

Hyperbolicity and rigidity for fibred partially hyperbolic systems

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Abstract Every volume-preserving accessible centre-bunched fibred partially hyperbolic system with 2-dimensional centre either (a) has two distinct centre Lyapunov exponents, or (b) exhibits an invariant continuous line field (or pair of line fields) tangent to the centre leaves, or (c) admits a continuous conformal structure on the centre leaves invariant under both the dynamics and the stable and unstable holonomies. The last two alternatives carry strong restrictions on the topology of the centre leaves: (b) can only occur on tori, and for (c) the centre leaves must be either tori or spheres. Moreover, under some additional conditions, such maps are rigid, in the sense that they are topologically conjugate to specific algebraic models. When the system is symplectic (a) implies that the centre Lyapunov exponents are non-zero, and thus the system is (non-uniformly) hyperbolic.

Keywords hyperbolicity, rigidity, partially hyperbolic system, volume-preserving diffeomorphism, symplectic diffeomorphism, holonomy map

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1 Introduction

In this paper we investigate the behaviour of the Lyapunov exponents of volume-preserving and symplectic diffeomorphisms under small modifications of the dynamical system. We are especially concerned with the following pair of fundamental questions: *How often can one perturb the diffeomorphism in order to make it (non-uniformly) hyperbolic? Which obstructions are there to the existence of such perturbations?*

In a nutshell, we conclude that non-uniformly hyperbolic systems are prominent in the contexts we deal with and, indeed, diffeomorphisms that are not approximated by non-uniformly hyperbolic ones present very rigid features. Our arguments benefit from a combination of methods, both classical and new, that have much broadened this field of research in the last couple of decades or so.

The concept of Lyapunov exponents originated from the stability theory of differential equations developed by A.M. Lyapunov [48] around the end of the 19th century. Consider a differential equation of the form

$$x' = L(t)x + R(t, x), \quad (1)$$

where $L(t) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a linear map and $R(t, x)$ is a non-linear perturbation of order greater than 1. Fix any t_0 and for each $v \neq 0$ denote by $x_v(\cdot)$ the solution of the *linearised equation* $x' = L(t)x$ with initial condition $x_v(t_0) = v$. The stability theorem of Lyapunov asserts that if the *Lyapunov exponent function*

$$\lambda(v) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|x_v(t)\| \quad (2)$$

is negative for every $v \neq 0$ then, under an additional condition called *Lyapunov regularity*, the solution $x(t) \equiv 0$ is exponentially stable for the equation (1). See Barreira, Pesin [12] for a detailed presentation of this topic.

The flow of a differentiable vector field may always be written in the form (1) around any given trajectory. Furstenberg, Kesten [34] proved that the limit in (2) exists (for all $v \neq 0$) for almost every trajectory, relative to any measure¹ invariant under the flow. A few years later, Oseledets [53] proved that Lyapunov regularity also holds for almost every trajectory, again with respect to any invariant measure. Corresponding facts for discrete-time dynamical systems, such as diffeomorphisms on manifolds, follow easily. These results brought the subject of Lyapunov exponents firmly to the realm of ergodic theory, where it has prospered since.

The next major step, initiated by Pesin [54, 55], was to develop the non-linear theory, namely, to establish that *in the absence of vanishing Lyapunov exponents* the dynamical system must exhibit special geometric features, including the existence of stable and unstable sets that are smooth embedded disks at almost every point. See also Ruelle [61], Fathi, Herman, Yoccoz [32], and Pugh, Shub [56]. Such a geometric structure is at the basis of

¹ In this paper all measures are finite Borel measures.

several deep results about *non-uniformly hyperbolic systems*, that is, whose Lyapunov exponents are non-zero almost everywhere, by Pesin [55], Katok [43], Ledrappier [44], Ledrappier, Young [46], Barreira, Pesin, Schmeling [13], Shub, Wilkinson [62] and others.

All this points at the following central question: *Is every dynamical system approximated by a non-uniformly hyperbolic one?* In other words, can one always get rid of zero Lyapunov exponents by slightly perturbing the dynamical system?

Work of Herman (see Yoccoz [68, Section 4]) implies that the answer may fail to be positive in the context of volume-preserving diffeomorphisms. Also, results of Mañé, Bochi [49, 14] show that vanishing Lyapunov exponents are actually quite common among C^1 area-preserving surface diffeomorphisms. The latter was extended by Bochi, Viana [18, 15] to volume-preserving and symplectic diffeomorphisms in any dimension.

On the other hand, general perturbative techniques have been developed for removing vanishing Lyapunov exponents. See, in particular, Herman [37], Shub, Wilkinson [62], Avila, Bochi [5], Dolgopyat, Pesin [28], Baraviera, Bonatti [11], Bochi, Fayad, Pujals [16], and Marín [52]. For more references and an extended discussion, check Bochi, Viana [17] and Bonatti, Díaz, Viana [19, Section 12.5].

An alternative approach which has been particularly fruitful in recent years is to deal with the phenomenon of zero Lyapunov exponents by means of the so-called Invariance Principle, formulated by Ledrappier [45] and Bonatti, Gomez-Mont, Viana [20] for linear cocycles, and Avila, Viana [7] in the full non-linear setting. Roughly speaking, the Invariance Principle asserts that systems whose Lyapunov exponents do vanish must satisfy certain stringent conditions that can often be excluded a priori for different reasons, for instance, topological. Among the main applications, let us mention Hertz, Hertz, Tahzibi, Ures [60], Avila, Viana, Wilkinson [9, 10], and Viana, Yang [67].

In a related setting of fiber-bunched cocycles over volume-preserving partially hyperbolic diffeomorphisms, Kalinin, Sadovskaya [41, 42] proved that if the extremal Lyapunov exponents coincide then, up to finite covering, there exists a continuous invariant flag whose factor bundles carry continuous invariant conformal structures. More recently, Butler, Xu [27] studied the Lyapunov exponents of partially hyperbolic diffeomorphisms along the stable (respectively unstable) bundle, finding conditions under which the extremal exponents coincide.

Partial hyperbolicity (see Section 2.1 for definitions) provides a particularly convenient context for studying the persistence of zero Lyapunov exponents. On the one hand, in that setting one needs only consider the system's Lyapunov exponents along the centre bundle, E^c , as the exponents along the stable and unstable bundles, E^s and E^u , are clearly bounded from zero. On the other hand, the geometric structure exhibited by partially hyperbolic systems, especially their admitting invariant stable and unstable foliations tangent to E^s and E^u , respectively, makes them particularly suited to the approaches mentioned before, particularly to the application of the Invariance Principle.

In this paper we deal with *fibred partially hyperbolic systems* in the sense of Avila, Viana, Wilkinson [10, Section 3.2] (see also Avila, Viana [8, Section 6]). By this we mean that the diffeomorphism $f : N \rightarrow N$ is partially hyperbolic and there exists an f -invariant continuous fibre bundle $\pi : N \rightarrow B$ whose fibres are C^1 sub-manifolds tangent to the centre bundle E^c of f . In all the cases we shall consider here, the fibres will be modelled after a compact orientable surface S . Moreover, the diffeomorphism will be taken to be accessible and centre-bunched, and to preserve some measure in the Lebesgue class of N .

Our first conclusion is that the topology of the fibre S has profound implications on the dynamics of such maps and, in particular, on their Lyapunov exponents. Indeed, in Theorem A we find that if the genus of S is strictly greater than 1 then the centre Lyapunov exponents of f must be distinct. In particular, symplectic fibred partially hyperbolic systems are necessarily hyperbolic unless $S = \mathbb{S}^2$ or $S = \mathbb{T}^2$. A similar phenomenon was highlighted by Avila, Viana [7, Theorem 6.6] in a different setting.

The arguments in the proof of Theorem A also lead to information on the dynamics of the exceptional diffeomorphisms for which the centre Lyapunov exponents coincide. For this we restrict ourselves to a subclass of fibred partially hyperbolic systems, namely, perturbations of certain partially hyperbolic skew-product maps. By a *partially hyperbolic skew-product* we mean a partially hyperbolic centre-bunched diffeomorphism of the form

$$f_0 : M \times S \rightarrow M \times S, \quad (x, v) \mapsto (g_0(x), \Gamma_x(v)) \quad (3)$$

where $g_0 : M \rightarrow M$ is a transitive Anosov diffeomorphism on a compact manifold M , each $\Gamma_x : S \rightarrow S$ is a diffeomorphism on a compact orientable surface S , and the centre bundle of g coincides with the vertical direction $\{0\} \times TS \subset T(M \times S)$. It is assumed that f_0 preserves a measure μ in the Lebesgue class of $M \times S$, and the results that follow concern the dynamics of nearby μ -preserving maps.

In Theorem C we take S to be the torus \mathbb{T}^2 , and $f_0 : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2$ to be an *elliptic affine extension* of g_0 , that is, a diffeomorphism of the form

$$f_0(x, v) = (g_0(x), L_0 v + w_0(x)), \quad (4)$$

where L_0 is any elliptic element of $\mathrm{SL}(2, \mathbb{Z})$. We prove that every nearby accessible μ -preserving map f which is not hyperbolic must be topologically conjugate to an elliptic affine extension of g_0 , unless it admits some invariant line field or some pair of transverse line fields. We also check that the latter alternative may be excluded in some cases (Corollary D), but not always (Example 3).

In Theorem E we take S to be the sphere \mathbb{S}^2 , and $f_0 : M \times \mathbb{S}^2 \rightarrow M \times \mathbb{S}^2$ to be a *Möbius extension* of g_0 , that is, a diffeomorphism of the form

$$f_0(x, v) = (g_0(x), \zeta_x(v)) \quad (5)$$

where each $\zeta_x(v) = (a_x v + b_x)/(c_x v + d_x)$ is a Möbius transformation, viewed as a diffeomorphism of the sphere via the stereographic projection $P : \mathbb{S}^2 \rightarrow$

$\mathbb{C} \cup \{\infty\}$. Then we prove that if a nearby accessible μ -preserving map f is not hyperbolic then it must be topologically conjugate to a Möbius extension of g_0 .

This manuscript is organised as follows. In Section 2 we give the precise statements of all these results, including the formal definitions of the notions involved. Section 3 collects a few main tools from the literature that are used in the proofs of our results. Theorem A and Corollary B are proved in Section 4. In Section 5 we check that the assumptions of Theorem C are very common. The material in there is not used in the proof, and may be skipped at a first reading. Theorem C is proved in Section 6 and the proof of Corollary D is given in Section 7. Theorem E is proved in Section 8.

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2 Definitions and statement of results

This section contains the statements of our main results, including the precise definitions of the notions involved.

2.1 Partially hyperbolic diffeomorphisms

A diffeomorphism $f : N \rightarrow N$ on a compact manifold N is *partially hyperbolic* if the tangent space TN admits a non-trivial Df -invariant continuous splitting $TN = E^u \oplus E^c \oplus E^s$ such that:

- there are positive continuous functions $\chi, \hat{\chi}, \nu, \hat{\nu}, \gamma,$ and $\hat{\gamma}$ on M satisfying

$$\chi < \nu < 1 < \hat{\nu}^{-1} < \hat{\chi}^{-1} \text{ and } \nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1};$$

- there is a Riemannian norm $\|\cdot\|$ on M such that for any unit vector $v_p \in T_p(M)$, we have

$$\begin{aligned} \chi(p) < \|Df_p(v_p)\| < \nu(p) & \text{ if } v_p \in E^s(p) \\ \gamma(p) < \|Df_p(v_p)\| < \hat{\gamma}(p)^{-1} & \text{ if } v_p \in E^c(p) \\ \hat{\nu}(p)^{-1} < \|Df_p(v_p)\| < \hat{\chi}(p)^{-1} & \text{ if } v_p \in E^u(p). \end{aligned} \quad (6)$$

E^u and E^s are called, respectively, the *unstable bundle* and the *stable bundle* of f . There exist f -invariant foliations \mathcal{F}^u and \mathcal{F}^s tangent to E^u and E^s , respectively, at every point. Moreover, both the *unstable foliation* \mathcal{F}^u and the *stable foliation* \mathcal{F}^s are unique. See [23,40]. E^c is called the *centre bundle* of f . It need not be integrable, in general. One calls f *dynamically coherent* if there exist f -invariant foliations \mathcal{F}^{cu} and \mathcal{F}^{cs} tangent at every point to $E^u \oplus E^c$ and $E^c \oplus E^s$, respectively. Intersecting their leaves, one obtains an f -invariant *centre foliation* \mathcal{F}^c tangent to E^c at every point. See [35,26]. A

partially hyperbolic diffeomorphism $f : N \rightarrow N$ is said to be *accessible* if any two points p and q in N are connected by some *su-path* in N , that is, some piecewise differentiable oriented curve each of whose (finitely many) legs is contained in a single leaf of either \mathcal{F}^u or \mathcal{F}^s .

