Complete Constant Mean Curvature surfaces in homogeneous spaces

José M. Espinar†, Harold Rosenberg ‡

† Institut de Mathématiques, Université Paris VII, 175 Rue du Chevaleret, 75013 Paris, France; e-mail: jespinar@ugr.es
‡ Instituto de Matematica Pura y Aplicada, 110 Estrada Dona Castorina, Rio de Janeiro 22460-320, Brazil; e-mail: rosen@impa.br

Abstract

In this paper we classify complete surfaces of constant mean curvature whose Gaussian curvature does not change sign in a simply connected homogeneous manifold with a 4-dimensional isometry group.

1 Introduction

In 1966, T. Klotz and R. Ossermann showed the following:

Theorem [KO]: A complete $H-$surface in $\mathbb{R}^3$ whose Gaussian curvature $K$ does not change sign is either a sphere, a minimal surface, or a right circular cylinder.

The above result was extended to $S^3$ by D. Hoffman [H], and to $\mathbb{H}^3$ by R. Tribuzy [T] with an extra hypothesis if $K$ is non-positive. The additional hypothesis says that, when $K \leq 0$, one has $H^2 - K - 1 > 0$.

In recent years, the study of $H-$surfaces in product spaces and, more generally, in a homogeneous three-manifold with a 4-dimensional isometry group is quite active (see [AR, AR2], [CoR], [ER], [FM, FM2], [DH] and references therein).

The aim of this paper is to extend the above Theorem to homogeneous spaces with a 4-dimensional isometry group. These homogeneous space are denoted by $\mathbb{E}(\kappa, \tau)$, where $\kappa$ and $\tau$ are constant and $\kappa - 4\tau^2 \neq 0$. They can be classified as $\mathbb{M}^2(\kappa) \times \mathbb{R}$ if $\tau = 0$, with $\mathbb{M}^2(\kappa) =$

---

1The author is partially supported by Spanish MEC-FEDER Grant MTM2010-19821, and Regional J. Andalucia Grants P06-FQM-01642 and FQM325
\(S^2(\kappa)\) if \(\kappa > 0\) \((S^2(\kappa)\) the sphere of curvature \(\kappa)\), and \(M^2(\kappa) = \mathbb{H}^2(\kappa)\) if \(\kappa < 0\) \((\mathbb{H}^2(\kappa)\) the hyperbolic plane of curvature \(\kappa)\). If \(\tau\) is not equal to zero, \(E(\kappa, \tau)\) is a Berger sphere if \(\kappa > 0\), a Heisenberg space if \(\kappa = 0\) \((\text{of bundle curvature } \tau)\), and the universal cover of \(\text{PSL}(2, \mathbb{R})\) if \(\kappa < 0\). Henceforth we will suppose \(\kappa\) is plus or minus one or zero.

The paper is organized as follows. In Section 2, we establish the definitions and necessary equations for an \(H\)–surface. We also state here two classification results for \(H\)–surfaces. We prove them in Section 5 and Section 6 for the sake of completeness.

Section 3 is devoted to the classification of \(H\)–surfaces with non-negative Gaussian curvature,

**Theorem 3.1.** Let \(\Sigma \subset E(\kappa, \tau)\) be a complete \(H\)–surface with \(K \geq 0\). Then, \(\Sigma\) is either a rotational sphere \((\text{in particular, } 4H^2 + \kappa > 0)\), or a complete vertical cylinder over a complete curve of geodesic curvature \(2H\) on \(M^2(\kappa)\).

In Section 4 we continue with the classification of \(H\)–surfaces with non-positive Gaussian curvature.

**Theorem 4.1.** Let \(\Sigma \subset E(\kappa, \tau)\) be a complete \(H\)–surface with \(K \leq 0\) and \(H^2 + \tau^2 - |\kappa - 4\tau^2| > 0\). Then, \(\Sigma\) is a complete vertical cylinder over a complete curve of geodesic curvature \(2H\) on \(M^2(\kappa)\).

The above theorem is not true without the inequality; for example, any complete minimal surface in \(\mathbb{H}^2 \times \mathbb{R}\) that is not a vertical cylinder.

In the Appendix, we give a result, which we think is of independent interest, concerning differential operators on a Riemannian surface \(\Sigma\) of the form \(\Delta + g\), acting on \(C^2(\Sigma)\)–functions, where \(\Delta\) is the Laplacian with respect to the Riemannian metric on \(\Sigma\) and \(g \in C^0(\Sigma)\).

## 2 The geometry of surfaces in homogeneous spaces

Henceforth \(E(\kappa, \tau)\) denotes a complete simply connected homogeneous three-manifold with 4–dimensional isometry group. Such a three-manifold can be classified in terms of a pair of real numbers \((\kappa, \tau)\) satisfying \(\kappa - 4\tau^2 \neq 0\). In fact, these manifolds are Riemannian submersions over a complete simply-connected surface \(M^2(\kappa)\) of constant curvature \(\kappa\), \(\pi : E(\kappa, \tau) \longrightarrow M^2(\kappa)\), and translations along the fibers are isometries, therefore they generate a Killing field \(\xi\), called the \textit{vertical field}. Moreover, \(\tau\) is the real number such that \(\nabla_X \xi = \tau X \wedge \xi\) for all vector fields \(X\) on the manifold. Here, \(\nabla\) is the Levi-Civita connection of the manifold and \(\wedge\) is the cross product.

