STABLE ACCESSIBILITY WITH 2-DIMENSIONAL CENTER

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Abstract. For partially hyperbolic diffeomorphisms with 2-dimensional center, accessibility is C^1-stable. Moreover, for center bunched skew-products (stable) accessibility is C^∞-dense.

Contents

1. Introduction

A diffeomorphism $f : M \to M$ of a compact manifold M is *partially hyperbolic* if there exist: a continuous splitting of the tangent bundle $TM = E^u \oplus E^c \oplus E^s$ invariant under the derivative Df (all three sub-bundles are assumed to have positive dimension); a Riemannian metric $\|\cdot\|$ on M; and positive continuous functions ν, $\hat\nu$, γ, $\hat\gamma$ with ν, $\hat\nu < 1$ and $\nu < \gamma < \hat\gamma^{-1} < \hat\nu^{-1}$, such that

$$\begin{aligned} \|Df(p)v\| &< \nu(p) && \text{if } v \in E^s(p),\\ \gamma(p) < \|Df(p)v\| &< \hat\gamma(p)^{-1} && \text{if } v \in E^c(p),\\ \hat\nu(p)^{-1} < \|Df(p)v\| & && \text{if } v \in E^u(p). \end{aligned} \tag{1}$$

for any unit vector $v \in T_pM$. This is an open property in the space of C^1 diffeomorphisms. We will denote $d_* = \dim E^*$, for $* \in \{u, c, s\}$, and $d = \dim M$.

The stable bundle E^s and the unstable bundle E^u are uniquely integrable and their integral manifolds form two quasi transverse continuous foliations, $\mathcal{W}^u = \mathcal{W}^u_f$ and $\mathcal{W}^s = \mathcal{W}^s_f$, whose leaves are immersed submanifolds of the same class of differentiability as f. These are called the *strong unstable* and *strong stable* foliations of f. They are invariant under f, in the sense that $f(\mathcal{W}^*(x)) = \mathcal{W}^*(f(x))$ for any $x \in M$ and $* \in \{u, s\}$. Given $\epsilon > 0$ and $* \in \{u, s\}$, we represent by $\mathcal{W}^*_\epsilon(x) = \mathcal{W}^*_{f,\epsilon}(x)$ the ϵ-neighborhood of x inside $\mathcal{W}^*(x)$.

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Given two points $x, y \in M$, we say that x is accessible from y if there exists a C^1 path that connects x to y and is tangent at every point to the union $E^u \cup E^s$. The equivalence classes of this (equivalence) relation are called *f-accessibility classes.* The diffeomorphism f is called *accessible* if there exists a unique f-accessibility class, namely, the ambient M. Moreover, f is called *stably accessible* if it admits a C^1 neighborhood $\mathcal{U}$ such that every C^2 diffeomorphism $g \in \mathcal{U}$ is accessible.

For any $k \geq 1$, we denote by $\mathcal{PH}^k$ the space of C^k partially hyperbolic diffeomorphisms in M. Most of our results concern the subspace $\mathcal{PH}_2^k$ of diffeomorphisms $f \in \mathcal{PH}^k$ with 2-dimensional center bundle, that is, such that $d_c = 2$.

Theorem A. *If $f \in \mathcal{PH}_2^1$ is accessible then f is stably accessible.*

We say that an f-accessibility class C is *stable* if for every compact set $K \subset C$ there exists a C^1 neighborhood $\mathcal{U} = \mathcal{U}_K$ of f such that K is contained in a unique g-accessibility class for every C^2 diffeomorphism $g \in \mathcal{U}$. In particular, f is stably accessible if, and only if, the ambient M is a stable f-accessibility class.

Stable accessibility classes are open sets. Indeed, let p and q be two distinct points in C (for instance, in the same stable manifold). For any $r \in M$ close to q, let $h : M \to M$ be a diffeomorphism C^∞ close to the identity, such that $h(p) = p$ and $h(r) = q$. Then $g = h \circ f \circ h^{-1}$ is close to f. Taking $K = \{p, q\}$, the assumption implies that p and q are in the same g-accessibility class. This means that p and q are in the same f-accessibility class, that is, $r \in C$. So, C contains a whole neighborhood of q.

Here we prove that the converse is also true, at least when the center bundle is 2-dimensional:

Theorem B. *If $f \in \mathcal{PH}_2^1$ then any open f-accessibility class is stable.*

Theorem A is a direct consequence of Theorem B. The main technical step in the proof of Theorem B is a result on approximation of general paths in accessibility classes by a certain class of paths for which a continuation exists for every nearby diffeomorphism. This result is stated in Section 4 (Theorem 4.1), where we also explain how it leads to Theorem B.

In Sections 6–5 we state and prove a result about density of stable accessibility (Theorem 6.1), for a class of fibered partially hyperbolic diffeomorphisms with 2-dimensional center bundle. It contains a claim made in Section 7 of our paper [2], that was used for proving Theorem H in that paper. After our research had been completed, we learned from V. Horita and M. Sambarino that they had independently obtained a similar result, in a paper that appeared in [11].

When the center dimension $d_c = 1$, the accessibility property is always stable [5]. The present work extends that fact to center dimension equal to 2. Recently, and also in the 2-dimensional case, J. Rodriguez-Hertz and C. Vasquez [9] proved that accessibility classes are immersed submanifolds, which implies Theorem A.

When the center bundle is one-dimensional, the (stable) accessibility property is known to be C^r dense among partially hyperbolic diffeomorphisms [3, 8]. Without any hypothesis on the dimension of the central bundle, Dolgopyat and Wilkinson [6] proved that stable accessibility is C^1 dense.

2. Deformations paths

In this section, all maps are assumed to be C^1 and proximity is always meant in the C^1 topology. We introduce a class of paths, that we call deformation paths, contained in accessibility classes and having a useful property of persistence under variation of the diffeomorphism and the base point. This also provides a kind of parametrization for accessibility classes:

Theorem 2.1. *For every $f \in \mathcal{PH}^1$, there exist $k \geq 1$, a neighborhood $\mathcal{V}$ of f and a sequence $P_l : \mathcal{V} \times M \times \mathbb{R}^{k(d_u+d_s)l} \to M$ of continuous maps such that, for any $g \in \mathcal{V}$,*

(1) $P_m(g, \cdot\,,, w) \circ P_l(g, \cdot\,, v) = P_{l+m}(g, \cdot\,, (v, w))$ *for every $l \geq 1$ and $m \geq 1$ and $v \in \mathbb{R}^{k(d_u+d_s)l}$ and $w \in \mathbb{R}^{k(d_u+d_s)m}$;*
(2) $\zeta \mapsto P_l(g, \zeta, v)$ *is a homeomorphism from M to M, with $P_l(g, \cdot, 0) = \mathrm{id}$, for every $l \geq 1$ and $v \in \mathbb{R}^{k(d_u+d_s)l}$;*
(3) $\cup_{n\geq1} P_n(\{(g, z)\} \times \mathbb{R}^{k(d_u+d_s)n})$ *is the g-accessibility class of each $z \in M$.*

A *deformation path based on* (f, z) is a (continuous) map $\gamma : [0, 1] \to M$ such that there exist $l \geq 1$ and a continuous map $\Gamma : [0, 1] \to \mathbb{R}^{k(d_u+d_s)l}$ satisfying $\gamma(t) = P_l(f, z, \Gamma(t))$. Notice that any deformation path based on (f, z) is contained in the f-accessibility class of z. It follows immediately from the definition that deformation paths are persistent, in the following sense:

Corollary 2.2. *If $\gamma : [0, 1] \to M$ is a deformation path based on (f, z) then, for any g close to f and any w close to z, there exists a deformation path based on (g, w) that is uniformly close to γ.*

In the remainder of this section we prove Theorem 2.1. Let $I = [-1, 1]$. We need the following particular case of [10, Theorem 4.1]:

Lemma 2.3. *For every $f \in \mathcal{PH}^1$ and $\zeta \in M$, there exists a neighborhood $\mathcal{V}$ of f and a continuous map $\psi = \psi_{f,\zeta} : \mathcal{V} \times I^d \to M$ such that for every $g \in \mathcal{V}$,*

(1) $\psi(g, 0) = \zeta$,
(2) $x \mapsto \psi(g, x)$ *is a homeomorphism,*
(3) $\psi(g, x, y) \in W^u_g(\psi(g, 0, y))$ *for every $x \in I^{d_u}$ and $y \in I^{d-d_u}$.*

Lemma 2.4. *For every $f \in \mathcal{PH}^1$ there exist a neighborhood $\mathcal{V}$ of f, numbers $k \geq 1$ and $\epsilon > 0$ and continuous maps $\Phi_u : \mathcal{V} \times M \times \mathbb{R}^{kd_u} \to M$ and $\Phi_s : \mathcal{V} \times M \times \mathbb{R}^{kd_s} \to M$ such that:*

(1) $x \mapsto \Phi_u(g, x, v)$ *is a homeomorphism, for every $g \in \mathcal{V}$ and $v \in \mathbb{R}^{kd_u}$;*
(2) $\mathcal{W}^u_{g,\epsilon}(x) \subset \Phi_u(\{g\} \times \{x\} \times \mathbb{R}^{kd_u}) \subset W^u_g(x)$ *for every $g \in \mathcal{V}$ and $x \in M$,*

and analogously for Φ_s.

