PROPERLY IMMERSED MINIMAL SURFACES IN A SLAB OF $\mathbb{H} \times \mathbb{R}$, $\mathbb{H}$ THE HYPERBOLIC PLANE

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Abstract. We prove that the ends of a properly immersed simply or one connected minimal surface in $\mathbb{H} \times \mathbb{R}$ contained in a slab of height less than $\pi$ of $\mathbb{H} \times \mathbb{R}$, are multi-graphs. When such a surface is embedded then the ends are graphs. When embedded and simply connected, it is an entire graph.

1. Introduction

A fundamental problem in surface theory is to understand surfaces of prescribed curvature in homogenous 3-manifolds. Simply connected properly embedded surfaces are the simplest to consider and after the compact sphere, the plane is next. There are some unicity results. A proper minimal embedding of the plane in $\mathbb{R}^3$ is a flat plane or a helicoid [11]. A proper embedding of the plane in hyperbolic 3-space as a constant mean curvature one surface (a Bryant surface) is a horosphere [4]. Also, a proper minimal embedding of an annulus $S^1 \times \mathbb{R}$ in $\mathbb{R}^3$ is a catenoid [3] and a proper minimal embedding of an annulus with boundary, $-S^1 \times \mathbb{R}^+\pm$, is asymptotic to an end of a catenoid, plane or helicoid [11, 1]. This is true for Bryant annular ends in $\mathbb{H}^3$. A proper constant mean curvature one embedding of an annulus $S^1 \times \mathbb{R}$ in $\mathbb{H}^3$ is a catenoid cousin. Such an annulus with compact boundary in $\mathbb{H}^3$ is asymptotic to an end of a catenoid cousin or a horosphere.

In this paper we consider proper minimal embeddings (and immersions) of the plane and the annulus in $\mathbb{H} \times \mathbb{R}$.

Contrary to $\mathbb{R}^3$, there are many such surfaces in $\mathbb{H} \times \mathbb{R}$. Given any continuous rectifiable curve $\Gamma \subset \partial_\infty(\mathbb{H} \times \mathbb{R})$, $\Gamma$ a graph over $\partial_\infty(\mathbb{H})$, there is an entire minimal graph asymptotic to $\Gamma$ at infinity [12, 13]. Also if $\Gamma$ is an ideal polygon of $\mathbb{H}$, there are necessary and sufficient conditions on $\Gamma$ which ensure the existence of a minimal graph over the interior of $\Gamma$, taking values plus and minus infinity on alternate sides of $\Gamma$ [5]. This graph is then a simply connected minimal surface in $\mathbb{H} \times \mathbb{R}$. We will see there are many minimal embeddings of the plane that are not graphs (aside from the trivial example of a (geodesic of $\mathbb{H}$) $\times \mathbb{R}$; a vertical plane.

In this paper we will give a condition which obliges a properly embedded minimal plane to be an entire graph. More generally, we will give a condition which obliges properly immersed minimal surfaces of finite topology to have multi-graph ends.

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We will see that slabs $S$ of height less than $\pi$ play an important role. There are two reasons for this. First, complete vertical rotational catenoids exist precisely when their height is less than $\pi$. Secondly, for $h > \pi$, there are vertical rectangles of height $h$ at infinity; i.e., in $\partial_\infty(H \times \mathbb{R})$, that bound simply connected minimal surfaces $M(h)$ (the rectangle is the asymptotic boundary of $M(h)$). These surfaces are invariant by translation along a horizontal geodesic and we discuss them in detail later; [9, 14, 15].

Let $\epsilon > 0$ and $S = \{(p, t) \in H \times \mathbb{R}; |t| \leq (\pi - \epsilon)/2\}$. We will prove that simply or one-connected properly immersed minimal surfaces in $H \times \mathbb{R}$, $\Sigma \subset S$ have multi-graph ends. In fact, an annular properly immersed minimal surface in $S$, with compact boundary, has a multi-graph subend. More precisely we prove:

**Slab Theorem 1.** Let $S \subset H \times \mathbb{R}$ be a slab of height $\pi - \epsilon$ for some $\epsilon > 0$. Assume $\Sigma$ is a properly immersed minimal surface in $H \times \mathbb{R}$, $\Sigma \subset S$. If $\Sigma$ is simply connected and embedded then $\Sigma$ is an entire graph.

More generally;

1. If $\Sigma$ is of finite topology and embedded with one end then $\Sigma$ is simply connected and an entire graph.
2. If $\Sigma$ is homeomorphic to $S^1 \times \mathbb{R}^+$, then an end of $\Sigma$ is a multi-graph. If this annular surface is embedded then an end is a graph.

In particular, by (2), if $\Sigma$ is of finite topology then each end of $\Sigma$ is a multi-graph.

Results of this nature have been obtained by Colding and Minicozzi in their study of embedded minimal disks in balls of $\mathbb{R}^3$, whose boundary is on the boundary of the ball; see proposition III. 1.1 of [2]. We have been inspired here by their ideas; in particular using foliations by catenoids to control minimal surfaces.

**Remark 1.1.** We will give an example of an Enneper-type minimal surface in $S$. This is a properly immersed minimal surface in $H \times \mathbb{R}$, $\Sigma \subset S$ that is simply connected and whose end is a 3-fold covering graph over the complement of a compact disc in $H$.

**Remark 1.2.** Also there is an example of a properly embedded simply connected minimal surface in a slab of height $\pi$ that is not a graph. We will describe this surface after the proof of the Slab Theorem. Thus $\pi$ is optimal for the slab theorem.