Let \(\Sigma\) be a complete \(H\)–surface immersed in \(E(\kappa, \tau)\). By passing to a 2–sheeted covering space of \(\Sigma\), we can assume \(\Sigma\) is orientable. Let \(N\) be a unit normal to \(\Sigma\). In terms of a conformal
parameter \( z \) of \( \Sigma \), the first, \( \langle \cdot, \cdot \rangle \), and second, \( II \), fundamental forms are given by
\[
\langle \cdot, \cdot \rangle = \lambda |dz|^2 \\
II = p dz^2 + \lambda H \, |dz|^2 + \overline{p} \, dz^2,
\] (2.1)
where \( p \, dz^2 = \langle -\nabla_{\partial_z} N, \partial_z \rangle \, dz^2 \) is the Hopf differential of \( \Sigma \).

Set \( \nu = \langle N, \xi \rangle \) and \( T = \xi - \nu N \), i.e., \( \nu \) is the normal component of the vertical field \( \xi \), called the angle function, and \( T \) is the tangent component of the vertical field.

First we state the following necessary equations on \( \Sigma \) which were obtained in [FM].

**Lemma 2.1.** Given an immersed surface \( \Sigma \subset \mathbb{E}(\kappa, \tau) \), the following equations are satisfied:
\[
\begin{align*}
K &= K_e + \tau^2 + (\kappa - 4\tau^2) \nu^2 \quad (2.2) \\
p_{\bar{z}} &= \frac{\lambda}{2} (H_{\bar{z}} + (\kappa - 4\tau^2) \nu A) \quad (2.3) \\
A_{\bar{z}} &= \frac{\lambda}{2} (H + i\tau) \nu \quad (2.4) \\
\nu_z &= -(H - i\tau) A - \frac{2}{\lambda} p A \quad (2.5) \\
|A|^2 &= \frac{1}{4} \lambda (1 - \nu^2) \quad (2.6) \\
A_z &= \frac{\lambda}{\lambda} A + p \nu \quad (2.7)
\end{align*}
\]
where \( A = \langle \xi, \partial_{\bar{z}} \rangle \), \( K_e \) the extrinsic curvature and \( K \) the Gauss curvature of \( \Sigma \).

For an immersed \( H \)-surface \( \Sigma \subset \mathbb{E}(\kappa, \tau) \) there is a globally defined quadratic differential, called the Abresch-Rosenberg differential, which in these coordinates is given by (see [AR2]):
\[
Q \, dz^2 = (2(H + i\tau) p - (\kappa - 4\tau^2) A^2) \, dz^2,
\]
following the notation above.

It is not hard to verify this quadratic differential is holomorphic on an \( H \)-surface using (2.3) and (2.4).

**Theorem 2.1** ([AR],[AR2]). \( Q \, dz^2 \) is a holomorphic quadratic differential on any \( H \)-surface in \( \mathbb{E}(\kappa, \tau) \).

Associated to the Abresch-Rosenberg differential we define the smooth function \( q : \Sigma \longrightarrow [0, +\infty) \) given by
\[
q = \frac{4|Q|^2}{\lambda^2}.
\]
By means of Theorem 2.1, \( q \) either has isolated zeroes or vanishes identically. Note that \( q \) does not depend on the conformal parameter \( z \), hence \( q \) is globally defined on \( \Sigma \).

We continue this Section establishing some formulae relating the angle function, \( q \) and the Gaussian curvature.
Lemma 2.2. Let $\Sigma$ be an $H$–surface immersed in $E(\kappa, \tau)$. Then the following equations are satisfied:

$$\| \nabla \nu \|^2 = \frac{4H^2 + \kappa - (\kappa - 4\tau^2) \nu^2}{4(\kappa - 4\tau^2)} \left(4(H^2 - K_e) + (\kappa - 4\tau^2)(1 - \nu^2) \right) - \frac{q}{\kappa - 4\tau^2}$$

(2.8)

$$\Delta \nu = - \left(4H^2 + 2\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2) - 2K_e \right) \nu.$$  

(2.9)

Moreover, away from the isolated zeroes of $q$, we have

$$\Delta \ln q = 4K.$$  

(2.10)

Proof. From (2.5)

$$|\nu_z|^2 = \frac{4|p|^2 |A|^2}{\lambda^2} + (H^2 + \tau^2)|A|^2 + \frac{2(H + i\tau)}{\lambda} pA - \frac{2(H - i\tau)}{\lambda} \overline{p}A^2,$$

and taking into account that

$$|Q|^2 = 4 \left(H^2 + \tau^2 \right) |p|^2 + (\kappa - 4\tau^2)|A|^4 - (\kappa - 4\tau^2) \left(2(H + i\tau)pA + 2(H - i\tau)\overline{p}A^2 \right),$$

we obtain, using also (2.6), that

$$|\nu_z|^2 = (H^2 + \tau^2)|A|^2 + (H^2 - K_e)|A|^2 + (\kappa - 4\tau^2) \frac{|A|^4}{\lambda}$$

$$+ 4 \left( \frac{H^2 + \tau^2}{\kappa - 4\tau^2} \right) \frac{|p|^2}{\lambda} - \frac{|Q|^2}{(\kappa - 4\tau^2)\lambda}$$

where we have used that $4|p|^2 = \lambda^2(H^2 - K_e)$ and $\kappa - 4\tau^2 \neq 0$. Thus

$$\| \nabla \nu \|^2 = \frac{4}{\lambda} |\nu_z|^2 = \left(2H^2 - K_e + \tau^2 \right)(1 - \nu^2) + \frac{\kappa - 4\tau^2}{4}(1 - \nu^2)^2$$

$$+ 4 \left( \frac{H^2 + \tau^2}{\kappa - 4\tau^2} \right) \left(H^2 - K_e \right) - \frac{q}{\kappa - 4\tau^2},$$

finally, re-ordering in terms of $H^2 - K_e$ we have the expression.

