1 Topology of Algebraic Varieties

Class 1: Singular Homology and Cohomology

- Definition of singular homology, functoriality and homotopy invariance.
- Relative homology, long exact sequence of a pair.
- Relation between the first homology group and the fundamental group. Examples from algebraic geometry.
- Homology with coefficients in a group, cohomology with coefficients in a group, universal coefficient theorems.

Class 2: Singular Homology and Cohomology II

- Künneth formula, compact support cohomology groups, Poincaré duality.
- Axioms for a homology theory, Eilenberg-Steenrod theorem, long exact sequence for triples, Mayer-Vietoris sequence.
- Leray-Thom-Gysin isomorphism, Leray-Thom-Gysin exact sequence associated to a pair, intersection map.

Class 3: Lefschetz Theorems

- Statement of Lefschetz main theorem (relative homology of the affine part of a projective variety relative to a hyperplane section).
- Consequences: Homology of affine varieties, properties of the intersection map, Lefschetz hyperplane section theorem, homology of complete intersections and affine complete intersections.
- Hard Lefschetz theorem, primitive homology, Lefschetz decomposition.
Class 4: Proof of Lefschetz Theorem
- Ehresman fibration theorem, strong deformation retracts.
- Relative homology of a one parameter family with isolated critical points relative to a smooth fiber, reduction to the local case, proof in the non-degenerated case.
- Lefschetz pencils, weak version of Lefschetz main theorem.

Class 5: Proof of Lefschetz Theorem II
- Lefschetz thimbles, vanishing cycles.
- Proof of Lefschetz main theorem in the degenerate local case, Milnor number of a singularity.
- Application: Middle homology of a smooth hypersurface of the projective space.

2 \(L^2\)-Hodge Theory

Class 6: Review of Complex Geometry
- Complexified cotangent bundle, De Rham cohomology, differential \((p, q)\)-forms, Dolbeault cohomology.
- Hermitian and Kähler manifolds, projective varieties are Kähler with the Fubini-Study metric.
- Poincaré-De Rham isomorphism, integral and rational classes, Kodaira embedding theorem.

Class 7: Harmonic Forms
- Volume form on a Hermitian manifold, induced metric on exterior powers of the complexified cotangent bundle.
- \(L^2\) product on the space of \((p, q)\)-forms over a compact manifold, Hodge \(\star\) operator, \(\bar{\partial}\), harmonic forms.
- Finiteness theorem for elliptic operators, Hodge theorem on harmonic forms, Kodaira-Serre duality, Poincaré duality.
Class 8: Harmonic Forms on Kähler Manifolds

• Kähler metrics are constant up to second order approximations.

• Lefschetz operator and its formal adjoint, commutators with $\partial$ and $\bar{\partial}$, $\Delta_\partial = 2\Delta_{\bar{\partial}}$ for compact Kähler manifolds.

• Cohomology groups of type $(p,q)$, Hodge decomposition theorem.

Class 9: Hodge Conjecture

• Lefschetz operator commutes with the Laplacian, hard Lefschetz theorem.

• Hodge cycles and classes, fundamental class of an algebraic subvariety, algebraic cycles are Hodge cycles, Hodge conjecture.

• Statement of Lefschetz (1,1) theorem, Hodge conjecture is true for dimension at most 3, Hodge conjecture for hypersurfaces reduces to the middle homology of even dimensional hypersurfaces.

Class 10: Polarized Hodge Structures on Compact Kähler Manifolds

• Primitive $k$-forms, the $\star$ of a primitive form, Lefschetz decomposition in cohomology.

• Polarization and its compatibility with Lefschetz and Hodge decompositions, Hodge index theorem.

• Hodge filtration, intermediate Jacobians, the first Jacobian is an Abelian variety.

3 Coherent Sheaves on Projective Varieties

Class 11: Derived Functors

• Abelian categories with enough injectives, the derived object of a complex is well defined, simple complex associated to a double complex resolution, every complex admits a derived object, the derived objects are defined in the derived category.

