Minimal Surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Barbara Nelli - Harold Rosenberg

Abstract

In $\mathbb{H}^2 \times \mathbb{R}$ one has catenoids, helicoids and Scherk-type surfaces. A Jenkins-Serrin type theorem holds here. Moreover there exist complete minimal graphs in $\mathbb{H}^2$ with arbitrary continuous asymptotic values. Finally, a graph on a domain of $\mathbb{H}^2$ cannot have an isolated singularity.

1 Introduction

In this paper we consider minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$; particularly, surfaces which are vertical graphs over domains in $\mathbb{H}^2$. When a convex domain $D \subset \mathbb{H}^2$ is bounded by geodesic arcs $A_1, \ldots A_n, B_1, \ldots, B_m$, together with strictly convex arcs $C_1, \ldots, C_s$, we obtain necessary and sufficient conditions (in terms of the lengths of the boundary arcs of $D$) which assure the existence of a unique function $u$ defined in $D$, whose graph is a minimal surface of $\mathbb{H}^2 \times \mathbb{R}$, and which takes the values $+\infty$ on the arcs $A_1, \ldots A_n$, $-\infty$ on the arcs $B_1, \ldots, B_m$, and arbitrary prescribed continuous data on the arcs $C_1, \ldots, C_s$. In $\mathbb{R}^2 \times \mathbb{R}$, this is the theorem of Jenkins and Serrin [JS].

For example, let $D$ be a domain whose boundary is a regular geodesic octagon with sides $A_1, B_1, \ldots, A_4, B_4$, and suppose the interior angles are $\frac{\pi}{2}$. Our theorem yields a function $u$ in $D$, whose graph is minimal, taking the values $+\infty$ on each $A_i$, and $-\infty$ on each $B_j$. The graph of $u$ is bounded by the eight vertical geodesics passing through the vertices of $D$. Rotation of each $\mathbb{H}^2 \times \{t\}$, by $\pi$ about each vertex of $D \times \{t\}$, extends the graph of $u$ to a complete embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ (one continues the rotation about all the vertical geodesics that arise). One can take the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by various Fuchsian groups to obtain interesting quotient surfaces. For example, one can obtain an 8-punctured sphere in the quotient of total curvature $-12\pi$; with four top ends and four bottom ends.

One also obtains graphs over ideal polygons with vertices at infinity. For example, consider the polygon which is the boundary of the convex hull of the $n$ roots of unity, $n$ even and at least four. Then, there is a minimal graph over the interior of this polygon taking the values plus and minus infinity on adjacent edges (cf. Figure 2(b)).

We prove the existence of entire minimal graphs over $\mathbb{H}^2$ (Bernstein’s theorem fails here). In the model $\{0 \leq x_1^2 + x_2^2 < 1\}$ of $\mathbb{H}^2$, the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$ is $\{x_1^2 + x_2^2 = 1\} \times \mathbb{R}$. For any Jordan curve $\Gamma$ in the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$ that
has a simple projection on \( \{ x_1^2 + x_2^2 = 1 \} \), there is a minimal graph over \( \mathbb{H}^2 \) having \( \Gamma \) as asymptotic boundary.

In [DN], the existence of such minimal graphs is established when \( \Gamma \) is the boundary value of a function with very small \( C^3 \)-norm on the disk.

Finally, we prove a theorem for minimal graphs defined over a punctured disk in \( \mathbb{H}^2 \): the graph extends smoothly to the puncture.

2 Preliminaries

In the three dimensional manifold \( \mathbb{H}^2 \times \mathbb{R} \), we take the disk model for \( \mathbb{H}^2 \). Let \( x_1, x_2 \) denote the coordinates in \( \mathbb{H}^2 \) and \( x_3 \) the coordinate in \( \mathbb{R} \). The metric in \( \mathbb{H}^2 \times \mathbb{R} \) is

\[
d\sigma^2 = \frac{dx_1^2 + dx_2^2}{F} + dx_3^2
\]

where

\[
F = \left( \frac{1 - x_1^2 - x_2^2}{2} \right)^2
\]

The graph of a function \( u \) defined over a domain in \( \mathbb{H}^2 \) has constant mean curvature \( H \) if and only if \( u \) satisfies the following equation:

\[
(1) \quad \text{div} \left( \frac{\nabla u}{\tau_u} \right) = \frac{2H}{F}
\]

where \( \tau_u = \sqrt{1 + F|\nabla u|^2} \) and the divergence is the divergence in \( \mathbb{R}^2 \). We list the principal steps for the computation of (1).

The Christoffel symbols for the metric \( d\sigma^2 \) are the following:

\[
\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{x_1}{\sqrt{F}}
\]

\[
\Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = \frac{x_2}{\sqrt{F}}
\]

\[
\Gamma^2_{11} = -\frac{x_2}{\sqrt{F}}, \quad \Gamma^1_{22} = -\frac{x_1}{\sqrt{F}}
\]

The other \( \Gamma^k_{ij} \) are identically zero.

Let \( e_1, e_2, e_3 \), be the canonical basis of \( \mathbb{R}^3 \) and set \( \varepsilon_1 = \sqrt{F} e_1, \varepsilon_2 \sqrt{F} e_2, \varepsilon_3 = e_3 \), so that \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) is an orthonormal basis for \( \mathbb{H}^2 \times \mathbb{R} \). Finally let \( \bar{\nabla} \) be the connection of the metric \( d\sigma^2 \). We have:

\[
\bar{\nabla}_{e_1} e_1 = -x_2 e_2, \quad \bar{\nabla}_{e_2} e_2 = -x_1 e_1,
\]
\[ \nabla \epsilon_1 \epsilon_2 = x_2 \epsilon_1, \quad \nabla \epsilon_2 \epsilon_1 = x_1 \epsilon_2. \]

The coordinate vector fields on the graph of \( u \) are \( X_1 = \frac{1}{\sqrt{F}} \epsilon_1 + u_1 \epsilon_3 \), \( X_2 = \frac{1}{\sqrt{F}} \epsilon_2 + u_2 \epsilon_3 \) and \( N = \tau^{-1}(-u_1 \sqrt{F} \epsilon_1 - u_2 \sqrt{F} \epsilon_2 + \epsilon_3) \) is the upward unit normal. The induced metric on the graph is:

\[ g_{11} = \frac{1}{F} + u_1^2, \quad g_{12} = u_1 u_2, \quad g_{22} = \frac{1}{F} + u_2^2. \]

The coefficients of the second fundamental form are:

\[ b_{11} = \langle \nabla X_1 X_1, N \rangle = \frac{1}{\tau} \left( -\frac{x_1 u_1}{\sqrt{F}} + \frac{x_2 u_2}{\sqrt{F}} + u_{11} \right) \]
\[ b_{12} = \langle \nabla X_1 X_2, N \rangle = \frac{1}{\tau} \left( -\frac{x_2 u_1}{\sqrt{F}} - \frac{x_1 u_2}{\sqrt{F}} + u_{12} \right) \]
\[ b_{22} = \langle \nabla X_2 X_2, N \rangle = \frac{1}{\tau} \left( \frac{x_1 u_1}{\sqrt{F}} - \frac{x_2 u_2}{\sqrt{F}} + u_{22} \right) \]

where \( \langle, \rangle \) is the scalar product for the metric \( d\sigma^2 \).

Equation (1) is obtained by substituting the quantities just calculated in the following identity:

\[ 2H = \frac{b_{11} g_{22} + b_{22} g_{11} - 2 b_{12} g_{12}}{g_{11} g_{22} - g_{12}^2}. \]

### 3 Catenoids and Helicoids

**Catenoids.** We construct a family of minimal rotational surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) (see also [PR]).

Let \( \pi \) be a vertical geodesic plane containing the origin and let \( \gamma \) be a curve in \( \pi \). Assume \( \gamma \) to be a graph over the \( x_3 \) axis. Let \( r \) be the Euclidean distance between the point of \( \gamma \) at height \( t \) and the \( x_3 \) axis: \( r = r(t) \) is a parametrization of the curve \( \gamma \). Consider the surface of revolution \( S \) obtained by rotating \( \gamma \) about the \( x_3 \) axis. \( S \) is minimal if and only if \( r = r(t) \) satisfies the following differential equation:

\[ 4r(t) r''(t) - 4r'(t)^2 - (1 - r(t)^4) = 0. \]

A first integral for equation (2) is
\[ J(t) = \frac{r'^2}{r^2} - \frac{1 + r^4}{4r^2} = -C \]

where \( C \) is a positive constant (\( i.e. \frac{dJ(t)}{dt} = 0 \) along the curve \( \gamma \)). Hence we obtain:

\[ r' = \pm \sqrt{Cr^2 - \frac{1 + r^4}{4}} \]

The allowed values for \( C \) and \( r \) are

\[ C > \frac{1}{2} \]

\[ 2C - \sqrt{4C^2 - 1} \leq r^2 < 1 \leq 2C + \sqrt{4C^2 - 1} \]

i.e.

\[ r_{\text{min}} = \sqrt{\frac{2C + 1}{2}} - \sqrt{\frac{2C - 1}{2}} \leq r < 1 \]

Remark that as \( C \rightarrow \frac{1}{2} \) then \( r_{\text{min}} \rightarrow 1 \) hence the curve \( \gamma \) disappears at infinity, while as \( C \rightarrow \infty \) then \( r_{\text{min}} \rightarrow 0 \).