On the other hand, by differentiating (2.5) with respect to $\bar{z}$ and using (2.7), (2.4) and (2.3), one gets

$$\nu_{\bar{z}} = - (\kappa - 4\tau^2) \nu |A|^2 - \frac{2}{\lambda} |p|^2 \nu - \frac{H^2 + \tau^2}{2} \lambda \nu.$$

Then, from (2.6),

$$\nu_{\bar{z}} = - \frac{\lambda \nu}{4} \left( (\kappa - 4\tau^2)(1 - \nu^2) + \frac{8|p|^2}{\lambda^2} + 2(H^2 + \tau^2) \right).$$

4
thus
\[ \Delta \nu = \frac{4}{\lambda} \nu_{\bar{z}z} = - ((\kappa - 4\tau^2)(1 - \nu^2) + 2(H^2 - K_e) + 2(H^2 + \tau^2)) \nu. \]

Finally,
\[ \Delta \ln q = \Delta \ln \frac{4|Q|^2}{\lambda^2} = -2\Delta \ln \lambda = 4K, \]
where we have used that $Qdz^2$ is holomorphic and the expression of the Gaussian curvature in terms of a conformal parameter.

\textbf{Remark 2.1.} Note that (2.9) is nothing but the Jacobi equation for the Jacobi field $\nu$.

Next, we recall a definition in these homogeneous spaces.

\textbf{Definition 2.1.} We say that $\Sigma \subset \mathbb{E}(\kappa, \tau)$ is a vertical cylinder over $\alpha$ if $\Sigma = \pi^{-1}(\alpha)$, where $\alpha$ is a curve on $\mathbb{M}^2(\kappa)$.

It is not hard to verify that if $\alpha$ is a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$, then $\Sigma = \pi^{-1}(\alpha)$ is complete and has constant mean curvature $H$. Moreover, these cylinders are characterized by $\nu \equiv 0$.

We now state two results about the classification of $H$–surfaces. They will be used in Sections 3 and 4, but we prove them in Section 5 and Section 6 for the sake of clarity. The first one concerns $H$–surfaces for which the angle function is constant. However, we need to introduce a family of surfaces that appear in the classification:

\textbf{Definition 2.2.} Let $S_{\kappa, \tau}$ be a family of complete $H$–surfaces, in $\mathbb{E}(\kappa, \tau)$, $\kappa < 0$, satisfying for any $\Sigma \in S_{\kappa, \tau}$:

- $4H^2 + \kappa < 0$.
- $q$ vanishes identically on $\Sigma \in S_{\kappa, \tau}$, i.e., $\Sigma$ is invariant by a one parameter family of isometries.
- $0 < \nu^2 < 1$ is constant along $\Sigma$.
- $K_e = -\tau^2$ and $K = (\kappa - 4\tau^2)\nu^2 < 0$ are constants along $\Sigma$.

An anonymous referee indicated to us the preprint "Hypersurfaces with a parallel higher fundamental form", by S. Verpoort who observed that we mistakenly omitted the surfaces $S_{\kappa, \tau}$ in a first draft of this paper.

\textbf{Theorem 2.2.} Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H$–surface with constant angle function. Then $\Sigma$ is either a vertical cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$, a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$, or $\Sigma \in S_{\kappa, \tau}$ with $\kappa < 0$.  

5
Remark 2.2. Theorem 2.2 improves [ER, Lemma 2.3] for surfaces in $H^2 \times \mathbb{R}$.

Of special interest for us are those $H$–surfaces for which the Abresch-Rosenberg differential is constant.

Theorem 2.3. Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H$–surface with $q$ constant.

- If $q = 0$ on $\Sigma$, then $\Sigma$ is either a slice in $H^2 \times \mathbb{R}$ or $S^2 \times \mathbb{R}$ if $H = 0 = \tau$, or $\Sigma$ is invariant by a one-parameter group of isometries of $\mathbb{E}(\kappa, \tau)$.

Moreover, the Gauss curvature of these examples is

- If $4H^2 + \kappa > 0$, then $K > 0$ they are the rotationally invariant spheres.
- If $4H^2 + \kappa = 0$ and $\nu \equiv 0$, then $K \equiv 0$ and $\Sigma$ is either a vertical plane in $\text{Nil}_3$, or a vertical cylinder over a horocycle in $H^2 \times \mathbb{R}$ or $\text{PSL}(2, \mathbb{C})$.
- There exists a point with negative Gauss curvature in the remaining cases.

- If $q \neq 0$ on $\Sigma$, then $\Sigma$ is a vertical cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$.

3 Complete $H$–surfaces $\Sigma$ with $K \geq 0$

Here we prove

Theorem 3.1. Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H$–surface with $K \geq 0$. Then, $\Sigma$ is either a rotational sphere (in particular, $4H^2 + \kappa > 0$), or a complete vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.

Proof. The proof goes as follows: First, we prove that $\Sigma$ is a topological sphere or a complete non-compact parabolic surface. We show that when the surface is a topological sphere then it is a rotational sphere. If $\Sigma$ is a complete non-compact parabolic surface, we prove that it is a vertical cylinder by means of Theorem 2.3.

