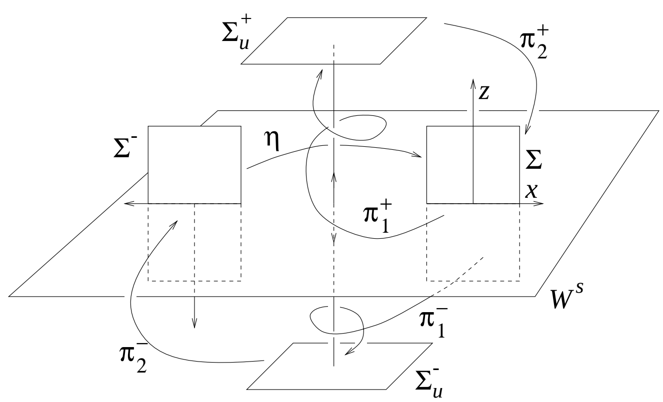

Figure 5: Constructing the first-return map

From now on we restrict to the case $\alpha < 1$. In order to analyse the dynamics of the flow in the vicinity of the homoclinic connections we follow the standard procedure of considering the first-return map F to some convenient cross-section Σ. We take Σ intersecting one of the homoclinic trajectories and the local stable manifold of O; we also consider auxiliary cross-sections Σ^-, symmetric to Σ, and $\Sigma_u^\pm$, intersecting the two local unstable separatrices. Then $\pi_1^\pm$, $\pi_2^\pm$, and η denote corresponding Poincaré maps, as indicated in Figure 5, and we let

$$F(x,z) = \begin{cases} \pi_2^+ \circ \pi_1^+(x,z) & \text{if } z > 0 \\ \eta \circ \pi_2^- \circ \pi_1^-(x,z) & \text{if } z < 0 \end{cases}$$

(F is not defined on the line $\{z = 0\}$ of intersection between Σ and the local stable manifold of the singularity). We assume our flows to be linearizable near the equilibrium (a generic condition), so that it is easy to calculate $\eta, \pi_1^\pm$ explicitly and to see that the images of $\pi_1^\pm$ are spiraling regions in $\Sigma_u^\pm$ accumulating at the points $W^u_{loc}(O) \cap \Sigma_u^\pm$. On the other hand, $\pi_2^\pm$ are diffeomorphisms. Then, under open conditions on $\pi_2^\pm$, the image of Σ under F consists of two spiraling regions contained in the interior of Σ, see Figure 6.

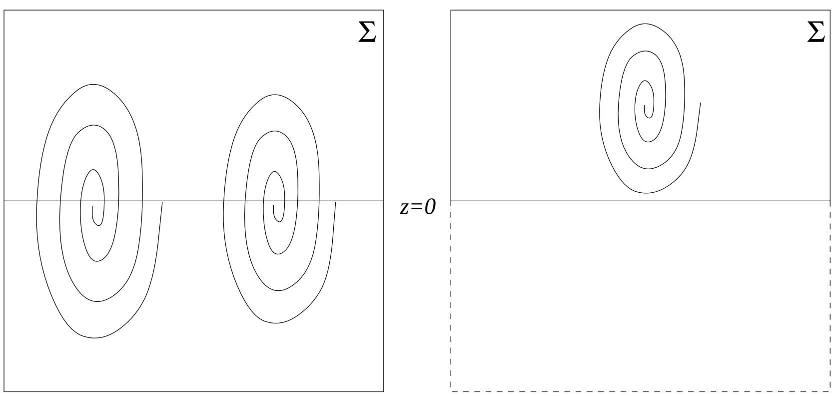

Figure 6: Trapping regions for double and for single connections

Furthermore, this implies that $F_\mu(\Sigma) \subset \Sigma$ for all small μ, where F_μ is the return map to Σ associated with the flow $(\varphi^t_\mu)_\mu$. Then all the asymptotic dynamics of points in this "trapping region" Σ is concentrated inside the maximal invariant set $\Lambda_\mu = \cap_{n\geq 0} F^n_\mu(\Sigma)$. In general, Λ_μ need not be dynamically indivisible, indeed it may contain several different types of dynamical behaviour. However, we have