We can write equation (4) as follows:

\[ \frac{dt}{dr} = \pm \frac{2}{\sqrt{4Cr^2 - (1 + r^4)}} \]

In order to study the curve \( \gamma \) we can choose the positive sign in (5), i.e. \( t > 0 \). In fact by the symmetries of equation (4) and (5), the curve \( \gamma \) for \( t < 0 \) will be the reflection with respect to the plane \( x_3 = 0 \) of the curve \( \gamma \) for \( t > 0 \). With the choice of the positive sign, we have the following properties.

\[ (i) \ \frac{dt}{dr} > 0 \] hence \( t \) is an increasing function of \( r \).

\[ (ii) \ For \ r = r_{\text{min}} \ we \ have \ \frac{dt}{dr} = \infty, \ i.e. \ the \ tangent \ to \ the \ curve \ \gamma \ at \ the \ point \ r = r_{\text{min}} \ is \ parallel \ to \ the \ x_3 \ axis \ and \ this \ is \ the \ only \ point \ where \ this \ happens. \]

\[ (iii) \ As \ r \rightarrow 1, \ we \ have: \]

\[ \frac{dt}{dr} \rightarrow \frac{\sqrt{2}}{\sqrt{2C - 1}}. \]

Hence, when \( C \) varies between \( \frac{1}{2} \) and \( +\infty \) the asymptotic angle of \( \gamma \) varies between \( \frac{\pi}{2} \) and \( 0 \).

\[ (iv) \ \frac{d^2t}{dr^2} < 0, \ hence \ the \ concavity \ does \ not \ change. \]
(v) Consider the change of variables $w = r^2$. Equation (5) becomes

$$dt = \pm \frac{dw}{\sqrt{w(2C + \sqrt{4C^2 - 1} - w)(w - 2C + \sqrt{4C^2 - 1})}}$$

Hence

$$t(w) = \pm \int_{2C - \sqrt{4C^2 - 1}}^{w} \left( s(2C + \sqrt{4C^2 - 1} - s)(s - 2C + \sqrt{4C^2 - 1}) \right)^{-\frac{1}{2}} ds$$

This is an elliptic integral. By the properties of elliptic functions $\lim_{C \to \infty} t(w)$ is independent of $w$. For every value of the constant $C$ we have $t(4C - \sqrt{4C^2 - 1}) = 0$, hence for the limit value $C = \infty$, the surface is a horizontal plane (doubly covered).

Using (i)-(v) we have the following theorem (see Figure 1).

**Theorem 1.** Let $\Gamma_{\pm}(t)$ be the two circles at infinity of $\mathbb{H}^2 \times \mathbb{R}$ defined by $\{x_1^2 + x_2^2 = 1, \ x_3 = \pm t\}$. Then, for each $t \in (0, \frac{\pi}{2})$ there exists a rotational surface (catenoid) $C(t)$, whose asymptotic boundary is $\Gamma_{\pm}(t) \cup \Gamma_{-}(t)$. As $t \to 0$, $C(t)$ converges to the doubly covered plane $\mathbb{H}^2$ with a singularity at the origin. As $t \to \infty$, $C(t)$ diverges to the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$. Furthermore for any angle $\alpha \in ]0, \frac{\pi}{2}[\, there is a $C(t)$ whose asymptotic normal vector at boundary points forms an angle equal to $\alpha$ with the $x_3$ axis.

Added in proof: see Proposition 5.1 in [5], Theorem 15 in [12]).

**Helicoids.** Let $\alpha$ be a horizontal geodesic passing through the $x_3$ axis. Consider the surface $\mathcal{E}$ obtained by translating $\alpha$ vertically and rotating it around the $x_3$ axis. A parametrization for $\mathcal{E}$ is the following:
\( X(u, v) = (v \cos \theta(u), v \sin \theta(u), u) \)

\( v \in (-1, 1), u \in \mathbb{R} \) and \( \theta : \mathbb{R} \rightarrow \mathbb{R} \) is a \( C^2 \) function representing the angle between \( \alpha \) and the \( x_1 \) axis at the level \( u \).

\( \mathcal{E} \) is a minimal surface if and only if

\[ \theta_{uu} = 0. \]

Hence the solutions are:

(i) \( \theta(u) = a, a \in \mathbb{R} \). In this case \( \mathcal{E} \) is a vertical plane forming an angle \( a \) with the \( x_1 \) axis.

(ii) \( \theta(u) = au, a \in \mathbb{R} \setminus \{0\} \). In this case the surface \( \mathcal{E} \) is congruent to a Euclidean helicoid.

## 4 Scherk Type Surfaces

Let \( \mathcal{P} \) be a regular \( 2k \)-gon in \( \mathbb{H}^2 \) with (open) edges \( A_1, B_1, \ldots, A_k, B_k \) (see Figure 2(a), \( k = 2 \)). We will construct a minimal graph \( \Sigma \) over the domain \( D \) bounded by \( \mathcal{P} \) such that the boundary values are alternatively \( +\infty \) on the edges \( A_i \) and \( -\infty \) on the edges \( B_i \). \( \Sigma \) will be called a Scherk type surface. Also, the Scherk surface exists if the vertices of \( A_i \) and \( B_j \) are at infinity; so that \( \mathcal{P} \) is a ideal polygon whose vertices are the \( 2k \) roots of unity (see Figure 2(b), \( k = 2 \)).

Choose one edge of \( \mathcal{P} \) where the desired value is \( +\infty \) and call it \( A \). Let \( B \) and \( C \) be the two geodesic arcs passing through the center of \( \mathcal{P} \) and the vertices of \( A \). Let \( T \)
be the (open) triangle with sides $A$, $B$ and $C$. Let $\Gamma(n)$ be the curve obtained by the union of the following geodesics arcs: $B$, $C$ together with the arcs obtained by raising $A$ to height $n$, and the vertical geodesics joining the vertices of the raised $A$ with the vertices of $A$.

Let $\Sigma_n$ be a solution of Plateau’s problem for $\Gamma(n)$. Rado’s theorem is true in $\mathbb{H}^2 \times \mathbb{R}$, since vertical translation is an isometry, hence $\Sigma_n$ is the graph of a function $u_n$ defined in the triangle $T$ and

$$ u_{n|A} = +n, \quad u_{n|B} = u_{n|C} = 0. $$

**Theorem 2.** The sequence $\{u_n\}$ converges to a minimal solution $u$ defined in $T$ such that

$$ u_{|A} = +\infty, \quad u_{|B} = u_{|C} = 0. $$

Moreover, the gradient of $u$ diverges as one approaches the side $A$.

**Proof.** The sequence $\{u_n\}$ is non decreasing and positive. Hence, to show that the function $u$ exists, we will prove that the sequence $\{u_n\}$ is uniformly bounded on compact subsets $K$ of $T$.

We start by constructing a barrier over the graph of the $u_n$ in $K$.

The following construction is represented in Figure 3.

Let $\alpha$ be the horizontal geodesic containing the side $A$. Denote by $q_1$ and $q_2$ the vertices of $A$. For $i = 1, 2$, let $p_i$ the point on $\alpha \setminus A$ at a distance $\varepsilon > 0$ from $q_i$. Denote by $a_\varepsilon$ the geodesic arc between $p_1$ and $p_2$. Let $\tau$ be the horizontal geodesic orthogonal to the edge $A$ passing through the mid-point of $A$ (in order to simplify Figure 3 we assume that $\tau$ passes through the origin of $\mathbb{H}^2$). Let $p_\infty$ be the point at infinity of $\tau$ contained in the same halfplane as $T$ defined by $\alpha$. 

Figure 3
Finally denote by $\beta_\varepsilon$ the horizontal geodesic through $p_1$ and $p_\infty$ and by $\gamma_\varepsilon$ the horizontal geodesic through $p_2$ and $p_\infty$.

Consider a horizontal geodesic $\delta$ orthogonal to $\tau$ and let $p_3 = \delta \cap \beta_\varepsilon$, $p_4 = \delta \cap \gamma_\varepsilon$. Call $d$ the geodesic arc on $\delta$ between $p_3$ and $p_4$, $b_\varepsilon$ the geodesic arc on $\beta_\varepsilon$ between $p_1$ and $p_3$ and $c_\varepsilon$ the geodesic arc on $\gamma_\varepsilon$ between $p_2$ and $p_4$. We can chose $||b_\varepsilon||$ and $||c_\varepsilon||$ large enough such that the quadrilateral with edges $a_\varepsilon$, $b_\varepsilon$, $c_\varepsilon$, $d$ contains the triangle $T$.

Let $h$ be a positive number and call $\ast(h)$ each object obtained by translating vertically to height $h$ an object of $\mathbb{H}^2 \times \{0\}$.

