Minimal surfaces of finite total curvature in \( \mathbb{H} \times \mathbb{R} \)

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We dedicate this work to Renato Tribuzy, on his 60’th birthday.

1 Introduction

We consider complete minimal surfaces \( \Sigma \) in \( \mathbb{H} \times \mathbb{R} \), \( \mathbb{H} \) the hyperbolic plane. Let \( C(\Sigma) \) denote the total curvature of \( \Sigma \), \( C(\Sigma) = \int_{\Sigma} K dA \), \( K \) the intrinsic curvature of \( \Sigma \). We shall prove that \( C(\Sigma) \) is an integer multiple of \( 2\pi \), when it is finite. We give examples of such \( \Sigma \) with total curvature \( -2\pi m \), \( m \) any non-negative integer.

In \( \mathbb{R}^3 \), complete minimal surfaces of finite total curvature have total curvature an integer multiple of \(-4\pi \). This results from the Gauss map of the surface, that extends meromorphically to the conformal compactification. In \( \mathbb{H} \times \mathbb{R} \), we have no conformal Gauss map. We have the holomorphic quadratic differential of the harmonic height function: projection on the \( \mathbb{R} \) factor of \( \mathbb{H} \times \mathbb{R} \).

Now we describe a simply connected example. Let \( \Gamma \) be an ideal polygon of \( \mathbb{H} \) with \( m+2 \) vertices at infinity, \( 2m+2 \) sides, \( A_1, B_1, A_2, B_2, ..., A_{m+1}, B_{m+1} \). Let \( D \) be the convex hull of \( \Gamma \).

In [3], the authors find necessary and sufficient conditions on the ”lengths” of the \( A_i \) and \( B_j \) which ensure the existence of a minimal graph \( u : D \rightarrow \mathbb{R} \), taking the values \( +\infty \) on each \( A_i \) and \( -\infty \) on each \( B_j \).

They prove the graph of such a \( u \) is complete and of total curvature \(-2\pi m \). The \( \Gamma \) obtained from the \( m+2 \) roots of unity satisfies the ”length” conditions. Thus this gives examples of total curvature \(-2\pi m \) for each integer \( m \geq 1 \).
For $m = 0$, take $\Sigma = \gamma \times \mathbb{R}$, $\gamma$ a complete geodesic of $\mathbb{H}$. It would be interesting to construct non-simply connected examples of finite total curvature. For example an annulus of total curvature $-4\pi$.

2 Preliminaries

We consider $X : \Sigma \to \mathbb{H} \times \mathbb{R}$ a minimal surface conformally embedded in $\mathbb{H} \times \mathbb{R}$, $\mathbb{H}$ the hyperbolic plane. We denote by $X = (F, h)$ the immersion where $F : \Sigma \to \mathbb{H}$ is the vertical projection to $\Sigma = \Sigma \times (0)$, and $h : \Sigma \to \mathbb{R}$ the horizontal projection. We consider local conformal parameters $z = x + iy$ on $\Sigma$. The metric induced by the immersion is of the form $ds^2 = \lambda^2(z)|dz|^2$.

If $\mathbb{H}$ is isometrically embedded in $\mathbb{L}^3$ the Minkowski space, the mean curvature vector is (see B.Lawson [8], page 8)

$$2 \vec{H} = (\Delta X)^{T_H(\mathbb{H} \times \mathbb{R})} = ((\Delta F)^{T_H}, \Delta h) = 0$$

Then $F$ is a harmonic map and $h$ is a real harmonic function. In the following, we will use the unit disk model for $\mathbb{H}$. We will note $(D, \sigma^2(u)|du|^2)$ the disk with the hyperbolic metric $\sigma^2(u)|du|^2$. We will denote $|v|_\sigma^2 = \sigma^2|v|^2$, $\langle v_1, v_2 \rangle_\sigma = \sigma^2(v_1, v_2)$ where $|v|$ and $\langle v_1, v_2 \rangle$ stands for the standard norm and inner product in $\mathbb{R}^2$. The harmonic map equation in the complex coordinate $u = u_1 + iu_2$ of $D$ (see [12], page 8) is

$$F_{zz} + 2(\log \sigma \circ F)_u F_z F_{\overline{z}} = 0 \quad (1)$$

where $2(\log \sigma \circ F)_u = 2 \tilde{F}(1 - |F|^2)^{-1}$. In the theory of harmonic maps there are two global objects to consider. One is the holomorphic quadratic Hopf differential associated to $F$:

$$Q(F) = (\sigma \circ F)^2 F_z \overline{F_z}(dz)^2 := \phi(z)(dz)^2 \quad (2)$$