It is clear that one may always find $\alpha \in (0, 1]$ such that f is α -*pinched*:

$$\nu < \min\{\gamma\chi^\alpha, \gamma\widehat{\chi}^\alpha\} \text{ and } \widehat{\nu} < \min\{\widehat{\gamma}\chi^\alpha, \widehat{\gamma}\widehat{\chi}^\alpha\}. \quad (7)$$

Then (check Hirsch, Pugh [39, § 6] and Pugh, Shub, Wilkinson [58]), as long as we take f to be C^2 , the centre bundle E^c is α -Hölder. Let α be fixed from now on, uniformly on a neighborhood of f . We call f *centre-bunched* if the functions in (6) and (7) may be chosen to satisfy

$$\nu^\alpha < \gamma\widehat{\gamma} \text{ and } \widehat{\nu}^\alpha < \gamma\widehat{\gamma}. \quad (8)$$

2.2 Lyapunov exponents

The *Lebesgue class* of a compact manifold N is the set of measures on N equivalent to one given by the integration of any volume form (this does not depend on the choice of the form). We call a diffeomorphism $f : N \rightarrow N$ *volume-preserving* if it preserves some measure μ in the Lebesgue class, and we call it μ -*preserving* if we want to specify that measure.

The theorem of Oseledets [53] asserts that for μ -almost every point $p \in N$ there exist $k(p) \in \mathbb{N}$, real numbers $\lambda_1(p) > \lambda_2(p) > \dots > \lambda_{k(p)}(p)$ and a Df -invariant splitting $T_pM = E_p^1 \oplus E_p^2 \oplus \dots \oplus E_p^{k(p)}$ such that

$$\lim_{|n| \rightarrow \infty} \frac{1}{n} \log \|Df_p^n(v_p)\| = \lambda_j(p) \text{ for all non-zero } v_p \in E_p^j.$$

The numbers $\lambda_j(p)$ are called the *Lyapunov exponents* and the E_p^j are the *Oseledets spaces* of f at p . When the system (f, μ) is ergodic the functions $p \mapsto \kappa(p), \lambda_j(p), \dim E_p^j$ are constant on a full μ -measure set. We use $\theta_1(p) \geq \dots \geq \theta_{\dim N}(p)$ to denote the Lyapunov exponents counted with multiplicity: the *multiplicity* of each $\lambda_j(p)$ is the dimension of the subspace E_p^j , and so this means that we have $\theta_i(f, p) = \lambda_j(f, p)$ for $\dim E_p^j$ different values of i . The map f is said to be *non-uniformly hyperbolic* for μ if the set of points where the Lyapunov exponents are all non-zero has full μ -measure.

A *symplectic form* on the manifold N is a closed non-degenerate differential 2-form ω on N . Such a form exists only if the dimension of N is even, and then $\omega^{\dim N/2}$ is a volume form on N . The associated volume measure will be denoted as vol . A diffeomorphism $f : N \rightarrow N$ is said to be ω -*symplectic* if it preserves ω and, consequently, the measure vol . In this setting, the Lyapunov exponents have the following symmetry property: at vol -almost every point,

$$\theta_j(f, q) + \theta_{\dim N + 1 - j}(f, q) = 0 \text{ for every } j = 1, \dots, \dim N/2. \quad (9)$$

If $f : N \rightarrow N$ is partially hyperbolic and symplectic then (see [17, Section 4]) $\dim E^u = \dim E^s$ and $E^u \oplus E^s$ coincides with the symplectic orthogonal of the centre bundle E^c :

$$\omega_p(u_1, u_2) = 0 \text{ for all } u_2 \in E_p^c \iff u_1 \in E_p^u \oplus E_p^s.$$

In particular, the restriction $\omega^c = \omega|_{E^c}$ is non-degenerate at every point. Thus, assuming the diffeomorphism is dynamically coherent, ω^c defines a symplectic form (an area form) on each centre leaf. Clearly, this symplectic structure is preserved by the restriction of f to centre leaves.

2.3 Fibred partially hyperbolic systems

Recall that we call *fibred partially hyperbolic system* on a compact manifold N any partially hyperbolic diffeomorphism $f : N \rightarrow N$ such that there exists an f -invariant continuous fibre bundle $\pi : N \rightarrow B$ whose fibres are C^1 sub-manifolds tangent to the centre bundle E^c of f . Then f is dynamically coherent, with the fibration as a centre foliation. In all the cases we consider, the fibres are modelled after a compact orientable surface S .

Given $r \geq 1$ and a measure μ in the Lebesgue class, we shall denote by $\mathcal{F}_\mu^r(N, S)$ the space of all μ -preserving centre-bunched C^r fibred partially hyperbolic systems. Analogously, given $r \geq 1$ and a symplectic form ω , we shall denote by $\mathcal{F}_\omega^r(N, S)$ the space of all ω -symplectic centre-bunched C^r fibred partially hyperbolic systems. It is clear that $\mathcal{F}_\omega^r(N, S) \subset \mathcal{F}_\mu^r(N, S)$ if $\mu = \text{vol}$ is the volume measure induced by ω .

Example 1 Let $f_0 : M \times S \rightarrow M \times S$ be a C^r partially hyperbolic skew-product, as defined in (3). If f_0 preserves a probability measure μ in the Lebesgue class of $M \times S$ then f_0 is in the interior of $\mathcal{F}_\mu^r(M \times S, S)$ among all μ -preserving maps. Analogously, if f_0 preserves a symplectic form ω on $M \times S$ then f_0 is in the interior of $\mathcal{F}_\omega^r(M \times S, S)$ among all ω -symplectic maps.

Indeed, let $f : M \times S \rightarrow M \times S$ be any diffeomorphism in a small C^r -neighbourhood of f_0 . By normal hyperbolicity theory (see [40, Theorem 4.1]), f is partially hyperbolic and dynamically coherent, with a centre foliation \mathcal{F}^c whose leaves are uniformly C^r -close to the leaves of the centre foliation $\{\{x\} \times S : x \in M\}$ of the unperturbed diffeomorphism f_0 . Moreover (see [40, Theorems 7.1 and 7.4]), the two centre foliations are conjugate, in the sense that there exists a homeomorphism $H_f : M \times S \rightarrow M \times S$ that sends each $\{x\} \times S$ to a centre leaf \mathcal{F}_x^c of f , in such a way that

$$f(H_f(\{x\} \times S)) = H_f(f_0(\{x\} \times S)). \quad (10)$$

The *leaf conjugacy* H_f is not unique, but the correspondence $x \mapsto \mathcal{F}_x^c$ defined in this way does not depend on the choice of H_f . Moreover, the restriction of H_f to each $\{x\} \times S$ is a C^1 diffeomorphism onto \mathcal{F}_x^c , and these leaf derivatives vary continuously on $M \times S$. See [59]. In particular, the leaves of \mathcal{F}^c are the fibres of a continuous fibre bundle $\pi : M \times S \rightarrow M$. Since centre-bunching is

a C^1 -open condition, we also have that every diffeomorphism f close to f_0 is centre-bunched. This proves the claim.

Related remarks were made in [10, Section 3.2], [6, Example 2.7], and [52, Section 3]. We also point out that, by [8, Theorem 6.1], stably accessible diffeomorphisms are dense in a neighbourhood of f_0 .

Theorem A *If μ is a measure in the Lebesgue class of N then every accessible $f \in \mathcal{F}_\mu^r(N, S)$, $r \geq 2$ satisfies at least one of the following conditions:*

- (1) *the centre Lyapunov exponents of f are distinct, and they are continuous at f as functions of the dynamical system;*
- (2) *$S = \mathbb{T}^2$ and the centre bundle E^c of f contains an invariant continuous line field or an invariant pair of transverse continuous line fields;*
- (3) *$S = \mathbb{S}^2$ or $S = \mathbb{T}^2$ and there exists a continuous conformal structure on the centre leaves invariant under f and under its stable and unstable holonomies.*

A few comments are in order concerning the three alternatives in the conclusion of the theorem. Firstly, under additional assumptions (pinching, existence of periodic points) one can ensure that every diffeomorphism in $\mathcal{F}_\mu^r(N, S)$ or $\mathcal{F}_\omega^r(N, S)$ is C^r -approximated by one that satisfies condition (1). This follows from the methods developed in [52, Theorems A and B] and [47, Corollary 1] but we shall not detail the arguments here.

Our next main goal will be to characterise the third alternative in the theorem more precisely. Indeed, we shall see in Sections 2.4 and 2.5 that diffeomorphisms as in (3) are quite rigid. For that we shall restrict our setting somewhat in each of the two cases $S = \mathbb{S}^2$ and $S = \mathbb{T}^2$. Example 3 below shows that alternative (2) may also occur. At this point, it is not clear whether a rigidity statement holds in that setting as well. But in Section 7 we show how this alternative can be excluded in some cases.

Theorem A applies, in particular, to the volume measure vol associated to any given symplectic form ω . Besides, in the symplectic case the conclusions are a bit stronger:

Corollary B *If $f \in \mathcal{F}_\omega^r(N, S)$, $r \geq 2$ for some symplectic form ω , alternative (1) in Theorem A implies that f is non-uniformly hyperbolic for the invariant measure vol , and alternative (3) implies that there exists a continuous Riemannian metric on the centre leaves invariant under f and under the stable and unstable holonomies.*

2.4 Rigidity - the torus case

Initially, we deal with the case $S = \mathbb{T}^2$. Let $f_0 : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2$ be an elliptic affine extension of an Anosov diffeomorphism $g_0 : M \rightarrow M$, as defined in (4). We speak of an L_0 -affine extension instead when we want to specify the choice of the elliptic element L_0 of $\text{SL}(2, \mathbb{Z})$. In Section 5 we observe that stable accessibility is dense among the diffeomorphisms of the form (4).

Remark 1 Every elliptic $L_0 \in \mathrm{SL}(2, \mathbb{Z})$ has finite order, which must be 1, 2, 3, 4 or 6. Indeed, let $e^{\pm i\theta}$ be the eigenvalues. The characteristic equation $x^2 - \mathrm{trace}(L_0)x + 1 = 0$ gives that $\mathrm{trace}(L_0) = 2 \cos \theta$. As the trace is an integer, it follows that $\cos \theta \in \{-1, -1/2, 0, 1/2, 1\}$, which implies the claim.

We also assume that f_0 preserves a given measure μ in the Lebesgue class of $M \times \mathbb{T}^2$. That is the case, for example, if f preserves a symplectic form ω on $M \times \mathbb{T}^2$ and, more specifically, when the Anosov diffeomorphism g_0 itself preserves a symplectic form ω_M on M : then it suffices to take $\omega = \omega_M \times \omega_S$, where ω_S is the standard area form on \mathbb{T}^2 .

Theorem C *There exists a neighbourhood \mathcal{U}_T of f_0 in the space of C^r , $r \geq 2$ diffeomorphisms of $M \times \mathbb{T}^2$ such that every accessible μ -preserving $f \in \mathcal{U}_T$ satisfies at least one of the following conditions:*

- (1) *the centre Lyapunov exponents of f are distinct, and they are continuous at f as functions of the dynamical system;*
- (2) *the centre bundle E^c contains an invariant continuous line field or an invariant pair of transverse continuous line fields;*
- (3) *f is topologically conjugate to an L_0 -affine extension $(x, v) \mapsto (g_0(x), L_0v + w(x))$ of g_0 .*

In the symplectic case, the alternative (1) implies that f non-uniformly hyperbolic. Note that the L_0 -elliptic extension in the alternative (3) is only claimed to be a homeomorphism, as our methods can only prove that w is continuous (Hölder regularity of the solutions of the Beltrami equation, cf. [3, Theorem 8], can probably be used to prove that w is Hölder). It would be interesting to know whether this can be upgraded to differentiability.

