Continuity of Lyapunov exponents

Marcelo Viana
(joint work with C. Bocker and with A. Avila)

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Consider $A_1, \ldots, A_N \in \text{GL}(d)$ and $p_1, \ldots, p_N > 0$ with $\sum_j p_j = 1$. Let $(B_n)_n$ be identical independent random variables in $\text{GL}(d)$ with probability distribution $\mu = \sum_j p_j \delta_{A_j}$. The Lyapunov exponents

$$\lambda_+(\mu) = \lim_{n} \frac{1}{n} \log \|B_n \cdots B_1\|$$

$$\lambda_-(\mu) = \lim_{n} -\frac{1}{n} \log \|(B_n \cdots B_1)^{-1}\|$$

exist almost surely.
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exist almost surely.

**Theorem (Carlos Bocker, MV)**

For $d = 2$, the functions $(A_{i,j}, p_j)_{i,j} \mapsto \lambda_\pm$ are continuous.
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with the following topology:

$\nu$ is close to $\mu$ if $\nu$ is weak*-close to $\mu$ and $\text{supp}\, \nu \subset B_\varepsilon(\text{supp} \, \mu)$.
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"Theorem"

For any $d \geq 2$, the functions $\mu \mapsto \lambda_{\pm}(\mu)$ are continuous on $G(d)$. 
Example

For $A_1 = \begin{pmatrix} 2^{-1} & 0 \\ 0 & 2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we have $\lambda_+ = 0$ if $p_2 > 0$ but $\lambda_+ = \log 2$ if $p_2 = 0$. 

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Mañé-Bochi-V:
For any ergodic measure preserving transformation $f : M \to M$, the continuity points for Lyapunov exponents in the space of $GL(d)$-cocycles over $f$ are very special cocycles: the Oseledets splitting is dominated.
Given $\mu \in G(d)$:

- $L < \mathbb{R}^d$ is $\mu$-invariant if $g(L) = L$ for every $g \in \text{supp } \mu$.
- A probability $\eta$ in $\mathbb{PR}^d$ is $\mu$-stationary if $\int g_* \eta \, d\mu(g) = \eta$. 

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- A probability $\eta$ in $\mathbb{P}\mathbb{R}^d$ is $\mu$-stationary if $\int g_\ast \eta \, d\mu(g) = \eta$.

Equivalently, $\Psi(x, g_1, g_2, \ldots, g_n, \ldots) = (g_1(x), g_2, \ldots, g_n, \ldots)$ preserves $\eta \times \mu^\mathbb{N}$. We call $\eta$ ergodic if $\eta \times \mu^\mathbb{N}$ is ergodic for $\Psi$. 

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Representation of exponents

Given $\mu \in \mathcal{G}(d)$:

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**Theorem (Furstenberg-Kifer)**

There exist $r \geq 0$ and numbers $\beta_0 > \beta_1 > \cdots > \beta_r$ and $\mu$-invariant subspaces $L_0 > L_1 > \cdots > L_r > L_{r+1}$, with $\beta_0 = \lambda_+$ and $L_0 = \mathbb{R}^d$ and $L_{r+1} = 0$, such that for every $v \in L_i \setminus L_{i+1}$ and $0 \leq i \leq r$,

$$\lim_{n \to \infty} \frac{1}{n} \log \| (g_n \cdots g_1)(v) \| = \beta_i \quad \mu^\mathbb{N}\text{-almost surely}.$$
Define $\phi : \text{GL}(d) \times \mathbb{PR}^d \rightarrow \mathbb{R}, \phi(g, v) = \log \left( \frac{\|g(v)\|}{\|v\|} \right)$. Then:

The $\beta_i$ are the possible values of $\int \phi \, d\mu \, d\eta$ when $\eta$ varies in the set of all $\mu$-stationary ergodic measures.

$L_i = \text{largest subspace such that } \eta(L_i) = 0 \text{ for every } \mu\text{-stationary ergodic measure } \eta \text{ with } \int \phi \, d\mu \, d\eta > \beta_i$.

$\int \phi \, d\mu \, d\eta > \beta_i$ for every $\mu\text{-stationary measure } \eta, \text{ ergodic or not, such that } \eta(L_i) = 0$. 