Since $K \geq 0$ and $\Sigma$ is complete, [KO, Lemma 5] implies that $\Sigma$ is either a sphere or non-compact and parabolic.

If $\Sigma$ is a sphere, then it is a rotational example (see [AR2] or [AR]). Thus, we can assume that $\Sigma$ is non-compact and parabolic.

We can assume that $q$ does not vanish identically in $\Sigma$. If $q$ does vanish, then $\Sigma$ is either a vertical cylinder over a straight line in $\text{Nil}_3$ or a vertical cylinder over a horocycle in $H^2 \times \mathbb{R}$ or $\text{PSL}(2, \mathbb{C})$. Note that we have used here that $K \geq 0$ and Theorem 2.3.

On the one hand, from the Gauss equation (2.2)

$$0 \leq K = K_e + \tau^2 + (\kappa - 4\tau^2)\nu^2 \leq K_e + \tau^2 + |\kappa - 4\tau^2|,$$

6
then
\[ H^2 - K_e \leq H^2 + \tau^2 + |\kappa - 4\tau^2|. \] (3.1)

On the other hand, using the very definition of \( Q dz^2 \), (3.1) and the inequality \(|\xi_1 + \xi_2|^2 \leq 2(|\xi_1|^2 + |\xi_2|^2)\) for \( \xi_1, \xi_2 \in \mathbb{C} \), we obtain
\[
\frac{q}{2} = \frac{2|Q|^2}{\lambda^2} \leq 4(H^2 + \tau^2)\frac{|p|^2}{\lambda^2} + (\kappa - 4\tau^2)^2 \frac{4|A|^2}{\lambda^2} \\
= 4(H^2 + \tau^2)(H^2 - K_e) + \frac{(\kappa - 4\tau^2)^2}{4} (1 - \nu^2)^2 \\
\leq 4(H^2 + \tau^2)(H^2 - K_e) + \frac{(\kappa - 4\tau^2)^2}{4} \\
\leq 4(H^2 + \tau^2)(H^2 + \tau^2 + |\kappa - 4\tau^2|) + \frac{(\kappa - 4\tau^2)^2}{4}.
\]

So, from (2.10), \( \Delta \ln q = 4K \geq 0 \) and \( \ln q \) is a bounded subharmonic function on a non-compact parabolic surface \( \Sigma \) and since the value \(-\infty\) is allowed at isolated points (see [AS]), \( q \) is a positive constant (recall that we are assuming that \( q \) does not vanishes identically). Therefore, Theorem 2.3 gives the result.

\[ \square \]

4 Complete \( H^- \)surfaces \( \Sigma \) with \( K \leq 0 \)

**Theorem 4.1.** Let \( \Sigma \subset \mathbb{E}(\kappa, \tau) \) be a complete \( H^- \)surface with \( K \leq 0 \) and \( H^2 + \tau^2 - |\kappa - 4\tau^2| > 0 \). Then, \( \Sigma \) is a complete vertical cylinder over a complete curve of geodesic curvature \( 2H \) on \( \mathbb{M}^2(\kappa) \).

**Proof.** We divide the proof in two cases, \( \kappa - 4\tau^2 < 0 \) and \( \kappa - 4\tau^2 > 0 \).

**Case** \( \kappa - 4\tau^2 < 0 \):

On the one hand, since \( K \leq 0 \), we have
\[ H^2 - K_e \geq H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2 \geq H^2 + \kappa - 3\tau^2, \]
from the Gauss Equation (2.2). Therefore, from (2.8) and \( \kappa - 4\tau^2 < 0 \), we obtain:
\[
q \geq 4(H^2 + \tau^2)(H^2 - K_e) + (\kappa - 4\tau^2)(1 - \nu^2) \left( H^2 + \tau^2 + H^2 - K_e + \frac{\kappa - 4\tau^2}{4} (1 - \nu^2) \right) \\
= (H^2 - K_e) \left( 4H^2 + 4\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2) \right) \\
+ (H^2 + \tau^2)(\kappa - 4\tau^2)(1 - \nu^2) + \frac{(\kappa - 4\tau^2)^2}{4} (1 - \nu^2)^2 \\
\geq (H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2) \left( 4H^2 + 4\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2) \right) \\
+ (H^2 + \tau^2)(\kappa - 4\tau^2)(1 - \nu^2) + \frac{(\kappa - 4\tau^2)^2}{4} (1 - \nu^2)^2,
\]
note that the last inequality holds since \(4H^2 + 4\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2) \geq 4H^2 + \kappa > 0\). \(4H^2 + \kappa > 0\) follows from
\[
0 < 4(H^2 + \tau^2) - |\kappa - 4\tau^2| = 4H^2 + \kappa.
\]

Set \(a := H^2 + \tau^2\) and \(b := \kappa - 4\tau^2\). Define the real smooth function \(f : [-1, 1] \to \mathbb{R}\) as
\[
f(x) = (a + bx^2)(4a + b(1 - x^2)) + ab(1 - x^2) + \frac{b^2}{4}(1 - x^2)^2.
\]
(4.1)

Note that \(q \geq f(\nu)\) on \(\Sigma\). \(f(\nu)\) is just the last part in the above inequality involving \(q\). It is easy to verify that the only critical point of \(f\) in \((-1, 1)\) is \(x = 0\). Moreover,
\[
f(0) = (4a + b)^2/4 > 0 \quad \text{and} \quad f(\pm 1) = 4a(a + b) > 0.
\]