## 2. The Dragging Lemma

**Dragging Lemma 1.** Let $g : \Sigma \to N$ be a properly immersed minimal surface in a complete 3-manifold $N$. Let $A$ be a compact surface (perhaps with boundary) and $f : A \times [0, 1] \to N$ a $C^1$-map such that $f(A \times \{t\}) = A(t)$ is a minimal immersion for $0 \leq t \leq 1$. If $\partial(A(t)) \cap g(\Sigma) = \emptyset$ for $0 \leq t \leq 1$ and $A(0) \cap g(\Sigma) \neq \emptyset$, then there is a $C^1$ path $\gamma(t)$ in $\Sigma$, such that $g \circ \gamma(t) \in A(t) \cap g(\Sigma)$ for $0 \leq t \leq 1$. Moreover we can prescribe any initial value $g \circ \gamma(0) \in A(0) \cap g(\Sigma)$. 
Remark 2.1. To obtain a $\gamma(t)$ satisfying the Dragging lemma that is continuous (not necessarily $C^1$) it suffices to read the following proof up to (and including) Claim 1.

Proof. When there is no chance of confusion we will identify in the following $\Sigma$ and its image $g(\Sigma)$, $\gamma \subset \Sigma$ and $g \circ \gamma$ in $g(\Sigma) \subset N$. In particular when we consider embeddings of $\Sigma$ there is no confusion.

Let $\Sigma(t) = g(\Sigma) \cap A(t)$ and $\Gamma(t) = f^{-1}(\Sigma(t))$, $0 \leq t \leq 1$ the pre-image in $A \times [0,1]$. When $g : \Sigma \to N$ is an immersion, we consider $p_0 \in g(\Sigma) \cap A(0)$, and pre-images $z_0 \in g^{-1}(p_0)$ and $(q_0,0) \in f^{-1}(p_0)$. We will obtain the arc $\gamma(t) \in \Sigma$ in a neighborhood of $z_0$ by a lift of an arc $\eta(t)$ in a neighborhood of $(q_0,0)$ in $\Gamma([0,1])$ i.e. $g \circ \gamma(t) = f \circ \eta(t)$.

We will extend the arc continuously by iterating the construction.

Since $\Gamma(t)$ represents the intersection of two compact minimal surfaces, we know $\Gamma(t)$ is a set of a finite number of compact analytic curves $\Gamma_1(t), ..., \Gamma_k(t)$. These curves $\Gamma_i(t)$ are analytic immersions of topological circles. By hypothesis, $\Gamma(t) \cap (\partial A \times [0,1]) = \emptyset$ for all $t$. The maximum principle assures that the immersed curves can not contain a small loop, nor an isolated point. Since $A(t)$ is compact and has bounded curvature, a small loop in $\Gamma(t)$ would bound a small disc $D$ in $\Sigma$ with boundary in $A$. Since $A$ is locally a stable surface, we can consider a local foliation around the disc and find a contradiction with the maximum principle. We say in the following that $\Gamma(t)$ does not contain small loops.

Claim 1: We will see that for each $t$ with $\Gamma(t) \neq \emptyset$, $t < 1$ there is a $\delta(t) > 0$ such that if $(q,t) \in \Gamma(t)$, then there is a $C^1$ arc $\eta(t)$ defined for $t \leq \tau \leq t + \delta(t)$ such that $\eta(t) = (q,t)$ and $\eta(\tau) \in \Gamma(\tau)$ for all $\tau$ (there may be values of $t$ where $\gamma'(t) = 0$).

Since $\Gamma(0) \neq \emptyset$, this will show that the set of $t$ for which $\eta(t)$ is defined is a non empty open set. This defines an arc $\gamma(t)$ as a lift of $f \circ \eta(t) \subset A(\tau)$ in a neighborhood of $\gamma(t) \in \Sigma$.

First suppose $(q,t) \in \Gamma(t)$ is a point where $A(t) = f(A \times \{t\})$ and $g(\Sigma)$ are transverse at $f(q,t)$. Let us consider the $C^1$ immersions $F : A \times [0,1] \to N \times [0,1]$ with $F(q,t) = (f(q,t),t)$ $G : \Sigma \times [0,1] \to N \times [0,1]$ with $G(z,t) = (g(z),t)$.

Let $\hat{M} = F(A \times [0,1]) \cap G(\Sigma \times [0,1])$ and $M = F^{-1}(\hat{M})$. $F(A \times [0,1])$ and $G(\Sigma \times [0,1])$ are transverse at $p = F(q,t)$. Thus $\hat{M}$ is a 2-dimensional surface of $N \times [0,1]$ near $p$. We consider $X(t)$ a tangent vector field along $\Gamma(t)$ and $JX(t)$ an orthogonal vector field to $X(t)$ in $T_{(q,t)}$. If $\partial/\partial t \perp T_{p}\hat{M}$, then $T_{p}\hat{M} = T_{f(q,t)}A(t) = T_{f(q,t)}g(\Sigma)$ and $(q,t)$ would be a non transverse point of intersection of $A(t)$ and $g(\Sigma)$. Thus $< \partial t, JX(t) > \neq 0$ and we can find $\eta(t)$ a smooth path, defined for $\tau \in [t-\delta(q), t+\delta(q)]$ such that $\eta(t) = (q,t)$ and $\eta'(t) = JX(t)$ is transverse to $\Gamma(t)$ at $(q,t)$.

By transversality and $f$ being $C^1$ in the variable $t$, we have a $\delta(q) > 0$ such that for $t-\delta(q) \leq \tau \leq t+\delta(q)$, $A(\tau)$ intersects $f \circ \eta(t)$ in a unique point and this point varies continuously with $t-\delta(q) \leq \tau \leq t+\delta(q)$. With a fixed initial point in $\Sigma$, a lift of $f \circ \eta(t)$, defines $\gamma(\tau) \in \Sigma$. 