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Examples

Example

For $A_1 = \begin{pmatrix} 3^{-1} & 0 \\ 0 & 3 \end{pmatrix}$, $A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2^{-1} \end{pmatrix}$, $p_1 = p_2 = 1/2$
we have $r = 1$ and $L_1 = X$ - axis and $\beta_1 = \log 2/3$

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we have $r = 0$.

So: $r > 0$ means that there exists some $\mu$-invariant subspace (reducibility) which, in addition, is “mostly contracting”.

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Theorem (Furstenberg-Kifer)

If \( r = 0 \) then \( \mu \) is a continuity point for \( \lambda_+ \).

**Proof:** Given \( \mu_n \to \mu \), take \( \mu_n \)-stationary ergodic measures \( \eta_n \) such that \( \lambda_+(\mu_n) = \int \phi \ d\mu_n d\eta_n \).

Suppose that \( \eta_n \to \eta \). Then \( \eta \) is \( \mu \)-stationary and \( \int \phi \ d\mu_n d\eta_n \) converges to \( \int \phi \ d\mu \ d\eta \).

The hypothesis \( r = 0 \) implies that \( \int \phi \ d\mu \ d\eta = \lambda_+(\mu) \).
A partial result

**Theorem (Furstenberg-Kifer)**

If $r = 0$ then $\mu$ is a continuity point for $\lambda_+$. 

**Proof:** Given $\mu_n \to \mu$, take $\mu_n$-stationary ergodic measures $\eta_n$ such that

$$\lambda_+ (\mu_n) = \int \phi \, d\mu_n d\eta_n.\]

Suppose that $\eta_n \to \eta$. Then $\eta$ is $\mu$-stationary and

$$\int \phi \, d\mu_n d\eta_n \to \int \phi \, d\mu \, d\eta.$$

The hypothesis $r = 0$ implies that

$$\int \phi \, d\mu \, d\eta = \lambda_+ (\mu).$$

If there exists at most one $\mu$-invariant subspace then $r = 0$ either for the cocycle or for its inverse, and the conclusion follows just the same.

What about the general case?

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there are two $\mu$-stationary ergodic measures in $\mathbb{P}\mathbb{R}^2$, namely, the Dirac masses at the $X$-axis and the $Y$-axis. They correspond to different $\beta_j$. 

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Continuity of Lyapunov exponents
Probabilistic repellers

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there are two $\mu$-stationary ergodic measures in $\mathbb{PR}^2$, namely, the Dirac masses at the $X$-axis and the $Y$-axis. They correspond to different $\beta_j$.

The invariant subspace $L_1 = X$-axis is a probabilistic repeller. The ideology of the proof is that such probabilistic repellers should be unstable under most perturbations of the probability distribution $\mu$. 
"Theorem"
Suppose $r > 0$. For every $\epsilon > 0$ there is $\delta > 0$ and a neighborhood $V \subset \mathcal{G}(d)$ of $\mu$ such that for every $\nu \in V$ and every $\nu$-stationary ergodic measure $\eta$, either $\eta(B_\delta(L_1)) < \epsilon$ or $\eta(B_\delta(L_1)) = 1$.

Idea: in the last case, $\eta$ is not a candidate for realizing $\lambda_+(\nu)$. 
Instability of probabilistic repellers

“Theorem”

Suppose $r > 0$. For every $\epsilon > 0$ there is $\delta > 0$ and a neighborhood $V \subset G(d)$ of $\mu$ such that for every $\nu \in V$ and every $\nu$-stationary ergodic measure $\eta$, either $\eta(B_\delta(L_1)) < \epsilon$ or $\eta(B_\delta(L_1)) = 1$.

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The original Bocker-V approach in $d = 2$ is based on a careful discretization of the phase space $\mathbb{PR}^2$.

With Artur Avila, we have been trying with a more direct analysis of the random walk in continuum space, based on certain energy estimates.
A nonlinear setting

Let $M$ be a compact Riemannian manifold (examples: $\mathbb{P}\mathbb{R}^d$, Grassmannian manifolds) and $\mathcal{M}$ be the space of probability measures on $M$.

Let $G < \text{Diff}^1(M)$ (e.g. $G = \text{GL}(d)$) and $\mathcal{G}$ be the space of compactly supported probability measures on $G$. 

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A point $v \in M$ is $\mu$-invariant if $g(v) = v$ for every $g \in \text{supp} \, \mu$.

Then, $\mu^N$-almost surely, $L(\mu, \dot{v}) = \lim_n \frac{1}{n} \log \|D(g_n \cdots g_1)(v) \dot{v}\|$ exists for every non-zero $\dot{v} \in T_v M$.