Consider the following curves (see Figure 4):

$$L^h_1 = b_\varepsilon \cup (p_1 \times [0, h]) \cup b_\varepsilon(h) \cup (p_3 \times [0, h]),$$

$$L^h_2 = c_\varepsilon \cup (p_2 \times [0, h]) \cup c_\varepsilon(h) \cup (p_4 \times [0, h]).$$

We claim that there exists a least area, hence stable, minimal annulus bounded by $L^h_1 \cup L^h_2$, if $||b_\varepsilon||$ is sufficiently large. A sufficient condition is given by the Douglas criteria for the Plateau problem: if there is an annulus bounded by $L^h_1 \cup L^h_2$ with area smaller than the sum of the areas of the flat geodesic domains bounded by $L^h_1$ and $L^h_2$, then there exists a least area minimal annulus bounded by $L^h_1 \cup L^h_2$.

By a straightforward computation we obtain that the sum of the areas of the flat geodesic domains bounded by $L^h_1$ and $L^h_2$ is equal to $2||b_\varepsilon||h$ (as $||c_\varepsilon|| = ||b_\varepsilon||$).

Now, consider the annulus that is the union of the four geodesic domains bounded by the following quadrilaterals (see Figure 5):

$$Q_1 = a_\varepsilon \cup (p_1 \times [0, h]) \cup a_\varepsilon(h) \cup (p_2 \times [0, h]),$$

$$Q_2 = d \cup (p_3 \times [0, h]) \cup d(h) \cup (p_4 \times [0, h]).$$
The area of this annulus is at most \(2\pi + 2||a_\epsilon||h\), as the area of a hyperbolic triangle is always smaller than \(\pi\).

Then, in order to satisfy the Douglas condition, we need:

\[
2\pi + 2||a_\epsilon||h < 2||b_\epsilon||h
\]

that is verified as soon as we choose the edge \(b_\epsilon\) long enough. Hence, there exists a least area minimal annulus \(A^h_\epsilon\) bounded by \(L^h_1\) and \(L^h_2\), for any \(h\). By the maximum principle \(A^h_\epsilon\) is contained in the convex hull of \(L^h_1 \cup L^h_2\).

For each \(n\) the annulus \(A^h_\epsilon\) is above the surface \(\Sigma_n\) (the graph of \(u_n\)); by above we mean that if a vertical geodesic meets both surfaces, then the point of \(\Sigma_n\) is below the points of \(A^h_\epsilon\).

To see this, translate vertically \(A^h_\epsilon\) to height \(n\) (so, every point of \(A^h_\epsilon\) is above height \(n\)). Then lower the translated \(A^h_\epsilon\) back to height zero. By the maximum principle, there is no interior contact point between \(A^h_\epsilon\) and \(\Sigma_n\) before returning to the original position of \(A^h_\epsilon\). Moreover, if \(\epsilon > 0\), the boundaries of the two surfaces do not touch.

Letting \(\epsilon \rightarrow 0\), we conclude that \(A^h_0 = A^h\) is above \(\Sigma_n\) and, by the boundary maximum principle at each interior point of the vertical geodesics \(q_1 \times [0, h]\) and \(q_2 \times [0, h]\) the tangent plane to \(A^h\) is “outside” the tangent plane to \(\Sigma_n\) (i.e. the angle between the tangent plane to \(\Sigma_n\) and the geodesic plane containing either \(L^h_1\) or \(L^h_2\) is bigger than the angle between this last plane and the tangent plane to \(A^h\)).

The barrier \(A^h\) shows that the sequence \(\{u_n\}\) is uniformly bounded on compact subsets \(K\) of \(T\) such that \(K\) is contained in the horizontal projection of \(A^h\). The idea is to show that the horizontal projections of \(A^h\) exhaust \(T\) as \(h \rightarrow \infty\).
For $k > h$, one can use $A^h$ as barrier to solve the Plateau problem to find a stable annulus $A^k$ with boundary $L^k_1$, $L^k_2$. So, translating $A^h$ vertically, one sees that the two surfaces are never tangent (neither at interior points, nor at boundary points). Hence as $k \to \infty$, the angle the tangent plane of $A^k$ makes along the vertical boundary segments is controlled by that of $A^h$.

Now, for each $n$ let $M^n$ be the surface $A^{2n}$ translated down a distance $n$. As each $M_n$ is stable, one has local uniform area bounds and uniform curvature estimates (see [Sc]). So, a subsequence of $\{M^n\}$ converges to a minimal surface $M^\infty$. By the maximum principle, one can translate $A^h$ up to $+\infty$ and down to $-\infty$ without ever touching $M^\infty$. Then, there is some component $M$ of $M^\infty$ whose boundary is the union of the two vertical geodesics $q_1 \times \mathbb{R}$ and $q_2 \times \mathbb{R}$. Furthermore the distance between $M$ and $A \times \mathbb{R}$ is bounded. In fact this distance is uniformly bounded.

Now, we have to prove that $M = A \times \mathbb{R}$.

In $\mathbb{H}^2$, consider the family of equidistant circles $\{C_t\}_{t \geq 0}$ defined as follows: $C_0 = \alpha$, each $C_t$ is the circle equidistant from the geodesic $\alpha$, whose curvature vector points towards the halfplane $P_+$ determined by $\alpha$, containing the triangle $T$ (see Figure 6). The family of surfaces $C_t \times \mathbb{R}$ foliates $P_+ \times \mathbb{R}$. When $t$ is large one has

$$(C_t \times \mathbb{R}) \cap M = \emptyset.$$ 

Now decrease $t$: By the maximum principle, one cannot have a first point of contact between $M$ and $C_t \times \mathbb{R}$ before $t = 0$. Then $M = A \times \mathbb{R}$ and we are through.

Thus $\{u_n\}$ has a subsequence converging to a minimal solution $u$ defined on $T$. The convergence is uniform on compact subsets of $T$. Furthermore, as we desired:

$$u|_A = \infty, \quad u|_B = u|_C = 0.$$
The last assertion of Theorem 2 follows from a more general fact proved in Lemma 1 of next section.

\[\square\]

**Remark 1.** We notice that our construction can be done for any angle, smaller than \(\pi\), between the two geodesic arcs where the boundary value is zero. In a 2\(k\)-gon \(P\) as above, such angles are \(\frac{\pi}{k}\), \(k = 2, 3, \ldots\). Then, we make the symmetry of the graph of \(u\) with respect to one of the two geodesic arcs where \(u\) is zero and keep on going with such symmetries in order to close the surface. The surface thus obtained is the Scherk type surface \(\Sigma\), which we were looking for.

Also one can show that the Scherk solutions \(u\) in \(T\) converge to a Scherk solution in the ideal triangle \(T^*\) obtained as limit of the triangles \(T\) when the length of the sides \(B\) and \(C\) tend to infinity. Reflection then gives a Scherk surface graph over the interior of a 2\(k\)-gon \(P\).

**Remark 2.** When interior angles of \(P\) are choosen to be \(\frac{\pi}{2}\), \((k > 2)\), then \(\Sigma\) extends to a complete embedded minimal surface in \(\mathbb{H}^2 \times \mathbb{R}\). In fact the surface \(\Sigma\) is bounded by the 2\(k\) vertical geodesics through the vertices of \(P\) and one extends \(\Sigma\) by rotation of \(\pi\) about all the vertical geodesics that arise.

**Remark 3.** Let \(P\) be the regular 2\(k\)-gon with \(\frac{\pi}{2}\) angles. Consider the symmetry of \(P\) about each of its vertices. This produces 2\(k\) new 2\(k\)-gons isometric to \(P\), each having a vertex in common with \(P\). Consider the hyperbolic isometries identifying alternate sides of \(P\) (that is the translation along the edge between the two chosen sides). The quotient of the surface \(\Sigma\) by these translations gives a 2\(k\)-punctured sphere whose total curvature is \(-4\pi(1 - k)\).

**Remark 4.** There is another natural way to obtain a complete surface from \(\Sigma_n\) when the interior angles of \(P\) are equal to \(\frac{\pi}{2}\) (\(\Sigma_n\) is the graph of \(u_n\) over the triangle \(T\)). Assume that the polygon \(P\) has 2\(k\) sides. Do the symmetry of \(\Sigma_n\) about all the geodesic arcs of its boundary. Then continue extending the surface by symmetry in the geodesic arcs of the boundary. This yields a complete embedded minimal surface in \(\mathbb{H}^2 \times \mathbb{R}\) that is invariant by vertical translation by 2\(n\). The quotient of the surface by this translation gives a compact surface of genus \(k\).
5 Jenkins-Serrin Types Theorems

We give necessary and sufficient conditions to solve the Dirichlet problem for the minimal surface equation in $\mathbb{H}^2 \times \mathbb{R}$, over a convex domain of $\mathbb{H}^2$, allowing infinite boundary values on some arcs of the boundary of the domain.

Let us fix some notation.

We consider an open bounded convex domain $D$ whose boundary $\partial D$ contains two sets of (open) geodesic arcs $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ with the property that no two $A_i$ and no two $B_j$ have a common endpoint. The remaining part of $\partial D$ is the union of open convex arcs $C_1, \ldots, C_h$ and all endpoints.