Proof. We will only go through the details of the construction of Φ_u, the case of Φ_s being analogous. Let $h_t : I \to I$ be the flow satisfying $(dh_t/dt)(x) = 1 - h_t(x)^2$. Let $H : I^d \to I$ be given by $H(v) = (1 - v^2_{d_u+1})...(1 - v^2_d)$. For $v \in \mathbb{R}^{d_u}$, let $h_v : I^d \to I^d$ be given by

$$h_v(x) = \big(h_{H(x)v_1}(x_1), ..., h_{H(x)v_{d_u}}(x_{d_u}), x_{d_u+1}, ..., x_d\big).$$

Pick points $\zeta_i \in M$, $1 \leq i \leq k$ so that the interiors of the images $\psi_i(\{f\} \times I^d)$ cover M, where $\psi_i = \psi_{f,\zeta_i} : \mathcal{V}_i \times I^d \to M$ are the maps given by Lemma 2.3. Let $\mathcal{V}$ be a neighborhood of f contained in $\cap_i \mathcal{V}_i$ and ϵ be a positive number such that for every $g \in \mathcal{V}$ and $z \in M$ there exist i and y such that

$$W^u_{g,\epsilon}(z) \subset \psi_i\big(\{g\} \times \mathrm{inter}(I^{d_u}) \times \{y\}\big).$$

Let $\Phi_i : \mathcal{V} \times M \times \mathbb{R}^{d_u} \to M$ be given by

$$\Phi_i(g, \psi_i(g,\zeta), v) = \psi_i(g, h_v(\zeta)) \quad \text{for } \zeta \in I^d$$
$$\Phi_i(g, z, v) = z \quad \text{if } z \notin \psi_i(\{g\} \times I^d).$$

Then define $\Phi^{(i)} : \mathcal{V} \times M \times \mathbb{R}^{id_u} \to M$, $1 \le i \le k$ by

$$\Phi^{(1)} = \Phi_1 \quad \text{and} \quad \Phi^{(i+1)}(g, \cdot, (w_i, w)) = \Phi_{i+1}(g, \cdot, w) \circ \Phi^{(i)}(g, \cdot, w_i)$$

and take $\Phi_u = \Phi^{(k)}$. Claim (1) follows from part (2) of Lemma 2.3, by composition. The lower bound in claim (2) follows from the choice of ϵ and the upper bound is a consequence of part (3) of Lemma 2.3. □

Proof of Theorem 2.1. Define $P_l : \mathcal{V} \times M \times \mathbb{R}^{k(d_u+d_s)l} \to M$, $l \in \mathbb{N}$ by letting

$$P_1(g, \cdot, (w_u, w_s)) = \Phi_s(g, \cdot, w_s) \circ \Phi_u(g, \cdot, w_u)$$

for $w_u \in \mathbb{R}^{kd_u}$ and $w_s \in \mathbb{R}^{kd_s}$ and

$$P_l(g, \cdot, (w_1, \dots, w_l)) = P_1(g, \cdot, w_l) \circ \cdots \circ P_1(g, \cdot, , w_1)$$

for $w_1, \dots, w_l \in \mathbb{R}^{k(d_u+d_s)}$. Property (1) in Theorem 2.1 is a direct consequence of this definition. Property (2) follows from part (2) of Lemma 2.3, by composition. Finally, Lemma 2.4 gives that $\cup_{n \ge 1} P_n(g, z, \mathbb{R}^{k(d_u+d_s)n})$ is the g-accessibility class of z, as claimed in part (3) of the theorem. □

Theorem 2.5. *For every $l \ge 1$ there exists $m \ge 1$ and for every $v \in \mathbb{R}^{k(d_u+d_s)l}$ there exists $v^* \in \mathbb{R}^{k(d_u+d_s)m}$, depending linearly on v, such that*

$$P_m(g, \cdot, v^*) = P_l(g, \cdot, v)^{-1} \quad \textit{for any } g \in \mathcal{V}.$$

The initial step in the proof of this theorem is:

Lemma 2.6.

(a) *For every $v \in \mathbb{R}^{kd_u}$ there exists $v^* \in \mathbb{R}^{k(d_u+d_s)k}$ such that*

$$P_k(g, \cdot, v^*) = \Phi_u(g, \cdot, v)^{-1} \quad \textit{for any } g \in \mathcal{V}.$$

(b) *For every $v \in \mathbb{R}^{kd_s}$ there exists $v^* \in \mathbb{R}^{k(d_u+d_s)k}$ such that*

$$P_k(g, \cdot, v^*) = \Phi_s(g, \cdot, v)^{-1} \quad \textit{for any } g \in \mathcal{V}.$$

Proof. Write $v = (v_1, \dots, v_k)$ with $v_j \in \mathbb{R}^{d_u}$ for $j = 1, \dots, k$. Then define

$$v^* = (v_k^*, 0, v_{k-1}^*, 0, \dots, v_1^*, 0) \in \mathbb{R}^{k(d_u+d_s)k},$$

where $0 \in \mathbb{R}^{kd_s}$ and each $v_j^* \in \mathbb{R}^{kd_u}$ is defined by

$$v_j^* = (v_{j,1}^*, \dots, v_{j,k}^*) \quad \text{with} \quad v_{j,i}^* = \begin{cases} 0 & \text{if } i \ne j \\ -v_j & \text{if } i = j. \end{cases}$$

Then, by the definition of Φ_i,

$$\Phi_u(g, \cdot, v_i^*) = \Phi_i(g, \cdot, -v_i) = \Phi_i(g, \cdot, v_i)^{-1}. \tag{2}$$

By the definition of Φ_u,

$$\Phi_u(g, \cdot, v) = \Phi_k(g, \cdot, v_k) \circ \cdots \circ \Phi_1(g, \cdot, v_1). \tag{3}$$

By the definition of P_k,

$$\begin{aligned} P_k(g, \cdot, v^*) &= P_1(g, \cdot, (v_1^*, 0)) \circ \cdots \circ P_1(g, \cdot, (v_k^*, 0)) \\ &= \Phi_u(g, \cdot, v_1^*) \circ \cdots \circ \Phi_u(g, \cdot, v_k^*). \end{aligned} \tag{4}$$

Claim (a) in the lemma is a direct consequence of (2) – (4).

For claim (b), take $v^* = (0, v_k^*, 0, v_{k-1}^*, \dots, 0, v_1^*) \in \mathbb{R}^{k(d_u+d_s)k}$, where $0 \in \mathbb{R}^{kd_u}$ and each $v_j^* \in \mathbb{R}^{kd_s}$ is defined just as before. Then argue as in the previous case. $\square$

Proof of Theorem 2.5. For $l = 1$, take $m = 2k$ and for each $v = (v_u, v_s) \in \mathbb{R}^{k(d_u+d_s)}$ take $v^* = (v_s^*, v_u^*) \in \mathbb{R}^{k(d_u+d_s)m}$, where v_u^* and v_s^* are the vectors in $\mathbb{R}^{k(d_u+d_s)k}$ given by Lemma 2.6. Then

$$\begin{aligned} P_m(g, \cdot, v^*) &= P_k(g, \cdot\,, v_u^*) \circ P_k(g, \cdot\,, v_s^*) = \Phi_u(g, \cdot\,, v_u)^{-1} \circ \Phi_s(g, \cdot\,, v_s)^{-1} \\ &= \left[\Phi_s(g, \cdot\,, v_s) \circ \Phi_u(g, \cdot\,, v_u)\right]^{-1} = P_1(g, \cdot\,, v)^{-1}. \end{aligned}$$

In general, for any $l \geq 1$, take $m = 2kl$ and for each $v = (v_{u,1}, v_{s,1}, \dots, v_{u,l}, v_{s,l})$ in $\mathbb{R}^{k(d_u+d_s)l}$ consider $v^* = (v_{s,l}^*, v_{u,l}^*, \dots, v_{s,1}^*, v_{u,1}^*)$ in $\mathbb{R}^{k(d_u+d_s)m}$. Then

$$P_m(g, \cdot\,, v^*) = P_{2k}(g, \cdot\,, (v_{s,1}^*, v_{u,1}^*)) \circ \cdots \circ P_{2k}(g, \cdot\,, (v_{s,l}^*, v_{u,l}^*))$$

By the previous paragraph, we may rewrite the right-hand side of this equality as

$$P_1(g, \cdot\,, v_1)^{-1} \circ \cdots \circ P_1(g, \cdot\,, v_l)^{-1} = \left[P_1(g, \cdot\,, v_l) \circ \cdots \circ P_1(g, \cdot\,, v_1)\right]^{-1}.$$

It follows that $P_m(g, \cdot\,, v^*) = P_l(g, \cdot\,, v)^{-1}$, as claimed. $\square$

3. An intersection property

The following result lies at the heart of the proof of Theorem B:

Theorem 3.1. *Let f be a partially hyperbolic diffeomorphism with 2-dimensional center. Let D be a 2-dimensional disk transverse to $E^s \oplus E^u$ and η_u, η_s be smooth paths in D intersecting transversely at some point. Then, for every C^1 diffeomorphism g close to f and any continuous paths γ_u, γ_s uniformly close to η_u, η_s, there are points x_u, x_s in the images of γ_u, γ_s such that $\mathcal{W}_g^u(x_u)$ intersects $\mathcal{W}_g^s(x_s)$.*

For the proof we need the following lemma. Let 0^d denote the origin of $\mathbb{R}^d$. Recall that if M, N, P are compact orientable smooth manifolds and $f : M \to P$ and $g : N \to P$ are continuous maps, then we can define the intersection number of f and g as the intersection number of the map $(f, g) : M \times N \to P \times P$, $(f, g)(x, y) = (f(x), g(y))$ with the diagonal in $P \times P$. Clearly, the intersection number is a homotopy invariant of f and g.