Again by transversality, we can find a neighborhood of \((q, t)\) in \(\Gamma(t)\) and a \(\delta > 0\) so that the above path \(\gamma(\tau)\) exists for \(t - \delta \leq \tau \leq t + \delta\), through each point in the neighborhood of \(q\). It suffices, to look for a local immersion of a neighborhood of 0 in \(T_pM\) into \(M\), to obtain a \(C^1\) diffeomorphism \(\psi : B(0) \subset T_pM \to M\). \(M\) has the structure of a \(C^1\) manifold in a neighborhood of points of transversality and this structure extends to \(F^{-1}(M) \subset A \times [0,1]\).
We will find a $\delta > 0$ that works in a neighborhood of a singular point $(q,t) \in \Gamma(t)$, where there is a $z \in \Sigma$ such that $f(q,t) = g(z)$ and $T_{f(q,t)}A(t) = T_{g(z)}g(\Sigma)$. We consider singularities of $\Gamma(t)$ where $A(t)$ and $g(\Sigma)$ are tangent. Near a singularity $(q,t) \in \Gamma(t)$, $\Gamma(t)$ contains $2k$ analytic curves intersecting at $q$ at equal angles, $k \geq 1$.

Let $V$ be a neighborhood of $q$ in $A$. The set $\Gamma(t) \cap V$ is $2k$ analytic curves. Let $\alpha : [-\epsilon,\epsilon] \to V \cap \Gamma(t)$ be a regular parametrization of one curve with $\alpha(0) = q$ and $\alpha(\pm \epsilon) \in \partial V$. By transversality as discussed in the previous paragraph, $\partial JX(t)/\partial t \neq 0$ at $\alpha(s)$ for $s \neq 0$ and $JX(t)$ can be integrated as a curve on $M$ for $t - \delta(s) \leq \tau \leq t + \delta(s)$. Here $\delta(s)$ is a $C^1$ function which can be chosen increasing with $\delta(0) = \delta'(0) = 0$.

There exists a $C^1$ diffeomorphism $\phi : \Omega = \{(s,\tau) \in \mathbb{R}^2; -\epsilon \leq s \leq 0, t - \delta(s) \leq \tau \leq t + \delta(s)\} \to M$ such that $\phi(s,t) = \alpha(s)$ for $s \in [-\epsilon,\epsilon]$ and $\phi(s,\tau) \in \Gamma(\tau)$ for $t - \delta(s) \leq \tau \leq t + \delta(s)$. We consider a function $\tau : [-\epsilon,\epsilon] \to \mathbb{R}$, such that $(s,\pm \tau(s)) \in \Omega$ and $\tau$ is increasing, $\tau(0) = \tau'(0) = 0$ and $\tau(\epsilon) = t + \delta(\epsilon)$.

Now we can construct a path $\eta(\tau) \in \Gamma(\tau)$ which joins $(q,t)$ to a point in $\Gamma(t + \delta(\epsilon))$. The $C^1$ arc $f \circ \eta(\tau)$, $t \leq \tau \leq t + \delta(\epsilon)$ is locally parametrized by $\phi(s,\tau(s)), s \in [0,\epsilon]$ and continuously extends to $f(q,t)$ when $\tau \to t$. Each point $\alpha(s)$, can be connected $C^1$, by the arc $\phi(s,\tau), t \leq \tau \leq \tau(s)$ from $\alpha(s)$ to $\phi(s,\tau(s))$, and next a subarc of $\eta(\tau)$ for $\tau(s) < \tau \leq t + \delta(\epsilon)$ (see figure 1). The constant $\delta(\epsilon)$ depends only on $\alpha(\epsilon) = q_1$, and we note $\delta(q_1) = \delta(\epsilon)$.

Now there are a finite number of arcs $\alpha$ in $V - (q)$, with end points $q$ and a collection of $q_1, q_2, \ldots, q_{2k}$. So one has a $0 < \delta$ with $\delta < \delta(q_i)$ that works in a neighborhood of $q$. The claim is proved.

To complete the proof of the Dragging Lemma, it suffices to prove that $\gamma(t)$ extends $C^1$ for any value of $t \in [0,1]$. Assume that there is a point $t_0$ such that the arc $\gamma(t)$ is defined in a $C^1$ manner for $t < t_0$. By compactness of $A$, the arc accumulates at a point $(q,t_0) \in \Gamma(t_0)$. Remark that the structure of $M$ along $\Gamma(t_0)$ gives easily the existence of a continuous extension to $t_0$. To ensure a $C^1$ path through $t_0$, we need a more careful analysis at $(q,t_0)$.

Claim 2: Suppose the path $\gamma(t)$ satisfies the conditions of the Dragging lemma for $0 \leq t \leq t_0 < 1$. Then $\gamma(t)$ can be extended to $0 < t < t_0 + \delta$, to be $C^1$ and satisfy the conditions of the Dragging lemma, for some $\delta > 0$.

If $(q,t_0)$ is a transversal point, $M$ has a structure of a manifold and if $t_0 - \delta(t_0) < t_1 < t_0$ and $\eta(t_1) = (q_1,t_1)$ is in a neighborhood of $(q,t_0)$, we can find a $C^1$ arc that joins $\eta(t_1)$ to $(q,t_0) \in \Gamma(t_0)$. Next we extend the arc for $t_0 \leq t \leq t_0 + \delta(t_0)$.

If $(q,t_0)$ is a singular point, we consider a neighborhood $V \subset A$ of $q$ and $\Gamma(t_0)$ intersects $\partial V$ in $2k$ transversal points $q_1, \ldots, q_{2k}$. We consider $V \times [t_1, t_0]$ with $t_0 - \delta(t_0) < t_1 < t_0$. By transversality at $(q_1,t_0), \ldots, (q_{2k},t_0)$, the analytic set $\Gamma(t_1)$ intersects $\partial V$ in $2k$ points and $V$ in $k$ analytical arcs $\alpha_1, \ldots, \alpha_k$. We suppose that $\eta(t_1) \in \alpha_1 \subset V \times \{t_1\}$.