The function $\phi$ depends on $z$, whereas $Q(F)$ does not. An other object is the complex coefficient of dilatation (see Alhfors [1]) of a quasi-conformal map, which does not depend on $z$, a conformal parameter on $\Sigma$:

$$a = \frac{F_{\overline{z}}}{F_z}$$
Since we consider conformal immersions, we have
\[ |F_x|^2 + (h_x)^2 = |F_y|^2 + (h_y)^2 \]
\[ \langle F_x, F_y \rangle + h_x h_y = 0 \]
hence \((h_z)^2(dz)^2 = -Q(F)\) (see [10]).
Then the zeroes of \(Q\) are double and we can define \(\eta\) as the holomorphic one form \(\eta = \pm 2i\sqrt{Q}\). The sign is chosen so that:
\[ h = \text{Re} \int \eta \]  
(3)

When \(X\) is a conformal immersion then the unit normal vector \(n\) in \(\mathbb{H} \times \mathbb{R}\) has third coordinate:
\[ \langle n, \frac{\partial}{\partial t} \rangle = n_3 = \frac{|g|^2 - 1}{|g|^2 + 1} \]
where
\[ g^2 := \frac{F_z}{F_{\bar{z}}} = -\frac{1}{a} \]  
(4)

Then we define the function \(\omega\) on \(\Sigma\) (which has poles where \(\Sigma\) is horizontal) by \(n_3 = \tanh \omega\). By identification we have
\[ \omega = \frac{1}{2} \ln \frac{|F_z|}{|F_{\bar{z}}|} \]  
(5)

Using the equations above (2),(4) we can express the differential \(dF\) independently of \(z\) by:
\[ dF = F_{\bar{z}}d\bar{z} + F_z dz = \frac{1}{2\sigma \circ F} g^{-1} \eta - \frac{1}{2\sigma \circ F} g \eta \]  
(6)
The metric \(ds^2 = \lambda |dw|^2\) is given [10] in a local coordinate \(z\) by:
\[ ds^2 = (|F_z|_\sigma + |F_{\bar{z}}|_\sigma)^2 |dz|^2 \]  
(7)

Thus combining equations 6 and 7, we derive the metric in terms of \(g\) and \(\eta\) by
\[ ds^2 = \frac{1}{4} (|g|^{-1} + |g|)^2 |\eta|^2 = 4 \cosh^2 \omega |Q| \]  
(8)
We remark that the zeroes of $Q$ correspond to the poles of $\omega$ so that the immersion is well defined. Moreover the zeroes of $Q$ are points of $\Sigma$, where the tangent plane is horizontal.

It is a well known fact (see [12] page 9) that harmonic mappings satisfy the Böchner formula:

$$\triangle_0 \log \frac{|F_z|}{|F_{\bar{z}}|} = -2K \mathcal{H} J(F)$$ (9)

where $J(F) = \sigma^2 \left(|F_z|^2 - |F_{\bar{z}}|^2\right)$ is the Jacobian of $F$ with $|F_z|^2 = F_z F_{\bar{z}}$.

Hence taking into account (2), (4), (5) and (9):

$$\triangle_0 \omega = 2 \sinh(2\omega) |Q|$$ (10)

where $\triangledown_0$ denote the laplacian in the euclidean metric $dz^2$. From this we deduce

$$\Delta_{\Sigma} \omega = n_3$$

where $\Delta_{\Sigma}$ is the Laplacian in the metric $ds^2$.

