PROPERLY EMBEDDED SURFACES WITH CONSTANT MEAN CURVATURE

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1. INTRODUCTION

In this paper we derive some global properties of properly embedded surfaces in \( \mathbb{R}^3 \) of non-zero constant mean curvature \( H \). We call such a surface an \( H \)-surface. Our main result is a maximum principle at infinity for these \( H \)-surfaces.

**Theorem 1.** Let \( M_1 \) and \( M_2 \) be connected disjoint \( H \)-surfaces in \( \mathbb{R}^3 \). Then \( M_2 \) is not on the mean convex side of \( M_1 \).

The surface \( M_1 \) separates \( \mathbb{R}^3 \) into two connected components since \( M_1 \) is properly embedded. The mean convex side of \( M_1 \) is the component \( W_1 \) of \( \mathbb{R}^3 - M_1 \), towards which points the mean curvature vector of \( M_1 \).

In the minimal case, the Halfspace Theorem implies that two disjoint proper immersed minimal surfaces must be parallel planes, see [3].

Assume \( H \neq 0 \) and \( M_2 \subset W_1 \). If there exist points \( x \in M_1 \) and \( y \in M_2 \) whose distance is \( \text{dist}(M_1, M_2) \), then the theorem above can be proved directly as follows. There are no focal points of \( M_1 \) along the interior of the line segment from \( x \) to \( y \), since the segment minimizes distance between \( M_1 \) and \( M_2 \), and after a small translation of \( M_2 \) we can assume that there are not focal points of \( M_1 \) in the segment \( xy \). So the equidistant surfaces to \( M_1 \) are non-singular along this segment, starting with a small neighborhood of \( x \) on \( M_1 \). Their mean curvature is strictly increasing when one goes from \( x \) to \( y \). But the surface at \( y \) touches \( M_2 \) at its mean convex side, a contradiction.

If the above points \( x \) and \( y \) do not exist, we can take divergent sequences \( x_n \in M_1 \), \( y_n \in M_2 \) such that \( |x_n - y_n| \to \text{dist}(M_1, M_2) \) and \( x_n - y_n \) converge to a vector \( v \). So the surface \( M_2 + v \) lies in the mean convex side of \( M_1 \), \( M_2 + v \subset W_1 \), and touches \( M_1 \) at infinity, that is, \( \text{dist}(M_1, M_2 + v) = 0 \). Theorem 1 says that this can not happen, which explains why we call it the maximum principle at infinity.

We will also study \( H \)-surfaces \( M \) in a slab of \( \mathbb{R}^3 \); between two horizontal planes say. An unsolved problem is whether such a surface admits a horizontal plane of

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symmetry. It is natural to attack this problem by the technique of Alexandrov reflection \cite{1}, starting with horizontal planes coming down from the top of the slab. This method requires that the part of surface above the moving plane be a graph over a horizontal planar domain and that its reflected image with respect to this plane lies in the mean convex side $W$ of $M$. If there is a first plane where the symmetry of $M$ above the plane touches the part of $M$ below, then the usual maximum principle says $M$ is symmetric about this plane. In the case $M$ is compact, we deduce in this way that the surface has a lot of mirror symmetries and so it must be a round sphere, \cite{1}. However in our setting this does not appear to work for several reasons. First, one might not be able to get started. In fact, if we suppose $M$ has unbounded curvature near the plane $x_3 = a$, $a$ being the supremum of $x_3$ restricted to $M$, then moving this plane down till $x_3 = a - \varepsilon$, for small $\varepsilon > 0$, the part of $M$ above this new plane may not be a vertical graph. If one assumes $M$ has bounded curvature then this phenomena does not occur, so at least one can begin to do symmetry of the part of $M$ above the plane. Even then, one quickly encounters another difficulty: it may be the case (and Pascal Collin has constructed $H$-surfaces with boundary doing this \cite{2}) that at the last allowed position of the moving plane, the part of $M$ below this plane and the symmetry of the part above touch for the first time at infinity. Then we can not proceed.

This leads us to study the class $S$ of (non-necessarily connected) properly embedded $H$-surfaces in $\mathbb{R}^3$ satisfying the following conditions: $M$ lies in a horizontal slab, is symmetric about the plane $P = \{x_3 = 0\}$ and $M_+ = M \cap \{x_3 > 0\}$ is a graph over a open set in $P$. This is the class of surfaces we would obtain if Alexandrov reflection technique could be applied to proper $H$-surfaces in a slab. There are many such examples. The Delaunay surfaces are in $S$ and their width varies from $2/H$ (attained for the limit case of a stack of spheres) to $1/H$ (for the cylinder). Kapouleas \cite{9} has constructed examples of finite topology in $S$ which look like a sphere with $n$ horizontal Delaunay ends, the directions of the ends symmetrically placed, like the $n$ roots of unity. For further results on surfaces in $S$ with finite topology see Grosse-Brauckmann, Kusner and Sullivan \cite{8}. Lawson \cite{11} constructed doubly periodic $H$-surfaces in $S$ and Ritoré \cite{17} and Grosse-Brauckmann \cite{7} have given more doubly periodic examples of this type. In particular, Ritoré constructs examples whose width tends to zero (for $H$ fixed): these surfaces look like two close parallel planes connected by a doubly periodic family of catenoidal necks, the distance between two neighbor necks being small. Notice that each surface in $S$ has width at most $2/H$ by Theorem 6.