Lemma 3.2. *Let $n, u, s \in \mathbb{N}$ with $n = u + 2 + s$. There exists $\epsilon > 0$ with the following property. Let W^u and W^s be foliations with C^1 leaves in $\mathbb{R}^n$, tangent to continuous distributions E^u and E^s of u- and s-dimensional planes. Assume that E_x^u is ϵ-close to $\mathbb{R}^u \times \{0^{2+s}\}$ and E_x^s is ϵ-close to $\{0^{u+2}\} \times \mathbb{R}^s$ for every x in the unit ball B^n of $\mathbb{R}^n$. Let $\gamma_u, \gamma_s : [-1, 1] \to \mathbb{R}^n$ be continuous paths ϵ-close to the paths $\eta_u, \eta_s : [-1, 1] \to \mathbb{R}^n$ given by $\eta_u(t) = (0^u, t, 0, 0^s)$ and $\eta_s(t) = (0^u, 0, t, 0^s)$. Then there exist $t_u, t_s \in (-1, 1)$ such that $W^u(\gamma_u(t_u))$ intersects $W^s(\gamma_s(t_s))$.*

Proof. For $k \in \mathbb{N}$, consider the sphere S^k as the one point compactification of $\mathbb{R}^k$. Let $\rho_u = \gamma_u - \eta_u$ and $\hat\rho_u$ be a continuous extension of ρ_u to $\mathbb{R}$, with compact support and $\|\hat\rho_u\|_0 = \|\rho_u\|_0$. Let $\phi_u : [-1/4, 1/4]^{u+1} \to \mathbb{R}^{2+s}$ be the only continuous map such that $\phi_u(0^u, t) = 0$ and $x \mapsto (x, \phi_u(x, t)) + \gamma_u(t)$ is a C^1 map from $[-1/4, 1/4]^u$ to $W^u(\gamma_u(t))$, for every $t \in [-1/4, 1/4]$. Let $\hat\phi_u$ be a continuous extension of ϕ_u to $\mathbb{R}^{u+1}$, with compact support and $\|\hat\phi_u\|_0 = \|\phi_u\|_0$. The map

$$\mathbb{R}^u \times \mathbb{R} \to \mathbb{R}^u \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^s, \quad (x, t) \mapsto (x, \hat\phi_u(x, t)) + (0^u, t, 0, 0^s) + \hat\rho_s(t)$$

coincides with $(x,t) \mapsto (x,t,0,0^s)$ outside some compact set and, thus, admits a continuous extension $\Phi_u : S^u \times S^1 \to S^u \times S^1 \times S^1 \times S^s$. By construction, $\Phi_u(x,t) = (x, \phi_u(x,t)) + \gamma_u(t)$ for every $(x,t) \in [-1/4, 1/4]^{u+1}$. Since ϵ is small, Φ_u is uniformly close (and, hence, homotopic) to the map $(x,t) \mapsto (x,t,0,0^s)$.

Define analogous objects ρ_s, $\hat{\rho}_s$, ϕ_s, $\hat{\phi}_s$ and Φ_s. Then Φ_u and Φ_s have intersection number 1. Consequently, there exist $(x_u, t_u) \in S^u \times S^1$ and $(t_s, x_s) \in S^1 \times S^s$ such that $\Phi_u(x_u,t_u) = \Phi_s(t_s,x_s)$. Since $\Phi_u(x_u,t_u)$ is close to $(x_u,t_u,0,0^s)$ and $\Phi_s(t_s,x_s)$ is close to $(0^u,0,t_s,x_s)$, all x_u,t_u,t_s,x_s are small. Thus, we have

$$\Phi_u(x_u,t_u) = (x_u, \phi_u(x_u,t_u)) + \gamma_u(t_u) \in \mathcal{W}^u(\gamma_u(t_u)),$$
$$\Phi_s(t_s,x_s) = (\phi_s(t_s,x_s), x_s) + \gamma_s(t_s) \in \mathcal{W}^s(\gamma_s(t_s)).$$

and both coincide, giving the conclusion. □

Proof of Theorem 3.1. Consider a local change of coordinates close to the transverse intersection of η_s and η_u and apply Lemma 3.2. □

4. An approximation property

The other main ingredient in the proof of Theorem B is:

Theorem 4.1. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism and U be an open f-accessibility class. Then, for every $z \in U$, the set of deformation paths based on (f,z) is dense in $C^0([0,1],U)$.*

The proof of this theorem is contained in the three lemmas that follow. Let $z \in M$ be such that the f-accessibility class U of z is open.

Lemma 4.2. *There exist $l_0 \geq 1$ and $v_0 \in \mathbb{R}^{k(d_u+d_s)l_0}$ such that $y_0 = P_{l_0}(f,z,v_0)$ is in the interior of $P_{l_0}(f,z,V)$ for every neighborhood V of v_0.*

Proof. For $l, m \geq 1$, let $K(l,m) = P_l(f,z,[-m,m]^{k(d_u+d_s)l})$. Each $K(l,m)$ is compact and, hence, closed in M. Since the union $U = \cup_{l,m} K(l,m)$ is an open set and M is a Baire space, there exist $l_0, m_0 \geq 1$ such that $K(l_0,m_0)$ has nonempty interior. Let l_0 be fixed and consider the map $\varphi : \mathbb{R}^{k(d_u+d_s)l_0} \to M$ given by $\varphi(v) = P_{l_0}(f,z,v)$. A point $y \in M$ is a *regular value* if every $v_0 \in \varphi^{-1}(y)$ is a *regular point*, meaning that $\varphi(v_0)$ is in the interior of $\varphi(V)$ for every neighborhood V of v_0. Since the set of regular values is residual in M (see [1, Proposition 7.6]), there exists some regular value $y_0 \in \varphi(\mathbb{R}^{k(d_u+d_s)l_0})$. Any $v_0 \in \varphi^{-1}(y_0)$ satisfies the conclusion of the lemma. □

Lemma 4.3. *For any compact set $K \subset U$ and any $\epsilon > 0$ there is $\delta > 0$ such that, given any $y', y'' \in K$ with $d(y',y'') \leq \delta$, there exists a deformation path $\gamma : [0,1] \to M$, satisfying $\gamma(0) = y'$ and $\gamma(1) = y''$ and* $\operatorname{diam} \gamma([0,1]) < \epsilon$.

Proof. Let l_0, v_0 and $y_0 = P_{l_0}(f,z,v_0)$ be as in Lemma 4.2. For each $y \in K$, choose $q \geq 1$ and $u \in \mathbb{R}^{k(d_u+d_s)q}$ such that $P_q(f,y_0,u) = y$. By Theorem 2.1, $P_{l_0+q}(f,z,(v_0,u)) = y$ and the map $y_0' \mapsto P_q(f,y_0',u)$ defines a homeomorphism from a neighborhood of y_0 to a neighborhood of y. It follows that the image of any small ball around v_0 under the map $v_0' \mapsto P_{l_0+q}(f,z,(v_0',u))$ has diameter less than ϵ and contains some neighborhood W_y of y. Let $\delta > 0$ be a Lebesgue number for the cover $\{W_y : y \in K\}$ of K obtained in this way. Given y' and y'' as in the statement, take $y \in K$ such that W_y contains both y' and y'' and, hence, there are v_0' and v_0''

close to v_0 such that $P_{l_0+q}(f,z,(v_0',u)) = y'$ and $P_{l_0+q}(f,z,(v_0'',u)) = y''$. Then define $\gamma(t) = P_{l_0+q}(f,z,\Gamma(t))$, with $\Gamma(t) = ((1-t)v_0' + tv_0'',u))$ for $t \in [0,1]$. $\square$