The Gauss curvature is given by:

$$K_{\Sigma} = K(X_x, X_y) + K_{ext} = -\tanh^2 \omega - \frac{\|
abla \omega\|^2}{4 \cosh^2 \omega |Q|}$$

This formula follows from the fact that the sectional curvature of the tangent plane to $\Sigma$ at a point $z$ is $-n_3^2$ and the second fundamental form is

$$II = \frac{\omega_x}{\cosh \omega} (dx)^2 - \frac{\omega_x}{\cosh \omega} (dy)^2 + 2 \frac{\omega_y}{\cosh \omega} (dxdy)$$

The total curvature is defined by

$$C(\Sigma) = \int_{\Sigma} K_{\Sigma} dA$$

3 Minimal surfaces of finite total curvature

**Theorem 3.1.** Let $X$ be a complete minimal immersion of $\Sigma$ in $\mathbb{H} \times \mathbb{R}$ with finite total curvature. Then

- a) $\Sigma$ is conformally $\overline{\mathcal{M}} - \{p_1, ..., p_n\}$, a Riemann surface punctured in a finite number of points.
b) \( Q \) is holomorphic on \( \Sigma \) and extends meromorphically to each puncture. If we parameterize each puncture \( p_i \) by the exterior of a disk of radius \( R_0 \), and if \( Q(z) = z^{2m_i}(dz)^2 \) at \( p_i \) then \( m_i \geq -1 \).

c) The third coordinate of the unit normal vector \( n_3 \rightarrow 0 \) uniformly at each puncture.

d) The total curvature is a multiple of \( 2\pi \):

\[
\int (-KdA) = 2\pi(2 - 2g - 2k - \sum_{i=1}^{n} m_i)
\]

**Proof.** The proof of this theorem uses arguments of harmonic diffeomorphisms theory as can be found in the work of Han, Tam, Treibergs and Wan [4], [13], [5] and Minsky [11].

The conformal type is an application of Huber’s theorem ([7]). \( \Sigma \) is conformally a compact Riemann surface minus a finite number of points (the ends).

We consider \( M(r_0) = M - \cup_i D(p_i, r_0) \); the surface minus a finite number of disks removed around the punctures \( p_i \). Around each puncture we consider a conformal parametrization of the punctured disk \( D^*(p_i, r_0) \). We parametrize these ends by the exterior of the disk of radius \( R_0 \) in \( \mathbb{C} \). In this parameter we express the metric as \( ds^2 = \lambda^2|dz|^2 \) with \( \lambda^2 = 4\cosh^2\omega|\phi| \) in a conformal parameter \( z \). Then \( -K\lambda^2 = 2\Delta_0 \ln \lambda \) where \( \Delta_0 = 4\partial^2_{zz} \).

Let us define \( u = \ln \cosh^2 \omega \), a subharmonic function by Böchner’s formula:

\[
\Delta_0 u = 8\sinh^2 \omega|\phi| + \frac{2|\nabla \omega|^2}{\cosh^2 \omega} \geq 0
\]

The function \( u \) is globally defined, since \( \omega \) is globally defined on \( \Sigma \).

**Step 1:** We prove that the holomorphic quadratic differential \( Q \) has a finite number of zeroes on \( \Sigma \).

Since the zeroes are isolated, \( Q \) has a finite number of zeroes on the compact part \( M(r) \). Then we assume that there is a disk \( D^*(p_i, r) \) which contains an infinite number of zeroes of \( Q \), \( \{z_i\} \). We parametrize conformally this disk on the exterior of the disk of radius \( R_0 \). In this parameter \( Q(z) = \phi(z)(dz)^2 \) and if \( \Delta_0 \) is the laplacian in the flat metric \( |dz|^2 \) at the puncture:

\[
\Delta_0 \ln |\phi| = \sum 2\pi \delta_{z_i}
\]
Then with $-K\lambda^2 - \frac{1}{2}\Delta_0 \ln |\phi| = \frac{1}{2}\Delta_0 u$ we have on the annulus $C(R) = \{R_0 \leq |z| \leq R\}$:

$$\int_{C(R)}(-KdA) - m\pi = \frac{1}{2} \int_{C(R)} \Delta_0 u \geq 0$$

Then $m$ has to be finite and $\int_{C(r)} \Delta_0 u \leq C_0$

Step 2: An upper bound.