We will give a structure result for $H$-surfaces in the class $S$ contained between two parallel planes at distance smaller than $1/H$: Assuming that $M$ has bounded curvature, we prove that the shape of the surface is close to the doubly periodic $H$-surfaces constructed by Lawson \cite{11}.

In this section we give a priori estimates for $H$-surfaces in different contexts. These results will be used in the proof of our main results in the sections §3 and §4 below. We give a bound, by modifying arguments of Fisher-Colbrie [4], for the radius of a geodesic ball contained in the interior of a stable $H$-surface and we extend Serrin’s height estimate for compact $H$-graphs, [18], to $H$-graphs over arbitrary domains.

Let $M$ be an $H$-surface in $\mathbb{R}^3$. We will consider on $M$ the unit normal vector field $n$ which makes $H > 0$. Equivalently, $n$ will be the normalized mean curvature vector.

An $H$-surface $M$ is said to be stable (in the strong sense) if for any function $u$ with compact support in $M$ we have that

$$\int_M \left| \nabla u \right|^2 - |A|^2 u^2 \geq 0,$$

where $\nabla u$ and $A$ denote the gradient of $u$ and the shape operator of $M$ respectively. Stability is equivalent to the existence of a positive solution of the equation $\Delta v + |A|^2 v = 0$ on $M$, see [5]. In particular any graph is stable, because the third coordinate of the unit normal vector $n_3$ satisfies the equation above (for $H$-surfaces there is another natural notion of stability, weaker than the one considered in this paper, which is related with the isoperimetric problem: we ask that the integral inequality holds for any $u$ with $\int_M u = 0$).

A fundamental fact about stable $H$-surfaces is that we have an estimate (depending only on $H$) of the largest geodesic ball contained in the interior of the surface. The proof of the next result follows from the ideas of Fischer-Colbrie [4] and it is implicit in Lópex and Ros [12].

**Theorem 2.** Let $M$ be a stable $H$-surface. Then the (intrinsic) distance of any point of $M$ to $\partial M$ is smaller than or equal to $\pi/H$.

**Proof.** The operator $L = \Delta + |A|^2 = \Delta + (4H^2 - 2K)$, $K$ being the Gauss curvature of $M$, has index zero and, so, there is a positive function $u$ on $M$ with $Lu = 0$, (see Proposition 1 of Fisher-Colbrie [4]).

Consider the new metric $ds^2 = u^2 ds^2$ and let $p \in M$ and $R > 0$ be such that the open $ds^2$-geodesic ball $B = B(R,p) \subset M$ centered at $p$ is relatively compact in $M$. It is enough to prove that $R \leq \pi/H$.

Let $\gamma$ be a minimizing geodesic for the metric $d\tilde{s}^2$, joining $p$ to $\partial B$. As the $ds^2$-distance between $p$ and any point of $\partial B$ is $R$, then if we denote by $a$ the $ds^2$-length of $\gamma$ we have $a \geq R$. Parameterize $\gamma$ by arclength $s$ in the $ds^2$ metric, $0 \leq s \leq a$.

Since $\gamma$ is minimizing for $d\tilde{s}^2$, the second variation of length yields:

$$0 \leq \int_0^R \left( \frac{d\phi}{ds} \right)^2 - \tilde{K} \phi^2 ) d\tilde{s},$$

for all $\phi$ with $\phi(0) = \phi(R) = 0$, $\tilde{R}$ being the $d\tilde{s}^2$-length of $\gamma$ and $\tilde{K}$ the Gauss curvature of the metric $d\tilde{s}^2$. 
We have that 
\[ \frac{d\phi}{ds} = \frac{d\phi}{d\tilde{s}} \frac{ds}{d\tilde{s}} = \frac{1}{u} \frac{d\phi}{ds} \]
and
\[ \tilde{K} = \frac{1}{u^2} (K - \Delta \log u). \]
Moreover, as \( u \) is a Jacobi function on \( \mathcal{B} \), we can write
\[ 0 = Lu = \Delta u - Ku + \left( 2H^2 + |A|^2 \right) u \geq \Delta u - Ku + 2H^2 u. \]
Therefore, if \( c = 2H^2 \) and \( u'(s) = d(u \circ \gamma)/ds \), we obtain
\[ \Delta \log u = \frac{u\Delta u - |\nabla u|^2}{u^2} \leq K - c - \frac{(u')^2}{u^2} \]
where we have used that \( |u'| \leq |\nabla u| \).
Then (1) and (2) yield:
\[ \int_0^a \left( c\psi^2 + (\psi')^2 + 2\psi''\psi \right) ds \leq 0. \]
Finally, we take
\[ \psi(s) = \sin \left( \frac{\pi s}{a} \right), \quad 0 \leq s \leq a, \]
and so (4) becomes:
\[ \int_0^a \left( \frac{\pi^2}{a^2} \cos^2 \left( \frac{\pi s}{a} \right) + \left( c - \frac{2\pi^2}{a^2} \right) \sin^2 \left( \frac{\pi s}{a} \right) \right) u(\gamma(s)) ds \leq 0. \]
Thus \( c < \frac{2\pi^2}{a^2} \) and therefore \( R \leq a < \frac{\pi}{H} \), which proves the theorem.