Another interesting open question concerning alternative (3) of the theorem is whether the conjugacy may be taken to be differentiable, at least when M is a surface. That would imply that the L_0 -elliptic extension is differentiable, of course. The ideas in [9, Section 7.3] suggest that progress is perhaps possible also on the way of the converse.

In the corollary that follows it is assumed that g_0 has some fixed point. By a result of Sondow [64] (see also Franks [33], Hirsch [38] and Manning [50, 51]), that is automatic if M is the quotient G/F of a compact, connected Lie group G by any finite subgroup F . In particular, the assumption is automatically satisfied if $M = \mathbb{T}^d$ for any $d \geq 2$.

Corollary D *In the setting of Theorem C, assume that g_0 has a fixed point and the order of L_0 is greater than 2. Then, up to reducing the neighbourhood \mathcal{U}_T if necessary, every accessible μ -preserving $f \in \mathcal{U}_T$ satisfies that either the two centre Lyapunov exponents are distinct, or f is topologically conjugate to an L_0 -affine extension $(x, v) \mapsto (g_0(x), L_0v + w(x))$ of g_0 .*

In other words, alternative (2) of Theorem C may be excluded when the order of L_0 is greater than 2. Example 3 shows that this need no longer be true when $L_0 = \pm \mathrm{Id}$.

2.5 Rigidity - the sphere case

Now let us consider the case $S = \mathbb{S}^2$. Take $f_0 : M \times \mathbb{S}^2 \rightarrow M \times \mathbb{S}^2$ to be a Möbius extension of a transitive Anosov diffeomorphism $g_0 : M \rightarrow M$, as defined in (5). It is assumed that f_0 preserves a given measure μ in the Lebesgue class of $M \times \mathbb{S}^2$.

Theorem E *There exists a neighbourhood \mathcal{U}_S of f_0 in the space of C^r , $r \geq 2$ diffeomorphisms of $M \times \mathbb{S}^2$ such that every accessible μ -preserving $f \in \mathcal{U}_S$ satisfies one of the following conditions:*

- (1) *the centre Lyapunov exponents of f are distinct, and they are continuous at f as functions of the dynamical system;*
- (2) *f is topologically conjugate to some Möbius extension of g_0 .*

It is an interesting question, raised by the anonymous reviewer, whether in the alternative (2) the Möbius transformations are necessarily rotations. The next example addresses that kind of question under stronger symplecticity assumptions. We also point out that in the symplectic case the alternative (1) implies that f is non-uniformly hyperbolic.

Example 2 Let $g_0 : M \rightarrow M$ be an Anosov diffeomorphism preserving some symplectic form ω_M on M . Let $\omega = \omega_M \times \omega_S$, where ω_S is the standard area form on the sphere, induced by the Euclidean volume form in \mathbb{R}^3 through

$$\omega_{S,p}(u, v) = (dx_1 \wedge dx_2 \wedge dx_3)(u, v, p)$$

for any $p \in \mathbb{S}^2$ and $u, v \in T_p\mathbb{S}^2$. It is clear that a map f_0 as in (5) preserves ω if and only if every ζ_x preserves ω_S . We claim that the latter happens if and only if ζ_x is a rotation, that is, the restriction of a rigid motion of \mathbb{R}^3 that preserves the unit sphere. Since the ‘if’ part is obvious, we only have to check that if ζ_x preserves ω_S then it is a rotation. That can be done as follows.

Let $P : \mathbb{S}^2 \rightarrow \mathbb{C} \cup \{\infty\}$ be the stereographic projection. As observed by Arnold, Rogness [4] and Siliciano [63], for every Möbius transformation $\zeta : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ there exists a unique rigid motion $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ mapping \mathbb{S}^2 to a sphere $T(\mathbb{S}^2)$ whose ‘north pole’ lies in the upper half-space – this ensures that the stereographic projection $Q : T(\mathbb{S}^2) \rightarrow \mathbb{C} \cup \{\infty\}$ is well-defined – such that

$$\zeta = P^{-1} \circ Q \circ T : \mathbb{S}^2 \rightarrow \mathbb{S}^2.$$

It follows that ζ is area-preserving if and only if the map $P^{-1} \circ Q : T(\mathbb{S}^2) \rightarrow \mathbb{S}^2$ is area-preserving, meaning that its Jacobian $J(P^{-1} \circ Q)$ with respect to the standard area-forms on the two spheres is constant equal to 1.

The Jacobian of $P^{-1} : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{S}^2$ with respect to the standard area forms on the plane and the sphere is

$$JP^{-1}(z) = \frac{4}{(1 + |z|^2)^2}.$$

Thus the level sets of JP^{-1} are the circles about the origin, and the level sets of JP are the parallels of \mathbb{S}^2 , that is, the intersections of the sphere with the horizontal planes. Analogously, the level sets of JQ^{-1} are the circles about the point $x_0 + iy_0 \in \mathbb{C}$, where $(x_0, y_0, z_0) = T(0, 0, 0)$ is the centre of $T(\mathbb{S}^2)$ (keep in mind that $z_0 > -1$). For the Jacobian of $P^{-1} \circ Q$ to be constant, the level sets must coincide, and so we must have $x_0 = y_0 = 0$. Observing that

$$JP^{-1}(0) = 4 \text{ and } JQ^{-1}(0) = \frac{4}{(1+z_0)^2},$$

we see that we must also have $z_0 = 0$. Then $T(\mathbb{S}^2) = \mathbb{S}^2$ and so $Q = P$, showing that $\zeta = T$ is a rotation.

3 Preliminaries

In this section we collect a few tools from the literature that will be used in our arguments.

3.1 Conformal barycentres

Recall that a *conformal structure* on a vector space V is an inner product up to multiplication by a positive scalar or, more formally, a projective class of inner products on V . We consider only real vector spaces and linear maps. We call *canonical* the conformal structure on $\mathbb{C} = \mathbb{R}^2$ associated to the Euclidean inner product.

Let $\mathbb{H} \subset \mathbb{C}$ be the Poincaré upper half-plane. A construction of Douady, Earle [29, Section 2] associates to each probability measure m on the boundary $\partial\mathbb{H}$ with no atom of mass $1/2$ or greater a *conformal barycentre* $B(m) \in \mathbb{H}$ which is invariant with respect to the conformal automorphisms of the half-plane:

$$B(\phi_*m) = \phi(B(m)) \text{ for every } \phi \in \text{PSL}(2, \mathbb{R}). \quad (11)$$

We associate to m the unique conformal structure on \mathbb{C} which is preserved by the stabiliser $\text{Stab}(B(m))$ of the conformal barycentre, that is, which is invariant under every linear isomorphism

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{C} \rightarrow \mathbb{C}$$

such that $\phi_A(B(m)) = B(m)$ for the automorphism $\phi_A \in \text{PSL}(2, \mathbb{R})$ defined by

$$\phi_A(z) = \frac{az + b}{cz + d}.$$

Remark 2 The stabiliser of the imaginary unit i is the subgroup of automorphisms $\phi \in \mathrm{PSL}(2, \mathbb{R})$ of the form

$$\phi(z) = \frac{az + b}{cz + d} \quad \text{with } a = d \text{ and } b = -c.$$

The corresponding linear isomorphisms $A : \mathbb{C} \rightarrow \mathbb{C}$ preserve the standard conformal structure on \mathbb{C} and, moreover, that is the only conformal structure on the plane invariant under every linear isomorphism A such that $\phi_A \in \mathrm{Stab}(i)$. The stabiliser of any other $w \in \mathbb{H}$ coincides with $\phi_w \mathrm{Stab}(i) \phi_w^{-1}$, where $\phi_w \in \mathrm{PSL}(2, \mathbb{R})$,

$$\phi_w(z) = \frac{a_w z + b_w}{c_w z + d_w},$$

is such that $\phi_w(i) = w$. Hence $\mathrm{Stab}(w)$ also preserves a unique conformal structure on \mathbb{C} , namely, the push-forward of the standard conformal structure under the linear isomorphism

$$\begin{pmatrix} a_w & b_w \\ c_w & d_w \end{pmatrix} : \mathbb{C} \rightarrow \mathbb{C}.$$

Remark 3 It follows from the construction in [29, Section 2] that the conformal barycentre varies continuously with the probability measure relative to the weak* topology. Indeed, for each m , consider the vector field $\xi_m : \mathbb{H} \rightarrow \mathbb{R}$ defined by

$$\xi_m(w) = \frac{1}{\phi'_w(w)} \int_{\partial\mathbb{H}} \phi_w(z) dm(z)$$

where ϕ_w is the conformal automorphism of the half-plane such that $\phi_w(w) = i$. It is clear that ξ_m is real-analytic and varies continuously with m relative to the weak* topology. By construction, the conformal barycentre $B(m)$ is the only zero of ξ_m , and it has index 1, meaning that the winding number of ξ_m along any small simple closed curve c around $B(m)$ is equal to 1. Fix an arbitrarily small c . The winding number of ξ_m along c remains 1 for any m' close to m , and that implies that $\xi_{m'}$ has some zero in the inside of c . By uniqueness, this means that $B(m')$ is in the inside of c , for any m' close to m , which proves the claim.

3.2 Measurable Riemann mapping theorem

We quote a few useful facts from conformal mapping theory. See Ahlfors, Bers [3] and Ahlfors [1, 2] for more details and references.

A map f on a Riemannian manifold is said to be *conformal* if the derivative at (almost) every point preserves angles, relative to the given Riemannian metric. For any domain U of the plane, the Riemannian metric may always be written as $ds = \lambda|dz + \mu d\bar{z}|$ where $\lambda = \lambda(z)$ is a positive number and $\mu = \mu(z)$

is a complex number with $|\mu| < 1$. Then conformality with respect to this metric (called μ -conformality) is expressed by the *Beltrami equation*

$$\partial_{\bar{z}}f = \mu \partial_z f, \quad (12)$$

where

$$\partial_z f = \frac{1}{2}(\partial_x f - i\partial_y f) \text{ and } \partial_{\bar{z}} f = \frac{1}{2}(\partial_x f + i\partial_y f)$$

The measurable Riemann mapping theorem asserts that if μ is measurable and $\sup|\mu| < 1$ then a μ -conformal map f does exist. More precisely, we shall use the following existence and uniqueness result, which is contained in [3, Theorem 6]:

Theorem 1 *Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function such that $\sup|\mu| < 1$. Then there exists a unique homeomorphism $f : \mathbb{C} \rightarrow \mathbb{C}$ which is μ -conformal and fixes the points $0, 1$ and ∞ .*

We shall also need the fact that the homeomorphism f depends continuously on μ in the sense of the following result, which is contained in [3, Theorem 8]:

Theorem 2 *Let $k < 1$ be fixed. Then for any compact set $K \subset \mathbb{C}$ there exists $C(K) > 0$ such that*

$$\sup_{z \in K} |f_1(z) - f_2(z)| \leq C(K) \sup |\mu_1 - \mu_2|$$

for any measurable functions $\mu_1, \mu_2 : \mathbb{C} \rightarrow \mathbb{C}$ with $\sup|\mu_i| < k$ for $i = 1, 2$, where $f_i : \mathbb{C} \rightarrow \mathbb{C}$ denotes the μ_i -conformal homomorphism that fixes the points $0, 1$, and ∞ (given by Theorem 1).

3.3 Invariance Principle

Next we recall a few useful ideas from Ledrappier [45], Bonatti, Gomez-Mont, Viana [20], Avila, Viana [7] and Avila, Santamaria, Viana [6]. Additional related information can be found in Viana [65].