We want to find a solution $u$ of the minimal surface equation in $D$ such that

\[ u|_{A_i} = +\infty, \quad u|_{B_j} = -\infty, \]

\[ i = 1, \ldots, k, \quad j = 1, \ldots, l \]

and $u$ takes assigned continuous data on each arc $C_s$, $s = 1, \ldots, h$.

The existence of such a solution depends on a relation between the lengths of the geodesic arcs of the boundary and the perimeter of polygons inscribed in $\partial D$ whose vertices are chosen among the vertices of $A_i, B_j$.

Let $P$ be such a polygon and let

\[ \alpha = \sum_{A_i \subset P} ||A_i||, \quad \beta = \sum_{B_j \subset P} ||B_j||, \quad \gamma = \text{Perimeter}(P). \]

**Theorem 3.** Let $D$ be a domain as above and let $f^s : C_s \rightarrow \mathbb{R}$ be continuous functions. If $\{C_s\} \neq \emptyset$, then the Dirichlet problem in $D$ with boundary values

\[ u|_{A_i} = +\infty, \quad u|_{B_j} = -\infty, \quad u|_{C_s} = f^s \]

has a solution if and only if

\[ 2\alpha < \gamma, \quad 2\beta < \gamma \]

for each polygon $P$ as above. If $\{C_s\} = \emptyset$ the result is the same except that if $P = \partial D$, then condition (9) should be replaced by $\alpha = \beta$.

If it exists, the solution is unique; in the case $\{C_s\} = \emptyset$ uniqueness is up to a constant.

**Remark 5.** We notice that two convex arcs $C_s$ may have a common endpoint $p$; there may be a discontinuity of the data $f^s$ at $p$. It will be clear from the proof of Theorem 3...
that the minimal surface obtained in this case will contain the vertical segment through
$p$, between the two limit values at $p$, of the continuous boundary data.

This result is analogous to that of Jenkins and Serrin for minimal graphs in $\mathbb{R}^3$ (cf. [JS]).
We prove Theorem 3 in 6 steps. Each step, especially 1 and 3, is an interesting result
on its own.

**Step 1.** Existence when $\partial \mathcal{D}$ contains only one geodesic arc $A$, and one strictly convex
arc $C$. The function $f : C \to \mathbb{R}$ is continuous and positive.

**Step 2.** Existence when $\partial \mathcal{D}$ contains geodesic arcs $A_1, \ldots, A_k$ and strictly convex arcs
$C_1, \ldots, C_h$. The functions $f^*_s : C_s \to \mathbb{R}$ are continuous and positive.

**Step 3.** The same as Step 2, with $C_1, \ldots, C_h$ convex arcs (not necessarily strictly convex).

**Step 4.** Existence when $\partial \mathcal{D}$ contains geodesic arcs $A_1, \ldots, A_k$, $B_1, \ldots, B_l$ and convex
arcs $C_1, \ldots, C_h$ with $h \geq 1$.

**Step 5.** Existence when $\partial \mathcal{D}$ contains only geodesic arcs $A_1, \ldots, A_k$, $B_1, \ldots, B_l$.

**Step 6.** Uniqueness.

**Proof of Step 1.** Let $u_n : \mathcal{D} \to \mathbb{R}$ be the minimal solution with boundary values

$$u_{n|A} = +n, \quad u_{n|C} = \min(n, f).$$

Let us prove that $u_n$ exists. Define $\Gamma(n)$ to be the union of the following geodesic
arcs: the geodesic arc $A$ raised to height $n$, the graph of the function $\min(n, f)$ and
the vertical geodesic arcs joining the endpoints of the curves just described. Let $\Sigma(n)$
be the solution of the Plateau problem for the curve $\Gamma(n)$. By Rado’s theorem, $\Sigma(n)$
is the graph of a function $u_n$ defined in $\mathcal{D}$, with the desired boundary values. By the
maximum principle, $\{u_n\}$ is an increasing sequence.

We need to prove that the sequence $\{u_n\}$ is uniformly bounded on compact subsets of
$\mathcal{D}$.

Given a complete geodesic $\alpha$ in $\mathbb{H}^2$, $E$ one of the components of $\mathbb{H}^2 \setminus \alpha$, there exists
a minimal graph $h$ defined on $\mathcal{D}$, asymptotic to $+\infty$ on $\alpha$ and to zero on $\partial_\infty(E)$ (see
Figure 7 in the disk model of $\mathbb{H}^2$). This function was found independently by U.
Abresch and R. Sa Earp (see [9]).

In the halfspace model of $\mathbb{H}^2$ with $E = \{x > 0, \; y > 0\}$:

$$h(x, y) = \ln \left( \frac{\sqrt{x^2 + y^2} + y}{x} \right) \quad x > 0, \; y > 0$$

Let $K$ be a compact subset of $\mathcal{D}$. Let $\alpha$ be a complete geodesic, disjoint from the
geodesic containing $A$, intersecting $C$ in two points, such that the region of $\mathcal{D}$ bounded
by $\alpha$ and $A$ is disjoint from $K$. The geodesic $\alpha$ separates the circle at infinity in two
arcs; let $B$ denote the arc at infinity such that the disk $E$, bounded by $\alpha \cup B$, contains
Let $h$ be the minimal graph on $E$ which is $+\infty$ on $\alpha$ and on $B$ it is the maximum of $f$ on $K \cap C$. By the maximum principle, each $u_n$ is bounded by $h$ on $K$. Then, the sequence $\{u_n\}$ converges to a minimal solution $u$ on $D$.

The existence of Scherk’s type surface in a triangle, guarantees that $u$ takes the right boundary values, as in [3].

Remark 6. Let $C$ be a strictly convex arc and denote by $\mathcal{C}(C)$ the (open) convex hull of $C$. Let $u$ be a minimal solution in $\mathcal{C}(C)$ with bounded values on $C$. As a result of the previous proof, $u$ is bounded on every compact set of $\mathcal{C}(C)$ depending only on the values of $u$ on $C$ and on the distance of the compact set to the boundary of $\mathcal{C}(C)$.

Assertion: if $u$ is a minimal solution that is unbounded on $C$, then $u$ is unbounded in $\mathcal{C}(C)$.

This assertion implies that for solving the Dirichlet problem one can not assign infinite data on a strictly convex arc of the boundary of the domain.

For the proof of the assertion we use a modification of the argument of step 1. Let $A$ be the geodesic arc in the boundary of $\mathcal{C}(C)$. For each $n \in \mathbb{N}$, define the function $u_n = \min\{n, u\}$. Let $\Delta$ be a horizontal geodesic triangle containing $\mathcal{C}(C)$ with sides $a$, $b$, $c$, such that the side $a$ contains $A$ in its interior. Let $\varphi_+$ be the Scherk type solution equal to $+\infty$, on $a$, and zero on $b$ and $c$.

Let $K$ be a compact set in $\mathcal{C}(C) \cup C$. On $\partial K$ we have

$$\min_{K \cap C} u_n \leq u_n \leq \max_{K \cap C} u_n + \varphi_+$$

By the maximum principle, the previous inequality holds in $K$. Letting $n \to \infty$, one sees that $u$ is unbounded on $\mathcal{C}(C)$. 

---

Figure 7
For the proof of Step 2, we need several preliminary results. The first depends on the fact that the tangent plane to the graph of a minimal solution is almost vertical at points near to a geodesic arc of the boundary where the solution diverges to infinity. Let us be more precise.

Denote by $S$ the graph of a minimal solution $u : \mathcal{D} \rightarrow \mathbb{R}$ and let

$$ (\nu)_u = ((\nu)_1)_u, (\nu)_2, (\nu)_3) $$

be the inward unit conormal to the boundary of $S$.

Let $(x_1(s), x_2(s), x_3(s))$ be an arc length parametrization of the boundary of $S$. A straightforward computation yields:

$$ (\nu)_3 = -\frac{\partial x_1}{\partial s} \frac{u_2}{\tau} + \frac{\partial x_2}{\partial s} \frac{u_1}{\tau}. $$

Then $|(\nu)_3| < 1$ and $(\nu)_3$ is integrable on arcs of $\partial \mathcal{D}$ regardless of the boundary behaviour of $u$ on such arcs. The behaviour of the flux of $(\nu)_3$ on geodesic arcs of the boundary is established in the following Lemma.

**Lemma 1.** Let $\mathcal{D}$ be a domain and let $A$ be a geodesic arc of the boundary of $\mathcal{D}$. (i) Let $u : \mathcal{D} \rightarrow \mathbb{R}$ be a minimal solution such that $u|_A = \infty$. Then

$$ \int_A (\nu)_3 ds = ||A||. $$

(ii) Let $\{u_n\}$ be a sequence of minimal solutions in $\mathcal{D}$ continuous in $\mathcal{D} \cup A$. If $\{u_n\}$ diverges uniformly to infinity on compact subsets of $A$ and remains uniformly bounded in compact subsets of $\mathcal{D}$, then

$$ \lim_{n \rightarrow \infty} \int_A (\nu)_n ds = ||A||, $$

where $(\nu)_n$ is the third component of the unit conormal to the boundary of the graph of $u_n$.