$$\int_{C(R)}\Delta_0 u = \int_{\partial C(R)} \frac{\partial u}{\partial n} = \int_0^{2\pi} \frac{\partial u}{\partial R} Rd\theta - \int_0^{2\pi} \frac{\partial u}{\partial R} R_0 d\theta$$

$$= R \frac{d}{dR} \int_0^{2\pi} u(R, \theta) d\theta - R_0 \int_0^{2\pi} \frac{\partial u}{\partial r} d\theta \leq 2C_0$$

Now let $I(r) := \int_0^{2\pi} u(r, \theta) d\theta$. Then

$$\frac{d}{dR} I(R) \leq \frac{C_1}{R}$$

$$I(R) - I(R_0) \leq 2C_1 \ln \frac{R}{R_0}$$

Then for $R >> R_0$ large we have, with $a > 0, b > 0$:

$$I(r) \leq a \ln r + b$$

Step 3: Since $\phi$ has a finite number of zeroes, we prove at each puncture

$$\cosh^2 \omega |\phi| \leq \beta |z|^\alpha |\phi|.$$ To prove $\phi$ extends meromorphically to the punctures, we will use a theorem of Osserman [9] (recall that the metric is complete). For $R > R_0$, $\phi$ is without zeroes and for $|z| = R$ large enough, $u$ is subharmonic, hence

$$u(z) \leq \frac{4}{\pi R^2} \int_{B(z,R/2)} u$$

$$\leq \frac{4}{\pi |z|^2} \int_{B(0,3|z|/2)-B(0,|z|/2)} u$$

$$\leq \frac{4}{\pi |z|^2} \int_{|z|/2}^{3|z|/2} I(r) r dr \leq \frac{4}{\pi |z|^2} \int_{|z|/2}^{3|z|/2} (a \ln r + b) r dr$$

$$\leq a \ln |z| + \beta$$
Then

\[ 2 \ln \lambda = u + \ln |\phi| \leq \alpha \ln |z| + \beta + \ln |\phi| \]

and

\[ \lambda^2 = \cosh^2 \omega |\phi| \leq e^\beta |z|^\alpha |\phi| \]

Thus the function \( \phi \) extends meromorphically to the puncture by Osserman [9].

Step 4: We now prove that the function \( \phi(z) \equiv z^{2m} \), with \( m \geq -1 \) at each puncture.

If \( m \leq -2 \), then we can conformally parametrize the end on the punctured disk by \( w = 1/z \). Then \( Q(w) = \psi(w)(dw)^2 \) with \( \phi(1/w) = w^4 \psi(w) \), where \( \psi(w) \) has a pole of order \( 2m + 4 \). If \( 2m + 4 \leq 0 \), the following integral is finite:

\[ \int_{D(p,r)} |\phi|dz < \infty \]

Now, by the finite total curvature hypothesis we will show the area of the end is finite:

\[ \int_D K_\Sigma dA = \int_D 2\Delta_0 \ln \lambda = \int_D 8 \sinh^2 \omega |\phi| + \int_D \frac{2|\nabla \omega|^2}{\cosh^2 \omega} \]

\[ = \int_D 8 \cosh^2 \omega |\phi| - \int_D 8 |\phi| + \int_D \frac{2|\nabla \omega|^2}{\cosh^2 \omega} \]

Hence

\[ \text{Area}(D) = \int_D \cosh^2 \omega |\phi| < \infty. \]

But a complete end of \( \Sigma \) has infinite area by the monotonicity formula (see [2]).

c) Now we prove that \( n_3 \to 0 \) uniformly at each puncture. We adapt estimates on positive solutions of \( \sinh \)-Gordon equations by Minsky [11], Wan [13] and Han [5] to our context.
At each puncture we can choose $R_0$ such that $\phi(z)$ is without zeroes on $|z| \geq R_0/2$. Since $\phi(z)$ is without zeroes, the minimal surface is transverse to horizontal sections $\mathbb{H} \times \{c\}$ and we parametrize locally simply connected subdomains of the end by $w = \frac{i}{2}(x_3 + ix^*_3) = \int \sqrt{\phi}dz$ so that $|dw|^2 = |\phi(z)||dz|^2$ is a flat metric. If we consider $z \in C_{R_0}$, then on the disk $D(z, |z|/2)$, we have the conformal coordinate $w = \sqrt{\phi(z)}dz$, with the flat metric $|dw|^2 = |\phi||dz|^2$. In this metric, under the hypothesis that $m \geq -1$, the disk $D(z, |z|/2)$ contains a ball of radius at least $c \ln |z|$, where $c$ is independent of $z$.