Now we consider \( H \)-surfaces \( M \) which are vertical graphs (in short, \( H \)-graphs). The unit normal vector of a graph is never horizontal and so, on each connected component of \( M \), it points either down or up. Recall also that \( H \)-graphs are stable.
Lemma 3. Let $M$ be an $H$-surface given as the graph of a smooth positive function $u$ defined over a domain $\Omega \subset \{x_3 = 0\}$. If $M$ is properly embedded in the halfspace $x_3 > 0$ then,

a) $M$ is contained in the slab $\{0 < x_3 < 2\pi/H\}$,

b) $u$ extends continuously to zero on the boundary of $\Omega$, and

c) the normal vector of $M$ points down.

Proof. As any $H$-graph is stable, it follows from Theorem 2 that $x_3 < 2\pi/H$ on $M$: Otherwise we can find a geodesic ball of radius $\pi/H$ contained in the interior of $M$. This proves a). In particular, we conclude that $u$ extends continuously, with zero boundary values, to the topological boundary of $\Omega$. So it remains to prove c). Take a vertical line $l$ which intersects a connected component $M'$ of $M$ and a sphere $S$ of mean curvature $H$ centered at a high point of $l$ so that the sphere is disjoint from $M$ and $l \cap S$ is above $l \cap M$. Move $S$ down till we have a first contact with $M'$. If the mean curvature vector of $M'$ points up, then the maximum principle would imply that the graph $M$ contains a sphere, which is impossible.

From the Lemma above we conclude that for an $H$-graph $M \subset \{x_3 > 0\}$, $M$ is properly embedded in the upper halfspace if and only if $u$ extends continuously to zero on the boundary of $\Omega$.

We will consider limits of $H$-graphs. The following propositions justify the existence of such limits. First we prove that an $H$-graph $M$ in the upper halfspace with zero boundary values satisfies an interior curvature estimate in $\{x_3 > 0\}$. This is a known fact but we include a proof for the sake of completeness.

Proposition 4. There is a positive constant $C$, depending only on $H > 0$, such that any $H$-surface $M$ properly embedded in $\{x_3 > 0\}$ given as the graph of a function $u \in C^\infty(\Omega)$, with $\Omega \subset \{x_3 = 0\}$, satisfies

$$|A| \leq C/x_3,$$

$|A|$ being the length of the second fundamental form of $M$.

Proof. This can be shown as follows: In case the estimate fails, we can consider a sequence $p_k$, $k = 1, 2, \ldots$, of points in the surfaces $M_k$, satisfying the hypothesis of the assertion such that $|A_k(p_k)|x_3(p_k) > k$, where $A_k$ is the second fundamental form of $M_k$. Let $B(p, r) = \{x \in \mathbb{R}^3 / |x - p| < r\}$ and $r_k = x_3(p_k)/2$. It follows that for $q \in M_k \cap B(p_k, r_k)$, the expression $(r_k - |q - p_k|)|A_k(q)|$ attains its maximum at a point $q_k$ (as it vanishes when $q$ approaches to the boundary). Moreover if $r_k' = r_k - |q_k - p_k|$, then we have $S_k = M_k \cap B(q_k, r_k') \subset M_k \cap B(p_k, r_k)$ and $R_k = r_k'|A_k(q_k)| > k/2$. These points $q_k$ are called points of almost maximal curvature and the translated rescaled surfaces $\Sigma_k = |A_k(q_k)|/S_k - q_k\}$ converge, up to a subsequence, to a nonflat complete surface $\Sigma_\infty$ in the Euclidean 3-space with mean curvature 0. To see that we observe that the surfaces $\Sigma_k$ pass through the origin.
and \( \partial \Sigma_k \) is contained in the boundary the ball \( B(0, R_k) \) whose radius \( R_k \) converges to \( \infty \). Moreover in the ball \( B(0, R_k/2) \) the length the second fundamental form of \( \Sigma_k \) is bounded by 4, and equals 1 at the origin. The mean curvature of \( \Sigma_k \) equals \( H/|A_k(q_k)| \).

This permits one to construct a subsequence of \( \Sigma_k \) that converges to a complete minimal surfaces \( \Sigma_\infty \) passing through the origin and with curvature 1 at the origin. As the Gauss map of \( \Sigma_\infty \) lies in the closed lower hemisphere, it follows that the minimal surface \( \Sigma_\infty \) is stable and therefore it must be flat, see [5]. This contradiction proves the assertion. \( \square \)

**Proposition 5.** Let \( \{M_n\} \) be a sequence of \( H \)-surfaces, \( H > 0 \), such that \( M_n \) is the graph of a positive function \( u_n \) over an open subset \( \Omega_n \subset \{x_3 = 0\} \) that extends continuously to zero on the boundary of \( \Omega_n \). Assume that there are points \( p_n \in M_n \) with \( p_n \to p \) and \( x_3(p) > 0 \). Then there exists a subsequence of \( \{M_n\} \) which converges on compact subsets of \( x_3 > 0 \) to the graph of a positive function \( u : \Omega \to \mathbb{R} \), over an open subset \( \Omega \), which extends continuously to the closure of \( \Omega \) with zero boundary values.