If $\{u_n\}$ diverges uniformly to infinity in compact subsets of $\mathcal{D}$ and remains uniformly bounded on compact subsets of $A$, then

$$ \lim_{n \rightarrow \infty} \int_A (\nu)_n ds = -||A||. $$

**Proof.** (i) First we prove that the tangent plane to $S$ at points $(z, u(z))$ with $z$ near to $A$ is almost vertical. Let $N = (N_1, N_2, N_3)$ be the upward unit normal to $S$, then $|(\nu)_u| = \sqrt{1 - N_3^2}$ at boundary points. We extend $\nu_3$ to the interior points of $\mathcal{D}$
by setting $|(\nu_3)_u| = \sqrt{1 - N^2}$ and choosing the sign that makes $\nu_3$ continuous at the boundary (where it is already defined).

At points where the tangent plane is almost vertical, $N_3$ approaches zero, hence $(\nu_3)_u$ approaches one. In other words the tangent plane at points $(z, u(z))$ with $z$ next to $A$ is almost vertical if and only for any $\varepsilon > 0$ there is a neighborhood of $A$ in $D$ such that

\begin{equation}
|(\nu_3)_u| > 1 - \varepsilon
\end{equation}

at each point of the neighborhood.

A minimal graph is stable, so one has Schoen’s curvature estimates for the surface $S$:

let $p$ be a point of $S$ and let $D(p, R)$ be a disk contained in $S$ centered at $p$ of intrinsic radius $R$,

\begin{equation}
|A(q)| \leq \kappa \quad \forall q \in D \left( p, \frac{R}{2} \right)
\end{equation}

where $A$ is the second fundamental form of $S$ and $\kappa$ is an absolute constant (see [Sc]).

Now, assume by contradiction that there is a sequence of points $\{z_m\}$ in $D$ approaching $A$ (i.e. $u_m(z_m) \to \infty$ as $m \to \infty$) such that (10) does not hold. Then, there is a radius $R$ independent on $m$ such that $D(p_m, R) \subset S$, where $p_m = (z_m, u(z_m))$. Hence, by the curvature estimate (11), around each $p_m$ the surface $S$ is a graph over a disk $D(p_m, r)$ of the tangent plane at $p_m$, and the graph has bounded distance from the disk $D(p_m, r)$.

The radius of the disk depends only on $R$, hence it is independent of $m$. It is clear that, if $z_m$ is close enough to $A$, then the horizontal projection of $D(p_m, r)$ and thus of the surface $S$ is not contained in $D$. Contradiction. Hence (10) holds in a neighborhood of $A$.

Now, fix $\varepsilon > 0$ and let $\delta \leq \varepsilon$. Let $q_1, q_2$ be the points of $A$ at distance $\delta$ from the endpoints of $A$ and call $A_\delta$ the subarc of $A$ bounded by $q_1, q_2$. We construct a neighborhood of $A_\delta$ in $D$ (see Figure 8). Let $\tau$ be a horizontal geodesic orthogonal to $A$ passing through the mid-point of $A$. We can assume that $\tau$ passes through the origin. For $i = 1, 2$, let $\alpha_i$ be a horizontal geodesic through $q_i$ forming an angle of $\pi/2$ with $A$. Finally, let $\beta$ be the horizontal geodesic orthogonal to $\tau$, at distance $\delta$ from $A$ and call $q_3 = \beta \cap \alpha_1$, $q_4 = \beta \cap \alpha_2$. Denote by $Q_\delta$ the geodesic quadrilateral having vertices at points $q_1, q_2, q_3, q_4$.

The form $(\nu_3)_uds$ is exact, hence:

\[
0 = \int_{A_\delta} (\nu_3)_uds + \int_{Q_\delta \setminus A_\delta} (\nu_3)_uds.
\]

Then, if $\varepsilon$ is small enough, using (10) we obtain:

\[
\int_{A_\delta} (\nu_3)_uds \geq -2\varepsilon + (1 - \varepsilon)||A_\delta||.
\]
Letting $\varepsilon$ (and so $\delta$) tend to zero yields:

$$\int_{A} (\nu_3)_u ds \geq ||A||.$$

The opposite inequality is obvious, so (i) follows.
For the proof of (ii) one makes the obvious modifications of the arguments in (i).

Let us prove another useful result.

**Lemma 2.** Let $u : \mathcal{D} \to \mathbb{R}$ be a minimal solution continuous on a convex arc $C$ of the boundary of $\mathcal{D}$. Then:

$$\int_{C} (\nu_3)_u ds < ||C||.$$

**Proof.** It is enough to prove the result for a closed subarc of $C$, say $\tilde{C}$. Then we can assume that $u$ is defined in a convex set $\tilde{\mathcal{D}}$ with continuous boundary data. Denote by $v$ the solution of the Dirichlet problem in $\tilde{\mathcal{D}}$ such that:

$$v|_{\partial \tilde{\mathcal{D}} \setminus \tilde{C}} = u, \quad v_{\tilde{C}} = u + a$$

where $a$ is a constant to be fixed later. Let $w = v - u$, then

$$w|_{\partial \tilde{\mathcal{D}} \setminus \tilde{C}} = 0, \quad w_{\tilde{C}} = a$$

By the maximum principle $w$ is uniformly bounded in $\tilde{\mathcal{D}}$.
Using Stokes' theorem (together with a standard approximation argument at the points of discontinuity) we obtain:

$$\int_{\partial \tilde{\mathcal{D}}} w[(\nu_3)_u - (\nu_3)_v] ds = \int \int_{\tilde{\mathcal{D}}} \left[ w_1 \left( \frac{v_1}{\tau_v} - \frac{u_1}{\tau_u} \right) + w_2 \left( \frac{v_2}{\tau_v} - \frac{u_2}{\tau_u} \right) \right] dx_1 dx_2.$$
The argument of the last integral is equal to the following expression:

\[
\left( \frac{\tau u + \tau v}{2} \right) \left[ \left( \frac{u_1}{\tau u} - \frac{v_1}{\tau v} \right)^2 + \left( \frac{u_2}{\tau u} - \frac{v_2}{\tau v} \right)^2 + \frac{1}{F} \left( \frac{1}{\tau u} - \frac{1}{\tau v} \right)^2 \right].
\]

Hence, it is non negative and not identically zero in \( \bar{D} \). Then we have:

\[
a \int_{\bar{C}} [(\nu_3)_u - (\nu_3)_v] ds > 0.
\]

choosing alternatively \( a = \pm 1 \) we obtain the result. \( \square \)

**Remark 7.** We point out that the results of Lemma 1 and 2 hold for non convex domains as well.

*Proof of Step 2.* We prove that the first condition in (9) is sufficient and necessary for existence. We start by sufficiency.

Let \( u_n : \bar{D} \rightarrow \mathbb{R} \) be the minimal solution with the following boundary values

\[
u_n = +n, \quad u_n|_{C_s} = \min(n, f^s).
\]

By Remark 6, \( \{u_n\} \) is uniformly bounded in compact sets contained in each of the convex hulls \( C(C_s) \), \( s = 1, \ldots, h \). Hence, passing to a subsequence, \( \{u_n\} \) converges on compact subsets of

\[
\bigcup_{s=1}^{h} C(C_s)
\]

to a minimal solution \( u \) defined in an open set \( U \) containing \( \bigcup_{s=1}^{h} C(C_s) \). Furthermore \( \{u_n\} \) diverges uniformly on compact subsets of \( D \setminus U \) and \( u \) is a continuous function with values in \( \mathbb{R} \cup \{\infty\} \).

Let \( V = D \setminus U \). We claim that \( V = \emptyset \). We start by showing that \( \partial V \) has a very special structure, when \( V \) is not empty.

**Lemma 3.** With the notation above, one has:

(i) \( \partial V \) consists only of geodesic chords of \( D \) and parts of the boundary of \( D \);

(ii) two chords of \( \partial V \) cannot have a common endpoint;

(iii) the endpoints of chords of \( \partial V \) are among the vertices of the geodesic arcs \( A_i \);

(iv) a component of \( V \) cannot consist only of an interior chord of \( D \).

*Proof.* It is clear by Remark 6 that each arc of \( \partial V \) must be geodesic and that no vertex of \( \partial V \) lies in \( D \), then (i) follows. Now assume by contradiction that (ii) does not hold.

Let \( K_1, K_2 \) be two arcs of \( \partial V \) having a common endpoint \( q \in \partial D \). Choose two points \( q_1 \in K_1 \) and \( q_2 \in K_2 \) such that the triangle \( T \) with vertices \( q, q_1, q_2 \) lies in \( D \). We have:
\[
\int_{\partial T} (\nu_3)_n ds = 0
\]
where \((\nu_3)_n\) is defined as in Lemma 1 at interior points of \(D\). The triangle \(T\) may be either in \(U\) or in \(V\). Assume the former is true, then, by the first equality in (ii) of Lemma 1, choosing correctly the orientation, we have:

\[
\lim_{n \to \infty} \int_{\bar{qq}1} (\nu_3)_n ds = ||\bar{qq}1||, \quad \lim_{n \to \infty} \int_{\bar{qq}2} (\nu_3)_n ds = ||\bar{qq}2||.
\]

Here \(\bar{\pi}\) indicates the geodesic arc between two points and \((\nu_3)_n\) is defined as in Lemma 1 at interior points of \(D\).