We construct below a monotonous $C^1$ arc from $\eta(t_1)$ to a point $(q,t_2)$ on $\partial V \times \{t_2\}$ for some $t_1 < t_2 < t_0$ and by transversality an arc from $(q,t_2)$ to a point $(q',t_0) \in \Gamma(t_0)$.
\[ V \times \{t_0\}, \text{ using the fact that } t_0 - \delta(t_0) < t_2. \] Next we can extend the arc in a \( C^1 \) manner from \((q', t_0)\) to some point in \( \Gamma(t_0 + \delta(t_0)) \).

We consider \((\tilde{q}_1, t_1), \ldots, (\tilde{q}_\ell, t_1)\) singular points of \( \Gamma(t_1) \cap V \times \{t_1\} \) and we denote by \( W_1, \ldots, W_\ell \) neighborhoods of \( \tilde{q}_1, \ldots, \tilde{q}_\ell \) in \( A \cap V \). The arc \( \alpha_1 \) cannot have double points in \( V \) without creating small loops. Hence \( \alpha_1 \) passes through each \( W_1, \ldots, W_\ell \) at most one time, before joining a point of \( \partial V \) (We can restrict \( V \) in such a way that there are no small loops in \( V \)).

First we assume that there is \( t_2 \) such that for any \( t \in [t_1, t_2] \), the curve \( \Gamma(t) \) has exactly one isolated singularity in each neighborhood \( W_i \times \{t\} \) with the same type as \( \tilde{q}_i \in \Gamma(t_1) \) \((i = 1, \ldots, \ell)\) and \( t_2 < t_1 + \delta(t_1) \). If we parametrize \( \alpha_1 : [s_0, s_{2\ell+1}] \rightarrow \Gamma(t_1) \), we can find \( s_1, \ldots, s_{2\ell} \) such that \( \alpha_1(s_{2k-1}), \alpha_1(s_{2k}) \in \partial W_k \) and \( I_k = [s_{2k-2}, s_{2k-1}] \) are intervals parametrizing transversal points in \( \Gamma(t_1) \).

The manifold structure of \( M \) gives an immersion \( \psi_j : I_j \times [t_1, t_1 + \delta] \rightarrow M, \quad t_1 + \delta < t_2 \) and \( j = 1, \ldots, \ell + 1 \). In the construction of \( \eta \) up to \( t_1 \), the singular points are isolated; then we can assume \( \eta(t_1) \) is a regular point of \( \Gamma(t_1) \), hence is contained in an \( \alpha_1(I_j) \). We construct the beginning of the arc \( \eta(\tau) \) as the graph parametrized by \( \phi_j(s, \tau(s)) \) with \( \tau \) an increasing function from \( t_1 \) to \( t_1 + \delta/n \) as \( s \) varies from \( \hat{s} \in I_j \), corresponding to the initial point \( \eta(t_1) = \alpha_1(\hat{s}) \), to \( s_{2j-1} \). Next we pass through the singularity \((\tilde{q}_j, t_1 + 2\delta/n)\) by constructing an arc which joins the point \( \phi_j(s_{2j-1}, t_1 + \delta/n) \in \Gamma(t_1 + \delta/n) \cap \partial W_j \) to the point \( \phi_{j+1}(s_{2j}, t_1 + 3\delta/n) \in \Gamma(t_1 + 3\delta/n) \cap \partial W_j \) (see figure 2). For a suitable value of \( n \) we can iterate this construction, passing through the singularities \( \tilde{q}_j, \tilde{q}_{j+1}, \ldots \), until we join a point \((\hat{q}, t_2)\) of \( \partial V \times \{t_2\} \) and then we extend the arc up to \( t_0 \) by transversality outside \( V \).
Now we look for this interval $[t_1, t_2]$. Let $t_1 < t'_1 < t_0$ and $\Gamma(t'_1)$ have several singularities in some neighborhood $W_k$, or a unique singularity of index less the one of the $\tilde{q}_k$. We consider in this $W_k$ a finite collection of neighborhoods of isolated singularities $W'_{k,1}, ..., W'_{k,k'}$. We observe, by transversality that there are the same number of components of $\Gamma(t_1)$ and $\Gamma(t'_1)$ in $W_k$ (see figure 3). Hence each $W'_{k,j}$ contains a number of components of $\Gamma(t'_1)$ strictly less than the number of components of $\Gamma(t_1)$ in $W_k$. The index of the singularity is strictly decreasing along this procedure. We can iterate this analysis up to a point where each singularity can not be reduced to a simple one. This gives the interval $[t_1, t_2]$.

\[ \Box \]

3. Proof of The Slab theorem

Assume $\Sigma$ is either simply connected ($\partial \Sigma = \emptyset$) or $\Sigma$ is an annulus with compact boundary: $\Sigma$ homeomorphic to $S^1 \times \mathbb{R}^+$. Also assume $\Sigma$ is properly minimally immersed in $\mathbb{H} \times \mathbb{R}$ and $\Sigma \subset S = \{(p, t) \in \mathbb{H} \times \mathbb{R}; |t| < (\pi - \epsilon)/2 \}$. We fix $h \geq 2\pi$ sufficiently large so that there are points of $\Sigma$ in the geodesic ball $B$ of radius $h$ of $\mathbb{H} \times \mathbb{R}$ with center at a point $p_0$ in $\mathbb{H} \times \{0\}$, and $\partial \Sigma \subset B$ (if $\partial \Sigma \neq \emptyset$).

Let $\text{Cat}(p_0)$ denote a compact part of a rotational catenoid, a bi-graph over $t = 0$, bounded by two circles outside of the slab $S$, $p_0 \in \text{Cat}(p_0)$. We assume $h$ is sufficiently large so that $B$ also contains $\text{Cat}(p_0)$; see figure 4.