Lemma 4.4. *The deformation paths based on (f,z) are dense in $C^0([0,1],U)$.*

Proof. Given $c \in C^0([0,1],U)$ and $\epsilon > 0$, consider $K = c([0,1])$ and let $\delta > 0$ be given by Lemma 4.3. We may take $\delta < \epsilon$, of course. Fix $N \geq 1$ such that $d(x_{i-1},x_i) < \delta$ for $i = 1,\dots,N$, where $x_i = c(i/N)$. By Lemma 4.3, for each $i = 1,\dots,N$ there exists $l_i \geq 1$ and a continuous map $\Gamma_i : [0,1] \mapsto \mathbb{R}^{k(d_u+d_s)l_i}$ such that the deformation path $\gamma_i(t) = P_l(f,z,\Gamma_i(t))$ satisfies $\gamma_i(0) = x_{i-1}$ and $\gamma_i(1) = x_i$ and $\operatorname{diam}\gamma_i([0,1]) < \epsilon$. For each $i = 1,\dots,N$, let $m_i \geq 1$ be associated to l_i, in the sense of Theorem 2.5. Take $\gamma : [0,1] \mapsto M$ to be defined by $\gamma(t) = P_L(f,z,\Gamma(t))$, where $L = l_1 + m_2 + l_2 + \dots + m_N + l_N$ and

$$\Gamma(t) = (\hat{\Gamma}_1(t), \Gamma_2(0)^*, \hat{\Gamma}_2(t), \dots, \Gamma_N(0)^*, \hat{\Gamma}_N(t))$$

with

$$\hat{\Gamma}_i(t) = \begin{cases} \Gamma_i(0) & \text{if } t \leq (i-1)/N \\ \Gamma_i(Nt-i+1) & \text{for } (i-1)/N \leq t \leq i/N \\ \Gamma_i(1) & \text{if } t \geq i/N. \end{cases}$$

Note that Γ is continuous, as each of the $\hat{\Gamma}_i$ is continuous. We claim that

$$\gamma(t) = \gamma_i(Nt-i+1) \quad \text{for every } t \in [(i-1)/N, i/N] \text{ and } i = 1,\dots,N. \tag{5}$$

By the properties of $\gamma_i(t)$, this implies that $d(\gamma(t),c(t)) \leq 2\epsilon$ for every $t \in [0,1]$. Thus, the theorem will follow once we have proved our claim.

To this end, observe that, for any $t \in [(i-1)/N, i/N]$,

$$\begin{aligned}\Gamma(t) = \big(\Gamma_1(1), \Gamma_2(0)^*, \dots, &\Gamma_i(0)^*, \Gamma_i(Nt-i+1), \\ &\Gamma_{i+1}(0), \Gamma_{i+1}(0)^*, \dots, \Gamma_N(0)^*, \Gamma_N(0)\big)\end{aligned}$$

By Theorem 2.1(1) and Theorem 2.5,

$$P_{l_j+m_j}\big(f,z,(\Gamma_j(0)^*,\Gamma_j(0))\big) = P_{l_j}\big(f, P_{m_j}(f,z,\Gamma_j(0)^*), \Gamma_j(0)\big) = z$$

for $j = i+1,\dots,N$. Similarly,

$$\begin{aligned} P_{l_j+m_{j+1}}\big(f,z,(\Gamma_j(1),\Gamma_{j+1}(0)^*)\big) &= P_{m_{j+1}}\big(f, P_{l_j}(f,z,\Gamma_j(1)), \Gamma_{j+1}(0)^*\big) \\ &= P_{m_{j+1}}\big(f, x_j, \Gamma_{j+1}(0)^*\big) \\ &= P_{m_{j+1}}\big(f, P_{l_{j+1}}(f,z,\Gamma_{j+1}(0)), \Gamma_{j+1}(0)^*\big) = z \end{aligned}$$

for $j = 1,\dots,i-1$. Then Theorem 2.1(1) gives that

$$\gamma(t) = P_L(f,z,\Gamma(t)) = P_{l_i}(f,z,\Gamma_i(Nt-i+1)) = \gamma_i(Nt-i+1)$$

for $t \in [(i-1)/N, i/N]$, as we claimed. $\square$

We are ready to conclude the proof of Theorem B:

Proof of Theorem B. It suffices to prove that for every $x,y \in U$ there exists a neighborhood V of y and a neighborhood $\mathcal{V}$ of f such that z is in the g-accessibility class of x, for every $z \in V$ and $g \in \mathcal{V}$. Fix a small open 2-disk $D \subset U$ through x transverse to $E^s \oplus E^u$. Consider C^1 paths η_s and η_u in D intersecting transversely at a unique point. By Theorem 4.1, there exists a deformation path based on (f,x) which is uniformly close to η_s and there exists a deformation path based on (f,y) which is uniformly close to η_u. Then, by Theorem 2.1, for each g close to f and each z close to y there exists

- a deformation path γ_s based on (g, x) which is still close to η_s and
- a deformation path γ_u based on (g, z) which is still close to η_u.

Applying Theorem 3.1, we find points x_s, x_u in the images of γ_s, γ_u such that $W^s_g(x_s)$ intersects $W^u_g(x_u)$. Since x_s is in the g-accessibility class of x and x_u is in the g-accessibility class of z, the conclusion follows. □

5. Connected subgroups of surface diffeomorphisms

The goal in the remainder of the paper is to prove that stable accessibility is dense inside certain classes of partially hyperbolic diffeomorphisms with 2-dimensional center bundle. For the sake of simplicity, we focus on the case of diffeomorphisms for which the center foliation is an invariant fibration. The precise statement will be given in Theorem 6.1. Towards the proof, in this section we analyze certain properties of connected subgroups of surface diffeomorphisms.

Let M be a C^1 compact surface and G be a path-connected subgroup of the group $\mathrm{Diff}^1(M)$ of C^1 diffeomorphisms of M. Along the way, we will make a few additional assumptions on G. The *G-orbit* of a point $x \in M$ is the set $G(x) = \{g(x) : g \in G\}$. It is said to be *trivial* if $G(x) = \{x\}$.

Proposition 5.1. *Assume that no G-orbit is trivial. Then for every $x \in M$, either $G(x)$ is open or every compact, connected and locally path-connected set $Z \subset G(x)$ is either a point, a C^1-embedded segment or a C^1-embedded circle.*

Proof. The strategy is borrowed from Rodriguez-Hertz [7, Section 5].

If $G(x)$ is a neighborhood of some point, then using the obvious fact that G acts transitively on $G(x)$, we conclude that it is a neighborhood of every point. In other words, if $G(x)$ has non-empty interior then it is actually open. In what follows we assume that the interior of $G(x)$ is empty.

Fix $\delta < \operatorname{diam} G(x)$. Then there exists $\epsilon \in (0, \delta/2)$ such that $\operatorname{diam} G(y) > \delta$ for every y in the ϵ-neighborhood of x. We claim that if Z is contained in the ϵ-neighborhood of x then Z is a tree, that is, for every two distinct z_0 and z_1 there exists a single embedded segment contained in Z that connects z_0 to z_1.

To prove the existence of such a segment, let $\gamma : [0,1] \to Z$ be a path such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$. Let U be a maximal open subset of $[0,1]$ such that $\gamma(t)$ is constant on the boundary of each connected component of U. Then $\gamma([0,1] \setminus U)$ is a topologically embedded segment (it is a homeomorphic image of the quotient of $[0,1]$ by the map that collapses each connected component of U to a point) connecting z_0 to z_1.

To prove uniqueness, notice that if two such segments exist then their union contains some embedded circle C. Since C is contained in Z, which is contained in the $\delta/2$-neighborhood of x, it follows that one of the connected components of $M \setminus C$ is contained in the $\delta/2$-neighborhood of x. Let D be this connected component. Then $\operatorname{diam} D < \delta$ and, since D is also contained in the ϵ-neighborhood of x, $\operatorname{diam} G(y) > \delta$ for every $y \in D$. It follows that $G(y)$ intersects $C \subset Z \subset G(x)$ for every $y \in D$, and so $G(x)$ contains D. This contradicts the assumption that $G(x)$ has no interior, and this contradiction completes the proof of the claim.

Now assume that Z is contained in the $\epsilon/2$-neighborhood of x. Then Z is a tree and so, either it is a point or an embedded segment, or it has a subset $h(Y)$ homeomorphic via h to the figure Y. We are going to exclude this third alternative.

Let y be the 3-valent vertex of $h(Y)$. Let y_1, y_2, y_3 be points in each of the three connected components I_1, I_2, I_3 of $h(Y) \setminus \{y\}$ and $g_1, g_2, g_3 \in G$ be such that $g_j(y) = y_j$. We claim that g_j may be chosen arbitrarily close to the identity if y_j is close enough to y. This can be seen as follows.