Actually, \(f : \mathbb{R} \to \mathbb{R}\) has two others critical points, \(x = \pm \sqrt{\frac{4a + b}{3|b|}}\), but here, we have used that
\[
\frac{4a + b}{3|b|} > 1,
\]
since
\[
0 < 4(H^2 + \kappa - 3\tau^2) = (4H^2 + \kappa) - 3|\kappa - 4\tau^2| = (4a + b) - 3|b|.
\]
So, set \(c = \min \{f(0), f(\pm 1)\} > 0\), then
\[
q \geq f(\nu) \geq c > 0.
\]

Now, from (2.10) and \(q \geq c > 0\) on \(\Sigma\), it follows that \(ds^2 = \sqrt{q}I\) is a complete flat metric on \(\Sigma\) and
\[
\Delta ds^2 \ln q = \frac{1}{\sqrt{q}} \Delta \ln q = \frac{4K}{\sqrt{q}} \leq 0.
\]

Since \(q\) is bounded below by a positive constant and \((\Sigma, ds^2)\) is parabolic, then \(\ln q\) is constant which implies that \(q\) is a positive constant (recall \(q\) is bounded below by a positive constant). Thus, the result follows from Theorem 2.3. The case \(\kappa - 4\tau^2 < 0\) is proved.

**Case \(\kappa - 4\tau^2 > 0\):**

Set \(w_1 := 2(H + i\tau)\frac{p}{\lambda}\) and \(w_2 := (\kappa - 4\tau^2)\frac{A^2}{\lambda}\), i.e., \(q = 4|w_1 - w_2|^2\). Then
\[
|w_1|^2 = (H^2 + \tau^2)(H^2 - K_e) \geq (H^2 + \tau^2)^2
\]
\[
|w_2|^2 = \frac{(\kappa - 4\tau^2)^2}{16}(1 - \nu^2)^2 \leq \left(\frac{\kappa - 4\tau^2}{4}\right)^2,
\]
where we have used that \(H^2 - K_e \geq H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2 \geq H^2 + \tau^2\), since \(K \leq 0\) and \(\kappa - 4\tau^2 > 0\).
Let us recall a well known inequality for complex numbers, let \( \xi_1, \xi_2 \in \mathbb{C} \) then \( |\xi_1 + \xi_2|^2 \geq ||\xi_1| - |\xi_2||^2 \). Thus,

\[
\frac{1}{4} q \geq ||w_1| - |w_2||^2 \geq \left| (H^2 + \tau^2) - \frac{|\kappa - 4\tau^2|}{4} \right|^2
\]

\[
= \frac{1}{16} \left| 4(H^2 + \tau^2) - |\kappa - 4\tau^2| \right|^2 > 0.
\]

So, \( q \) is bounded below by a positive constant then, arguing as in the previous case, \( q \) is constant. Thus, the result follows from Theorem 2.3. The case \( \kappa - 4\tau^2 > 0 \) is proved.

\[ \square \]

**Remark 4.1.** Note that in the above Theorem, in the case \( \kappa - 4\tau^2 > 0 \), we only need to assume \( 4(H^2 + \tau^2) - |\kappa - 4\tau^2| > 0 \).

## 5 Complete \( H \)–surfaces with constant angle function

We classify here the complete \( H \)–surfaces in \( \mathbb{E}(\kappa, \tau) \) with constant angle function. The purpose is to take advantage of this classification result in the next Section.

**Theorem 2.2.** Let \( \Sigma \subset \mathbb{E}(\kappa, \tau) \) be a complete \( H \)–surface with constant angle function. Then \( \Sigma \) is either a vertical cylinder over a complete curve of curvature \( 2H \) on \( \mathbb{M}^2(\kappa) \), a slice in \( \mathbb{H}^2 \times \mathbb{R} \) or \( \mathbb{S}^2 \times \mathbb{R} \), or \( \Sigma \in \mathcal{S}_{\kappa, \tau} \) with \( \kappa < 0 \) (see Definition 2.2).

**Proof.** We can assume that \( \nu \leq 0 \). We will divide the proof in three cases:

- \( \nu = 0 \): In this case, \( \Sigma \) must be a vertical cylinder over a complete curve of geodesic curvature \( 2H \) on \( \mathbb{M}^2(\kappa) \).
- \( \nu = -1 \): From (2.4), \( \tau = 0 \) and \( H = 0 \), then \( \Sigma \) is a slice in \( \mathbb{H}^2 \times \mathbb{R} \) or \( \mathbb{S}^2 \times \mathbb{R} \).
- \( -1 < \nu < 0 \): We prove here that \( \Sigma \in \mathcal{S}_{\kappa, \tau} \) with \( \kappa < 0 \). From (2.5), we have

\[
(H - i\tau)A = -\frac{2p}{\lambda}A
\]

then

\[
H^2 + \tau^2 = \frac{4|p|^2}{\lambda^2} = H^2 - K_e
\]

since \( |A|^2 \neq 0 \) from (2.6), so \( K_e = -\tau^2 \) on \( \Sigma \).
Thus, from (2.9), we have
\[ 4H^2 + 4\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2) = 0. \tag{5.2} \]