The function $\omega$ satisfies the $\sinh$-Gordon equation

$$\Delta_{|\phi|} \omega = 2 \sinh 2\omega$$

where $\Delta_{|\phi|}$ is the laplacian in the flat metric $|dw|^2$. For $|z| \geq R_0$, we can find a disk of radius at least $r = c \ln |z|$ around $z$ in the $|dw|^2$ metric. When $z$ is large, the radius $r$ diverges to $+\infty$.

Then for $R_0$ large enough we can find a disk with radius 1 in the $|dw|^2$ metric around any point $z$ with $|z| \geq R_0$. On this disk $D_{|\phi|}(z, 1)$, we consider the hyperbolic metric given by ($w$ is $w - z$ in the following step):

$$d\sigma^2 = \mu^2|dw|^2 = \frac{4}{(1 - |w|^2)^2}|dw|^2$$

Then $\mu$ take infinite values on $\partial D_{|\phi|}(z, 1)$ and since the curvature of this metric is $K = -1$, the function $\omega_2 = \ln \mu$ satisfies the equation

$$\Delta_{|\phi|} \omega_2 = e^{2\omega_2} \geq e^{2\omega_2} - e^{-2\omega_2} = 2 \sinh 2\omega_2$$

Now we apply a maximum principle to bound $\omega$ above as in Wan [13]. The same holds with ($\tilde{\omega} = -\omega$):

Let $\eta = \omega - \omega_2$. Then

$$\Delta \eta = e^{2\omega} - e^{-2\omega} - e^{2\omega_2} = e^{2\omega_2}(e^{2\eta} - e^{-2\omega_2}e^{-2\eta} - 1)$$

which can be written in the metric $d\tilde{\sigma}^2 = e^{2\omega_2}|dw|^2$, as

$$\Delta \tilde{\sigma} \eta = e^{2\eta} - e^{-4\omega_2}e^{-2\eta} - 1$$

Since $\omega_2$ goes to $+\infty$ on the boundary of the disk, the function $\eta$ is bounded above and attains its max at an interior point $p_0$, $\eta(p_0) = \bar{\eta}$ and $\Delta \bar{\eta} \leq 0$. At this point we have

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\[ e^{2\eta} - e^{-2\omega} e^{-2\eta} \leq 1 \]

Hence
\[ e^{2\eta} \leq \frac{1 + \sqrt{1 + a^2}}{2} \]

where \( a = e^{-2\omega(p_0)} \leq \sup \frac{1}{\mu} \leq \frac{1}{3} \). Then at any point of the disk
\[ \omega \leq \omega_2 + \frac{1}{2} \ln \frac{1 + \sqrt{1 + 1/4}}{2} \]

The same estimate holds with \( \tilde{\omega} = -\omega \). Then at \( z \) (i.e. \( w = 0 \)) we have
\[ |\omega(z)| \leq \ln 4 + \frac{1}{2} \ln \frac{1 + \sqrt{1 + 1/4}}{2} := K_0 \]

uniformly on \( R \geq R_0 \). Using this estimate we can apply a maximum principle as in Minsky [11]. For \( |z| \) large, we can find a disk \( D_{\phi}(z, r) \) with \( r \) large too.

We consider the function
\[ F(x, y) = \frac{K_0}{\cosh r} \cosh \sqrt{2}x \cosh \sqrt{2}y \]

Then \( F \geq K_0 \geq \omega \) on \( \partial D_{\phi}(z, r) \). Since \( \Delta F = 4F \), we apply the maximum principle to have \( \omega \leq F \). If \( p_0 \) is a point where \( \omega(p_0) \geq F(p_0) \) is a minimum of \( F - \omega \), then \( 0 \leq \omega(p_0) \leq \sinh \omega(p_0) \) and
\[ \Delta(F - \omega) = 4F - 2 \sinh 2\omega \leq 4(F(p_0) - \omega(p_0)) \leq 0 \]