Class 12: Sheaf Cohomology

• Derived long exact sequence associated to a short exact sequence of complexes.

• Divisible groups are injective, the category of sheaves of abelian groups have enough injectives, sheaf cohomology.
• Čech cohomology for coverings, Čech resolution of a sheaf, Leray theorem, long exact sequence in Čech cohomology, flabby sheaves are Čech acyclic, Čech cohomology as limit of refinements is isomorphic to sheaf cohomology.

Class 13: Stein Varieties
• Fine sheaves are acyclic, De Rham theorem, Dolbeault theorem.
• Definition of analytic varieties, coherent analytic sheaves, Oka coherence, Cartan theorem, coherent sheaves form an abelian category, Oka theorem.
• Stein varieties, closed subvarieties of a Stein variety are Stein, the extended polydisc is a Stein variety.

Class 14: Cartan-Serre Finiteness
• Holomorphically convex varieties with separating holomorphic functions are Stein, intersection of Stein varieties is Stein, every analytic variety admits a good Stein covering.
• Frechet spaces, compact operators, Schwartz theorem.
• Frechet structure on the sections of a coherent sheaf induced by the uniform convergence on compact subsets, Cartan-Serre finiteness theorem.

Class 15: Algebraic Coherent Sheaves
• Definition of complex algebraic varieties and morphisms, definition of algebraic quasi-coherent and coherent sheaves, Serre’s theorem on the cohomology of affine varieties (quasi-coherent sheaves are acyclic).
• Projective varieties, twisted sheaves, Serre’s theorem about coherent algebraic sheaves over projective varieties (are quotients of finite direct sums of twisted sheaves).
• Cohomology of the projective space with coefficients in the twisted sheaves, Serre’s theorem on the cohomology of projective varieties with coefficients in a coherent sheaf (finiteness and vanishing after a high enough twisting).

Class 16: Serre’s GAGA Correspondence
• Analytification functor, holomorphic functions are flat over regular functions, first GAGA theorem (correspondence of sheaf cohomology groups).
• Sheaf of morphisms between coherent sheaves is coherent, second GAGA theorem (every analytic morphism comes from an algebraic morphism).
• Third GAGA theorem (every analytic coherent sheaf comes from an algebraic coherent sheaf), Chow theorem.
4 Cohomology of Hypersurfaces

Class 17: Hypercohomology

- Definition of hypercohomology as derived functor and some properties.
- Hypercohomology relative to a open covering, long exact sequence in hypercohomology with respect to a covering and Leray theorem in hypercohomology.
- Naive filtration in hypercohomology, spectral sequence, criterion for degeneration at $E_r$.

Class 18: Analytic De Rham Cohomology

- Degeneration at $E_1$ of the spectral sequence associated to the naive filtration.
- Analytic De Rham cohomology as hypercohomology with coefficients in the analytic De Rham complex, correspondence of the naive filtration with the Hodge filtration in the projective case.
- Meromorphic forms and meromorphic forms with logarithmic poles, local description of meromorphic forms with logarithmic poles, Atiyah-Hodge and Deligne theorems.

Class 19: Algebraic Hodge Filtration

- Algebraic differential forms, algebraic De Rham cohomology, algebraic Hodge filtration for smooth projective varieties.
- Poincaré residue sequence, algebraic De Rham cohomology of affine varieties with logarithmic forms, algebraic Hodge filtration for affine varieties.
- Carlson-Griffiths lemma (operator reducing the order of the pole).

Class 20: Griffiths Basis

- Euler’s short exact sequence, Bott formula and corollary.
- First Griffiths theorem (about generators of the Hodge filtration), second Griffiths theorem (about the kernel of the residue map).
- Carlson-Griffiths theorem (about the explicit values of the residue map on Griffiths basis).