Since $\Sigma$ is properly immersed, the set $B \cap \Sigma$ has a finite number of compact connected components and there is a compact $K$ in $\mathbb{H} \times \mathbb{R}$, such that any two points of $\Sigma$ in $B$ can be joined by a path of $\Sigma$ in $K$.

Now suppose $p \in \Sigma - K$ has a vertical tangent plane and let $M = \alpha \times \mathbb{R}$ be tangent to $\Sigma$ at $p$, $\alpha$ a geodesic of $\mathbb{H}$. We will prove $M$ must intersect $K$. Suppose this were not the case. $M$ separates $\mathbb{H} \times \mathbb{R}$ in two components $M(+)$ and $M(-)$; assume $K \subset M(+)$.

The local intersection of $M$ and $\Sigma$ near $p$ consists of $2k$ curves through $p$, $k \geq 1$, meeting at equal angles $\pi/k$ at $p$.

Let $\Sigma_1(+)$ and $\Sigma_2(+)$ be distinct local components at $p$ of $\Sigma - M$, that are contained in $M(+)$. Observe that $\Sigma_1(+) \cap M(+)$. Otherwise we could find a path $\alpha_0$ in $\Sigma \cap M(+)$, joining a point $x \in \Sigma_1(+) \cap M(+)$. Then join $x$ to $y$ by a local path $\beta_0$ in $\Sigma$ going through $p$, but $\beta_0 \subset M(+)$ except at $p$; see figure 5.

Let $\Gamma = \alpha_0 \cup \beta_0 \subset M(+)$. If $\Sigma$ is simply connected $\Gamma$ bounds a compact disk $D$ in $\Sigma$. If $\Sigma$ is an annulus there are two cases. $\Gamma \cup \partial \Sigma$ bounds an immersed compact annulus (we also call $D$) or $\Gamma$ bounds a compact disk in the annulus. By construction of $\Gamma$, $D$ contains points of $M(-)$. $D$ is compact and minimal, $\partial D \subset M(+)$. $\Gamma$ is strictly decreasing along this procedure. We can iterate this analysis up to a point where each singularity can not be reduced to a simple one. This gives the interval $[t_1, t_2]$.

Thus $\Sigma_1(+) \cap M(+)$. Now let $\mu(\epsilon)$ be the geodesic of length $\epsilon$ starting at $p$, normal to $\Sigma$ at $p$, and contained
in \( M(+) \). We will now also denote by \( \alpha \), the geodesic \( \alpha \) translated vertically to pass through \( p \). Let \( \alpha(\epsilon) \) be the complete geodesic of \( \mathbb{H} \) obtained from \( \alpha \) by translating \( \alpha \) along \( \mu(\epsilon) \) to the endpoint of \( \mu(\epsilon) \) distinct from \( p \). The distance between \( \alpha \) and \( \alpha(\epsilon) \) diverges at infinity; the two geodesics have distinct end points at infinity.

Choose \( \epsilon \) small so that \( M(\epsilon) = \alpha(\epsilon) \times \mathbb{R} \) meets both \( \Sigma_1(+) \) and \( \Sigma_2(+) \), at points \( x \in \Sigma_1, y \in \Sigma_2 \). If \( \Sigma \) is simply connected then no connected component of \( M(\epsilon) \cap \Sigma \) can be compact. Hence \( x \) and \( y \) are in non-compact components \( C(x) \subset \Sigma_1 \cap M(\epsilon) \) and \( C(y) \subset \Sigma_2 \cap M(\epsilon) \).

We claim this also holds when \( \Sigma \) is an annulus; more precisely:

**Claim:** If \( \Sigma \) is an annulus then the components \( C(x) \) of \( x \) in \( \Sigma_1 \cap M(\epsilon) \) and \( C(y) \) of \( y \) in \( \Sigma_2 \cap M(\epsilon) \) are both non compact.

**Proof of the Claim.** First suppose \( C(x) \) and \( C(y) \) are compact. Neither can be null homotopic in \( \Sigma \) by the maximum principle, hence \( C(x) \cup C(y) \) bound an immersed compact annulus \( D \) in \( \Sigma \). But \( \partial D \subset M(\epsilon) \) which is impossible.

It remains to show one of the \( C(x), C(y) \) can not be non compact. Suppose \( C(x) \) is not compact and \( C(y) \) is compact. So \( C(y) \cup \partial \Sigma \) bound an immersed annulus \( D \) in \( M(\epsilon)(+) \). The distance between \( M \) and \( M(\epsilon) \) diverges and \( \Sigma \) is proper so there are points \( z \) of \( C(x) \) arbitrarily far from \( M \).

Choose such a \( z \) so that one can place a vertical catenoid \( \text{Cat}(z) \) in \( M(+) \) which is a horizontal translation of \( \text{Cat}(p_0) \) and contains \( z \). The boundary \( \partial(\text{Cat}(z)) \cap S = \emptyset \).

Let \( \eta \) be a geodesic joining \( z \) to the point \( p_0 \in B \). Apply the Dragging Lemma to the translation of \( \text{Cat}(z) \) along \( \eta \) from \( z \) to \( p_0 \), so that the translation of \( \text{Cat}(z) \) is contained in \( B \), at the end of the movement.

We obtain a path in \( \Sigma \cap M(+) \) joining \( z \) to a point \( \omega \) of \( \Sigma \) in \( B \). Then join \( \omega \) to a point of \( \partial \Sigma \) in \( \Sigma \cap K \). This path contradicts that \( \Sigma_1 \) and \( \Sigma_2 \) are in distinct components of \( \Sigma - M \). Hence the claim is proved and both \( C(x) \subset \Sigma_1 \cap M(\epsilon) \) and \( C(y) \subset \Sigma_2 \cap M(\epsilon) \) are non compact.