Hence \( \omega \leq F \) on the disk. We have \( |\omega| \leq F \) by considering the same argument with \( F + \omega \). Hence
\[ |\omega(z)| \leq \frac{K_0}{\cosh r} \]

And \( |\omega| \to 0 \) uniformly at the puncture i.e. the tangent plane become vertical.

d) Now we compute the total curvature. We apply Gauss-Bonnet on the compact piece \( M(r) = M - \sum_{1 \leq i \leq k} D(p_i, r) \) and we obtain
\[ \int_{M(r)} K_2 dA + \int_{\partial M(r)} k_g = 2\pi(2 - 2g - k) \]
Here $k_g$ is the geodesic curvature of $\partial M(r) = \Gamma_1 \cup \ldots \Gamma_k$ on the surface $M(r)$. Now consider a puncture $p_i$ parameterized on $R \geq R_0$. We consider $w = x + iy$ a parametrization of the punctured disk (with $w = \int \sqrt{\phi} dz$). In the $w$-plane, if $\phi(z) = z^{2m}$ there are $2m + 2$ horizontal asymptotic directions i.e. directions with $\text{Im}(w) = 0$ (diverging curves at zero level) which define some angular sector in $R \geq R_0$. Now for $C_1 \gg 0$ large, we consider the "polygon" $\Gamma(C_1)$ which is the union of segments of curves $\text{Re}(w) = \pm C_1$ and $\text{Im}(w) = \pm C_1$, alternatively. At each change of direction the exterior angle is $\pi/2$. These curves, with $\Gamma_i = \{R = R_0\}$ bound an annulus $\Omega(r, C_1, p_i)$ and

$$\int_{\Omega} K_S dA + \int_{\Gamma(C_1)} k_g - \int_{\Gamma_i} k_g = -(2m + 2)\pi$$

Now we let $C_1 \to 0$. If we prove $\int_{\Gamma(C_1)} k_g \to 0$, we will establish that

$$\int_M K_S dA = 2\pi (2 - 2g - 2k - \sum_i m_i)$$

where $\phi(z) = z^{2m_i}$ at each $p_i$.

Now we prove $\int_{\Gamma(C_1)} k_g \to 0$. This fact comes from the exponential decreasing property of the function $\omega$. First we prove

$$\int_{\text{Im}(w) = C_1} k_g ds \to 0$$

The curve $\text{Im}(w) = C_1$ is a horizontal curve at level $C_1$, parameterized by $\text{Re}(w) = x$. In Hauswirth [6], we find an expression of the curvature of the curve as function of $\omega$. In the $w$ variable (recall that the Hopf differential is $Q = \frac{1}{4}(dw)^2$):

$$k_g(x) = \frac{-\omega_y}{\cosh \omega}$$

Now we need a gradient estimative of $\omega$. Schauder’s estimate gives (with the exponential decreasing property of $\omega$ proved above):

$$|u|_{2,\alpha} \leq C(|\sinh \omega|_{0,\alpha} + |\omega|_0) \leq Ce^{-R}.$$ 

On the curve $x + iC_1$, we have $|\nabla \omega| \leq Ce^{-|C_1|}e^{-\sqrt{2}C_1^{2}+1}$ and

$$\int_{\text{Im}(w) = C_1} |k_g| ds = \int_{-\infty}^{+\infty} |\omega_y| dx \leq C|C_1|e^{-|C_1|}$$
which is converging to zero as $|C_1| \to +\infty$. Now we prove

$$\int_{\text{Re}(w)=C_1} k_g ds \to 0.$$ 

Now suppose the curve is not horizontal. If $\gamma(y) = (F(C_1, y), y)$ where $F$ is the harmonic map into $\mathbb{H}$, we have $\gamma'(y) = (F_y, 1)$ and $|\gamma'(y)|^2 = 1 + \sinh^2 \omega$ (conformal coordinate). Then the curvature of the curve in $\mathbb{H} \times \mathbb{R}$ is horizontal. This curvature can be expressed as a function of $\omega$ by

$$k_g(y) = \frac{\omega_x}{\sinh \omega}$$

Now one can argue as above to prove the result.

References


[8] B. Lawson. *Lectures on minimal submanifolds*; T1, Mathematics Lecture Series; 009, Publish or Perish.