3.3.1 Cocycles and exponents

Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism on a compact manifold and $\pi : \mathcal{V} \rightarrow M$ be a continuous finite-dimensional vector bundle. A *linear cocycle* over f is a continuous map $F : \mathcal{V} \rightarrow \mathcal{V}$ such that $\pi \circ F = f \circ \pi$ and F acts on the fibres by linear isomorphisms $F_x : \mathcal{V}_x \rightarrow \mathcal{V}_{f(x)}$. A theorem of Furstenberg, Kesten [34] asserts that the *extremal Lyapunov exponents*

$$\lambda_+(F, x) = \lim_n \frac{1}{n} \log \|F_x^n\| \text{ and } \lambda_-(F, x) = \lim_n -\frac{1}{n} \log \|(F_x^n)^{-1}\|$$

exist at μ -almost every point $x \in M$, relative to any f -invariant probability measure μ . Note that $\lambda_-(F, x) \leq \lambda_+(F, x)$ whenever they are defined. If the system (f, μ) is ergodic then the functions $x \mapsto \lambda_{\pm}(F, x)$ are constant on a full μ -measure set.

The *projectivisation* of $\pi : \mathcal{V} \rightarrow M$ is the continuous fibre bundle $\pi : \mathbb{P}\mathcal{V} \rightarrow M$ whose fibres are the projective quotients of the fibres of \mathcal{V} . Note that the fibres are smooth manifolds modelled after some projective space $\mathbb{P}\mathbb{R}^k$. The *projective cocycle* associated to $F : \mathcal{V} \rightarrow \mathcal{V}$ is the smooth cocycle $\mathbb{P}F : \mathbb{P}\mathcal{V} \rightarrow \mathbb{P}\mathcal{V}$ whose action $\mathbb{P}F_x : \mathbb{P}\mathcal{V}_x \rightarrow \mathbb{P}\mathcal{V}_{f(x)}$ on the fibres is given by the projectivisation of F_x . The *extremal Lyapunov exponents* of $\mathbb{P}F$ are the numbers

$$\lambda_+(\mathbb{P}F, x, \xi) = \lim_n \frac{1}{n} \log \|D(\mathbb{P}F)_x^n(\xi)\| \text{ and}$$

$$\lambda_-(F, x, \xi) = \lim_n -\frac{1}{n} \log \|D(\mathbb{P}F)_x^n(\xi)^{-1}\|.$$

They are defined at m -almost every point $(x, \xi) \in \mathbb{P}\mathcal{V}$, for any $\mathbb{P}F$ -invariant measure m . Moreover,

$$\begin{aligned} \lambda_+(F, x) - \lambda_-(F, x) &\geq \lambda_+(\mathbb{P}F, x, \xi) \\ &\geq \lambda_-(\mathbb{P}F, x, \xi) \geq \lambda_-(F, x) - \lambda_+(F, x) \end{aligned}$$

whenever they are defined.

3.3.2 Invariant holonomies

We call a *stable holonomy* for the projective cocycle $\mathbb{P}F$ a family of homeomorphisms $H_{x,y}^s : \mathbb{P}\mathcal{V}_x \rightarrow \mathbb{P}\mathcal{V}_y$ defined for all x and y in the same stable leaf of f and satisfying (cf. Definition 2.4 in Avila, Santamaria, Viana [6]),

- (a) $H_{y,z}^s \circ H_{x,y}^s = H_{x,z}^s$ and $H_{x,x}^s = \text{id}$;
- (b) $\mathbb{P}F_y \circ H_{x,y}^s = H_{f(x),f(y)}^s \circ \mathbb{P}F_x$;
- (c) $(x, y, \xi) \mapsto H_{x,y}^s(\xi)$ is continuous when (x, y) varies in the set of pairs of points in the same local stable leaf;
- (d) there are $C > 0$ and $\eta > 0$ such that $H_{x,y}^s$ is (C, η) -Hölder for every x and y in the same local stable leaf.

The concept of *unstable holonomy* for $\mathbb{P}F$ is analogous, replacing (local) stable leaves with (local) unstable leaves.

Let us assume that the linear cocycle F is α -Hölder for some $\alpha \in (0, 1]$. Then, by Propositions 3.2 and 3.3 in Avila, Santamaria, Viana [6], stable and unstable holonomies for $\mathbb{P}F$ do exist if F is *fibre-bunched*, meaning that for some choice of a norm $\|\cdot\|$ on \mathcal{V} we have

$$\|F_x\| \|(F_x)^{-1}\| \nu(x)^\alpha < 1 \text{ and } \|F_x\| \|(F_x)^{-1}\| \hat{\nu}(x)^\alpha < 1, \quad (13)$$

where $\nu(\cdot)$ and $\hat{\nu}(\cdot)$ are as in (6). See also Section 3 in Marín [52].

We shall apply these ideas to the centre derivative cocycle $F = D^c f$ of a C^2 partially hyperbolic diffeomorphism f as in the statement of our results. Note

that F is indeed Hölder since, as we recalled in Section 2.1, the centre bundle E^c is α -Hölder for some $\alpha \in (0, 1]$. Moreover, it is clear from the definitions (8) and (13) that F is fibre-bunched if and only if f is centre-bunched. Then, by the previous remarks, F admits stable holonomy and unstable holonomy.

3.3.3 Invariant disintegrations

Since $\mathbb{P}F$ is continuous and the base space M is compact, for any f -invariant measure μ there exist $\mathbb{P}F$ -invariant measures m with $\pi_* m = \mu$. Fix any such measure m . By the Rokhlin disintegration theorem (see [66, Chapter 5]), there exists a *disintegration* of m into conditional probabilities along the fibres, that is, a measurable family $\{m_x : x \in M\}$ of probability measures such that $m_x(\mathbb{P}\mathcal{V}_x) = 1$ for μ -almost every x and

$$m(U) = \int_M m_x(U \cap \mathbb{P}\mathcal{V}_x) d\mu(x)$$

for every measurable set $U \subset \mathbb{P}\mathcal{V}$. The disintegration is essentially unique, meaning that any two coincide on some full μ -measure subset.

A disintegration is said to be *s-invariant* if

$$(H_{x,y}^s)_* m_x = m_y \text{ for every } x \text{ and } y \text{ in the same stable leaf.}$$

The notion of *u-invariance* is analogous, using *u*-holonomy instead. We say that the disintegration is *bi-invariant* if it is both *s*-invariant and *u*-invariant. A $\mathbb{P}F$ -invariant probability measure m is called an *su-state* if it admits a bi-invariant disintegration.

3.3.4 Invariance Principle

A subset of M is *s-saturated* if it consists of entire stable leaves, and *u-saturated* if it consists of entire unstable leaves. Moreover, we call it *bi-saturated* if it is both *s*-saturated and *u*-saturated.

Assuming accessibility, if the Lyapunov exponents $\lambda_{\pm}(F, \cdot)$ coincide then every $\mathbb{P}F$ -invariant measure that projects down to μ is an *su-state*. That is a consequence of the following version of the Invariance Principle, which is contained in Theorems B and C of [6]:

Theorem 3 *Let $f : M \rightarrow M$ be a C^2 partially hyperbolic centre-bunched diffeomorphism preserving a measure μ in the Lebesgue class of M . Let $F : \mathcal{V} \rightarrow \mathcal{V}$ be a linear cocycle over f admitting invariant stable and unstable holonomies, and suppose that $\lambda_-(F, x) = \lambda_+(F, x)$ at μ -almost every point.*

Then every $\mathbb{P}F$ -invariant probability m on $\mathbb{P}\mathcal{V}$ with $\pi_ m = \mu$ admits a disintegration $\{m_x : x \in M\}$ such that*

- (a) *the disintegration is bi-invariant over a full-measure bi-saturated subset M_F of M ;*

(b) if f is accessible then $M_F = M$ and the conditional probabilities m_x depend continuously on the base point $x \in M$, relative to the weak* topology.

Continuity dependence of the conditional probabilities is actually a consequence of bi-invariance, when the diffeomorphism is accessible, as shown in [6, Section 7].

4 Proof of Theorem A

The assumption that $f \in \mathcal{F}_\mu^r(M, S)$ includes that f is partially hyperbolic, centre-bunched, and dynamically coherent. Since f is also taken to be accessible, it follows from [57, Theorem A] that the system (f, μ) is ergodic. In particular, the Lyapunov exponents are constant on a full measure subset.

4.1 Invariant holonomies

The hypothesis that f is a fibred partially hyperbolic system also ensures (see [10, Section 3.2]) that it has *global* stable and unstable holonomies: for any $x, y \in M$ such that \mathcal{F}_x^c and \mathcal{F}_y^c are contained in the same centre-stable leaf there exists a homeomorphism (*stable holonomy map*)

$$h_{x,y}^s : \mathcal{F}_x^c \rightarrow \mathcal{F}_y^c \text{ such that } h_{x,y}^s(p) \in \mathcal{F}^s(p) \cap \mathcal{F}_y^c \text{ for all } p \in \mathcal{F}_x^c, \quad (14)$$

and for any $x, y \in M$ such that \mathcal{F}_x^c and \mathcal{F}_y^c are contained in the same centre-unstable leaf there exists a homeomorphism (*unstable holonomy map*)

$$h_{x,y}^u : \mathcal{F}_x^c \rightarrow \mathcal{F}_y^c \text{ such that } h_{x,y}^u(p) \in \mathcal{F}^u(p) \cap \mathcal{F}_y^c \text{ for all } p \in \mathcal{F}_x^c. \quad (15)$$

See also [6, Example 2.7] and [52, Section 3]. It is worth pointing out that these holonomy maps are C^1 , by [58, Theorem B].

As observed in Section 3.3.2, under our assumptions the projective derivative cocycle $\mathbb{P}(D^c f)$ admits stable holonomies

$$H_{p,q}^s : \mathbb{P}E_p^c \rightarrow \mathbb{P}E_q^c, \quad (16)$$

defined for $p \in \mathcal{F}_x^c$, $q = h_{x,y}^s(p)$, and any $x, y \in M$ such that \mathcal{F}_x^c and \mathcal{F}_y^c are contained in the same centre-stable leaf, and unstable holonomies

$$H_{p,q}^u : \mathbb{P}E_p^c \rightarrow \mathbb{P}E_q^c, \quad (17)$$

defined for $p \in \mathcal{F}_x^c$, $q = h_{x,y}^u(p)$, and any $x, y \in M$ such that \mathcal{F}_x^c and \mathcal{F}_y^c are contained in the same centre-unstable leaf.

Recall that we call *su-state* any $\mathbb{P}(D^c f)$ -invariant measure m on $\mathbb{P}E^c$ that projects down to μ on $M \times S$ and admits a disintegration $\{m_q : q \in M \times S\}$ along the fibres of $\mathbb{P}E^c$ which is bi-invariant, that is, which satisfies

$$(H_{p,q}^s)_* m_p = m_q \text{ and } (H_{p,q}^u)_* m_p = m_q, \quad (18)$$

respectively, for any p and q as in (16), and for any p and q as in (17).

Initially, let us assume that f has no su -state. Then, by the Invariance Principle (Theorem 3(b)), the two centre Lyapunov exponents must be distinct. Non-existence of su -states also implies that f is a continuity point for the centre Lyapunov exponents. That follows from [47, Proposition 4.8], which is itself a version of [7, Proposition 6.3]. In this way we get alternative (1) in the theorem.

4.2 Invariant disintegrations

From now on, assume that there does exist some su -state m , and let $\{m_q : q \in M \times S\}$ be a bi-invariant disintegration along the fibres of $\mathbb{P}E^c$. As observed previously (see also [6, Section 7]), the conditional probabilities m_q depend continuously on the base point q relative to the weak* topology. Since m is $\mathbb{P}(D^c f)$ -invariant, we have that

$$\mathbb{P}(D^c f)_* m_q = m_{f(q)} \text{ for } \mu\text{-almost every point } q.$$

This, together with continuity and the fact that μ is supported on the whole $M \times S$, ensures that

$$\mathbb{P}(D^c f)_* m_q = m_{f(q)} \text{ for every } q \in M \times S. \quad (19)$$

Suppose that there exists a point $p \in M \times S$ such that m_p admits an atom with largest mass, and this mass is greater than or equal to $1/2$. Then, since f is accessible and the disintegration is su -invariant, the same is true at every point $q \in M \times S$. Let v_q denote the corresponding atom. The map $q \mapsto v_q$ defines a continuous line bundle on $M \times S$ tangent to the centre leaves at every point. By the Poincaré–Hopf theorem this implies that the centre leaves are tori, and so $S = \mathbb{T}^2$. Moreover, the properties (18) and (19) give that this line bundle is Df -invariant and invariant under the stable and unstable holonomies.