Now, using the definition of \( q \), (5.1), (5.2) and \( K_e = -\tau^2 \), we have

\[
q = \frac{4|Q|^2}{\lambda^2} = 4(H^2 + \tau^2)\frac{4|p|^2}{\lambda^2} + (\kappa - 4\tau^2)\frac{4|A|^4}{\lambda^2} \\
- 4\kappa - 4\tau^2 \left( 2(H + i\tau)pA^2 + 2(H - i\tau)\overline{p}A^2 \right) \\
= 4(H^2 + \tau^2)(H^2 - K_e) + (\kappa - 4\tau^2)^2 \frac{(1 - \nu^2)^2}{4} + 2(\kappa - 4\tau^2)(1 - \nu^2)(H^2 + \tau^2) \\
= \frac{1}{4} \left( 4H^2 + (\kappa - 4\tau^2)(1 - \nu^2) + 4\tau^2 \right)^2 = 0
\]

that is, \( q \) vanishes identically on \( \Sigma \). Moreover, from (5.2), we can see that \( 4H^2 + \kappa < 0 \), that is, \( \kappa < 0 \). Therefore, \( \Sigma \in S_{\kappa, \tau}, \kappa < 0 \).

\[ \square \]

6 Complete \( H \)–surfaces with \( q \) constant

Here, we prove the classification result for complete \( H \)–surfaces in \( \mathbb{E}(\kappa, \tau) \) employed in the proof of Theorem 3.1 and Theorem 4.1.

**Theorem 2.3.** Let \( \Sigma \subset \mathbb{E}(\kappa, \tau) \) be a complete \( H \)–surface with \( q \) constant.

- If \( q = 0 \) on \( \Sigma \), then \( \Sigma \) is either a slice in \( \mathbb{H}^2 \times \mathbb{R} \) or \( \mathbb{S}^2 \times \mathbb{R} \) if \( H = 0 = \tau \), or \( \Sigma \) is invariant by a one-parameter group of isometries of \( \mathbb{E}(\kappa, \tau) \).

Moreover, the Gauss curvature of these examples is

- If \( 4H^2 + \kappa > 0 \), then \( K > 0 \) they are the rotationally invariant spheres.

- If \( 4H^2 + \kappa = 0 \) and \( \nu \equiv 0 \), then \( K \equiv 0 \) and \( \Sigma \) is either a vertical plane in \( \text{Nil}_3 \), or a vertical cylinder over a horocycle in \( \mathbb{H}^2 \times \mathbb{R} \) or \( \text{PSL}(2, \mathbb{C}) \).

- There exists a point with negative Gauss curvature in the remaining cases.

- If \( q \neq 0 \) on \( \Sigma \), then \( \Sigma \) is a vertical cylinder over a complete curve of curvature \( 2H \) on \( \mathbb{M}^2(\kappa) \).

10
The case \( q = 0 \) has been treated extensively when the target manifold is a product space, but is has not been established explicitly when \( \tau \neq 0 \). So, we assemble the results in [AR], [AR2] for the readers convenience.

**Lemma 6.1.** Let \( \Sigma \subset E(\kappa, \tau) \) be a complete \( H \)-surface whose Abresch-Rosenberg differential vanishes. Then \( \Sigma \) is either a slice in \( \mathbb{H}^2 \times \mathbb{R} \) or \( S^2 \times \mathbb{R} \) if \( H = 0 = \tau \), or \( \Sigma \) is invariant by a one-parameter group of isometries of \( E(\kappa, \tau) \).

Moreover, the Gauss curvature of these examples is

- If \( 4H^2 + \kappa > 0 \), then \( K > 0 \) they are the rotationally invariant spheres.
- If \( 4H^2 + \kappa = 0 \) and \( \nu \equiv 0 \), then \( K \equiv 0 \) and \( \Sigma \) is either a vertical plane in \( \text{Nil}_3 \), or a vertical cylinder over a horocycle in \( \mathbb{H}^2 \times \mathbb{R} \) or \( \widetilde{\text{PSL}}(2, \mathbb{C}) \).
- There exists a point with negative Gauss curvature in the remaining cases.

**Proof:** The idea of the proof for product spaces that we use below, can be found in [dCF] and [FM].

If \( H = 0 = \tau \), from the definition of the Abresch-Rosenberg differential, we have

\[
0 = -(\kappa - 4\tau)A^2,
\]

that is, \( \nu^2 = \pm 1 \) using (2.6). Thus, \( \Sigma \) is a slice in \( \mathbb{H}^2 \times \mathbb{R} \) or \( S^2 \times \mathbb{R} \).

If \( H \neq 0 \) or \( \tau \neq 0 \), we have

\[
2(H + i\tau)p = (\kappa - 4\tau^2)A^2,
\]

from where we obtain, taking modulus,

\[
H^2 - K_e = \frac{(\kappa - 4\tau^2)^2(1 - \nu^2)^2}{16(H^2 + \tau^2)}
\]

Replacing (6.1) in (2.5),

\[
(H + i\tau)\nu_z = -\frac{1}{4}(4H^2 + \kappa - (\kappa - 4\tau^2)\nu^2)A,
\]

and taking modulus,

\[
|\nu_z|^2 = g(\nu)^2|A|^2, \quad g(\nu) = \frac{4H^2 + \kappa - (\kappa - 4\tau^2)\nu^2}{4\sqrt{H^2 + \tau^2}}.
\]

Assume that \( \nu \) is not constant. Let \( p \in \Sigma \) be a point where \( \nu_z(p) \neq 0 \) and let \( \mathcal{U} \) be a neighborhood of that point \( p \) where \( \nu_z \neq 0 \) (we can assume \( \nu^2 \neq 1 \) at \( p \). In particular, \( g(\nu) \neq 0 \) in \( \mathcal{U} \) from (6.3). Now, replacing (6.3) in (2.6), we obtain

\[
\lambda = \frac{4|\nu_z|^2}{(1 - \nu^2)g(\nu)^2}.
\]
Thus, putting (6.2) and (6.4) in the Jacobi equation (2.9)

\[ \nu_{zz} = -2 \frac{\nu|\nu_z|^2}{1 - \nu^2}. \]  

(6.5)

So, define the real function \( s := \operatorname{arctgh}(\nu) \) on \( U \). Such a function is harmonic by means of (6.5), thus we can consider a new conformal parameter \( w \) for the first fundamental form so that \( s = \operatorname{Re}(w) \), \( w = s + it \).