Let $\gamma_j : [0,1] \to G$ be a continuous path with $\gamma_j(0) = \mathrm{id}$ and $\gamma_j(1) = g_j$. Let $\tau_j \geq 0$ be maximum such that $\gamma_j(\tau_j)(y) = y$ and define $\beta_j(t) = \gamma_j(t) \circ \gamma_j(\tau_j)^{-1}$ for $t \geq \tau_j$. By construction, $\beta_j(t)(y) = \gamma_j(t)(y) \in I_j$ for $t > \tau_j$ and $\beta_j(t) \to \mathrm{id}$ when t decreases to τ_j. The claim follows, replacing g_j and y_j with $\beta_j(t)$ and $\beta_j(t)(y)$ for t close τ_j.

Then $Z' = Z \cup g_1(Z) \cup g_3(Z)$ is contained in the ϵ-neighborhood of x. Moreover, $g_1(I_3)$ is an embedded segment C^0 close to I_3 and passing through y_1. Then, since Z' is a tree, $g_1(I_3)$ must be disjoint from I_3. Analogously, $g_3(I_1)$ is an embedded segment C^0 close to I_1, passing through y_3 and disjoint from I_1. Then $g_1(I_3)$ and $g_3(I_1)$ must intersect, which means that Z' cannot be a tree. This contradiction proves that the third alternative above is indeed impossible.

Now let Z be any compact locally path-connected subset of $G(x)$. Every $z \in Z$ has a compact path-connected neighborhood U_z inside Z with arbitrarily small diameter. Let $g_z \in G$ be such that $g_z(x) = z$. Then $V_z = g_z^{-1}(U_z)$ may be taken to be contained in the $\epsilon/2$-neighborhood of x. By the previous considerations, it follows that U_z is either a point or an embedded segment. Since this holds for every $z \in Z$, and we also assume that Z is connected, it follows that Z is either a point, an embedded segment or an embedded circle.

It remains to check that Z is C^1-embedded in the case it is not reduced to a point. Consider $Z' \supset Z$ defined as follows. If Z is an embedded segment with endpoints y, z, and $w \in Z \setminus \{y, z\}$, let g_y and g_z be elements of G such that $g_y(w) = y$ and $g_z(w) = z$. Then $Z \cup g_y(Z) \cup g_z(Z)$ contains an embedded open segment $Z' \supset Z$. If Z is an embedded circle, just let $Z' = Z$.

The set Z' is locally compact and C^1-homogeneous in the sense that for every $y_0, y_1 \in Z'$ there exists a C^1-diffeomorphism $g \in G$ and a neighborhood W of y_0 inside Z' such that $g(y_0) = y_1$ and $g(W)$ is a neighborhood of y_1 inside Z'. According to [13], this implies that Z' is a C^1 submanifold. □

Theorem 5.2. *Assume that no G-orbit is trivial. Then M is the disjoint union of an open set U and a compact set K such that K supports a lamination $\mathcal{L}$ with C^1 leaves and every G-orbit is either a connected component of U or a leaf of $\mathcal{L}$.*

Proof. Let K be the set of $x \in M$ whose G-orbit is not open. By Proposition 5.1, we can associate to each $x \in K$ a line field $l(x)$ tangent to the G-orbit at x. So to get the lamination structure we only have to prove that this line field is continuous.

As a first step we claim that for every $x \in M$ there exists a continuous path $\gamma_x : [0,1] \to G$ such that $\gamma_x([0,1])$ is close to the identity, $\gamma_x(0) = \mathrm{id}$ and $\gamma_x(1)(x) \neq x$. That can be seen as follows.

Since M is compact, the assumption implies that we may choose $g_1, ..., g_k \in G$ such that for every $x \in M$ there exists i such that $g_i(x) \neq x$. Let us choose also paths $\gamma_i : [0,1] \to G$ connecting id to g_i. Then, for every $x \in M$ and $n \geq 1$, there exist i and $0 \leq j \leq n-1$ such that $\gamma_i(j/n)(x) \neq \gamma_i((j+1)/n)(x)$. Notice that the path $\gamma_{i,j,n}(t) = \gamma_i((j+t)/n))^{-1} \circ \gamma_i(j/n)$ is contained in a small C^1-neighborhood of the identity if n is large. By construction, $\gamma_{i,j,n}(0) = \mathrm{id}$ and $\gamma_{i,j,n}(1)(x) \neq x$. This proves the claim.

It is clear that this construction is stable: given n, we may choose i and j uniform in a neighborhood of every $x \in M$. Thus, by compactness, the previous considerations prove that for every C^1-neighborhood $\mathcal{N}$ of the identity there exists $\delta > 0$ such that for every $x \in M$ there exists a continuous path $\gamma_{\mathcal{N},x} : [0,1] \to \mathcal{N}$ such that $d(x, \gamma_{\mathcal{N},x}(1)(x)) > 3\delta$. Let $\Gamma_{\mathcal{N},x} = \gamma_{\mathcal{N},x}([0,1])$ and $I_{\mathcal{N},x} = \Gamma_{\mathcal{N},x}(x)$.

For any $y, z \in I_{\mathcal{N},x}$, pick $g_y, g_z \in \Gamma_{\mathcal{N},x}$ such that $g_y(x) = y$ and $g_z(x) = z$. Then $l(z) = Dg(y) \cdot l(y)$, where $g = g_z \circ g_y^{-1}$. In particular, $l(y)$ is close to $l(z)$ if $\mathcal{N}$ is small. By Proposition 5.1, $I_{\mathcal{N},x}$ is a C^1-embedded manifold everywhere tangent to l. It follows that $I_{\mathcal{N},x}$ is an almost straight segment, of length greater than 3δ.

Let w be a point of $I_{\mathcal{N},x}$ at roughly the same distance from the two endpoints. Then let $g \in \Gamma_{\mathcal{N},x}$ be such that $g(x) = w$, and define $J_{\mathcal{N},x} = g^{-1}(I_{\mathcal{N},x})$. By construction, this is a C^1-embedded segment passing through x, such that both components of $J_{\mathcal{N},x} \setminus \{x\}$ have diameter at least δ, and such that $l(y)$ is close to $l(x)$ for every $y \in J_{\mathcal{N},x}$.

Moreover, if $y \notin J_{\mathcal{N},x}$ is close to x then $l(y)$ is close to $l(x)$ for otherwise $J_{\mathcal{N},y}$ would intersect $J_{\mathcal{N},x}$, which would contradict Proposition 5.1. This proves that the line field l is indeed continuous. □

Lemma 5.3. *Let K be a compact subset of a surface supporting a lamination $\mathcal{L}$ by C^1 leaves $\mathcal{L}$. Let $x_n, y_n \in K$ be points in the same leaf L_n such that $(x_n)_n$ and $(y_n)_n$ converge to the same point $p \in K$. If the leafwise distance d_n between x_n and y_n is bounded away from zero and infinity then the leaf L through p is an embedded circle of length at most* $\liminf d_n$.

Proof. Let ℓ_n be the leaf segment connecting x_n to y_n. Up to restricting to a subsequence, we may assume that the length of ℓ_n converges to $\inf_n d_n$ and, using Ascoli-Arzela, $(\ell_n)_n$ converges to a leaf segment ℓ of length $\inf_n d_n$ connecting $p = \lim_n x_x$ to $p = \lim_n y_n$. This proves the claims. □

Let $\mathrm{KS} \subset \mathrm{Diff}^1(M)$ denote the set of Kupka-Smale diffeomorphisms on M.

Theorem 5.4. *Let G, K and $\mathcal{L}$ be as in Theorem 5.2. Assume that for every $g \in G$ there exists a compact connected and locally connected set $C \subset G$ such that $g \in \overline{\mathrm{KS} \cap C}$. If $L \subset K$ is a non-isolated leaf of $\mathcal{L}$, then there exists a G-invariant, leafwise continuous vector field tangent to L.*

Proof. Let $x_0 \in L$ be arbitrary. It is enough to show that for every $g_0 \in G$ such that $g_0(x_0) = x_0$ we have $Dg_0(x_0)|T_{x_0}L = \mathrm{id}$. Since G is connected, all the elements of G preserve an (arbitrary) orientation of L, and so $Dg(x_0)|T_{x_0}L$ is the multiplication by some $\lambda > 0$. Suppose that λ is different from 1. Let C be as in the hypothesis. Then $C(x_0)$ has bounded leafwise length in L.

We claim that for every $g_1 \in C$ close to g_0 there exists a fixed point $x_1 \in L$ leafwise close to x_0. It is clear that $g_1(x_0) \in L$ is close to x_0 in the topology of M. Then, since $C(x_0)$ has bounded leafwise length, $g_1(x_0)$ must also be close to x_0 in the topology of L. Since $\lambda \neq 1$, there is a small open segment around x inside L strictly invariant under g_0 or g_0^{-1}. This segment is still strictly invariant under g_1 or g_1^{-1}, and so it must contain a fixed point x_1 of g_1. This proves the claim.