Similarly, suppose that some m_p admits a pair of atoms $\{u_p, v_p\}$ with masses equal to $1/2$. Then, just as before, the same must hold at every point $q \in M \times S$: let $\{u_q, v_q\}$ denote the corresponding pair of atoms. The map $q \mapsto \{u_q, v_q\}$ defines a pair of transverse continuous line fields. Again by the Poincaré–Hopf theorem, existence of such a pair implies that $S = \mathbb{T}^2$. Finally, the properties (18) and (19) ensure that this pair of transverse continuous line fields is Df -invariant and invariant under the stable and unstable holonomies.

The situations in the previous couple of paragraphs correspond to alternative (2) in the theorem.

4.3 Invariant conformal structures

We are left to consider the case when the conditional probability measures m_q have no atoms of mass greater than or equal to $1/2$. We are going to show that,

via the conformal barycentre construction in Section 3.1, each m_q determines a unique conformal structure on the corresponding centre subspace E_q^c . In that way, the leaves of the centre foliation \mathcal{F}^c are endowed with Riemann surface structures which are invariant under stable and unstable holonomies and the diffeomorphism itself. This goes as follows.

The boundary $\partial\mathbb{H}$ is naturally identified with the *real* projective space $\mathbb{P}(\mathbb{C}) = \mathbb{P}(\mathbb{R}^2)$ through the map $x \mapsto [x : 1]$. Consider any linear isomorphism $L : E_q^c \rightarrow \mathbb{C}$ and its projectivisation $\mathbb{P}L : \mathbb{P}E_q^c \rightarrow \mathbb{P}(\mathbb{C})$. Let \mathcal{M} be the push-forward of m_q under $\mathbb{P}L$. Consider the conformal structure defined on \mathbb{C} by the conformal barycentre $B(\mathcal{M})$, and transport it to E_q^c through L . The fact that the barycentre is conformally invariant ensures that the conformal structure thus defined on E_q^c does not depend on the choice of L , as we are going to explain.

Given any other linear isomorphism $L' : E_q^c \rightarrow \mathbb{C}$, consider

$$A = L' \circ L^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{C} \rightarrow \mathbb{C}.$$

The projectivisation $\mathbb{P}A : \mathbb{P}(\mathbb{C}) \rightarrow \mathbb{P}(\mathbb{C})$ is given in homogeneous coordinates by

$$[x : 1] \mapsto [\phi_A(x) : 1],$$

and so the probability measure \mathcal{M}' on $\partial\mathbb{H}$ associated to L' coincides with $(\phi_A)_*\mathcal{M}$. By conformal invariance, it follows that $B(\mathcal{M}') = \phi_A(B(\mathcal{M}))$ and so A maps the conformal structure defined by $B(\mathcal{M})$ to the one defined by $B(\mathcal{M}')$. Thus, the conformal structures defined on E_q^c through L and through L' coincide.

This completes the explanation of why the probability measure m_q defines a conformal structure on the vector space E_q^c , for each $q \in M \times S$. Remark 3 ensures that this conformal structure varies continuously with $q \in M \times S$. The fact that the disintegration $\{m_q : q \in M \times S\}$ is invariant under f and under the holonomies h^s and h^u , ensures that the conformal structures obtained in this way are invariant under the centre derivative $D^c f$, as well as under the holonomies H^s and H^u . In this way, every leaf of \mathcal{F}^c is endowed with a Riemann surface structure, and these Riemann surface structures are preserved by the dynamical system f and its holonomies h^s and h^u .

In particular, since f is accessible, the group of conformal automorphisms acts transitively on every centre leaf \mathcal{F}_x^c . According to [31, Theorem V.4], the only compact Riemann surfaces with that property are the sphere \mathbb{S}^2 and the torus \mathbb{T}^2 . This gives alternative (3) in the theorem.

The proof of Theorem A is now complete.

4.4 Proof of Corollary B

Assume that $f \in \mathcal{F}_\omega^r(M, S)$ for some symplectic form ω on $M \times S$. Then $\dim E^u = \dim E^s$, and we may use the symmetry property (9) to conclude

that the two centre Lyapunov exponents are symmetric to each other. Thus, the fact they are distinct, cf. alternative (1) in Theorem A, implies that they are non-zero, and so f is non-uniformly hyperbolic.

As observed in Section 2.2, the restriction $\omega^c = \omega|_{E^c}$ defines a symplectic form on the centre leaves which is, clearly, preserved by the restriction of f to the leaves. Together with the conformal structure, this symplectic form defines an inner product on each centre space E_q^c , varying continuously with q . Thus one gets a continuous Riemannian metric on each centre leaf.

It is clear that this Riemannian metric is invariant under f , because both the conformal structure and the symplectic form are. We have also seen that the conformal structure is invariant under the stable and unstable holonomies of f . The next lemma asserts that the same is true for the symplectic form ω^c . So, the Riemannian metric is also invariant under the stable and unstable holonomies of f , as claimed.

This means that we have reduced the proof of Corollary B to

Lemma 1 *For any $x, y \in M$ such that \mathcal{F}_x^c and \mathcal{F}_y^c are contained in the same centre-stable leaf and $p \in \mathcal{F}_x^c$,*

$$\omega_p^c(u, v) = \omega_q^c((Dh_{x,y}^s)_p u, (Dh_{x,y}^s)_p v), \quad q = h_{x,y}^s(p)$$

for any $u, v \in E_p^c$. A dual statement holds for the unstable holonomy $h_{x,y}^u$ when x and y are such that \mathcal{F}_x^c and \mathcal{F}_y^c are contained in the same centre-unstable leaf.

Proof Using local charts, for instance, we may identify the tangent spaces $T_p(M \times S)$ and $T_q(M \times S)$ at nearby points $p, q \in M \times S$ through linear isomorphisms that vary continuously with the points. Then, since the partially hyperbolic splitting $E^u \oplus E^c \oplus E^s$ is continuous, the projection along $E_q^u \oplus E_q^s$ induces a linear isomorphism $\Pi_{p,q} : E_p^c \rightarrow E_q^c$ uniformly close to the identity if the distance $\text{dist}(p, q)$ is small. We denote by $\Delta_{p,q}$ the determinant of $\Pi_{p,q}$ with respect to the symplectic form, characterised by

$$\omega_q^c(\Pi_{p,q} u, \Pi_{p,q} v) = \Delta_{p,q} \omega_p^c(u, v) \text{ for any } u, v \in E_p^c. \quad (20)$$

Since ω^c is continuous, $\Delta_{p,q}$ is uniformly close to 1 if $\text{dist}(p, q)$ is small.

Given any $x, y \in M$ such that \mathcal{F}_x^c and \mathcal{F}_y^c are in the same centre-stable leaf, and given $p \in \mathcal{F}_x^c$ and $q = h_{x,y}^s(p)$, we have that (compare [52, Section 3])

$$(Dh_{x,y}^s)_p = \lim_n (D^c f^{-n})_{f^n(q)} \circ \Pi_{f^n(p), f^n(q)} \circ (D^c f^n)_p \quad (21)$$

Thus, recalling that the centre derivative $D^c f$ preserves the symplectic form ω^c ,

$$\begin{aligned} & \omega_q^c((Dh_{x,y}^s)_p u, (Dh_{x,y}^s)_p v) \\ &= \lim_n \omega_q^c\left((D^c f^{-n})_{f^n(q)} \circ \Pi_{f^n(p), f^n(q)} \circ (D^c f^n)_p u, \right. \\ & \qquad \qquad \qquad \left. (D^c f^{-n})_{f^n(q)} \circ \Pi_{f^n(p), f^n(q)} \circ (D^c f^n)_p v\right) \\ &= \lim_n \omega_{f^n(q)}^c\left(\Pi_{f^n(p), f^n(q)} \circ (D^c f^n)_p u, \Pi_{f^n(p), f^n(q)} \circ (D^c f^n)_p v\right) \\ &= \lim_n \Delta_{f^n(p), f^n(q)} \omega_{f^n(p)}^c\left((D^c f^n)_p u, (D^c f^n)_p v\right) = \lim_n \Delta_{f^n(p), f^n(q)} \omega_p^c(u, v) \end{aligned}$$

for any $u, v \in E_p^c$. Since $\text{dist}(f^n(p), f^n(q))$ converges to zero as $n \rightarrow \infty$, because p and q are in the same stable leaf, the limit on the right hand side is equal to $\omega_p^c(u, v)$, and so the proof is complete.

5 Density of stable accessibility

A volume-preserving partially hyperbolic C^r , $r \geq 2$ diffeomorphism is said to be *stably accessible* if every volume-preserving C^r diffeomorphism in a C^1 -neighbourhood is accessible. In this section, we check that stable accessibility is dense among maps of the form (4) and so the accessibility assumption in Theorem C is quite mild. The proof of the theorem does not depend on this fact.

Theorem 4 *Every skew-product (4) may be approximated by another skew-product over g which is stably accessible in the space of volume-preserving maps.*

The special case of rotation extensions (that is, $L = \text{Id}$) was proved in [25], using also ideas from [23, 21, 22], and our arguments here are an adaptation. We outline how to deal with the presence of the elliptic coefficient L , referring the reader to the previous papers for more details. It is worth mentioning that for circle rotation extensions in dimension 3 any perturbation is accessible unless it (or some finite-order quotient) is smoothly conjugate to the product of an Anosov diffeomorphism with a rotation [24]. Another related result is [8, Theorem 6.1].

To better highlight the analogy to the previous papers, in this section we refer to a slightly more general setting. Namely, we consider maps of the form

$$f_w : M \times G \rightarrow M \times G, \quad f_w(x, \theta) = (g(x), w(x)L(\theta)), \quad (22)$$

where g is a (transitive) Anosov diffeomorphism on a compact manifold, G is a compact connected Lie group (the case we are most interested in is $G = \mathbb{T}^2$), and $L : G \rightarrow G$ is an elliptic group isomorphism: by *elliptic* we mean that $\|DL^n\|$ is bounded uniformly over all $n \in \mathbb{Z}$. We are going to explain that f_w is accessible for an open and dense subset of C^r maps $w : M \rightarrow G$.

functions w .

5.1 Holonomy maps

Let \mathcal{W}^u and \mathcal{W}^s denote, respectively, the unstable foliation and the stable foliation of the Anosov diffeomorphism g . The assumption on L ensures that f_w is partially hyperbolic and centre-bunched. It is also dynamically coherent: the leaves of the *centre-unstable foliation* \mathcal{F}^{cu} are the products $\mathcal{W}^u(x) \times G$, $x \in M$ and, analogously, the leaves of the *centre-stable foliation* \mathcal{F}^{cs} are the products $\mathcal{W}^s(x) \times G$, $x \in M$. Then the *centre foliation* \mathcal{F}^c coincides with the vertical fibration $\{\{x\} \times G : x \in M\}$. It also follows that the leaves of the *unstable foliation* \mathcal{F}^u are graphs over the unstable manifolds $\mathcal{W}^u(x)$ of the Anosov diffeomorphism g , and a corresponding fact holds for the *stable foliation* \mathcal{F}^s .

Given points x and y in the same leaf of \mathcal{W}^u , the projection along the leaves of \mathcal{F}^u defines a continuous map

$$h_{x,y}^u : \{x\} \times G \rightarrow \{y\} \times G.$$

Similarly, the projection along the leaves of \mathcal{F}^s defines a continuous map

$$h_{z,w}^s : \{z\} \times G \rightarrow \{w\} \times G$$

for any z and w in the same leaf of \mathcal{W}^s . These are, respectively, unstable and stable holonomies for f .