Theorem 5.1 *For families of vector fields $\{\mathcal{X}_\mu\}$ unfolding double saddle-focus connections as above, there exists a positive Lebesgue measure set $S \subset (-\delta, \delta)$ such that Λ_μ is a (global) chaotic attractor of F_μ for all $\mu \in S$.*

This means, in particular, that for these parameter values Λ_μ contains dense orbits with positive Lyapunov exponent. Of course, the basin of Λ_μ contains the whole section Σ. Successive images $F^n_\mu(\Sigma)$ give increasingly better approximations to the remarkably complex geometry of Λ_μ, recall Figure 6. Note that the suspension $\hat{\Lambda}_\mu = \text{closure}\,(\cup_{t\in\mathrm{R}}\varphi_\mu^t(\Lambda_\mu))$ contains the equilibrium point of the flow, in particular it can not be uniformly hyperbolic; $\hat{\Lambda}_\mu$ also contains criticalities and so it is not even a generalized uniformly hyperbolic attractor, recall Section 4.

Closing this section, we point out that our arguments to prove Theorem 5.1 apply also to the unfolding of flows with a unique saddle-focus connection, e.g. as in [Shi]. A main difference is that in this case one must consider large parameter values, i.e. far from the one for which the homoclinic connection occurs, in order to enforce invariance of Σ under the return map, cf. Figure 6.

5.2 Interval maps with infinitely many critical points

In this final section we give a brief discussion of the difficulties involved in the proof of Theorem 5.1 and of the methods we use to overcome them. A detailed presentation is to appear in [PRV1], [PRV2].

Explicit calculation of the return map along the lines sketched above leads to $F(x,z) = (1 + xg(z), xf(z))$ where

$$f(z) = \begin{cases} b_+|z|^\alpha \sin(\beta \log \frac{1}{|z|}) & \text{if } z > 0 \\ b_-|z|^\alpha \sin(\beta \log \frac{1}{|z|}) & \text{if } z < 0 \end{cases}$$

(we write $\alpha = -\lambda/\theta$, $\beta = \omega/\theta$, and the constants $b_\pm$ depend only on the flow) and g has a similar expression, with sin replaced by cos and $b_\pm$ replaced by constants $a_\pm$. Actually, these statements are accurate only for z close to zero (f must be modified away from the origin to create the trapping region), and then again only as a first-order approximation, but here we will allow ourselves this technical simplification.

A first difficulty arises from the fundamentally higher-dimensional nature of the system. It is not difficult to convince oneself, e.g. observing Figure 6, that F can not admit smooth invariant foliations, and so it can not be reduced in this way to a one-dimensional system; recall Section 4.1. To try to bypass this difficulty we assume the constants $|a_\pm|$ to be small. This is related with the small jacobian hypothesis in [BC2], but we observe that in our context having $a_\pm$ close to zero does not imply volume-dissipativeness (at least not if $\alpha < 1/2$). Then we have $F(1,z) \approx (1, f(z))$, which suggests that the dynamics of F may, to some extent, be mimed by that of the one-dimensional map f. This turns out to be only very roughly true, but it is indeed useful to study a version of our initial problem for such one-dimensional maps. In what follows we concentrate on discussing this simpler version, which is also interesting in itself, without further discussing the (considerable) work required to extend our conclusions back to the original setting to get Theorem 5.1.