**Figure 4. The compact \( K \) and the catenoid \( \text{Cat}(p_0) \)**
We now continue the proof that $M$ must intersect $K$. $\Sigma$ is either simply connected or an annulus and we have two non compact components $C(x) \subset \Sigma_1 \cap M(\epsilon), C(y) \subset \Sigma_2 \cap M(\epsilon)$ and $\Sigma_1, \Sigma_2$ are distinct components of $\Sigma - M$.

Now, as we argued in the proof of the claim above, we can take points $z_1 \in C(x), z_2 \in C(y)$, far enough away from $M$ so that one can place compact vertical catenoids $\text{Cat}(z_1)$ and $\text{Cat}(z_2)$, so $z_1 \in \text{Cat}(z_1), z_2 \in \text{Cat}(z_2), \partial \text{Cat}(z_i) \cap S = \emptyset, i = 1, 2$ and $\text{Cat}(z_i)$ are symmetric with respect to $t = 0$. Also $\text{Cat}(z_i) \subset M(+) , i = 1, 2$.

Take horizontal geodesics $\eta_1, \eta_2 \subset M(\epsilon)(+)$ from $\text{Cat}(z_1)$ to $p_0$ and from $\text{Cat}(z_2)$ to $p_0$. Apply the Dragging Lemma along $\eta_1, \eta_2$, to find a path in $\Sigma$ from $z_1$ to a point $\omega_1 \in \Sigma \cap B$, and another path from $z_2$ to $\omega_2 \in \Sigma \cap B$. Join $\omega_1$ to $\omega_2$ by a path in $\Sigma \cap K$. This contradicts that $z_1$ and $z_2$ are in distinct components of $\Sigma - M$. Hence $M$ intersects $K$.

To complete the proof of the theorem we will show that when $p \in \Sigma - K$ is far enough from $K$ then $p$ can not have a vertical tangent plane. We will do this by showing such a vertical tangent plane can not intersect $K$.

To do this we introduce comparison surfaces $M(h)$, first introduced by Hauswirth [9], then by Toubiana and Sa Earp and Toubiana [14], Daniel [6] and Mazet, Rodriguez, Rosenberg [15]. The surfaces $M(h)$ are all congruent in $\mathbb{H} \times \mathbb{R}$, they are complete minimal surfaces invariant by hyperbolic translation along a geodesic of $\mathbb{H}$. They exist for each $h > \pi$; we state the properties we will use.

1. Let $\beta$ be an equidistant of a geodesic $\gamma$ of $\mathbb{H} \times \{0\}$, whose distance $d_0$ to $\gamma$ is determined by $h$. There is an $M(h)(= M(h, \beta))$ which is a minimal bigraph over the domain $\Omega$ of $\mathbb{H}$ indicated in figure 6.
2. $M(h)$ has height $h/2$ over $\Omega \times \{0\}$,
3. $M(h)$ meets $\mathbb{H} \times \{0\}$ orthogonally along $\beta$,
4. $M(h)$ meets each $\mathbb{H} \times \{t\}$ in an equidistant of $\gamma \times \{t\}$ for $|t| < h/2$,
5. The asymptotic boundary of $M(h)$ is the vertical rectangle of $\mathbb{H} \times \mathbb{R}$:
$$\partial_\infty(M(h)) = (\partial_\infty(\beta) \times [-h, h]) \cup (\mu \times \{\pm h/2\}).$$
$\mu$ is the arc in $\partial_\infty(\mathbb{H})$ joining the end points of $\beta$, indicated in figure 6.

6. Given $q \in \mathbb{H}$ and a vector $v$ tangent to $\mathbb{H}$ at $q$, there exists an $M(h, \beta) = M(h)$ such that $q \in \beta$ and the tangent to $\beta$ at $q$ is orthogonal to $v$.

7. Now assume $h \geq 2\pi$ and $S$ is the slab of height $\pi - \epsilon$:
$$S = \{|t| \leq (\pi - \epsilon)/2\}.$$ Then there exists $d_2 > 0$ such that if $|t| < (\pi - \epsilon)/2$ and $T_t$ is a vertical translation by $t$ then $\text{dist}(T_t(M(h)) \cap S, \gamma \times \mathbb{R}) \leq d_2$. Here $\gamma$ is the geodesic of which $\beta$ is the equidistant. In fact, in any strict subslab $\tilde{S}$ of $\mathbb{H} \times [-h/2, h/2]$, $\tilde{S}$ symmetric about $t = 0$, $M(h)$ is a bigraph over the moon between $\beta$ and another equidistant $\tilde{\beta}$ of $\gamma$ and this moon is a bounded distance $d_2$ from $\gamma$.

8. If $\gamma_1$ and $\gamma_2$ are complete geodesics of $\mathbb{H}$ then $M_{\gamma_1}(h)$ is congruent to $M_{\gamma_2}(h)$ by a height preserving isometry of $\mathbb{H} \times \mathbb{R}$.

9. Let $p_0 = (0, y_0)$ and for $0 < y < y_0$, let $\beta$ be an equidistant curve of $\gamma$ such that $\beta$ is tangent at $(0, y)$ to the geodesic $\alpha$ through $(0, y)$ and $p_0$ (here $\beta$ is an equidistant of an $M(h)$). Then $\text{dist}_{\mathbb{H}}(p_0, \gamma) \to \infty$, as $y \to 0$; see figure 7.