On the other hand:

\[
\left| \int_{\bar{qq}2} (\nu_3)_n ds \right| \leq ||\bar{qq}2||.
\]

(12) and (13) together with the triangle inequality give a contradiction.

If \(T \subset V\), we make the same reasoning using the second equality in (ii) of Lemma 1. (iii) and (iv) are proved with analogous arguments, using Lemma 1. We leave this to the reader.

Now, we come back to the proof of Step 2. Assume by contradiction that \(V\) is not empty. The convex hull of each \(C_i\) is contained in \(U\), and each component of \(V\) is bounded by a geodesic polygon \(P\), whose vertices are among the endpoints of the \(A_i\). Denote by \(\hat{A}_i\) those edges of \(A_i\) that are contained in \(P\). In the notation of Theorem 3, \(||P|| = \gamma, \sum ||\hat{A}_i|| = \alpha\).

For each \(u_n\) we have:

\[
0 = \int_{P} (\nu_3)_n ds = \int_{\cup \hat{A}_i} (\nu_3)_n ds + \int_{P \setminus \cup \hat{A}_i} (\nu_3)_n ds.
\]

By (ii) of Lemma 1, we infer:

\[
\lim_{n \to \infty} \int_{P \setminus \cup \hat{A}_i} (\nu_3)_n ds = -(\gamma - \alpha).
\]

For every \(n\), \(|(\nu_3)_n| < 1\), so

\[
\left| \int_{\cup \hat{A}_i} (\nu_3)_n ds \right| \leq \sum ||\hat{A}_i|| = \alpha.
\]

Hence \(\alpha \geq \gamma - \alpha\), that contradicts the assumed conditions.

We are left with the proof of the necessity of the condition \(2\alpha < \gamma\). Let \(u\) be the minimal solution with the given boundary values and let \(P\) be a polygon as in the hypothesis of Theorem 3. We have:
\[
\int_{\cup A_i} (\nu_3)_u ds + \int_{P \setminus \cup \hat{A}_i} (\nu_3)_u ds = 0.
\]
Furthermore \(|(\nu_3)_u| < 1\) on \(P \setminus \cup \hat{A}_i\), hence
\[
\left| \int_{P \setminus \cup \hat{A}_i} (\nu_3)_u ds \right| < \gamma - \alpha
\]
and by (i) of Lemma 1, we have:
\[
\int_{\cup \hat{A}_i} (\nu_3)_u ds = \alpha
\]
Hence \(2\alpha < \gamma\).

Proof of Step 3. Let \(\{u_n\}\) be defined as in Step 2. First we prove that \(\{u_n\}\) is bounded at some point of \(D\). Assume that this is not the case, then \(V = D\) and we have:
\[
0 = \int_{\cup A_i} (\nu_3)_n ds + \int_{\cup C_i} (\nu_3)_n ds.
\]
(ii) of Lemma 1 implies
\[
\lim_{n \to \infty} \int_{\cup C_i} (\nu_3)_n ds = -\sum ||C_i|| \leq -(\gamma - \alpha).
\]
Here \(\gamma = ||\partial D||\) and \(\alpha = \sum ||A_i||\).
On the other hand, as \(|(\nu_3)_n| < 1\) for every \(n\), we have
\[
\left| \int_{\cup A_i} (\nu_3)_n ds \right| < \sum ||A_i|| = \alpha.
\]
Then \(\alpha \geq \gamma - \alpha\), a contradiction.
Hence the sequence \(\{u_n\}\) is bounded at some point of \(D\). In fact we will prove that there is a disk in \(D\) of radius independent on \(n\) where each \(u_n\) is uniformly bounded. Up to an isometry, we can assume that \(\{u_n\}\) is bounded at the origin \(\sigma \in \mathbb{H}^2\). We remark that by the maximum principle each \(u_n\) is positive in the domain of definition. Let \(m_n = u_n(\sigma)\). We assert that the gradient of \(u_n\) at \(\sigma\) is bounded depending only on the constant \(m_n\). In order to prove it, we will compare the gradient of \(u_n\) with that of a Scherk type surface.
Up to a rotation of \(x_1, x_2\) coordinates, we can assume that
\[
\frac{\partial u_n}{\partial x_1}(\sigma) > 0, \quad \frac{\partial u_n}{\partial x_2}(\sigma) = 0.
\]
Let \(\Delta(n)\) be a geodesic triangle contained in \(D\) with edges \(a, b, c\) such that the \(x_1\) axis bisects the edge \(a\) orthogonally and \(\Delta(n)\) is symmetric with respect to the \(x_1\) axis (see Figure 9).
Let $\varphi_{\Delta(n)}$ denote the Scherk type surface over $\Delta(n)$ with value $+\infty$ on $a$, value 0 on $b$, $c$ and $\varphi_{\Delta(n)}(\sigma) = m_n$ (we allow translations of $\Delta(n)$ along the $x_1$ axis in order to find such a Scherk type surface). Define $C(m_n) = |\nabla \varphi_{\Delta(n)}(\sigma)|$. We claim that:

\begin{equation}
|\nabla u_n(\sigma)| \leq C(m_n).
\end{equation}

In fact, assume by contradiction that (14) does not hold. Then, the symmetries of $\varphi_{\Delta(n)}$ imply:

$$\frac{\partial u_n}{\partial x_1}(\sigma) > \frac{\partial \varphi_{\Delta(n)}}{\partial x_1}(\sigma), \quad \frac{\partial u_n}{\partial x_2}(\sigma) = \frac{\partial \varphi_{\Delta(n)}}{\partial x_2}(\sigma) = 0.$$ 

Now, we move $\Delta(n)$ by hyperbolic translations along the $x_1$ axis, pushing the edge $a$ towards $\sigma$. As $\varphi_{\Delta(n)}$ and $\frac{\partial \varphi_{\Delta(n)}}{\partial x_1}$ diverge as one approaches the side $a$, there is a position of $\Delta(n)$ such that:

$$u_n(\sigma) < \varphi_{\Delta(n)}(\sigma)$$

and

$$\frac{\partial u_n}{\partial x_1}(\sigma) = \frac{\partial \varphi_{\Delta(n)}}{\partial x_1}(\sigma), \quad \frac{\partial u_n}{\partial x_2}(\sigma) = \frac{\partial \varphi_{\Delta(n)}}{\partial x_2}(\sigma) = 0.$$ 

Define $w = \varphi_{\Delta(n)} - u_n$. We have:

$$w(\sigma) = \chi > 0, \quad \nabla w(\sigma) = 0.$$ 

Then, there are at least four level lines of $w = \chi$ through $\sigma ([CM],[Se])$. These level lines divide every small neighborhood of $\sigma$ in at least four domains in which $w$ is alternately
greater than and less than \( \chi \). We prove that this yields a contradiction (our argument is analogous to [Se], we give it for the sake of completeness).

Let \( G \) be the subset of \( \Delta(n) \) whose points are at distance less than \( \varepsilon \) from the boundary of \( \Delta(n) \). The function \( u_n \) has bounded continuous gradient in \( \Delta(n) \), hence, using the form of the graph of \( \varphi_{\Delta(n)} \), one has that the set \( G \) is divided into two components by the conditions

\[ w > \chi, \quad w < \chi, \]

for suitably small \( \varepsilon \).

The first component is adjacent to edge \( a \), while the second is adjacent to \( b \) and \( c \) and the components themselves are separated by two level lines \( w = \chi \) exiting from the vertices of \( a \). By the maximum principle, each component of the set \( w > \chi \) must extend to the boundary of \( \Delta(n) \). It follows that the set \( w > \chi \) consists of one component. Then, any two regions near \( \sigma \) where \( w > \chi \) can be joined by a simple Jordan arc \( C_+ \) along which \( w > \chi \). Analogously any two regions near \( \sigma \) where \( w < \chi \) can be joined by a simple Jordan arc \( C_- \) along which \( w < \chi \). Then, the curves \( C_+ \) and \( C_- \) must intersect (see Figure 10). This is a contradiction.

Now let \( M = \sup_{n \in \mathbb{N}} u_n(\sigma) \). As \( m_n \leq M \), with the same reasoning used for proving (14), one has that \( C(m_n) \leq C(M) \). Hence the sequence \( \{u_n\} \) has uniformly bounded gradient at \( \sigma \).

Now, by Schoen’s curvature estimates (see (11)), one has that in a neighborhood of the point \( p_n = (\sigma, u_n(\sigma)) \) the surface is a graph of bounded height and slope over a disk \( D(p_n, R) \) of the tangent plane to the surface at \( p_n \), of radius \( R \) independent of \( n \). As \( |\nabla u_n(\sigma)| \) is uniformly bounded, the projection of each \( D(p_n, R) \) on the horizontal plane contains a disk of fixed radius and \( \{u_n\} \) is uniformly bounded there. Then, there exists an open set \( U \) in which \( \{u_n\} \) converges uniformly. Now we can apply the same reasoning as in Step 2, in order to prove that \( U = D \).