**Proof.** The curvature estimate in the Proposition above implies that, up to a subsequence, the surfaces \( M_n \) converge to an \( H \)-surface \( M \) properly immersed in \( x_3 > 0 \). If \( n_3 \) denotes the third coordinate of the unit normal vector of \( M \), we have that \( \Delta n_3 + |A|^2 n_3 = 0 \). As \( M \) is a limit of graphs we get from item c) in Lemma 3 that \( n_3 \leq 0 \) and the maximum principle implies that either \( n_3 < 0 \) in \( M \) or \( n_3 = 0 \) on a connected component of \( M \).

In the second case, that component must be a vertical circular cylinder intersected with the upper halfspace. By lemma 3, the height of the surfaces \( M_n \) is uniformly bounded, so this case is impossible.

Therefore \( n_3 < 0 \) and we claim that \( M \) is the graph of a positive function \( u \) over an open subset \( \Omega \subset \{x_3 = 0\} \). To prove this claim note that each point \( p \in M \) has a neighborhood \( U_p \) which is the graph of a smooth function over an open disc in the plane \( x_3 = 0 \). If there were two different points \( p, q \in M \) lying in the same vertical line, we deduce that the same would be true for the graph \( M_n \), for \( n \) large enough. This contradiction proves that \( M \) meets each vertical line at most once and so \( M \) is a graph of a function \( u \). From Lemma 3 it follows that \( u \) extends continuously to the closure of \( \Omega \) with zero boundary values. That proves the proposition. \( \square \)

If is well-known that compact \( H \)-graphs with zero boundary values satisfy a height estimate, see Serrin [18]. In the result below we extend this fact to arbitrary \( H \)-graphs with that boundary condition.

**Theorem 6.** Let \( M \) be an \( H \)-surface, \( H > 0 \), given by the graph of a positive function \( u \) on a planar domain \( \Omega \subset \{x_3 = 0\} \), where \( u \) extends continuously to the closure of \( \Omega \) with zero boundary values. Then \( M \) is contained in the slab \( 0 < x_3 \leq 1/H \).
Proof. First observe that Theorem 2 implies that $x_3$ is bounded on $M$. To prove the upper bound $x_3 \leq 1/H$ consider on $M$ the function
\[ \phi_M = \phi = H x_3 + n_3, \]
where $x_3$ and $n_3$ denote the third coordinate of the position and unit normal vectors on $M$ (note that $n_3 \leq 0$ on $M$). Then $\Delta \phi = -2(H^2 - K)n_3 \geq 0$.

If we assume that $\overline{\Omega}$ is smooth and compact, then the statement is a classical result of Serrin, [18]: to prove it, observe that $\phi$ is subharmonic and it attains its maximum at the boundary. On $\partial \Omega$, $\phi = n_3 \leq 0$ and therefore $\phi \leq 0$ in $\Omega$. Thus $H x_3 + n_3 \leq 0$ which implies
\[ x_3 \leq -\frac{n_3}{H} \leq \frac{1}{H}. \]

Now suppose $\overline{\Omega}$ is noncompact. If $\phi \leq 0$ on $M$ then we are done, so we can suppose that sup $\phi = c > 0$. Let $p_n \in M$ be a sequence with $\phi(p_n) \to c$, $x_3(p_n) \to x_\infty$, and $n_3(p_n) \to n_\infty$. Notice that $x_\infty > 0$, since if $x_\infty \leq 0$, $c = Hx_\infty + n_\infty \leq 0$.

Let $M_n$ be the horizontal translate of $M$ which places $p_n$ on the $x_3$-axis, intersected with the open half space $\{x_3 > 0\}$. From Proposition 5 we have that a subsequence (that we also denote by $M_n$) converges to an $H$-graph which is properly embedded in $\{x_3 > 0\}$. Let $p_\infty = \lim p_n$ and $M_\infty$ the connected component of the limit of $M_n$ which contains the point $p_\infty$. As $x_3(p_\infty) = x_\infty > 0$, we see that $p_\infty$ is an interior point of $M_\infty$. As the function $\phi_\infty = \phi_{M_\infty}$ achieves its maximum at $p_\infty$ we conclude from the maximum principle that $\phi_\infty$ is constant on $M_\infty$, which means that $M_\infty$ is a spherical cap. In particular $\phi_\infty \leq 0$ which contradicts that $\phi_\infty(p_\infty) = c > 0$.

The lower bound $0 < x_3$ is proved in a similar way using the equation $\Delta x_3 = 2Hn_3 < 0$. If $x_3$ is negative somewhere on $M$, then a suitable sequence of horizontal translated images of $M$ will converge in $\{x_3 < 0\}$ to an $H$-graph $M'_\infty$ properly embedded in the lower halfspace $\{x_3 < 0\}$, whose unit normal vector points down and such that $x_3$ attains its minimum at the interior, which contradicts the maximum principle. Thus $x_3 \geq 0$ on $M$ and the maximum principle again gives that $x_3 > 0$, as we claimed.

3. The Maximum Principle at Infinity

In this section we will prove our main result: Let $M_1$ and $M_2$ be two connected properly embedded $H$-surfaces in $\mathbb{R}^3$ ($H > 0$) such that $M_2$ lies in the mean convex side of $M_1$. Then we want to show that $M_2 = M_1$.