We call $v$ $\mu$-expanding if $L(\mu, \dot{v}) > 0$ for every $\dot{v} \neq 0$. 

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Theorem (Artur Avila, MV)
Suppose that $\nu$ is $\mu$-expanding and $(\mu_n)_n$ converges to $\mu$ in $G$. For each $n$, let $\eta_n \in M$ be a $\mu_n$-stationary measure having no atoms in a fixed neighborhood of $\nu$, and assume that $(\eta_n)_n$ converges to some $\eta \in M$. Then $\eta(\{\nu\}) = 0$.

This proves continuity of $\lambda_+$ for all $d \geq 2$ when $\dim L_1 = 1$. 
Given $\beta > 0$, the $\beta$-energy of a measure $\xi$ on $M \times M$ is

$$E_\beta(\xi) = \int d(x, y)^{-\beta} \ d\xi(x, y).$$

The map $\xi \mapsto E_\beta(\xi)$ is lower semicontinuous.
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Let $\eta_1, \eta_2$ be measures on $M$ with $\eta_1(M) = \eta_2(M)$:

A coupling of $(\eta_1, \eta_2)$ is a measure $\xi$ on $M \times M$ that maps to $\eta_j$ on the $j$th coordinate, for $j = 1, 2$.

Given $\beta > 0$, define $e_\beta(\eta_1, \eta_2) = \inf$imum of $\beta$-energy $E_\beta(\xi)$ over all couplings $\xi$. The infimum is attained.
Optimal self-couplings

Given a measure \( \eta \) on \( M \), define \( e_\beta(\eta) = e_\beta(\eta, \eta) \). The infimum is attained at some symmetric self-coupling, that is, one invariant under \((u, v) \mapsto (v, u)\). We call this a \( \beta \)-optimal self-coupling.
Optimal self-couplings

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The energy $e_\beta(\eta)$ is finite iff $\eta$ has no fat atoms:

$$
\eta(\{x\}) < \frac{1}{2} \eta(M) \text{ for every } x \quad \Rightarrow \quad e_\beta(\eta) < \infty
$$

$$
e_\beta(\eta) < \infty \quad \Rightarrow \quad \eta(\{x\}) \leq \frac{1}{2} \eta(M) \text{ for every } x
$$
Optimal self-couplings

Lemma

If $\nu$ is $\mu$-expanding then there exists a neighborhood $V$ of $\nu$, a weak* neighborhood $\mathcal{V}$ of $\mu$ and a constant $c > 0$ such that

$$\int d(g(x), g(y))^{-\beta} \, d\nu(g) < (1 - c\beta)d(x, y)^{-\beta}$$

for every $x \neq y$ in $V$, every $\nu \in \mathcal{V}$ and every small $\beta > 0$. 
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for every $x \neq y$ in $V$, every $\nu \in \mathcal{V}$ and every small $\beta > 0$.

Suppose $\eta(\{\nu\}) > 0$. Fix $U \subset V$ such that $\eta(\{\nu\}) > 0.9 \eta(U)$.

Notice: $e_\beta(\eta \mid U) = \infty$. 
Energy estimates

For each $n$, let $\xi_n$ be a $\beta$-optimal self coupling of $\eta_n \mid U$ and let $\tilde{\xi}_n$ be its push-forward:

$$\tilde{\xi}_n(A \times B) = \int \xi_n(g^{-1}(A) \times g^{-1}(B)) \, d\mu_n(g).$$
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Lemma $\Rightarrow E_\beta(\tilde{\xi}_n) < (1 - c\beta)E_\beta(\xi_n) = (1 - c\beta)e_\beta(\eta_n \mid U)$.

Moreover, $e_\beta(\eta_n \mid U) \leq C + E_\beta(\tilde{\xi}_n)$. 
For each $n$, let $\xi_n$ be a $\beta$-optimal self coupling of $\eta_n | U$ and let $\tilde{\xi}_n$ be its push-forward:

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Moreover, $e_{\beta}(\eta_n | U) \leq C + E_{\beta}(\tilde{\xi}_n)$.

Combining these inequalities: $e_{\beta}(\eta_n | U) \leq C / (c\beta)$ for all $n$.

Then $e_{\beta}(\eta | U) \leq C / (c\beta)$. Contradiction.