\( \square \)

**Proof of Step 4.** By the previous arguments we can find a minimal solution \( u^+: \mathcal{D} \rightarrow \mathbb{R} \) such that

\[
    u^+|_{A_i} = \infty, \quad u^+|_{B_j} = 0, \quad u^+|_{C_s} = \max \{0, f^s\}.
\]
Furthermore we can find a minimal solution \( u^- : \mathcal{D} \rightarrow \mathbb{R} \) such that
\[
    u^-|_{A_i} = 0, \quad u^-|_{B_j} = -\infty, \quad u^-|_{C_s} = \min\{0, f^s\}.
\]
\[
    \begin{cases} 
        2x - y = 0 \\
        y + kz = 0
    \end{cases}
\]
Then, define for each \( s \):
\[
    (f^s)_n = \begin{cases} 
        -n & \text{if } f^s < -n \\
        f^s & \text{if } |f^s| \leq n \\
        n & \text{if } f^s > n
    \end{cases}
\]
and let \( u_n : \mathcal{D} \rightarrow \mathbb{R} \) be the minimal solution such that
\[
    u_n|_{A_i} = n, \quad u_n|_{B_j} = -n, \quad u_n|_{C_s} = (f^s)_n.
\]
By the maximum principle:
\[
    u^- \leq u_n \leq u^+ \text{ in } \mathcal{D}.
\]
Hence, the sequence \( \{u_n\} \) is uniformly bounded on compact subsets of \( \mathcal{D} \) and there exists a subsequence converging to a minimal solution that takes the prescribed boundary values.
This proves existence under condition (9). The necessity of condition (9) is proved as in Step 2.

Proof of Step 5. We remark that in this case the number of edges \( A_i \) is equal to the number of edges \( B_j \), say \( k \). We need to construct some auxiliary sets and minimal solutions.
Let \( v_n : \mathcal{D} \rightarrow \mathbb{R} \) be the minimal solution such that
\[
    v_n|_{A_i} = n, \quad v_n|_{B_i} = 0.
\]
For \( c \in ]0, n[ \), we introduce the following subsets of \( \mathcal{D} \):
\[
    E_c = \{v_n > c\} \cap \mathcal{D}, \quad F_c = \{v_n < c\} \cap \mathcal{D}.
\]
Let \( E^i_c \) be the component of \( E_c \) whose closure contains the edge \( A_i \) and let \( F^i_c \) be the component of \( F_c \) whose closure contains the edge \( B_j \). By the maximum principle \( E_c = \bigcup_{i=1}^k E^i_c \) and \( F_c = \bigcup_{i=1}^k F^i_c \). We choose \( c \) close enough to \( n \) such that the \( E^i_c \) are disjoint and we define:
\[
    \mu(n) = \limsup\{c \in ]0, n[ \mid E^i_c \cap E^j_c = \emptyset \mid i \neq j\}.
\]
Of course there is at least one pair \( i, j \) such that
\[
E^i_{\mu(n)} \cap E^j_{\mu(n)} \neq \emptyset
\]
and this implies that for any given \( E^i_{\mu(n)} \), the set \( E^j_{\mu(n)} \) is disjoint from it. For each \( n \), we define the following minimal solution in \( \mathcal{D} \):
\[
u_n = v_n - \mu(n).
\]
In order to prove that the sequence \( \{\nu_n\} \) is uniformly bounded on compact subsets of \( \mathcal{D} \), let us define two auxiliary minimal solutions in \( \mathcal{D} \). Let \( u_i^+ \) and \( u_i^- \) be the minimal solutions in \( \mathcal{D} \) with the following boundary values:
\[
u_i^+|_{A_i} = \infty, \quad \nu_i^+|_{\partial \mathcal{D} \setminus A_i} = 0
\]
\[
u_i^-|_{B_j} = -\infty, \quad j \neq i, \quad \nu_i^-|_{\partial \mathcal{D} \setminus \cup_{j \neq i} B_j} = 0
\]
For every \( i = 1, \ldots, k \), \( u_i^+ \) and \( u_i^- \) exist by previous steps.
Finally, for any \( z \in \mathcal{D} \) we define:
\[
\nu^+(z) = \max_{1 \leq i \leq k} \{u_i^+(z)\}, \quad \nu^-(z) = \min_{1 \leq i \leq k} \{u_i^-(z)\}.
\]
We claim that at any point of \( \mathcal{D} \):
\[
u^- \leq \nu_n \leq \nu^+.
\]
Let \( p \in \mathcal{D} \) such that \( \nu_n(p) > 0 \), then \( p \) belongs to \( E^i_{\mu(n)} \) for some \( i \). On \( \partial E^i_{\mu(n)} \) one has \( \nu_n \leq \nu_i^+ \), then this inequality holds in \( E^i_{\mu(n)} \), and
\[
\nu_n(p) \leq \nu_i^+(p) \leq \nu^+(p).
\]
Since \( \nu^- \) is non positive, the left inequality in (15) is obvious at the point \( p \).
The proof of (15) at points where \( \nu_n \) is negative is analogous, using the set \( E^j_{\mu(n)} \).
Hence \( \{\nu_n\} \) has a subsequence converging to a minimal solution \( \nu : \mathcal{D} \to \mathbb{R} \). Let us prove that \( u \) takes the right boundary values.
Recall that:
\[
u_n|_{A_i} = n - \mu(n), \quad \nu_n|_{B_i} = -\mu(n),
\]
so we must prove that the sequences \( \{\mu(n)\} \) and \( \{n - \mu(n)\} \) both diverge to infinity.
We prove it for the sequence \( \{\mu(n)\} \); the proof will be analogous for the latter sequence.
The assumption that \( \{\mu(n)\} \) does not diverge will give a contradiction to the hypothesis \( \alpha = \beta \). By contradiction, take a subsequence (still denoted by \( \{\nu_n\} \)) such that \( \mu(n) \) tends to a finite limit \( \mu_0 \). Then:
\[ u_n \to \infty \text{ on } A_i, \quad u_n \to -\mu_0 \text{ on } B_i. \]

Then, for the limit function \( u \) we have:

\[ u|_{A_i} = \infty, \quad u|_{B_i} = -\mu_0. \]

Let \( (\nu_3)_u \) be the unit inward conormal to the boundary of the graph of \( u \). We finally obtain:

\[ \alpha = \int_{\cup A_i} (\nu_3)_u ds = -\int_{\cup B_i} (\nu_3)_u ds > -\beta, \]

where the first equality is given by (i) of Lemma 1. This is a contradiction.

The necessity of the condition \( \alpha = \beta \) is proved as in Step 2.

\textbf{Proof of Step 6 (Uniqueness).}

Let \( u \) and \( v \) be two minimal solutions assuming values +\( \infty \) on each \( A_i \), \( -\infty \) on each \( B_j \) and the same data on each geodesic arc \( C_s \).

Let \( M \) be a large constant and define:

\[
\psi = \begin{cases} 
-M & \text{if } u - v < -M \\
|u - v| & \text{if } |u - v| < M \\
M & \text{if } u - v > M 
\end{cases}
\]

Let \( 0 < \delta < \varepsilon \) and denote by \( D_{\delta\varepsilon} \) the subset of \( D \) whose distance from \( \partial \mathcal{D} \) is greater than \( \delta \) and whose distance from the vertex of each \( A_i \) and \( B_j \) is greater than \( \varepsilon \). It is clear that the boundary \( \Gamma \) of \( D_{\delta\varepsilon} \) consists of bounded arcs \( \tilde{A}_i, \tilde{B}_j, \tilde{C}_s \) adjacent to the \( A_i, B_j, C_s \) and circular arcs adjacent to the vertices of \( A_i, B_j \). Let \( (\nu_3)_u \) and \( (\nu_3)_v \) be defined as in Lemma 1 for the functions \( u \) and \( v \) respectively. Consider the following integral:

\[
\int_{\Gamma} \psi[(\nu_3)_u - (\nu_3)_v] ds.
\]

For \( \delta \) small enough, we have:

\[
\int_{\cup \tilde{C}_s} \psi[(\nu_3)_u - (\nu_3)_v] ds \leq 2 \int_{\cup \tilde{C}_s} |\psi| \leq 2\varepsilon \sum ||C_s||.
\]

We recall that next to boundary arcs where the solution is infinity, \( |\nu_3| \) is almost one, hence:

\[
\int_{\cup \tilde{A}_i} \psi[(\nu_3)_u - (\nu_3)_v] ds = \int_{\cup \tilde{A}_i} \psi[(\nu_3)_u - 1] ds - \int_{\cup \tilde{A}_i} \psi[(\nu_3)_v - 1] ds \leq 2\varepsilon M \sum ||\tilde{A}_i||
\]

In the same way we obtain:

\[
\int_{\cup \tilde{B}_j} \psi[(\nu_3)_u - (\nu_3)_v] ds \leq 2\varepsilon M \sum ||\tilde{B}_j||
\]

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By summing the previous inequalities, we infer:

\[(16) \quad \int_{\Gamma} \psi[(\nu_3)_u - (\nu_3)_v]ds \leq 2\varepsilon \sum(||C_s|| + M||\hat{A}_v|| + M||\hat{B}_j||) + 4\pi\sinh(\varepsilon)M(k + l)\]

where the last term is the contribution of the circular arcs next to vertices.