By the comments in the introduction, we can assume that $M_1$ is noncompact. We orient $M_1$ and $M_2$ by unit vector fields $N_1$ and $N_2$ whose direction is that of the mean curvature. The mean convex side $W_1$ of $M_1$ is the component of $\mathbb{R}^3 - M_1$ such that $N_1$ along $M_1$ points into $W_1$. So $M_2$ is contained in the closure of $W_1$.

Assuming that $M_2 \neq M_1$ we will obtain a contradiction. Clearly $M_2 \cap M_1 = \emptyset$ by the usual maximum principle. Denote by $W$ the component of $W_1 - M_2$ satisfying $\partial W = M_1 \cup M_2$. Thus the boundary of $W$ is not connected and the mean curvature
vector of $M_1$ points to $W$ (hence $N_1$ as well). If $W$ were mean convex, then the
halfspace theorem [3] would imply that $M_1$ and $M_2$ are parallel planes. Therefore
the mean curvature vector of $M_2$ (hence $N_2$ as well) points away from $W$.

Let $S$ be a relatively compact domain in $M_1$ with smooth boundary $\Gamma$. Assume
also that $S$ is unstable. We will show that there is a surface $\Sigma \subset W$ bounded by $\Gamma$ with constant mean curvature $H$ that is stable. Then by taking the domain $S$ in $M_1$ to be larger and larger, we will obtain a contradiction, using the fact that the distance between a point of a stable $H$-surface and its boundary is bounded.

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Let $F$ be the family of regions $Q \subset W$ enclosed by $S$ and surfaces $\Sigma \subset W$ with $\partial \Sigma = \Gamma$ and define the functional

$$\displaystyle F(Q) = A(\Sigma) + 2H V(Q),$$

where $A(\Sigma)$ is the area of $\Sigma$ and $V(Q)$ is the volume of $Q$. Our goal is to show
that there is $Q \in F$ minimizing the functional $F$ and such that the interior of $\Sigma$ is contained in the interior of $W$. This will imply that $\Sigma$ is a smooth surface with $\partial \Sigma = \Gamma$ and constant mean curvature $H$. Moreover the mean curvature vector of $\Sigma$ points outside $Q$ and $\Sigma$ is stable in the sense of §2.

We will need certain auxiliary surfaces and regions and some preliminary remarks.
Consider the least area surface $S_{min}$ in $W_1$ spanning the curve $\Gamma$ and homologous
to $S$. $S_{min}$ is a compact smooth minimal surface and by the maximum principle we
have $M_1 \cap S_{min} = \Gamma$. Denote by $Q_{min}$ the region in $W_1$ enclosed by $S \cup S_{min}$.

Let $S_0 = M_2 \cap B(\rho)$ be a relatively compact open set in $M_2$, where $\rho$ is a large
positive radius and $B(\rho) = \{x \in \mathbb{R}^3 / |x| < \rho\}$. There is an $\varepsilon > 0$ such that the sets $S_t = \{x \in \overline{W} | \text{dist}(x, S_0) = t\}$, $0 \leq t \leq \varepsilon$, are smooth surfaces parallel to $M_2$
foliating a neighborhood $Q_{par}$ of $S_0$ in $W$. We put $S_{par} = S_{\varepsilon}$. Denote by $Y$ the unit vector field, normal to the foliation $S_t$, and oriented by $N_2$ and let $H_t$ be the mean curvature of $S_t$. A calculation shows $H_t < H$ for $0 < t$. At any point of $S_t$ we have $\text{div} Y = -2H_t$, where $\text{div}$ is the divergence operator in $\mathbb{R}^3$. Therefore

(5) $$\text{div} Y > -2H, \quad \text{for} \quad t > 0.$$  

Finally, we consider $\varphi$ the first eigenfunction of the Jacobi operator of $S$. So $\varphi$
vanishes at $\Gamma$, is positive at the interior of $S$ and satisfies $\Delta \varphi + |A|^2 \varphi + \lambda_1 \varphi = 0$, where $\lambda_1$ is a negative constant (as $S$ is unstable). After a perturbation of $S$ we can assume that $0$ is not an eigenvalue of $\Delta + |A|^2$. Hence there is a smooth function $v$
on $S$, vanishing at $\Gamma$ and such that $\Delta v + |A|^2 v = 1$ in $S$. The boundary maximum principle implies that the derivative of $\varphi$ with respect to the outwards pointing normal vector is negative along $\Gamma$. Therefore we conclude that, for $a > 0$ small enough, $u = \varphi + av$ is positive at the interior of $S$.

For small $\varepsilon > 0$ and $0 < t \leq \varepsilon$, $u$ defines a normal deformation of $S$,

$$\displaystyle S_t' = \{x + tuN_1 | x \in S\} \subset W.$$
The surfaces \( S'_t, 0 < t < \varepsilon \), foliate an open set \( Q_{uns} \subset W \). Putting \( S_{uns} = S'_t \), we have \( \partial Q_{uns} = S \cup S_{uns} \), see Figure 3.