Choose $g_1 \in \mathrm{KS} \cap C$ and let $x_1 \in L$ be a fixed point close to x_0, as above. Since L is not isolated, we can choose a sequence $z_n \in K \setminus L$ converging to x_1. The distance from z_n to $w_n = g_1(z_n)$ along the corresponding leaf L_n cannot go to infinity. Here we use the local connectivity of C to break into finitely many connected pieces

C_j so that $C_j(z)$ has small diameter, and hence, by connectivity, small leafwise diameter, for all $z \in K$. The leafwise distance from z_n to w_n cannot go to zero either, for otherwise x_1 would be accumulated by fixed points of g_1, contradicting the Kupka-Smale hypothesis[1].

By Lemma 5.3, it follows that the leaf L is a circle. Then either every leaf through any point near L is a circle of bounded length (close to either the length of L or twice the length of L), or there is a leaf that spirals around L along one direction. The first case can be excluded because each such circle would have a fixed point under g_1 near x_1, which would also contradict the Kupka-Smale condition. So from now on we assume that there exists a leaf $\tilde{L}$ spiraling around L.

Let v be a unit vector field tangent to $\tilde{L}$ in the direction of the spiraling, that is, such that the corresponding flow F_t is such that $F_t(z)$ gets close to L as $t \to +\infty$ for any $z \in \tilde{L}$. We can extend this vector field to L in a unique way so that there is forward continuity, that is, in such a way that for every $z \in \tilde{L}$ if $t_n \to +\infty$ and $F_{t_n}(z) \to w \in L$ then $v(F_{t_n}(z)) \to v(w)$.

For any $g \in G$, there exists a unique continuous function $\phi_g : \tilde{L} \to \mathbb{R}$ such that $g(z) = F_{\phi_g(z)}(z)$. Then ϕ_g extends to L in a unique way, with forward continuity. Note that $(g, z) \mapsto \phi_g(z)$ is continuous on $G \times \tilde{L}$ and hence also on $G \times L$.

Now suppose g is a Kupka-Smale element of g and x is a fixed point of $g \mid L$ with $\phi_g(x) = 0$. Select $z \in \tilde{L}$ and $t_n \to \infty$ such that $F_{t_n}(z) \to x$. Then some small intervals around $F_{t_n}(z)$ are strictly invariant under either g or g^{-1}. This implies that x is accumulated by fixed points of g, which contradicts the Kupka-Smale.

We may not be able to apply this argument directly to $g = g_1$ and $x = x_1$ because $\phi_{g_1}(x_1)$ need not be zero: it may be any integer multiple of the length T of L (= minimal period of x_1 under the flow F_t). However, we are going to show that there exist $\gamma_n \in G$ such that, for every n large, x_1 is a fixed point of $\gamma_n g_1^n$ with $\phi_{\gamma_n g_1^n}(x_1) = 0$ and the derivative $D(\gamma_n g_1^n \mid L)$ is far from 1. Then any Kupka-Smale $g \in G$ near $\gamma_n g_1^n$ has a fixed point x near x_1 with $\phi_g(x) = 0$ and derivative far from 1. In this way, the general case will be reduced to the setting handled in the previous paragraph. We are left to find $(\gamma_n)_n$.

We claim that $\sup_{g \in G} \phi_g(z) = +\infty$ for every $z \in L$. Indeed, since $G(z)$ is non-trivial, we can always find $g \in G$ such that $\phi_g(z)$ is non-zero and, up to considering the inverse, it is no restriction to assume that $\phi_g(z) > 0$. By the compactness of L, it follows that $\inf_{z \in L} \sup_{g \in G} \phi_g(z) > 0$. Then, using the relation

$$\phi_{g_2 \circ g_1}(w) = \phi_{g_2}(g_1(w)) + \phi_{g_1}(w),$$

and composing appropriately, we get the claim. Considering the inverse, it follows that $\inf_{g \in G} \phi_g(z) = -\infty$. Thus $G \ni g \mapsto \phi_g(z) \in \mathbb{R}$ is surjective, for every $z \in L$.

Note that $\max_{z,w \in L} |\phi_g(z) - \phi_g(w)| \leq T$, since $g : L \to L$ is a homeomorphism. It follows that there is $h \in G$ such that ϕ_h has irrational translation number

$$\alpha = \lim_{n \to \pm\infty} \frac{\phi_{h^n}(z)}{nT} \text{ (uniformly over } z \in L).$$

To see this, take $\epsilon > 0$, $\tilde{z} \in L$ and $\tilde{g} \in G$ such that $\phi_{\tilde{g}}(\tilde{z}) > T + \epsilon$. Then $\inf_{z \in L} \phi_{\tilde{g}}(z) > \epsilon$ and so the the translation number of $\phi_{\tilde{g}}$ is at lest ϵ. Join the

[1]Indeed, since g_1 is Kupka-Smale, $Dg_1(x_1)$ is either a contraction or an expansion in the direction of L. Thus, by continuity, either g_1 or g_1^{-1} contracts the leaf segment joining z_n and w_n. Then the iterates, either forward or backward, of this segment must accumulate on a fixed point of g_1 close to x_1 in the same leaf as z_n and, thus, distinct from x_1.

identity to $\tilde{g}$ by some path in G. The translation numbers along the path must cover $[0, \epsilon]$ and so they do take irrational values.

Then $\lim_{n\to\pm\infty} |\log Dh^n|/n = 0$ uniformly on $z \in L$. Let $\gamma : [0,1] \to G$ be a continuous path with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = h$, and extend it to a (continuous) path $\gamma : \mathbb{R} \to G$ with $\gamma(t+1) = \gamma(t)h$ for every $t \in \mathbb{R}$. Writing $\gamma(t) = \gamma(t-[t])h^{[t]}$ we see that

$$\lim_{t\to\pm\infty} \frac{|\log D\gamma(t)|}{t} = \lim_{n\to\pm\infty} \frac{|\log Dh^n|}{n} = 0. \tag{6}$$

and

$$|\phi_{\gamma(t)}(z) - t\alpha T| \leq |\phi_{\gamma(t-[t])}(h^{[t]}(z))| + \big(t-[t]\big)\alpha T + |\phi_{h^{[t]}}(z) - [t]\alpha T|$$

is bounded.

The latter ensures that we may select $t_n \in \mathbb{R}$ such that $\phi_{\gamma(t_n)}(x_1) = -n\phi_{g_1}(x_1)$, that is, $\phi_{\gamma(t_n)g_1^n}(x_1) = 0$. Them x_1 is a fixed point of $\gamma(t_n)g_1^n$. Observe that $t_n \to \pm\infty$, depending on the sign of $\phi_{g_1}(x_1)$, which at this point we may take to be non-zero. So, (6) implies that the derivative of $\gamma(t_n)g_1^n$ at the point x_1 is far from 1 if n is large. To conclude it suffices to take $\gamma_n = \gamma(t_n)$. □

Corollary 5.5. *Let G, K and $\mathcal{L}$ be as in Theorem 5.4, and $G' \subset G$ be the set of commutators. Then G' acts trivially on any non-isolated leaf.*

Proof. Let L be a non-isolated leaf, and v be the vector field given by Theorem 5.4. Then every element of G acts on L as time-$t(g)$ map of the flow of v. In particular, any commutator restricts to the identity on L. □

Corollary 5.6. *Let G, K and $\mathcal{L}$ be as in Theorem 5.4, and assume further that some commutator has only isolated fixed points. Then K is a finite union of circles.*

Proof. Let g be a Kupka-Smale commutator. If there were a non-isolated leaf L, the previous corollary would give that $g \mid L = \mathrm{id}$, which would violate the Kupka-Smale condition. So all leaves are isolated, and the result follows. □

For any periodic point p of $g \in \mathrm{Diff}^1(M)$ which is a hyperbolic attractor or repeller, let $\lambda(g,p) \in [1,\infty)$ be the quotient of the logarithms of the norms of the eigenvalues of $Dg^n(p)$, where n is the period of p. We say that $g \in \mathrm{Diff}^1(M)$ is *non-resonant* if whenever $\lambda(g,p) = \lambda(g,q)$, either the periods of p and q are distinct, or p and q belong to the same orbit.

We say that $g' \in \mathrm{Diff}^1(M)$ is *transverse* to $g \in \mathrm{Diff}^1(M)$ if the stable manifolds of periodic saddles of g do not contain periodic points of g'.

Theorem 5.7. *G, K and $\mathcal{L}$ be as in Theorem 5.4. Assume that there exist $f, g \in G$ such that g and $g^k f$, $k \geq 1$ are Kupka-Smale and $fgf^{-1}g^{-1}$ has only isolated fixed points. Assume further that the non-resonant elements are dense in G, and for every $g \in G$ there exist $g', g'' \in G$ arbitrarily close to g with g'' transverse to g'. Then there is only one G-orbit.*

Proof. By Corollary 5.6, K consists of finitely many circles. We are going to show that under the current assumptions it is actually empty. Suppose otherwise and let L be any of the leaves of K. Let $g_0 = g$ and $g_k = g^k f$ for $k \geq 1$. We claim that $g_k \mid L$ has a periodic point for some $k \geq 0$. This can be seen as follows.