On the other hand, integrating by parts, we obtain:

\[
\int_{\Gamma} \psi[(\nu_3)_u - (\nu_3)_v]ds = \int \int \left[ \psi_1 \left( \frac{v_1}{\tau_v} - \frac{u_1}{\tau_u} \right) + \psi_2 \left( \frac{v_2}{\tau_v} - \frac{u_2}{\tau_u} \right) \right] dx_1 dx_2
\]

where the double integral is taken over the set \(D_{\delta\varepsilon} \cap \{|u - v| < M\}\). The argument of the last integral is non-negative and it is zero only at points where \(\nabla u = \nabla v\) (see the proof of Lemma 2).

Then, letting \(\varepsilon \rightarrow 0\) in (16), we obtain:

\[\psi_1 \left( \frac{v_1}{\tau_v} - \frac{u_1}{\tau_u} \right) + \psi_2 \left( \frac{v_2}{\tau_v} - \frac{u_2}{\tau_u} \right) = 0\]

at points where \(|u - v| < M\). Hence \(\nabla u = \nabla v\) in \(D\), since \(M\) can be taken arbitrarily large.

It follows that \(u = v + const\) in \(D\). If the family of boundary convex arcs \(\{C_s\}\) is empty, this proves the result. If not, the constant must be zero by the boundary condition on \(C_s\).

\[\square\]

6 Existence of Complete Minimal Graphs

**Theorem 4.** Let \(\Gamma\) be a continuous rectifiable Jordan curve in \(\partial_{\infty}\mathbb{H}^2 \times \mathbb{R}\), that is a vertical graph. Then, there exists a minimal vertical graph on \(\mathbb{H}^2\) having \(\Gamma\) as asymptotic boundary. The graph is unique.

**Proof.** We claim that, if we prove the Theorem for a \(C^\infty\) curve \(\Gamma\), then, the result is true for a continuous rectifiable \(\Gamma\). Let us prove it.

Consider two families of differentiable curves approximating \(\Gamma\), constructed as follows. For any \(\varepsilon > 0\), let \(\Gamma_{\varepsilon}^+ \subset \partial_{\infty}\mathbb{H}^2 \times \mathbb{R}\) be a differentiable vertical graph above \(\Gamma\), such that the vertical distance between \(\Gamma\) and \(\Gamma_{\varepsilon}^+\) is at most \(\varepsilon\). Then, let \(\Gamma_{\varepsilon}^- \subset \partial_{\infty}\mathbb{H}^2 \times \mathbb{R}\) be a differentiable vertical graph below \(\Gamma\), such that the vertical distance between \(\Gamma\) and \(\Gamma_{\varepsilon}^-\) is at most \(\varepsilon\).

For any \(\varepsilon > 0\) there exist a minimal graph \(M_{\varepsilon}^+\) (resp. \(M_{\varepsilon}^-\)) with asymptotic boundary \(\Gamma_{\varepsilon}^+\) (resp. \(\Gamma_{\varepsilon}^-\)).
By the maximum principle, each surface $M_n$ in the proof of Theorem 4 in [?] is above $M_{\Gamma^-}$ and below $M_{\Gamma^+}$, hence also the limit surface $M$. Now, let $\varepsilon$ go to zero: as both $\Gamma_{\varepsilon^+}$ and $\Gamma_{\varepsilon^-}$ converge to the curve $\Gamma$, the boundary of $M$ must be $\Gamma$.

Then, assume that $\Gamma$ is $C^\infty$.

In the model $D = \{0 \leq x_1^2 + x_2^2 < 1\}$ for $\mathbb{H}^2$, the curve $\Gamma$ is a graph over the circle $x_1^2 + x_2^2 = 1$. Consider an exhaustion of $D$ by disks $D_n$ centered at the origin, of Euclidean radius $1 - \frac{1}{n}$. For each $n$, let $\Gamma_n$ be a vertical $C^2$ graph over $\partial D_n$ converging to $\Gamma$ as $n \to \infty$. We choose the curves $\Gamma_n$ contained in the convex hull of $\Gamma$. The curves $\Gamma_n$ may be taken as the trace on $\partial D_n \times \mathbb{R}$ of the function whose graph is a $C^2$ extension of $\Gamma$ inside $D$. Let $M_n$ be the Plateau solution with boundary $\Gamma_n$; by Rado’s theorem $M_n$ is a vertical graph of a $C^2$ function $v_n : D_n \to \mathbb{R}$. The sequence $\{v_n\}$ is uniformly bounded on compact subsets of $D$, hence there is a subsequence converging to a minimal solution $v : D \to \mathbb{R}$, uniformly on compact subsets of $D$. Let $M$ be the graph of the function $v$. We have only to prove that the asymptotic boundary of $M$ is $\Gamma$. By definition of $M$, one has that $\Gamma \subset \partial_\infty M$. In order to prove the converse, we show that any point $p \not\in \Gamma$ is not contained in $\partial_\infty M$. Let $p$ such a point and assume that it lies above $\Gamma$ (the reasoning is analogous, when $p$ lies below $\Gamma$). We use a suitable Abresch-Sa Earp graph as barrier. One can choose $\alpha$ such that a suitable translation of the relative Abresch-Sa Earp graph has boundary above the boundary of $M_n$ for $n$ large, and its asymptotic boundary is below $p$. Hence, by the maximum principle, such Abresch-Sa Earp graph lies above $M_n$. Hence the asymptotic boundary of $M$ (that is the limit of the sequence $\{M_n\}$) does not contain the point $p$. Uniqueness follows from the maximum principle. Theorem 4 is proved.

\[\square\]

7 Isolated Singularities

The original idea of the proof of the following Theorem is from [11]. A removable singularities Theorem analogous to Theorem 5, for prescribed mean curvature graphs in Euclidean space and hyperbolic space is proved in [8] and [4] respectively.

**Theorem 5.** Let $D^*_\rho$ be a punctured disk of radius $\rho$ in $\mathbb{H}^2$. Let $u : D^*_\rho \to \mathbb{R}$ be a $C^2$ function such that the graph of $u$ is minimal in $\mathbb{H}^2 \times \mathbb{R}$. Then $u$ extends $C^2$ to the puncture.

**Proof.** First we prove that $u$ is bounded on $D^*_\rho$. Up to isometry, we can take $D^*_\rho$ centered at the origin of $\mathbb{H}^2$. Let $\varepsilon < \rho$ and consider the annulus $A_\varepsilon = D_\rho \setminus D_\varepsilon$. Let $C_\varepsilon$ the half catenoid that is a graph over $\mathbb{H}^2 \setminus D_\varepsilon$ with waist on $\partial D_\varepsilon$. The function $u$ is bounded on $A_\varepsilon$, hence there is a vertical translation of $C_\varepsilon$ that does not touch the graph of $u$. Now translate vertically $C_\varepsilon$ towards the graph of $u$. By the maximum principle the first contact point is on $\partial D_\rho \times \mathbb{R}$.

Letting $\varepsilon \to 0$, $C_\varepsilon$ tends to a plane, hence:
Now, let \( v : \mathcal{D}_\rho \rightarrow \mathbb{R} \) be the minimal solution with boundary values \( u_{\partial \mathcal{D}_\rho} \). We will show that \( u \equiv v \) in \( \mathcal{D}_\rho^* \).

Consider the form \( \theta \) defined in \( \mathcal{D}_\rho^* \) by:

\[
\theta = (u - v) \left( \frac{u_1}{\tau_u} - \frac{v_1}{\tau_v} \right) dx_2 - \left( \frac{u_2}{\tau_u} - \frac{v_2}{\tau_v} \right) dx_1
\]

We have:

\[
\int_{\partial \mathcal{D}_\rho} \theta = \int_{\partial \mathcal{A}_\varepsilon} \theta = \int_{\mathcal{A}_\varepsilon} d\theta
\]

where the first equality depends on the fact that \( u \equiv v \) on \( \partial \mathcal{D}_\rho \) and the last equality is by Stokes’ theorem.

The form \( \theta \) is bounded on \( \mathcal{D}_\rho^* \) because \( u, |u_i|\tau_u^{-1}, |v_i|\tau_v^{-1} \) are bounded for \( i = 1, 2 \). Then:

\[
(9) \quad \int_{\partial \mathcal{A}_\varepsilon} \theta \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0
\]

As in Lemma 2, we obtain that \( d\theta \) is non-negative and it is 0 if and only if \( u_i = v_i \), for \( i = 1, 2 \). Letting \( \varepsilon \longrightarrow 0 \) we obtain \( \nabla u \equiv \nabla v \) and so \( u \equiv v \) on \( \mathcal{D}_\rho^* \).

\[\square\]

References


Barbara Nelli
Universitá di L’Aquila
nelli@univaq.it

Harold Rosenberg
IMPA- Rio de Janeiro
rosenbergharold@gmail.com