If \( X \) is the unit normal vector field of the foliation \( S'_t \) oriented by \( N_1 \) and \( H'_t \) is the mean curvature of \( S'_t \), we have \( \text{div} X = -2H'_t \) as above. Moreover
\[
\frac{d}{dt} \bigg|_{t=0} 2H'_t = \Delta u + |A|^2 u = -\lambda_1 \varphi + a > 0,
\]
which implies, choosing \( \varepsilon \) small enough, that \( H'_t > H \). Therefore
\[
(6) \quad \text{div} X < -2H, \text{ for } 0 < t < \varepsilon.
\]

**Lemma 7.** Let \( Q \in \mathcal{F} \) be enclosed by \( S \) and \( \Sigma \). Assume that \( \Sigma \) is smooth at the interior of \( W \).

i) If \( Q \not\subset Q_{min} \), then \( F(Q \cap Q_{min}) < F(Q) \).

ii) If \( Q \cap Q_{par} \neq \emptyset \), then \( F(Q - Q_{par}) < F(Q) \).

iii) If \( Q_{uns} \not\subset Q \), then \( F(Q \cup Q_{uns}) < F(Q) \).

**Figure 1.** If \( \Sigma \) meets the exterior of the least area surface \( S_{min} \), then \( F(Q \cap Q_{min}) < F(Q) \).

**Proof.** The assertion in i) follows because \( Q' = Q \cap Q_{min} \in \mathcal{F} \) and satisfies \( V(Q') < V(Q) \) and \( A(\Sigma') \leq A(\Sigma) \), where \( \Sigma' = \partial Q' - S \), see Figure 1.

The claim in ii) is a consequence of the inequality \( \text{div} Y > -2H \) on \( Q_{par} \), see Figure 2. From the divergence theorem we obtain
\[
-2H V(Q \cap Q_{par}) < \int_{Q \cap Q_{par}} \text{div} Y = \int_{\partial(Q \cap Q_{par})} \langle Y, \nu \rangle = \int_{Q \cap S_{par}} \langle Y, \nu \rangle + \int_{\Sigma \cap Q_{par}} \langle Y, \nu \rangle,
\]
where \( \nu \) is the outer pointing unit normal to the boundary of \( Q \cap Q_{par} \). Using that \( \nu = -Y \) along \( Q \cap S_{par} \) and \( \langle Y, \nu \rangle \leq 1 \) at the points of \( \Sigma \cap Q_{par} \), we deduce, after rearrangement, that
\[
-2H V(Q \cap Q_{par}) + A(Q \cap S_{par}) < A(\Sigma \cap Q_{par}),
\]
and, therefore
\[
F(Q - Q_{par}) = 2H (V(Q) - V(Q \cap Q_{par})) + A(\Sigma - Q_{par}) + A(Q \cap S_{par})
\]
Figure 2. If $Q$ cuts the tubular neighborhood of $Q_{par}$ of $M_2$ (in gray), then $F(Q - Q_{par}) < F(Q)$.

$$< 2HV(Q) + A(\Sigma \cap Q_{par}) + A(\Sigma - Q_{par}) = F(Q),$$
as we claimed.

Figure 3. If $Q$ does not contain the shadowy region $Q_{uns}$, then $F(Q \cup Q_{uns}) < F(Q)$.

To prove $ii)$ we argue as in the proof of $i)$, but here we use that $\text{div} \ X < -2H$ on $Q_{uns}$, see Figure 3.

$$-2HV(Q_{uns} - Q) > \int_{Q_{uns} - Q} \text{div} \ X = \int_{\partial(Q_{uns} - Q)} \langle X, \nu \rangle = \int_{S_{uns} - Q} \langle X, \nu \rangle + \int_{\Sigma \cap Q_{uns}} \langle X, \nu \rangle.$$As $\nu = X$ on $S_{uns} - Q$ and $\langle X, \nu \rangle \geq -1$ at the other points of the boundary, we have

$$2HV(Q_{uns} - Q) + A(S_{uns} - Q) < A(\Sigma \cap Q_{uns}).$$Hence

$$F(Q \cup Q_{uns}) = 2H(V(Q) + V(Q_{uns} - Q)) + A(S_{uns} - Q) + A(\Sigma - Q_{uns})$$

$$< 2HV(Q) + A(\Sigma \cap Q_{uns}) + A(\Sigma - Q_{uns}) = F(Q),$$
as desired. \hfill \Box

**Proof of Theorem 1.** Assume 0 lies in $M_1$ and let $S(r)$ be the connected component of $M_1 \cap B(r)$, where $r > 0$, and $B(r)$ is the euclidean ball of radius $r$ centered at 0. Since $M_1$ is properly embedded, the boundary of $S(r)$ is contained in $\partial B(r)$. Let $\gamma$
be a path in $W$ joining 0 to a point $y$ in $M_2$. Choose $r$ large enough so that $\gamma$ is contained in $B(r)$ and $\text{dist}(\gamma, \partial B(r)) > 2\pi/H$.

From Theorem 2, and our choice of $r$, we know that $S = S(r)$ is unstable. Let $K$ be the compact connected domain contained in $Q_{\text{min}}$, bounded by $S$ and $M_2 \cap Q_{\text{min}}$, see Figure 4.