In what follows we sometimes identify a fibre $\{x\} \times G$ to the group G itself, in the obvious way. Thus we also view the $h_{x,y}^u$ and $h_{w,z}^s$ as transformations in G .

Lemma 2 *Every $h_{x,y}^u$ and $h_{z,w}^s$ is given by a left-translation on G .*

Proof For $n \in \mathbb{Z}$, let us write $f_w^n(x, \theta) = (g^n(x), \mathcal{G}_x^n(\theta))$. Then (compare [6, Proposition 3.2]),

$$h_{x,y}^u = \lim_n \mathcal{G}_{g^{-n}(y)}^n \circ \mathcal{G}_x^{-n}. \quad (23)$$

An induction argument gives that, for any $n \geq 1$ and $a, b \in M$,

$$\begin{aligned} \mathcal{G}_a^n &= w(g^{n-1}(a)) L(w(g^{n-2}(a))) \cdots L^{n-2}(w(g(a))) L^{n-1}(w(a)) L^n \\ \mathcal{G}_b^{-n} &= L^{-1}(w(g^{-n}(b))^{-1}) L^{-2}(w(g^{-n+1}(b))^{-1}) \cdots \\ &\quad \cdots L^{-n+1}(w(g^{-2}(b))^{-1}) L^{-n}(w(g^{-1}(b))^{-1}) L^{-n}. \end{aligned}$$

Taking $a = g^{-n}(y)$ and $b = x$, we see that

$$\begin{aligned} \mathcal{G}_{g^{-n}(y)}^n \circ \mathcal{G}_x^{-n}(\theta) &= w(g^{-1}(y)) L(w(g^{-2}(y))) \cdots \\ &\quad \cdots L^{n-2}(w(g^{-n+1}(y))) L^{n-1}(w(g^{-n}(y))) \cdot \\ &\quad \cdot L^{n-1}(w(g^{-n}(x))^{-1}) L^{n-2}(w(g^{-n+1}(x))^{-1}) \cdots \\ &\quad \cdots L^1(w(g^{-2}(b))^{-1}) w(g^{-1}(b))^{-1} \theta \end{aligned}$$

is a left-translation for every $n \geq 1$, and then so is the limit $h_{x,y}^u$. The argument for $h_{z,w}^s$ is analogous.

Next, for each $\tau \in G$, define the *right-translation maps*

$$\mathcal{R}_\tau : M \times G \rightarrow M \times G, \quad \mathcal{R}_\tau(x, \theta) = (x, \theta\tau).$$

Lemma 2 implies that the invariant holonomies commute with the right-translation maps:

$$\mathcal{R}_\tau \circ h_{x,y}^u = h_{x,y}^u \circ \mathcal{R}_\tau \quad \text{and} \quad \mathcal{R}_\tau \circ h_{z,w}^s = h_{z,w}^s \circ \mathcal{R}_\tau \quad (24)$$

for every x, y, z, w and τ . This means that the unstable foliation \mathcal{F}^u and the stable foliation \mathcal{F}^s are both preserved by every right-translation map \mathcal{R}_τ .

5.2 Holonomy groups

Let γ be any *su*-path in M , that is, any piecewise differentiable oriented curve such that each leg is contained in a leaf of either \mathcal{W}^u or \mathcal{W}^s . Denote by h_γ the composition of the unstable holonomies and the stable holonomies associated to the legs of γ , in the natural order. Clearly, h_γ is a left-translation and

$$\mathcal{R}_\tau \circ h_\gamma = h_\gamma \circ \mathcal{R}_\tau \quad (25)$$

for every τ and γ .

Let e denote the neutral element of the group G . For each $x \in M$, define

$$H_x^0 = \{h_\sigma(e) : \sigma \text{ is a homotopically null } su\text{-loop at } x\}.$$

Note that H_x^0 is a subgroup of G . Indeed, given any $\theta_1 = h_{\sigma_1}(e)$ and $\theta_2 = h_{\sigma_2}(\theta)$ in H_x^0 , it follows from (25) that

$$\theta_1\theta_2 = h_{\sigma_1}(e)\theta_2 = h_{\sigma_1}(\theta_2) = h_{\sigma_1}(h_{\sigma_2}(\theta)) = h_{\sigma_2*\sigma_1}(\theta)$$

where $\sigma_2*\sigma_1$ is the loop obtained by concatenating σ_2 with σ_1 (in this order). Since the concatenation is still homotopically null, this proves that $\theta_1\theta_2 \in H_x^0$. Similarly, given any $\theta = h_\sigma(e)$ denote by $-\sigma$ the loop obtained by reversing the orientation of σ . It is clear that $h_\sigma \circ h_{-\sigma} = \text{Id}$. Since h_γ is a left-translation, this means that

$$h_\sigma(e)^{-1} = h_{-\sigma}(e). \quad (26)$$

It is clear that $-\sigma$ is homotopically null, so this implies that $h_\sigma(e)^{-1} \in H_x^0$. That completes the proof that H_x^0 is a subgroup of G .

The *holonomy groups* H_x^0 and H_y^0 corresponding to two different points x and y in M are conjugate: there exists $\tau \in G$ such that

$$H_y^0 = \tau H_x^0 \tau^{-1}. \quad (27)$$

To see this, fix an *su*-path β from x to y in M and denote $\tau = h_\beta(e)$. The relation (26) gives that $h_{-\beta}(e) = \tau^{-1}$. For any $\theta \in H_x^0$, consider a homotopically null loop σ at x such that $\theta = h_\sigma(e)$, and let $\sigma' = \beta * \sigma * (-\beta)$. Since the holonomy maps are left-translations:

$$\begin{aligned} h_{\sigma'}(e) &= h_\beta(h_\sigma(h_{-\beta}(e))) = h_\beta(h_\sigma(\tau^{-1})) \\ &= h_\beta(h_\sigma(e)\tau^{-1}) = h_\beta(\theta\tau^{-1}) = \tau\theta\tau^{-1}. \end{aligned}$$

It is clear that σ' is a homotopically null loop at y , and so $h_{\sigma'}(e) \in H_y^0$. Thus this proves that $\tau H_x^0 \tau^{-1}$ is contained in H_y^0 . Reversing the roles of x and y , we get the other inequality, and thus (27) follows.

5.3 Stable accessibility

In particular, the following condition is independent of $x \in M$:

$$H_x^0 = G. \quad (28)$$

This is relevant because (28) implies that f is accessible: it clearly implies that every point in the fibre $\{x\} \times G$ is connected to the unit element (x, e) by an su -path in $M \times G$; for any other fibre $\{y\} \times G$, consider an su -path β from x to y in M and observe that (x, e) is connected to $h_\gamma(e) \in \{y\} \times G$ by some su -path in $M \times G$.

In fact, we have a much stronger fact (see [25, Theorem 9.1]): (28) implies that f is *stably accessible*. The main step is to show that if (28) holds then (x, e) is *centrally engulfed from* (x, e) : there exist a continuous map $\Psi : Z \times [0, 1] \rightarrow M \times G$ and a constant $N \geq 1$ such that

- Z is a compact, connected, orientable $\dim E^c$ -manifold with boundary;
- each curve $\Psi(z, \cdot)$ is an su -path on $M \times G$ with no more than N legs satisfying

$$\Psi(z, 0) = (x, e) \text{ and } \Psi(z, 1) \in \{x\} \times G;$$

- $\Psi(z, 1) \neq (x, e)$ for every $z \in \partial Z$, and the map

$$(Z, \partial Z) \rightarrow (\{x\} \times G, \{x\} \times (G \setminus \{e\})), \quad z \mapsto \Psi(z, 1)$$

has positive degree.

Then, a degree argument ensures that the accessibility property is stable under perturbations of the dynamical system. The arguments in Theorem 9.1 (see also Corollary 5.3) of [25] remain valid in our setting, so we refer the reader to that paper. Then, to prove our claim it suffices to check that (28) is a dense property among the maps of the form (4). This is given by [21, Proposition 2.3], whose proof can be outlined as follows (see [25, Theorem 9.8]).

In view of the observations, it is no restriction to assume that x is a periodic point. Recall that the periodic points of a (transitive) Anosov diffeomorphism are dense in the ambient manifold. Let $m = \dim E^c$ and then choose periodic points x_1, \dots, x_m close to x and such that they are all in distinct orbits. The local unstable manifold $W_{loc}^u(x)$ of x intersects the local stable manifold $W_{loc}^s(x_i)$ of x_i at a point z_i^0 , and the local unstable manifold $W_{loc}^u(x_i)$ of x_i intersects the local stable manifold $W_{loc}^s(x)$ of x at a point z_i^1 . The assumption that x, x_1, \dots, x_m are all in distinct orbits ensures that the orbits of the z_i^0 and z_i^1 are all distinct as well. Fix a neighbourhood U_i of each z_i^0 , small enough that none of the periodic points x, x_1, \dots, x_m is in U_i and, moreover, $f^k(z_j^a) \in U_i$ if and only if $j = i$, $k = 0$ and $a = 0$.

Let σ_i be the su -loop consisting of 4 short legs from x to z_i^0 , to x_i , to z_i^1 , and back to x . It is clear that σ_i is homotopically null. We denote the associated holonomy map as $h_{\sigma_i, w}$, to highlight the dependence on w . By construction, if $\tilde{w} : M \rightarrow G$ coincides with w outside U_i then

$$h_{\sigma_j, \tilde{w}}(x, e) = h_{\sigma_j, w}(x, e)$$

for every $j \neq i$. It is not difficult to find such perturbations \tilde{w} so that the point $h_{\sigma_j, \tilde{w}}(x, e)$ moves in any prescribed direction inside the fibre $\{x\} \times G \approx G$. Thus, by modifying w suitably inside each U_1, \dots, U_m , we can ensure that

$$\{h_{\sigma_j, \tilde{w}}(x, e) : j = 1, \dots, m\}$$

is not contained in any subgroup of G with dimension less than m . Since this set is contained in the holonomy group $H_{x, \tilde{w}}^0$, that implies that $H_{x, \tilde{w}}^0 = G$.

6 Proof of Theorem C

Let $f_0 : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2$ be a C^r , $r \geq 2$ elliptic affine extension of a transitive Anosov diffeomorphism $g_0 : M \rightarrow M$, as defined in (5). Assume that f_0 preserves a given measure μ in the Lebesgue class of $M \times \mathbb{T}^2$. It is clear that f_0 is a partially hyperbolic skew-product. In particular (cf. Example 1), every μ -preserving diffeomorphism f in a C^r -neighbourhood belongs to $\mathcal{F}_\mu^r(M, \mathbb{T}^2)$ and, thus, satisfies some of the three alternatives in Theorem A.

The first two alternatives correspond precisely to claims (1) and (2) in the present Theorem C; the second one will be further discussed in Section 7. So, we just need to upgrade the alternative (3) in Theorem A to the statement in claim (3) of Theorem C. In what follows we assume that the two centre Lyapunov exponents of $f : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2$ coincide, and the centre leaves are endowed with Riemann surface structures that vary continuously on $M \times \mathbb{T}^2$ and are invariant under both the dynamics and the invariant holonomies of f .