Let μ be any g-invariant probability measure supported in L. If $g_0 = g$ has no periodic points in L then $\operatorname{supp}\mu$ is either a Cantor set or the whole L. Keep in mind that all the elements of G preserve an (arbitrary) orientation of L. Since μ is g-invariant, the measure of a segment $[y, g(y)] \subset L$ is independent of y. If μ is f-invariant, then $\mu([y, f(y)])$ is also independent of y. Then

$$\begin{aligned}\mu\big([x, gf(x)]\big) &= \mu\big([x, f(x)]\big) + \mu\big([f(x), gf(x)]\big)\\ &\qquad= \mu\big([g(x), fg(x)]\big) + \mu\big([x, g(x)]\big) = \mu\big([x, fg(x)]\big).\end{aligned}$$

This implies that $gf(x) = fg(x)$ for every $x \in \operatorname{supp}\mu$, and so $fgf^{-1}g^{-1} = \mathrm{id}$ on the support of μ. That contradicts the hypothesis on the fixed points of $fgf^{-1}g^{-1}$, so μ cannot be f-invariant. In particular, there exists a segment $J \subset L$ with $\mu(f(J)) < \mu(J)$. We may choose J such that the endpoints are recurrent under g. Then we may find $k \geq 1$ in such a way that $g^k(f(J)) \subset J$, and so $g_k = g^k \circ f$ has a fixed point in J. This proves our claim.

By perturbing g_k we conclude that there exists some element $h_1 \in G$ which is non-resonant and has a non-zero number of periodic points in L which are all hyperbolic. If h_1 has a saddle, let $h = h_1$ or $h = h_1^{-1}$ so that the stable manifold of some periodic point $p \in L$ of h is contained in L. Let h' and h'' be diffeomorphisms close to h such that h'' is transverse to h'. Then the stable manifold of the continuation of p for h' contains a definite neighborhood of p in L, and this neighborhood must also contain the continuation of p for g''. This contradicts transversality, so the periodic points of h_1 in L cannot be saddles.

Now assume that all periodic points of h_1 are hyperbolic attractors or repellers, and let n be their period. Let p be an attractor and q be a repeller for h_1. Let $h_2 \in G$ be such that $h_2(p) = q$. Using that h_1 is non-resonant, we get that for suitable choices of $k, l \in \mathbb{Z}$ the diffeomorphism $h = h_2^{-1}h_1^{nk}h_2h_1^{nl}$ has a hyperbolic saddle at p whose stable manifold is contained in L. Then we can use the argument in the previous paragraph to reach a contradiction.

This completes the proof that K is empty. Hence all the G-orbits are open. By connectedness, it follows that there is a single orbit. □

6. Density of stable accessibility

Let $f : M \to M$ be a partially hyperbolic diffeomorphism of class C^k, with $k \geq 2$. Following Burns, Wilkinson [4], we say that f is *center bunched* if the functions ν, $\hat{\nu}$, γ, $\hat{\gamma}$ in (1) may be chosen to satisfy

$$\nu < \gamma\hat{\gamma} \quad \text{and} \quad \hat{\nu} < \gamma\hat{\gamma}. \tag{7}$$

Then the strong-stable (respectively, strong-unstable) holonomy maps of f inside each center-stable (respectively, center-unstable) leaf are C^1 (see [12]).

We say that f is *fibered*, if

- the quotient space $M/\mathcal{W}^c$ is compact and Hausdorff, and the canonical projection $\pi : M \to M/\mathcal{W}^c$ is a fiber bundle with C^1 fibers;
- the map $f_c : M/\mathcal{W}^c \to M/\mathcal{W}^c$ induced by f in the quotient space is a hyperbolic homeomorphism (in the sense of [14]).

By the stability theorem of Hirsch, Pugh, Shub [10, Theorem 6.8], this is a robust property, that is, any C^1-nearby diffeomorphism $g : M \to M$ is still fibered. Furthermore, g is topologically conjugate to a skew-product $(g_c, (g_x)_x)$, where g_c is a hyperbolic homeomorphism and each g_x is a diffeomorphism between two fibers. Moreover, this conjugacy varies continuously in a neighborhood of f.

Theorem 6.1. *Let $f \in \mathcal{PH}_2^k$ be a center bunched and fibered C^k diffeomorphism with 2-dimensional center bundle. Then stably accessible diffeomorphisms are C^k-dense in some neighborhood $\mathcal{U}$ of f.*

Let p be a periodic point of f_c and S_p be the corresponding fiber. Let G_p be the group of contractible su-loops in $M/\mathcal{W}^c$ based at p. Contractibility ensures that G_p is path-connected. The holonomy of the strong stable and strong unstable foliations of f yields a representation of G_p as a subgroup of $\mathrm{Diff}^1(S_p)$.

We call an su-loop $\gamma \in G_p$ *simple* if its corners are either periodic points or heteroclinic points associated to periodic points, and at least one of the corners is crossed only once by the loop. Similarly, we say that a pair of su-loops $\gamma_1, \gamma_2 \in G_p$ is *simple* if each one is simple and at least one of the corners is crossed only once by the union of the two loops.

Proposition 6.2. *There exists a C^1-neighborhood $\mathcal{U}$ of f and, for each $1 \leq k \leq \infty$ there exists a C^k-residual subset $\mathcal{R} \subset \mathcal{U} \subset \mathrm{Diff}^k(M)$ such that for $g \in \mathcal{R}$,*

1. *every simple su-loop corresponds to a Kupka-Smale diffeomorphism;*
2. *if p and q are fixed attractors or repellers of a simple loop γ then*
$$\lambda_1(p)/\lambda_2(p) \neq \lambda_1(q)/\lambda_2(q),$$
where λ_1, λ_2 denote the Lyapunov exponents, in decreasing order;
3. *for every simple pair (γ_1, γ_2) of su-loops, the stable manifolds of saddle points of γ_1 do not contain periodic points of γ_2;*
4. *for every simple pair, the fixed points of the commutator are isolated.*

Proof. Note that the holonomy associated to an su-loop can be decomposed as $\phi \circ \psi$, where ψ is the holonomy corresponding to the loop segment from p to q and ϕ is the holonomy over the loop segment from q to p. By considering perturbations localized around the fiber over q, the corresponding holonomy gets changed to $\phi \circ h \circ \psi$, where h is an arbitrary smooth perturbation of the identity.

Similarly, when considering a simple pair, we can perturb the dynamics so that one of the holonomy maps is unchanged, while the other changes from $\phi \circ \psi$ to $\phi \circ h \circ \psi$, where h is an arbitrary smooth perturbation of the identity.

The conclusion follows then from usual transversality arguments, but there is a caveat. Transversality statements usually show that a C^r map can be perturbed to another C^r map so to obtain, say, the Kupka-Smale condition. However here we have perturbations of a slightly more special type.

Recall that a periodic point p of period $\kappa \geq 1$ of a diffeomorphism φ is said to be *non-degenerate* if $D\varphi^\kappa(p) - \mathrm{id}$ is an isomorphism. The main point in the proof is to show that, by arbitrarily small perturbations of the dynamics, one can ensure that the holonomies have only non-degenerate periodic points.

We start with some abstract considerations. Let M be a compact smooth manifold of dimension d. Consider any Riemannian metric on M, and let $d(\cdot,\cdot)$ the associated distance function. It is no restriction to assume that the diameter is 1. The volume measure associated to the Riemannian metric will be denoted as vol or dx, indifferently.

Lemma 6.3. *If $g : M \to M$ is a C^1 map and $p \in M$ is a degenerate fixed point then there exists an increasing function $\omega : [0,1] \to [0,1]$ such that $\omega(\delta)/\delta^d \to \infty$ when $\delta \to 0$ and $\mathrm{vol}(\{x \in U : d(g(x), x)\| \leq \delta\}) \geq \omega(\delta)$ for all $\delta \in (0,1)$. Moreover,*

ω may be chosen depending only on a modulus of continuity for Dg and an upper bound for its norm.