![Figure 4](image)

Let $a$ be the infimum of $F$ on $\mathcal{F}$. If $(Q_n, \Sigma_n)$ is a minimizing sequence for $F$ (i.e. $\lim_{n \to \infty} F(Q_n, \Sigma_n) = a$) then by Lemma 7, we can cut and paste the $Q_n$ to form a new minimizing sequence $(\widetilde{Q}_n, \widetilde{\Sigma}_n)$ so that $\widetilde{Q}_n \subset K$ and $\text{int} \widetilde{\Sigma}_n \subset \text{int} K$.

By compactness [15, 5.5], there is a minimum $Q$ of $F$ in $K$ bounded by a rectifiable current $\Sigma$ with the support of $\Sigma$ contained in $W$, $\partial \Sigma = \partial S$, and $\Sigma$ disjoint from $M_2$. Regularity [14, Corollary 3.7] implies the part of $\Sigma$ in the interior of $K$ is a smooth surface of mean curvature $H$. This stable surface $\Sigma$ intersects $\gamma$ (since the union of $\Sigma$ and $S$ bounds $Q$). However, for $z$ in the intersection of $\gamma$ and $\Sigma$, we have $\text{dist}(z, \partial \Sigma) > 2\pi/H$. This contradicts Theorem 2 and completes the proof of Theorem 1.

\[ \Box \]


We now consider properly embedded $H$-surfaces $M$ with $\partial M = \emptyset$ and $H > 0$, which fit between two parallel planes $P_0$ and $P_1$. The width of $M$ is the infimum of the distance between such planes. The only compact connected $M$ is the sphere of width $2/H$, so the surfaces we consider are non compact. We know very little about these surfaces, even assuming bounded curvature. Assuming $M$ is connected, some questions we can not answer are:

If $M$ has bounded width, is it at most $2/H$?

If $M$ is between the planes $P_0$ and $P_1$, is there a parallel plane $P$ between $P_0$ and $P_1$ with $M$ symmetric by $P$?
We will study a special class of surfaces of bounded width. Let $\mathcal{S}$ be those (non-necessarily connected) properly embedded $H$-surfaces $M$, $H > 0$, which are symmetric with respect to the plane $P = \{x_3 = 0\}$ and such that $M_+ = M \cap \{x_3 > 0\}$ is a graph over an open subset $\Omega$ in $P$. Theorem 6 implies that each surface in $\mathcal{S}$ has width at most $2/H$. Among the surfaces in $\mathcal{S}$, the sphere, the Delaunay surfaces, and Kapouleas [9] finite topology examples have width at least $1/H$. Lawson doubly periodic $H$-surfaces [11] and the related ones constructed by Ritoré [17] and Grosse-Brauckmann [7] may have arbitrarily small width. For some of these doubly periodic surfaces the domain $\Omega$ consists of the plane $P$ where we have removed infinitely many pairwise disjoint compact convex disks. In this section we will prove that any surface in $\mathcal{S}$ with width smaller than $1/H$ (= the width of a cylinder of mean curvature $H$) looks like these doubly periodic surfaces.

**Theorem 8.** Suppose $M$ in $\mathcal{S}$ has width less than $1/H$ and $M_+$ is a graph over the open subset $\Omega \subset P$. Then the components of $P - \Omega$ are strictly convex. In particular $M$ is connected. If moreover $M$ has bounded curvature, then $P - \Omega$ is a countable disjoint union of strictly convex compact disks.

**Proof.** First we observe that if $M$ were contained in two non parallel slabs, then the slabs intersect and $M$ is cylindrically bounded. A theorem of Kusner, Korevaar and Solomon [10] would imply that the components of $M$ are spheres or Delaunay surfaces, which contradicts our width assumption. Therefore, $M$ is contained in the slab perpendicular to $P$.

The proof uses an operator $L$ that has its origins in a paper by Payne and Philippin [16] defined on any surface $M \in \mathcal{S}$. The operator is of the form

$$Lf = \Delta f + \langle \nabla f, X \rangle,$$

where $X$ is a tangent vector field to $M$, singular where $M$ is horizontal, and $\nabla f$ denote the gradient of a function $f$ on $M$.

As we have oriented $M$ so that $H > 0$, the maximum principle implies that $n_3 < 0$ in $M_+$. Consider

$$\psi = 2Hx_3 + n_3 \quad \text{on } M_+,$$

Clearly $\psi = 0$ on $\Gamma = \partial M_+ = P \cap M$ since $M$ is vertical along $\Gamma$. A simple calculation shows $\Delta \psi = 2Kn_3$, where $K$ is the Gauss curvature of $M$. Now if we look for a vector field $X$ such that $L\psi = 0$, then it suffices to find $X$ satisfying $\langle X, \nabla \psi \rangle = -2Kn_3$.