Since \( \nu = \tgh(s) \) by the definition of \( s \), we have that \( \nu \equiv \nu(s) \), i.e., it only depends on one parameter. Thus, we have \( \lambda \equiv \lambda(s) \) and \( T \equiv T(s) \) from (6.4) and (6.3) respectively, and \( p \equiv p(s) \) by the definition of the Abresch-Rosenberg differential. That is, all the fundamental data of \( \Sigma \) depend only on \( s \).

Now, let \( U \) be a simply connected domain on \( \Sigma \) and \( V \subset \mathbb{R}^2 \), a simply connected domain of a surface \( S \), so that \( \psi_0 : V \to U \subset \mathbb{E}(\kappa, \tau) \). We parametrize \( V \) by the parameters \( (s, t) \) obtained above. Then, the fundamental data (see [FM, Theorem 2.3]) \( \{\lambda_0, p_0, T_0, \nu_0\} \) of \( \psi_0 \) are given by

\[
\begin{align*}
\lambda_0(s, t) &= \lambda(s) \\
p_0(s, t) &= p(s) \\
T_0(s, t) &= a(s)\partial_s \\
\nu_0(s, t) &= \nu(s),
\end{align*}
\]

where \( a(s) \) is a smooth function.

Set \( \bar{t} \in \mathbb{R} \) and let \( i_{\bar{t}} : \mathbb{R}^2 \to \mathbb{R}^2 \) be the diffeomorphism given by

\[ i_{\bar{t}}(s, t) := (s, t + \bar{t}), \]

and define \( \psi_{\bar{t}} := \psi_0 \circ i_{\bar{t}} \). Then, the fundamental data \( \{\lambda_{\bar{t}}, p_{\bar{t}}, T_{\bar{t}}, \nu_{\bar{t}}\} \) of \( \psi_{\bar{t}} \) are given by

\[
\begin{align*}
\lambda_{\bar{t}}(s, t) &= \lambda(s) \\
p_{\bar{t}}(s, t) &= p(s) \\
T_{\bar{t}}(s, t) &= a(s)\partial_s \\
\nu_{\bar{t}}(s, t) &= \nu(s),
\end{align*}
\]

that is, both fundamental data match at any point \( (s, t) \in V \). Therefore, using [D, Theorem 4.3], there exists an ambient isometry \( I_{\bar{t}} : \mathbb{E}(\kappa, \tau) \to \mathbb{E}(\kappa, \tau) \) so that

\[ I_{\bar{t}} \circ \psi_0 = \psi_0 \circ i_{\bar{t}}, \]

for all \( \bar{t} \in \mathbb{R} \),

thus the surface is invariant by a one parameter group of isometries.

Let us prove the claim about the Gauss curvature. Using the Gauss Equation (2.2) in (6.2), one gets

\[
H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2 - K = \frac{(\kappa - 4\tau^2)^2(1 - \nu^2)^2}{16(H^2 + \tau^2)}. \]
Set \( a := 4(H^2 + \tau^2) \) and \( b := \kappa - 4\tau^2 \), then one can check easily that the above equality can be expressed as

\[
4aK = a^2 - b^2 + (2a + b)^2 - (2a + b(1 - \nu^2))^2.
\] (6.6)

So, if \( 4H^2 + \kappa > 0 \) then \( a > |b| \) and \( K > 0 \), that is, \( \Sigma \) is a topological sphere since it is complete. If \( 4H^2 + \kappa = 0 \), \( a = -b \) and the equation reads as

\[
4aK = a^2(1 - (1 + \nu^2)^2),
\]

that is, \( \Sigma \) has a point with negative Gauss curvature unless \( \nu \equiv 0 \).

If \( 4H^2 + \kappa < 0 \), one can check that \( a^2 - b^2 = (a-b)(a+b) < 0 \) since \( a+b > 0 \) and \( a-b < 0 \). So, if \( \inf_\Sigma \{\nu^2\} = 0 \) then, from (6.6), \( \Sigma \) has a point with negative curvature. Therefore, to finish this lemma, we shall prove that:

**Claim:** There are no complete surfaces with constant mean curvature \( 4H^2 + \kappa < 0 \) in \( \mathbb{E}(\kappa, \tau) \), \( \kappa < 0 \), with \( q \equiv 0 \), \( K \geq 0 \) and \( \inf_\Sigma \{\nu^2\} = c > 0 \).

**Proof of the Claim:** Assume such a surface \( \Sigma \) exists. Since we are assuming that \( K \geq 0 \) and \( \Sigma \) is parabolic and noncompact. If \( \Sigma \) were compact we would have a contradiction with the fact that \( \inf_\Sigma \{\nu^2\} = c > 0 \) and \( 4H^2 + \kappa < 0 \).