Proof. Let $v \in T_pM$ be a unit vector with $Dg(p)v = v$ and $\gamma : (-1,1) \to M$ be the geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Let $C > 1$ be an upper bound for $\|Dg\|$ and $\phi : [0,1] \to [0,\infty)$ be a modulus of continuity for $t \mapsto Dg(\gamma(t))\dot\gamma(t) - \dot\gamma(t)$ (interpret this expression in local coordinates). For each $\delta > 0$ small, let r be given by $\phi(r)r = \delta/2$. It is clear that $r \to 0$ when $\delta \to 0$, and so $r/\delta = 1/(2\phi(r)) \to \infty$ when $\delta \to 0$. Define $\omega(\delta) = 2r(\delta/(4C))^{d-1}$. Let V be the tubular neighborhood of radius $\delta/(4C)$ around the geodesic segment $\gamma([-r,r])$. On the one hand,

$$\|Dg(\gamma(t))\dot\gamma(t) - \dot\gamma(t)\| \le \phi(r) \text{ for every } t \in [-r,r],$$

and so $\|g(\gamma(t)) - \gamma(t)\| \le \phi(r)|t| \le \delta/2$ for every $t \in [-r,r]$. Thus, by the triangle inequality,

$$\|g(x) - x\| \le C(\delta/4C) + \delta/2 + (\delta/4C) < \delta \text{ for every } x \in V.$$

On the other hand, $\operatorname{vol}(V) \ge 2r(\delta/(4C))^{d-1} = \omega(\delta)$. $\square$

Observe also that the property in the conclusion implies that there exists a decreasing homeomorphism $\alpha : [1,\infty) \to (0,1]$ such that

$$\int_1^\infty \omega(\alpha(s))\,ds = \infty \text{ and } \int_1^\infty \alpha(s)^d\,ds < \infty. \tag{8}$$

This can be seen as follows. Let $\delta_j \in (0,1)$, $j \ge 1$, be a sequence such that $\delta_1 = 1$, $\delta_{j+1} < \delta_j/10$ and $\omega(\delta_{j+1}) \ge 2^j\delta_{j+1}^d$. Denote $\epsilon_j = j^{-2}\delta_j^{-d}$ for each $j \ge 1$. Notice that $\epsilon_{j+1} \ge 2\epsilon_j$. Now let α be a homeomorphism mapping each $[\epsilon_j, \epsilon_{j+1})$ to $(\delta_{j+1}, \delta_j]$ and such that $\int_{\epsilon_j}^{\epsilon_{j+1}} \alpha(s)^d ds$ is within a factor of two of the infimum for such homeomorphisms. Then

$$\int_1^\infty \alpha(s)^d ds \le 2\sum_{j=1}^\infty (\epsilon_{j+1} - \epsilon_j)\,\delta_{j+1}^d \le 2\sum_{j=1}^\infty \frac{1}{(j+1)^2} < \infty.$$

On the other hand,

$$\int_1^\infty \omega(\alpha(s))ds \ge \sum_{j=1}^\infty (\epsilon_{j+1} - \epsilon_j)\,\omega(\delta_{j+1}) \ge \sum_{j=1}^\infty \frac{2^j}{2(j+1)^2} = \infty.$$

Let $X_1, \ldots, X_r$ be smooth vector fields spanning TM, and $\Phi_1^t, \ldots, \Phi_r^t$ be the corresponding flows. For each $t = (t_1, ..., t_r) \in \mathbb{R}^r$, denote $\Phi^t = \Phi_r^{t_r} \circ \cdots \circ \Phi_1^{t_1}$. Fix $\epsilon > 0$ small enough that the map $t \mapsto \Phi^t(x)$ from $P = [-\epsilon,\epsilon]^r$ to M is a submersion for every $x \in M$. The Lebesgue measure on P is denoted by m or dt, indifferently.

Lemma 6.4. *Let $\psi : M \to M$ be a C^1 diffeomorphism. Then the fixed points of $\Phi^t \circ \psi^{-1}$ are non-degenerate for almost every $t \in P$.*

Proof. Fix $\omega : [0,1] \to [0,1]$ satisfying the conclusion of Lemma 6.3 for every $g = \Phi^t \circ \psi^{-1}$: this is possible because the derivatives of these maps are uniformly bounded and admit a uniform modulus of continuity. Then let $\alpha : [1,\infty) \to (0,1]$ be as in (8) and $\beta = \alpha^{-1} : (0,1] \to [1,\infty)$. Define

$$E(t) = \int_M \beta(d(\Phi^t(x), \psi(x))\,dx \text{ for each } t \in P.$$

The conclusion follows directly from the observations in the next two paragraphs.

Since ψ is a diffeomorphism, there exists $c_1 > 0$ such that, for any $\delta > 0$,

$$\operatorname{vol}\left(\left\{x : d(\Phi^t(x), \psi(x)) \leq \delta\right\}\right) \geq c_1 \operatorname{vol}\left(\left\{x : d(\Phi^t(\psi^{-1}(y)), y) \leq \delta\right\}\right).$$

So, if $t \in P$ is such that $\Phi^t \circ \psi^{-1}$ has some degenerate fixed point then,

$$\begin{aligned} E(t) = \int_M \beta(d(\Phi^t(x), \psi(x))\, dx &= \int_1^\infty \operatorname{vol}\left(\left\{x : \beta(d(\Phi^t(x), \psi(x))) \geq s\right\}\right)\, ds \\ &\geq c_1 \int_1^\infty \operatorname{vol}\left(\left\{y : \beta(d(\Phi^t(\psi^{-1}(y)), y))) \geq s\right\}\right)\, ds \\ &\geq c_1 \int_1^\infty \operatorname{vol}\left(\left\{y : d(\Phi^t(\psi^{-1}(y)), y) \leq \alpha(s)\right\}\right)\, ds \geq c_1 \int_1^\infty \omega(\alpha(s))\, ds = \infty. \end{aligned}$$

Since $t \mapsto \Phi^t(x)$ is a submersion, there exists $C_2 > 0$ such that

$$m\left(\left\{t : d(\Phi^t(x), \psi(x)) \geq \delta\right\}\right) \leq C_2 \delta^d$$

for every $x \in M$ and $\delta > 0$. Thus,

$$\begin{aligned} \int_P E(t)\, dt &= \int_M \int_P \beta(d(\Phi^t(x), \psi(x))\, dt\, dx \\ &= \int_M \int_1^\infty m\left(\left\{t : \beta(d(\Phi^t(x), \psi(x))) \geq s\right\}\right)\, ds \\ &= \int_M \int_1^\infty m\left(\left\{x : d(\Phi^t(x), \psi(x)) \leq \alpha(s)\right\}\right)\, ds \leq \int_M C_2 \alpha(s)^d\, ds. \end{aligned}$$

In particular, $\int_P E(t)\, dt$ is finite. □

At this stage, we can conclude that given a simple loop there is a perturbation that makes the holonomy have only non-degenerate fixed points. Such fixed points are finitely many, and they undergo no bifurcations under small perturbations. It is then easy to do an additional perturbation that makes the fixed points hyperbolic.

In order to deal with more general periodic points, we proceed by induction. We assume that we have already obtained a perturbation whose periodic orbits of period at most n are hyperbolic. We now consider periodic orbits of period $n+1$. These must be far from periodic orbits of period n (since those are hyperbolic). We proceed in a similar way as before, with more localized perturbations (by considering vector fields with small support). More precisely, given some small open set V which does not intersect its first n iterates, and a compact set $K \subset V$ we can consider vector fields supported in V and which span the tangent space over K. Then the iterate $n+1$ near K decomposes as a diffeomorphism $\phi \circ h \circ \psi$, where h is controlled by the vector fields, while ϕ and ψ remain fixed. It thus follows, as before, that we can eliminate degenerate periodic points of period $n+1$ over K. By a covering argument, we can thus eliminate degenerate periodic points of period $n+1$ over all of M (and make then hyperbolic by a further perturbation).

Now we can easily obtain the first item by applying again localized perturbations and covering arguments to eliminate tangencies between stable and unstable manifolds by the usual technique. Indeed, in charts the issue reduces to being given two graphs of C^1 functions $\mathbb{R} \to \mathbb{R}$ and being able to move smoothly one of them in order to achieve transversality with respect to the other: this can be achieved simply by vertical translations, applying Sard's Theorem to the difference of the two functions.

The second item presents no difficulties, using localized perturbations in the usual way. As for the third, we use a preliminary perturbation in order to make the sets of periodic points for the two diffeomorphisms (which are countable, by the first item) disjoint, then perturb again to move either the stable manifolds associated to periodic points of γ_1 or the periodic points of γ_2 so that they avoid each other.

For the fourth item, we may first perturb to ensure that γ_1 maps fixed points of γ_2 to non-fixed points of γ_2 and vice-versa. We can now make localized perturbations to make the fixed points of the commutator non-degenerate with an argument similar to the one we used to show non-degeneracy of the periodic points of a single simple loop above (since the localized perturbation will affect only one step of the composition). □

Proof of Theorem 6.1. Take $\mathcal{U}$ and $\mathcal{R}^k$ as in Proposition 6.2. The proposition ensures that for $g \in \mathcal{R}$ the representation of G_p in $\mathrm{Diff}^1(S_p)$ satisfies the conditions of Theorem 5.7. Hence, S_p is contained in an accessibility class. Saturating by stable and unstable leaves, we conclude that g is accessible. By Theorem B it is stably accessible. □

Let us conclude by pointing out that the same arguments also yield a version of Theorem 6.1 restricted to the subspace of volume-preserving diffeomorphisms. Actually, the definition of Kupka-Smale in the volume-preserving setting is slightly different, allowing for elliptic periodic points. However, that need not concern us here, because we only deal with fixed points contained in some invariant leaf, and those cannot be elliptic.

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