Here is one way to verify this last assertion. For $(0, y)$ and $h \geq 2\pi$ fixed, $\beta, \gamma$ and $M_\gamma$ are uniquely determined so that $M_\gamma$ is tangent to $\alpha \times \mathbb{R}$ at $(0, y)$ at a point of $\beta$. So it suffices to fix $(0, y)$, and let $p_\lambda = \lambda p_0$, with $\lambda \to \infty$. The minimizing geodesic from $p_\lambda$ to $\gamma$ tends to the vertical geodesic going up from the point of maximum $y$ coordinate on $\gamma$; see figure 7. Hence it’s length goes to $\infty$ as $\lambda \to \infty$; see figure 7.

Now we are ready to finish the proof of the theorem. Let $C > 0$ be such that if $\text{dist}_{\mathbb{H}}(p_0, (0, y)) \geq C$, then $\text{dist}_{\mathbb{H}}(p_0, \gamma) > d_2 + \text{diam}(K)$. Recall that $p_0$ is the center of the ball $B$. We write $p_0 = (0, y_0)$. We will prove that a point of $\Sigma$ at a distance at least $C$ from $p_0$, can not have a vertical tangent plane; this will prove the theorem.

Assume the contrary. Write $p = (0, y, t), p \in \Sigma$ is a point with a vertical tangent plane and $\text{dist}_{\mathbb{H}}(p_0, (0, y)) \geq C$. We vertically translate $M(h)$ by some $t, |t| < \pi/2,$
so that \( M_t(h) = T_t(M(h)) \) is tangent to \( \Sigma \) at \( p \). Let \( \gamma \) be the geodesic associated to \( M_t(h) \). Since \( \text{dist}_\mathbb{H}(p_0, (0, y)) \geq C \), we know that \( \text{dist}_\mathbb{H}(p_0, \gamma) > d_2 + \text{diam}(K) \).

Now if \( M_t(h) \) does not intersect \( K \), the same proof we gave using \( M_t(h) \) in place of \( M = \alpha \times \mathbb{R} \), shows we obtain a contradiction.

We explain this further. Let \( \eta \) be a geodesic of \( \mathbb{H} \) orthogonal to \( \gamma \), \( p \in \eta \). The set of all geodesics \( \gamma(s) \), \( s \in \mathbb{R} \), of \( \mathbb{H} \), orthogonal to \( \eta \), foliates \( \mathbb{H} \) and \( \gamma \) is a leaf of this foliation. For \( h > \pi \), the \( M(h, \gamma(s)) \) foliate the slab of \( \mathbb{H} \times \mathbb{R} \) between \( t = h \) and \( t = -h \). Hence if \( D \) is a compact minimal surface whose boundary is in some \( M(h, \gamma(s)) \), then \( D \) is contained in \( M(h, \gamma(s)) \). The other property of this foliation one uses is the following: If \( \Sigma \) is a properly immersed minimal surface in the slab \( S \) (height \( \pi - \epsilon \)) and the intersection of \( \Sigma \) and \( M(h, \gamma(\epsilon)) \) has a non compact component \( C \), for some \( \epsilon > 0 \), then \( \text{dist}(C, M(h, \gamma(0))) \) tends to infinity as one diverges in \( C \).

Hence it suffices to show \( M_t(h) \) can not intersect \( K \). Suppose there is a point \( w \in M_t(h) \cap K \). Then \( \text{dist}(\omega, p_0) \leq \text{diam}K \), and \( \text{dist}(\omega, \gamma) \leq d_2 \), so \( \text{dist}(p_0, \gamma) \leq d_2 + \text{diam}K \); a contradiction.

Now suppose \( \Sigma \) of the theorem is embedded. Let \( r > 0 \) and \( C(r) = \{ p \in \mathbb{H}; d(p, p_0) = r \} \). Define \( \text{Cyl}(r) = C(r) \times \mathbb{R} \); \( \text{Cyl}(r) \) is a vertical cylinder of radius \( r \).

Let \( r_0 > 0 \) be large so that \( \Sigma \) is a multigraph for \( r \geq r_0 \) (we proved \( \Sigma \) is not vertical for large \( r \)). \( \Sigma \) is proper, so \( \Sigma \cap \text{Cyl}(r) \) is a finite union of embedded Jordan curves, each a graph over \( C(r) \), for each \( r \geq r_0 \).

Let \( \beta(r) \) be one of the graphical components of \( \Sigma \cap \text{Cyl}(r) \), \( r \geq r_0 \). If \( \Sigma \) is simply connected (so \( \partial \Sigma = \emptyset \)), then the usual proof of Rado’s theorem shows \( \beta(r) \) bounds a unique compact minimal surface that is a graph over the disk bounded by \( C(r) \).

Hence \( \Sigma \) is an entire graph.

The same arguments shows that if \( \partial \Sigma = \emptyset \) and \( \Sigma \) has an annular end then \( \Sigma \cap \text{Cyl}(r) \) is one Jordan curve, a graph over \( C(r) \), that (by Rado’s theorem) bounds a unique compact minimal surface, a graph over the disk bounded by \( C(r) \). This proves (1) of the Slab theorem. Now suppose \( \Sigma \) is a properly immersed minimal annulus in \( S \),
with compact boundary. Choose \( r \) large so that \( \partial \Sigma \subset \text{Cyl}(r) \) and \( \Sigma \cap \text{Cyl}(r) \) is a finite union of multi-graphs. Since \( \Sigma \) has one end, there is exactly one multi-graph, and the end is a multi-graph. If \( \Sigma \) is embedded \( \Sigma \cap \text{Cyl}(r) \) is a graph and the end is a graph. This proves (2).