6.1 Uniformisation

Given any $\tau \in \mathbb{H}$, let $\mathbb{T}_\tau^2 = \mathbb{C}/L(1, \tau)$ be the quotient of the complex plane \mathbb{C} by the sub-lattice $L(1, \tau)$ generated by 1 and τ . Since every centre leaf $\mathcal{F}^c(q)$ is a topological torus, the corresponding Riemann surface is a complex torus, and so it admits a Riemann surface automorphism $\mathbb{T}_{\tau(q)} \rightarrow \mathcal{F}^c(q)$, for some $\tau(q) \in \mathbb{H}$. See [30, pp. 86–90]. The Riemann surfaces $\mathcal{F}^c(q)$ are all conformally equivalent, as they are mapped to one another by the stable and unstable holonomies, which preserve the Riemann surface structure. This implies that the different values of $\tau(p)$ all belong to the same orbit of the modular group $\mathrm{PSL}(2, \mathbb{Z})$. See [36, Section 2]. The following more precise statement will be useful in what follows:

Proposition 1 *There exist $\tau \in \mathbb{H}$ and a homeomorphism $\Psi : M \times \mathbb{T}_\tau^2 \rightarrow M \times \mathbb{T}^2$ whose restriction to each fibre $\{x\} \times \mathbb{T}_\tau^2$, $x \in M$ is a Riemann surface automorphism onto a leaf \mathcal{F}_x^c of the centre foliation of f .*

Proof The first step is to reduce the problem to the case of the unperturbed map f_0 . Let $H_f : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2$ be a leaf conjugacy as mentioned in Example 1. Thus H_f is a homeomorphism that maps each centre leaf $\{x\} \times \mathbb{T}^2$ of f_0 onto a centre leaf \mathcal{F}_x^c of f . Moreover, each restriction $H_f^x : \{x\} \times \mathbb{T}^2 \rightarrow \mathcal{F}_x^c$ of the leaf conjugacy is a C^1 diffeomorphism, and the (leaf) derivatives vary continuously on $M \times \mathbb{T}^2$. Endow each $\{x\} \times \mathbb{T}^2$ with the Riemann surface structure that turns H_f^x into a Riemann surface automorphism.

These Riemann surfaces are all conformally equivalent and, also by construction, their conformal structures vary continuously on $M \times \mathbb{T}^2$. Now to prove the proposition we only have to find a homeomorphism

$$\Phi : M \times \mathbb{T}_\tau^2 \rightarrow M \times \mathbb{T}^2, \quad (x, v) \mapsto (x, \Phi^x(v))$$

such that each $\Phi^x : \mathbb{T}_\tau^2 \rightarrow \mathbb{T}^2$ is a Riemann surface automorphism: then it suffices to take $\Psi = H_f \circ \Phi$.

Let $dz = dx + idy$ and $d\bar{z} = dx - idy$ be the canonical 1-forms on the torus $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ inherited from the complex plane \mathbb{C} through the canonical projection $\mathbb{C} \rightarrow \mathbb{T}^2$. Then let $\mu^x d\bar{z}/dz$ be the *Beltrami differential* of the Riemann surface structure on $\{x\} \times \mathbb{T}^2$ discussed in the previous paragraphs. In other words, $\mu^x : \mathbb{T}^2 \rightarrow \mathbb{D}$ is such that the metric $ds = |dz + \mu^x(v)d\bar{z}|$ belongs to the conformal structure at each point $(x, v) \in \{x\} \times \mathbb{T}^2$. The fact that these conformal structures vary continuously on $M \times \mathbb{T}^2$ ensures that

$$\mu : M \times \mathbb{T}^2 \rightarrow \mathbb{D}, \quad (x, v) \mapsto \mu^x(v)$$

is a continuous function. By compactness, it follows that $k = \sup |\mu|$ is strictly smaller than 1.

Via the canonical projection $\mathbb{C} \rightarrow \mathbb{T}^2$, we may view each μ^x as a continuous $\mathbb{Z} + i\mathbb{Z}$ -periodic function on the complex plane, with

$$\sup |\mu^x| \leq k < 1 \text{ for every } x \in M.$$

Then, by the measurable Riemann mapping theorem (Theorem 1), there exists a unique homeomorphism $w^x : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ that fixes 0, 1, and ∞ and satisfies the *Beltrami equation*

$$\partial_{\bar{z}} w^x = \mu^x \partial_z w^x, \quad (29)$$

which means that w^x maps the conformal structure defined by μ^x to the standard conformal structure on \mathbb{C} . Any other solution of (29) is obtained from w^x through post-composition with a Möbius automorphism of the complex plane. Moreover, using Theorem 2, the homeomorphism w^x depends continuously on the function μ^x in the sense that for any compact set $K \subset \mathbb{C}$ there exists $C(K) > 0$ such that

$$\sup_{z \in K} |w^x(z) - w^y(z)| \leq C(K) \sup |\mu^x - \mu^y| \text{ for any } x, y \in M. \quad (30)$$

Consequently, w^x depends continuously on $x \in M$, uniformly on each compact subset of \mathbb{C} .

We claim that

$$w^x(z+1) = w^x(z) + 1 \text{ and } w^x(z+i) = w^x(z) + \tau(x) \text{ for every } z \in \mathbb{C}, \quad (31)$$

where $\tau(x) = w^x(i)$. Keep in mind that $w^x(1) = 1$. Indeed, since μ^x is $\mathbb{Z} + i\mathbb{Z}$ -periodic, both $z \mapsto w^x(z+1)$ and $z \mapsto w^x(z+i)$ are solutions of the Beltrami equation that fix ∞ . Thus, there exist Möbius automorphisms $M_1(z) = a_1z + b_1$ and $M_i(z) = a_iz + b_i$ such that

$$w^x(z+1) = M_1 \circ w^x(z) \text{ and } w^x(z+i) = M_i \circ w^x(z)$$

for every z . It is clear that $b_1 = w^x(1) = 1$ and $b_i = w^x(i) = \tau(x)$. If $a_1 \neq 1$ then M_1 has a finite fixed point $p_1 = b_1/(1 - a_1)$. Let $z_1 = (w^x)^{-1}(p_1)$. Then

$$w^x(z_1 + n) = M_1^n(w^x(z_1)) = p_1 \text{ for every } n.$$

This contradicts the fact that w^x is injective, and that contradiction proves that $a_1 = 1$. The same argument proves that $a_i = 1$ and so the claim (31) is proved.

Now, (31) ensures that $w^x(\mathbb{Z} + i\mathbb{Z}) = L(1, \tau(x))$, and so w^x descends to a homeomorphism

$$W^x : \mathbb{T}^2 \rightarrow \mathbb{T}_{\tau(x)}.$$

Since the conformal structures defined by the μ^x are all conformally equivalent, as pointed out before, we have that the different values of $\tau(x)$ all belong to the same orbit of the modular group $\text{SL}(2, \mathbb{Z})$. See [36, Section 2]. It is also clear from the definition $\tau(x) = w^x(i)$ that $\tau(x)$ depends continuously on x . Since the modular group is discrete, it follows that the function $x \mapsto \tau(x)$ is constant. Denote by τ that constant.

Now just take $\Phi^x : \mathbb{T}_\tau^2 \rightarrow \mathbb{T}^2$ to be the inverse of W^x , for every $x \in M$.

6.2 Translation structures

The Riemann surface \mathbb{T}_τ^2 also carries a canonical structure of a translation surface, inherited from the complex plane \mathbb{C} . In what follows we consider on each centre leaf \mathcal{F}_x^c the translation structure transported from $\{x\} \times \mathbb{T}_\tau^2$ through the uniformisation map Ψ .

Lemma 3 *The stable and unstable holonomies of f are translations with respect to the translation structures on the centre leaves.*

Proof For any p and q in the same stable leaf of f , let $h_{p,q}^s : \mathcal{F}^c(p) \rightarrow \mathcal{F}^c(q)$ be the stable holonomy and

$$\mathbf{h}_{p,q}^s : \mathbb{T}_\tau^2 \rightarrow \mathbb{T}_\tau^2, \quad \mathbf{h}_{p,q}^s = \Psi_q^{-1} \circ h_{p,q}^s \circ \Psi_p$$

be its expression under the uniformisation provided by Proposition 1. This is a conformal homeomorphism of \mathbb{T}_τ^2 and so it lifts to a conformal automorphism $\hat{\mathbf{h}}_{p,q}^s : \mathbb{C} \rightarrow \mathbb{C}$, that is, a map of the form

$$\hat{\mathbf{h}}_{p,q}^s(z) = \alpha_{p,q}z + \beta_{p,q}$$

for some complex numbers $\alpha_{p,q}$ and $\beta_{p,q}$. Since the uniformisation map Ψ is a homeomorphism, $\hat{\mathbf{h}}_{p,q}^s(z)$ is a continuous function of p , q and z . That implies that $\alpha_{p,q}$ depends continuously on p and q . According to [36, Theorem 7], the set of all possible values of $\alpha_{p,q}$ is discrete. This implies that $\alpha_{p,q}$ is actually independent of p and q . On the other hand, as $d(p, q) \rightarrow 0$ the stable holonomy map $\mathbf{h}_{p,q}^s$ converges to the identity, and then so does its lift $\hat{\mathbf{h}}_{p,q}^s$. This means that $\alpha_{p,q} \rightarrow 1$ as $d(p, q) \rightarrow 0$, and so $\alpha_{p,q} = 1$ for every p and q . This proves that the stable holonomy map is a translation for any p and q in the same stable leaf.

The same argument applies to the unstable holonomies.

Let g be the map induced by f on the space of centre leaves, which we may view as the homeomorphism $g : M \rightarrow M$ defined by

$$f(\mathcal{F}_x^c) = \mathcal{F}_{g(x)}^c. \quad (32)$$

Since the leaf conjugacy $H_f : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2$ maps centre leaves of f_0 to centre leaves of f , it descends to a homeomorphism $h_f : M \rightarrow M$. The invariance property (10) means that this h_f conjugates g to g_0 :

$$g \circ h_f = h_f \circ g_0. \quad (33)$$

Recall that the uniformisation Ψ in Proposition 1 maps each fibre $\{x\} \times \mathbb{T}_\tau^2$ to the centre leaf \mathcal{F}_x^c . Thus,

$$\Psi^{-1} \circ f \circ \Psi : M \times \mathbb{T}_\tau^2 \rightarrow M \times \mathbb{T}_\tau^2, \quad (x, z) \mapsto (g(x), \mathbf{f}_x(z))$$

where $\mathbf{f}_x : \mathbb{T}_\tau^2 \rightarrow \mathbb{T}_\tau^2$ is given by

$$\mathbf{f}_x = (\Psi|_{\{g(x)\} \times \mathbb{T}_\tau^2})^{-1} \circ f \circ (\Psi|_{\{x\} \times \mathbb{T}_\tau^2}).$$

This is an invertible conformal map, and so its lift $\phi_x : \mathbb{C} \rightarrow \mathbb{C}$ is a conformal automorphism of the plane. It follows that $\phi_x(z) = a_x z + b_x$ for some $a_x, b_x \in \mathbb{C}$. In particular, the Jacobian of ϕ_x relative to the standard area form on \mathbb{C} is constant equal to $|a_x|^2$, and then so is the Jacobian of \mathbf{f}_x relative to the standard area form on \mathbb{T}_τ^2 (inherited from the complex plane). Since the torus has finite area, this implies that $|a_x| = 1$.

Next, consider the homeomorphism $A = h_f \times \lambda : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}_\tau^2$ where $\lambda : \mathbb{T}^2 \rightarrow \mathbb{T}_\tau^2$ is the map induced by the \mathbb{R} -linear isomorphism $\mathbb{C} \rightarrow \mathbb{C}$ that fixes 1 and sends i to τ (and which we also denote as λ). Then, denoting $Y = \Psi \circ A$,

$$Y^{-1} \circ f \circ Y : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2, \quad (x, v) \mapsto (g_0(x), \lambda^{-1}(a_x \lambda(v) + b_x)). \quad (34)$$

The fact that the affine map $\mathbb{C} \rightarrow \mathbb{C}$, $v \mapsto \lambda^{-1}(a_x \lambda(v) + b_x)$ descends to a homeomorphism of the torus means that the linear part

$$L_x : \mathbb{C} \rightarrow \mathbb{C}, \quad u \mapsto \lambda^{-1}(a_x \lambda(u))$$

preserves the lattice \mathbb{Z}^2 , and that means that $L_x \in \mathrm{SL}(2, \mathbb{Z})$. Since the latter is a discrete group, and the map $x \mapsto L_x$ is continuous, it follows that $x \mapsto L_x$ is actually constant, and then so is $x \mapsto a_x$. The spectrum of $L = L_x$ consists of $a = a_x$ and its conjugate, and so L is elliptic. By Remark 1, it follows that $L^n = \mathrm{Id}$ or, equivalently, $a^n = 1$ for some $n \in \{1, 2, 3, 4, 6\}$. Finally, we may write (34) as

$$Y^{-1} \circ f \circ Y : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2, \quad (x, v) \mapsto (g_0(x), Lv + w'(x)) \quad (35)$$

for some continuous function $w' : M \rightarrow \mathbb{T}^2$. Now, to complete the proof of claim (3) in the theorem we only need to explain why L may be taken equal to L_0 .