Since \( q \) vanishes identically on \( \Sigma \), \( \text{arctanh}(\nu) \) is a bounded harmonic function on \( \Sigma \) and so, \( \nu \) is constant. This implies that \( K \equiv 0 \) and \( c < \nu^2 < 1 \) is constant on \( \Sigma \). So, the projection \( \pi : \Sigma \to \mathbb{M}^2(\kappa) \) is a global diffeomorphism and a quasi-isometry. This is impossible since \( \Sigma \) is parabolic and \( \mathbb{M}^2(\kappa), \kappa < 0 \), is hyperbolic. Therefore, the Claim is proved and so, the lemma is proved.

**Proof of Theorem 2.3.** We focus on the case \( q \neq 0 \) because Lemma 6.1 gives the classification when \( q = 0 \).

Suppose \( \nu \) is not constant in \( \Sigma \). Since \( q = \nu^2 > 0 \), we can consider a conformal parameter \( z \) so that \( \langle \cdot, \cdot \rangle = |dz|^2 \) and \( Q dz^2 = c dz^2 \) on \( \Sigma \). Thus,

\[
Q = c = 2(H + i\tau)p - (\kappa - 4\tau^2)A^2.
\]

First, note that we can assume that \( H \neq 0 \) or \( \tau \neq 0 \), otherwise \( \nu \) would be constant. So, from (2.5), we have

\[
(H + i\tau)\nu_z = -(H^2 + \tau^2 + \frac{\kappa - 4\tau^2}{4}(1 - \nu^2))A - c\overline{A},
\]

where we have used \( 2(H + i\tau)p = c + (\kappa - 4\tau^2)A^2 \). That is,

\[
16(H^2 + \tau^2) \|\nabla \nu\|^2 = (g(\nu) + 4c)^2 (1 - \nu^2), \tag{6.7}
\]
where
\[ g(\nu) := 4H^2 + \kappa - (\kappa - 4\tau^2)\nu^2. \] (6.8)

From (2.10), \( \Sigma \) is flat and \( H^2 - K_e = H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2 \) by (2.2), joining this last equation to (2.8) we obtain using the definition of \( g(\nu) \) given in (6.8)
\[ \|\nabla \nu\|^2 = \frac{g(\nu)^2}{4(\kappa - 4\tau^2)} + \nu^2 g(\nu) - \frac{c^2}{\kappa - 4\tau^2}. \] (6.9)

Putting together (6.7) and (6.9) we obtain a polynomial expression in \( \nu^2 \) with coefficients depending on \( a := 4(\tau^2 + \tau^2), b := \kappa - 4\tau^2 \) and \( c \),
\[ P(\nu^2) := C(a, b, c)\nu^6 + \text{lower terms} = 0, \]
but one can easily check that the coefficient in \( \nu^6 \) is \( C(a, b, c) = -a^{-1}b^2 \neq 0 \), a contradiction. Thus \( \nu \) is constant, and so, by means of Theorem 2.2, \( \Sigma \) is a vertical cylinder over a complete curve of curvature \( 2H \).

7 Appendix

Let \( \Sigma \) be a connected Riemannian surface. We establish in this Appendix a result which we think is of independent interest, concerning differential operators of the form \( \Delta + g \), acting on \( C^2(\Sigma) \) functions, where \( \Delta \) is the Laplacian with respect to the Riemannian metric on \( \Sigma \) and \( g \in C^0(\Sigma) \).

**Lemma 7.1.** Let \( g \in C^0(\Sigma), \nu \in C^2(\Sigma) \) such that \( \|\nabla \nu\|^2 \leq h \nu^2 \) on \( \Sigma \), \( h \) is a non-negative continuous function on \( \Sigma \), and \( \Delta \nu + g\nu = 0 \) in \( \Sigma \). Then either \( \nu \) never vanishes or \( \nu \) vanishes identically on \( \Sigma \).

**Proof.** Set \( \Omega = \{p \in \Sigma : \nu(p) = 0\} \). We will show that either \( \Omega = \emptyset \) or \( \Omega = \Sigma \).

So, let us assume that \( \Omega \neq \emptyset \). If we prove that \( \Omega \) is an open set then, since \( \Omega \) is closed and \( \Sigma \) is connected, \( \Omega = \Sigma \). Let \( p \in \Omega \) and \( \mathcal{B}(R) \subset \Sigma \) be the geodesic ball centered at \( p \) of radius \( R \). Such a geodesic ball is relatively compact in \( \Sigma \).

Set \( \phi = \nu^2 / 2 \geq 0 \). Then
\[ \Delta \phi = \nu \Delta \nu + \|\nabla \nu\|^2 = -g\nu^2 + \|\nabla \nu\|^2 \leq -2(g - h)\phi, \]
that is,
\[ -\Delta \phi - 2(g - h)\phi \geq 0. \] (7.1)

Define \( \beta := \min \{\inf_{\Omega} \{2(g - h)\}, 0\} \leq 0 \). Then, \( \psi = -\phi \) satisfies
\[ \Delta \psi + \beta \psi = -\Delta \phi - \beta \phi \geq -\Delta \phi - 2(g - h)\phi \geq 0, \]
where we have used (7.1).

Since we are assuming that \( v \) has a zero at an interior point of \( B(R) \), \( \beta \leq 0 \) and \( \psi \) has a non-negative maximum at \( p \), the Maximum Principle [GT, Theorem 3.5] implies that \( \psi \) is constant and so \( v \) is constant as well, i.e., \( v \equiv 0 \) in \( B(R) \). Then \( B(R) \subset \Omega \), and \( \Omega \) is an open set. Thus \( \Omega = \Sigma \). \( \square \)

References