**Remark 3.1.** We now describe the surface discussed in Remark 1.2. Consider the surface \( M_\gamma(h), h > \pi \) and \( \gamma \) a geodesic of \( \mathbb{H} \). Fix a point \( p \) in \( \gamma \) and deform \( \gamma \) through equidistant curves \( \beta(t), 0 \leq t \leq 1 \), through \( p \), such that \( \beta(0) = \gamma \) and \( \beta(1) \) is a horocycle (do this by continuously deforming the endpoints of \( \gamma \) to make them converge to one point at infinity). Then the surfaces \( M_{\beta(t)}(h(t)) \) converge to a minimal surface \( \Sigma \) of height \( \pi \). The asymptotic boundary of \( \Sigma \) is a vertical segment of height \( \pi \). \( \Sigma \) is vertical along \( \beta(1) \), this simply connected embedded minimal surface shows the Slab Theorem fails for slabs of height \( \pi \). Benoit Daniel explicitly parametrized this surface; cf proposition 4.17 of [7].

4. An Enneper Type Minimal Surface

We construct a properly immersed simply connected minimal surface in \( \mathbb{H} \times \mathbb{R} \) contained in a slab \( S \) (of any height \( h \)) that is a 3-sheeted multi-graph outside of a compact set.

The idea comes from Enneper’s minimal surface \( \mathcal{E} \) in \( \mathbb{R}^3 \), whose Weierstrass data is \( (g, \omega) = (z, dz) \) on the complex plane \( \mathbb{C} \). We think of \( \mathcal{E} \) as constructed by solving Plateau problems for certain Jordan curves in \( \mathbb{R}^3 \); passing to the limit, and then reflecting about the two rays in the boundary. More precisely let \( C_n \) be a Jordan curve consisting of the segments on the \( x \) and \( y \) axes between 0 and \( n \), and the (slightly tilted up) large arc on the circle of radius \( n \) (centered at \( (0,0) \)) joining \( (n,0) \) to \( (0,n) \); see figure 8.

Let \( D_n \) be a disk of minimal area with \( \partial D_n = C_n \); see figure 8-Right.

One can choose \( C_n \) so the \( D_n \) converge to a minimal surface (not flat) with boundary the positive \( x \) and \( y \) axes. \( \mathcal{E} \) is then obtained by Schwarz reflection in the \( x \) and \( y \) axes. We now do this in \( \mathbb{H} \times \mathbb{R} \).
We use now the unit disk \( \{ x^2 + y^2 < 1 \} \) with the hyperbolic metric as a model for \( \mathbb{H} \). Let \( h > 0 \) and \( n \) an integer, \( n \geq 1 \). In \( \mathbb{H} \times \mathbb{R} \), let \( C_n \) be the Jordan curve; see figure 9:

\[
C_n = \{(x, 0); 0 \leq x \leq r_n \} \cup \{(0, y); 0 \leq y \leq r_n \} \\
\cup \{(r_n \cos \theta, r_n \sin \theta, h); \pi/2 \leq \theta \leq 2\pi \} \\
\cup ((r_n, 0) \times [0, h]) \cup ((0, r_n) \times [0, h])
\]

where \( r_n \to 1 \). Let \( D_n \) be a least area minimal disk with \( \partial D_n = C_n \). We claim a subsequence of \( D_n \) converges to a stable non trivial minimal surface with boundary the positive \( x \) and \( y \) axes. \( \Sigma \) is then obtained by reflection in the horizontal boundary geodesics.

First observe the \( C_n \) converge in the model of the disk to

\[
C = \{(x, 0); 0 \leq x \leq 1 \} \cup \{(0, y); 0 \leq y \leq 1 \} \\
\cup \{(\cos \theta, \sin \theta, h); \pi/2 \leq \theta \leq 2\pi \} \\
\cup ((1, 0) \times [0, h]) \cup ((0, 1) \times [0, h])
\]

The latter circular arc at infinity- \( \{(\cos \theta, \sin \theta, h); \pi/2 \leq \theta \leq 2\pi \} \) is the upper part of a vertical rectangle at infinity that bounds a minimal surface \( M(h_1) \), \( h_1 > \pi \). We remark that as \( h_1 \to \pi \), \( h_1 > \pi \) the \( \beta \) of \( M(h_1) \) tends to a horocycle. Hence \( d_2 = \text{dist}(\beta, \gamma) \to \infty \) as \( h_1 \to \pi \). This implies \( M(h_1) \) is a barrier for each of the \( D_n \), for \( h_1 \) sufficiently close to \( \pi \); see figure 10.

Translate \( M(h_1) \) down to be below height 0, then go back up to height \( h \). There can be no contact with \( D_n \). To prevent \( D_n \) from escaping in the sector \( 0 \leq \theta \leq \pi/2 \), one can place another \( M(h_1) \) in this sector, whose vertical rectangle at infinity has
horizontal arc of angle less than $\pi/2$ and whose vertical sides go from a height below zero to a height above $h$.

Let $F$ be the vertical solid cylinder in $\mathbb{H} \times \mathbb{R}$ over the circle of radius 1 in $\mathbb{H}$, centered at $(-2, -2)$ (hyperbolic distances). For $n > 2$, each $D_n$ intersects $F$ in a minimal surface of bounded curvature (bound independent of $n$). Hence a subsequence of these surfaces converges to a minimal surface, which is not part of a horizontal slice. The barriers $M(h_1)$ then show a subsequence of the $D_n$ converge to a minimal surface $\Sigma_1$ with boundary the positive $x$ and $y$ axes. To guaranty that $D_n$ does not escape in the sector $\{0 \leq \theta \leq \pi/2\}$, one puts a barrier $M(h_2)$ as in figure 11.

Now do the reflection about the positive $x$ and $y$ axis and then about the negative $x$ and $y$ axes. This gives a complete minimal (Enneper type) immersed surface. One checks the origin is not a singularity as follows. Reflecting $D_n$ four times gives an immersed minimal punctured disk that has the origin in its closure. Gulliver [8] then proved the origin is not a singularity.

**References**


MINIMAL SURFACES IN A SLAB OF $\mathbb{H} \times \mathbb{R}$

Figure 11. The barrier $M(h_2)$


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