On the one hand, it is clear that the restriction

$$\{x\} \times \mathbb{T}^2 \rightarrow \{g_0(x)\} \times \mathbb{T}^2, \quad (x, v) \mapsto (g_0(x), L_0 v + w_0(x)) \quad (36)$$

of f_0 to each centre leaf is isotopic to the linear automorphism $L_0 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$. The leaves \mathcal{F}_x^c of the centre foliation of f are uniformly close to the vertical fibres $\{x\} \times \mathbb{T}^2$, and so each may be identified with \mathbb{T}^2 via the horizontal projection $(x, v) \mapsto v$. In this way, every restriction

$$(f | \mathcal{F}_x^c) : \mathcal{F}_x^c \rightarrow \mathcal{F}_{g(x)}^c$$

may be viewed as a map $f_x : \mathbb{T}^2 \rightarrow \mathbb{T}^2$. By construction, these maps are uniformly close to (36), and so they are all isotopic to $L_0 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$. In particular, the action of every f_x on the homology of the torus is given by L_0 .

On the other hand, (35) gives that f_x is topologically conjugate to a map of the form $v \mapsto Lv + w(x)$, and so their actions on the homology of the torus are linearly conjugate. The latter map is isotopic to $L : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, and so its action on the homology of the torus is given by L . This shows that L and L_0 are linearly conjugate, that is, there exists $P \in \mathrm{SL}(2, \mathbb{Z})$ such that $L = P^{-1}L_0P$. Then, denoting $Z = (\mathrm{Id} \times P) \circ Y$,

$$Z^{-1} \circ f \circ Z : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2, \quad (x, v) \mapsto (g_0(x), L_0 v + w(x)), \quad (37)$$

with $w(x) = P^{-1}w'(x)$. This finishes the proof of Theorem C.

7 Invariant line fields

We begin by proving Corollary D. Then we present a simple example where invariant line fields as in alternative (2) in Theorem C do occur.

Let $x_0 \in M$ be a fixed point of g_0 . The transformation $\mathbb{T}^2 \rightarrow \mathbb{T}^2$, $v \mapsto L_0(v) + w_0(x_0)$ lifts to a map

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad v \mapsto L_0(v) + W_0$$

where $W_0 \in \mathbb{R}^2$ is any vector that projects to $w_0(x_0)$ under the covering map $\mathbb{R}^2 \rightarrow \mathbb{T}^2$. Our assumptions ensure that 1 is not an eigenvalue of L_0 , that is, the linear map $\text{Id} - L_0$ is invertible. Let v_0 be the projection to the torus \mathbb{T}^2 of the vector $(\text{Id} - L_0)^{-1}(W_0)$. Then $p_0 = (x_0, v_0)$ is a fixed point of f_0 , and it is easy to see that this fixed point is *simple*: the spectrum of

$$Df_0(p_0) = \begin{pmatrix} Dg_0(x_0) & 0 \\ Dw_0(x_0) & L_0 \end{pmatrix}$$

is the union of the spectra of $Dg_0(x_0)$ and L_0 , and thus does not contain 1. Consequently, every diffeomorphism f in a neighbourhood of f_0 has a unique fixed point p close to p_0 , and this fixed point is still simple. We refer to p as the *continuation* of the fixed point p_0 of f_0 .

Let $f \in \mathcal{U}_T$ be an accessible μ -preserving diffeomorphism, and suppose that its centre Lyapunov exponents coincide. Let m be any $\mathbb{P}(D^c f)$ -invariant measure on $\mathbb{P}E^c$ projecting down to μ on $M \times \mathbb{T}^2$. By the Invariance Principle (Theorem 3), m admits a continuous disintegration $\{m_q : q \in M \times \mathbb{T}^2\}$ invariant under the dynamics and under the stable and unstable holonomies, that is, satisfying (18) and (19).

Lemma 4 *Assuming f is close enough to f_0 , the conditional probabilities m_q can have no atoms of mass greater than or equal to $1/2$.*

Proof Clearly, the number of atoms of each m_q with mass greater than or equal to $1/2$ is at most 2. Moreover, that number does not depend on q , because the disintegration is holonomy invariant and f is assumed to be accessible. Let p be the continuation of the fixed point p_0 for f . Since, by (19), the probability measure m_p is invariant under $\mathbb{P}((D^c f)_p)$, any atoms with mass greater than or equal to $1/2$ must be periodic points of period 1 or 2. However, such periodic points cannot exist if f is close to f_0 , because then $\mathbb{P}((D^c f)_p)$ is close to $\mathbb{P}((D^c f_0)_{p_0}) = \mathbb{P}(L_0)$ which, by assumption, has no periodic points with period less than 3.

This means that the alternative (2) in Theorem C cannot occur in the present setting. Thus f must satisfy alternative (3), that is, it must be topologically conjugate to an L_0 -affine extension of g_0 . This proves Corollary D.

On the other hand, the alternative (2) may occur for volume-preserving, and even symplectic, diffeomorphisms arbitrarily close to f_0 when $L_0 = \text{Id}$:

Example 3 Consider $f_0 : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2$ given by $f_0(x, v) = (g_0(x), v)$. Assume that g preserves some symplectic form ω_M on M and let ω_S be the standard area form on \mathbb{T}^2 . Then f_0 preserves the symplectic form $\omega = \omega_M \times \omega_S$

on $M \times \mathbb{T}^2$. By Theorem 4, for any $r \geq 2$ there exist functions $w : M \rightarrow \mathbb{T}^2$ arbitrarily C^r -close to zero such that the diffeomorphism

$$f_w : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2, \quad f_w(x, v) = (g(x), v + w(x))$$

is stably accessible. Observe that f_w is ω -symplectic. Next, define

$$f : M \times \mathbb{T}^2 \rightarrow M \times \mathbb{T}^2, \quad f(x, (v_1, v_2)) = (g(x), w(x) + (v_1 + \alpha(v_2), v_2)),$$

where $\alpha : \mathbb{S}^1 \rightarrow \mathbb{R}$ is a smooth function. Assuming that α is C^r -close to zero, f is C^r close to f_w and, hence, it is accessible. It is clear that f can be made arbitrarily close to f_0 by picking both α and w sufficiently small. In particular, f is partially hyperbolic, centre-bunched and dynamically coherent. Moreover, f is itself a skew-product, and the vertical fibration $\{x\} \times \mathbb{T}^2$ is the centre foliation. It is also clear that f is ω -symplectic. By construction, the centre derivative of f is idempotent:

$$D^c f_{x,v} = \begin{pmatrix} 1 & \alpha'(v_2) \\ 0 & 1 \end{pmatrix}. \quad (38)$$

In particular, the horizontal line bundle $H_{x,v} \equiv (1, 0)$ is invariant under $D^c f$. Observe that $m = \text{vol} \times \delta_{H_{x,v}}$ is a $\mathbb{P}(D^c f)$ -invariant probability measure on $\mathbb{P}E^c$ that projects down to the volume measure on $M \times \mathbb{T}^2$.

8 Proof of Theorem E

Take $f_0 : M \times \mathbb{S}^2 \rightarrow M \times \mathbb{S}^2$ to be a C^r , $r \geq 2$ Möbius extension of a transitive Anosov diffeomorphism $g_0 : M \rightarrow M$, as defined in (5). Assume that f_0 preserves a given measure μ in the Lebesgue class of $M \times \mathbb{S}^2$. It is clear that f_0 is a partially hyperbolic skew-product. In particular (cf. Example 1), every μ -preserving diffeomorphism in a C^r -neighbourhood belongs to $\mathcal{F}_\mu^r(M, \mathbb{S}^2)$ and, thus, satisfies some of the three alternatives in Theorem A.

The first alternative corresponds precisely to the claim (1) in the present Theorem E. The second one is excluded here, because the sphere \mathbb{S}^2 supports neither continuous line fields nor continuous pairs of line fields. In the third one, the centre leaves are endowed with continuous Riemann surface structures invariant under f and under the stable and unstable holonomies. We are left to checking that this yields the claim (2) of Theorem E.

Proposition 2 *There exists a homeomorphism $\Psi : M \times \mathbb{S}^2 \rightarrow M \times \mathbb{S}^2$ whose restriction to each fibre $\{x\} \times \mathbb{S}^2$, $x \in M$ is a Riemann surface automorphism onto the centre leaf \mathcal{F}_x^c of f .*

Proof Let $H_f : M \times \mathbb{S}^2 \rightarrow M \times \mathbb{S}^2$ be a leaf conjugacy as mentioned in Example 1. Thus H_f is a homeomorphism that maps each centre leaf $\{x\} \times \mathbb{S}^2$ of f_0 onto a centre leaf \mathcal{F}_x^c of f . Moreover, each restriction $H_f^x : \{x\} \times \mathbb{S}^2 \rightarrow \mathcal{F}_x^c$ of the leaf conjugacy is a C^1 diffeomorphism, and the leaf derivatives vary continuously on $M \times \mathbb{S}^2$. Endow $\{x\} \times \mathbb{S}^2$ with the Riemann surface structure

that turns H_f^x into a Riemann surface automorphism. This structure may be represented by a function $\mu^x : \mathbb{S}^2 \rightarrow \mathbb{D}$, such that the metric $ds = |dz + \mu^x(v)d\bar{z}|$ belongs to the conformal structure at each point $(x, v) \in \{x\} \times \mathbb{S}^2$. The fact that these conformal structures vary continuously on $M \times \mathbb{S}^2$ ensures that the function

$$\mu : M \times \mathbb{S}^2 \rightarrow \mathbb{D}, \quad (x, v) \mapsto \mu^x(v)$$

is continuous.

By the measurable Riemannian mapping theorem (Theorem 1), for each $x \in M$ there is a unique homeomorphism $w^x : \bar{\mathbb{S}}^2 \rightarrow \bar{\mathbb{S}}^2$ that fixes 0, 1, and ∞ and satisfies the Beltrami equation

$$\partial_{\bar{z}} w^x = \mu^x \partial_z w^x, \quad (39)$$

which means that w^x maps the conformal structure defined on \mathbb{S}^2 by μ^x to the standard conformal structure on \mathbb{S}^2 . Any other solution of (39) is obtained from w^x through post-composition with a Möbius automorphism of the sphere. Moreover, by Theorem 2, w^x depends continuously on the function μ^x , uniformly on \mathbb{S}^2 . Consequently, w^x depends continuously on $x \in M$, uniformly on the sphere, and so the map

$$W : M \times \mathbb{S}^2 \rightarrow M \times \mathbb{S}^2, \quad (x, v) \mapsto (x, w^x(v))$$

is a homeomorphism. To complete the proof, define $\Psi = H_f \circ W^{-1} : M \times \mathbb{S}^2 \rightarrow M \times \mathbb{S}^2$. By construction, Ψ maps each $\{x\} \times \mathbb{S}^2$ conformally to the corresponding centre leaf \mathcal{F}_x^c .

It follows from Proposition 2 that we may write

$$\Psi^{-1} \circ f \circ \Psi : M \times \mathbb{S}^2 \rightarrow M \times \mathbb{S}^2, \quad (x, v) \mapsto (g(x), \tilde{\zeta}_x(v))$$

where $g : M \rightarrow M$ is the Anosov homomorphism in (32), and each $\tilde{\zeta}_x : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is a conformal automorphism of the sphere, that is, a Möbius transformation. Let $h_f : M \rightarrow M$ be the conjugacy between g and g_0 , as in (33). Then

$$(h_f \times \text{Id})^{-1} \circ \Psi^{-1} \circ f \circ \Psi \circ (h_f \times \text{Id}) : M \times \mathbb{S}^2 \rightarrow M \times \mathbb{S}^2, \quad (x, v) \mapsto (g_0(x), \zeta_x(v))$$

with $\zeta_x = \tilde{\zeta}_{h_f(x)}$. This completes the proof of Theorem E.

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Conflict of interest

The authors declare that they have no conflict of interest.

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