More precisely, we want to consider unfoldings of f by parametrized families

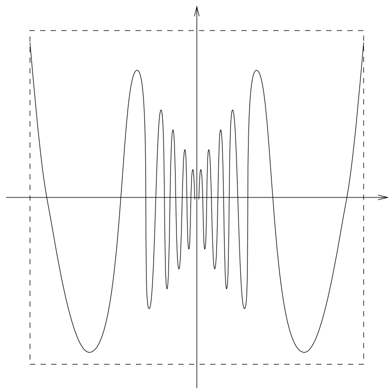

Figure 7: Maps with infinitely many critical points

of transformations of the interval $I = [-1, 1]$ to itself, of the form

$$f_\mu(z) = \begin{cases} f(z) + c_+\mu & \text{if } z > 0 \\ f(z) - c_-\mu & \text{if } z < 0 \end{cases}$$

where $c_\pm$ are positive constants and μ is a parameter close to zero. The way f_μ depends on this parameter is meant to emulate the unfolding of the homoclinic connections by the flows $(\varphi_\mu^t)_t$ or, more precisely, the way the second coordinate of $F_\mu(x, z)$ depends on μ. It is easy to check that $f = f_0$ has two sequences of critical points accumulating at zero, see Figure 7; we denote them by $x_k^\pm$, with $x_k^- < 0 < x_k^+$. Of course, every f_μ has the same set of critical points, and we denote the corresponding critical values by $z_k^\pm(\mu) = f_\mu(x_k^\pm)$. Then we prove that the maps f_μ have global chaotic behaviour in a measure-theoretically persistent way, in the sense of the following theorem.

Theorem 5.2 *There exist $\sigma > 1$ and a positive Lebesgue measure set S of values of μ for which*

1. $|(f_\mu^n)'(z_k^\pm)| \geq \sigma^n$ *for all k and all $n \geq 1$;*
2. *almost every $z \in I$ has positive Lyapunov exponent.*

An important difference between this and similar results for quadratic maps of the interval, [Ja], [BC1], lies on the nature of the initial parameter $\mu = 0$. Indeed, persistent chaotic behaviour for smooth unimodal maps is usually found at parameter values close to one for which the critical point is nonrecurrent (e.g. preperiodic): (almost) nonrecurrence allows the critical orbit to build-up expansion during initial iterates ant then one proceeds by induction to prove that this initial expanding behaviour is preserved in the subsequent iterates, as long as the parameter is chosen conveniently. In contrast, here $\mu = 0$ corresponds to the origin being fixed under the map, and the origin is a particularly nasty point: not only it is a point of nonsmoothness/discontinuity of the dynamics, it is also accumulated by critical points of f, f_μ.

This means that a first main step in the proof of Theorem 5.2 must be devoted to proving that *all the critical orbits do exhibit initial expansion, at least for a large set of parameters*. We fix constants $\tau > 0$, $\varepsilon > 0$, $\gamma \in (\alpha, 1)$ and for each $\mu \in (-\varepsilon, \varepsilon)$ and $z \in (-\varepsilon, \varepsilon)$ we let $j(\mu, z)$ be the smallest iterate j for which $f_\mu^j(z) \notin (-\varepsilon, \varepsilon)$. Then the main ingredient is to show that the set G of parameter values $\mu \in (-\varepsilon, \varepsilon)$ for which every critical value $z = z_k^\pm(\mu)$ satisfies

1. $|f_\mu^{i+1}(z)| \geq |f_\mu^i(z)|$ for $0 \leq i < j(\mu, z)$;
2. $|f_\mu^i(z) - x_l^\pm \geq \tau |x_l^\pm|$ for $0 \leq i < j(\mu, z)$ and every critical point $x_l^\pm$

has almost full measure in $(-\varepsilon, \varepsilon)$ if ε and τ are small. Condition 1 implies that the orbit of $z = z_k^\pm$ moves away from the origin very fast and condition 2 means that while doing it it avoids the neighbourhood of the critical points. We prove that under these assumptions the orbit of z is expanding during the time interval $[0, j(\mu, z))$ it spends near zero.

In a second step we proceed from this set of parameters G, using arguments inspired in [BC1], [BC2]. The main difficulties at this point, with respect to the quadratic case, come from the nonsmoothness of f_μ and, most important, from the fact that it has infinitely many critical points. However, we are able to prove that, for parameters in a positive measure subset of G, all these critical orbits exhibit exponential growth of the derivative at all times, as stated in the theorem.

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