Denote by $a$ the tangent part of $e_3 = (0, 0, 1)$. Then $\nabla x_3 = a$, $\nabla n_3 = -Aa$, where $A$ is the shape operator of $M$, and

$$\nabla \psi = 2H\nabla x_3 + \nabla n_3 = (2HI - A)a,$$

$I$ being the identity tensor. Using the basic matrix equality $A^2 - 2HA + KI = 0$, one has

$$A\nabla \psi = (2HA - A^2)a = Ka.$$
Consequently, if one defines
\[ X = -\frac{2n_3}{|a|^2} Aa, \quad \text{where} \quad a \neq 0, \]
we conclude
\[ \langle X, \nabla \psi \rangle = -\frac{2n_3}{|a|^2} \langle Aa, \nabla \psi \rangle = -\frac{2n_3}{|a|^2} \langle a, A \nabla \psi \rangle = -\frac{2n_3}{|a|^2} \langle a, K a \rangle = -2Kn_3 \]
and \( \psi \) is a solution of \( L \psi = 0 \) where \( a \neq 0 \). If \( \nu \) denotes the outer conormal to \( M_+ \) along \( \Gamma \), then by direct computation one has
\[ \frac{\partial \psi}{\partial \nu} = k_\Gamma, \]
where \( k_\Gamma \) is the curvature of \( \Gamma \) in \( P \) with respect to the plane unit normal pointing towards \( P - \Omega \). In particular if \( \frac{\partial \psi}{\partial \nu} > 0 \), then \( \Gamma \) is strictly convex towards \( P - \Omega \).

The maximum principle applied to \( L \psi = 0 \) yields directly the following properties:

\( (i) \) If \( \psi \) assumes an extremum at \( q \in \overline{M_+} \), then \( q \) is on \( \Gamma \) or the unit normal vector of \( M \) at \( q \) is vertical (i.e., \( q \) is a singular point of \( X \)).

\( (ii) \) If \( \psi \) has a local maximum at \( q \in \Gamma \), then either \( k_\Gamma(q) > 0 \) or the component of \( M \) passing through \( q \) is a cylinder.

As \( M \) is vertical along \( \Gamma \) we have that \( \psi = 0 \) on \( \Gamma \). We claim that \( \psi \leq 0 \) on \( M_+ \). Suppose first that \( \psi \) attains its maximum at \( q \). If \( q \notin \Gamma \), then by \( (i) \) we get that the normal vector at \( q \) is vertical. So \( n_3(q) = -1 \), and \( \psi(q) = 2Hx_3(q) - 1 \). Since the width of \( M \) is at most \( 1/H \), we have \( x_3(q) < 1/(2H) \), and \( \psi(q) < 0 \), which is impossible. So, \( q \in \Gamma \) and then \( \psi \) is nonpositive.

If \( \psi \) does not attain its maximum, let \( q_n \in M_+ \), \( \psi(q_n) \to \sup \psi \). If \( x_3(q_n) \to 0 \), then \( \psi(q_n) = 2Hx_3(q_n) + n_3(q_n) \leq 2Hx_3(q_n) \to 0 \), so \( \sup \psi \leq 0 \) and \( \psi \leq 0 \) on \( M_+ \). So we can assume \( x(q_n) \) converges to a number \( c > 0 \). Translate \( M \) horizontally so that \( q_n \) is over the origin, and let \( M_\infty \) be a limit \( H \)-surface of the translated surfaces in the open halfspace \( x_3 > 0 \). This surface is non empty because the point \( q_\infty \) obtained as limit of the translated images of \( q_n \) lies on \( M_\infty \). Moreover the function \( \psi_\infty \) constructed as in (7) on \( M_\infty \) attains its maximum at \( q_\infty \) and \( \psi_\infty(q_\infty) > 0 \). As \( M_\infty \) is contained in the slab \( 0 < x_3 < 1/(2H) \), reasoning as in the first case we obtain a contradiction. Hence \( \psi \leq 0 \) on \( M_+ \).

Now we can apply property \( (ii) \) to conclude that either \( M \) has a cylindrical component or \( k_\Gamma(q) > 0 \) for all \( q \in \Gamma \). Since we are assuming the width is strictly less than \( 1/H \), we have \( k_\Gamma > 0 \) at each point of \( \Gamma \). So the connected components of \( P - \Omega \) are strictly convex.

Next we prove that, if \( M \) has bounded curvature, then the planar curvature \( k_\Gamma \) is bounded away of zero. Suppose this were not the case; Then there is a sequence \( q_n \in \Gamma \), with \( k_\Gamma(q_n) \to 0 \). Translate \( M \) horizontally so that \( q_n \) transforms to a fixed point \( \sigma \). The curvature bounds allows us to take a limit surface \( M_\infty \) in the whole \( \mathbb{R}^3 \)
(not only in the half space \{x_3 > 0\}) of the translated surfaces. This surface \(M_\infty\) is not necessarily embedded (it could have tangential selfintersections at the level \(x_3 = 0\)) but retains any other properties of surfaces in \(S\). In particular, assertion (\(ii\)) applies to \(M_\infty\). Moreover its width is smaller than \(1/H\) and the function \(\psi_\infty\) constructed as in (7) on \((M_\infty)_+\) is nonpositive (because it is a limit on nonpositive functions) and vanishes at \(\sigma\). As the curvature of the curve \(M_\infty \cap P\) at \(\sigma\) is zero, we conclude from (\(ii\)) that one of the components of \(M_\infty\) must be a cylinder. This contradiction proves that \(k_\Gamma < 1/\rho\) for some positive constant \(\rho\). This implies that any connected component of \(\Gamma\) is a closed Jordan curve contained in a disk of radius \(\rho\). Therefore, if the number of components of \(P - \Omega\) where finite, then \(\Omega\) would contain arbitrarily large round disks, which is clearly impossible (use for instance Theorem 2). This completes the proof of the theorem. □

References


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