Stable Constant Mean Curvature Hypersurfaces

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Abstract

Let $N^{n+1}$ be a Riemannian manifold with sectional curvatures uniformly bounded from below. When $n = 3, 4$, we prove that there are no complete (strongly) stable $H$-hypersurfaces, without boundary, provided $|H|$ is large enough. In particular we prove that there are no complete strongly stable $H$-hypersurfaces in $\mathbb{R}^{n+1}$ without boundary, $H \neq 0$.

1 Introduction

Consider a Riemannian manifold $N$ of dimension $n + 1$ with sectional curvatures uniformly bounded from below; denote by $\text{sec}(N)$ the infimum of the sectional curvatures of $N$. Let $M$ be an immersed submanifold of codimension one and let $H$ be the mean curvature of $M$ in the metric induced by the immersion. If $H$ is constant, we call $M$ an $H$-hypersurface. We prove the following diameter estimate.

**Theorem 1** Let $M^n \subset N^{n+1}$ be a stable complete $H$-submanifold, $n = 3, 4$. There exists a constant $c = c(n, H, \text{sec}(N))$ such that for any $p \in M$ one has: $\text{dist}_M(p, \partial M) \leq c$ whenever $|H| > 2\sqrt{|\min\{0, \text{sec}(N)\}|}$.

For the definition of stability see Section 2. Particular cases of the previous Theorem in $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$ and any homogeneously regular three manifold are proved in [9], [5], [7], [8] respectively.

We wonder if Theorem 1 holds in all dimensions.

**Corollary 1** Let $M^n$ be a complete stable $H$-hypersurface of $N^{n+1}$. If $n = 3, 4$ and $|H| > 2\sqrt{|\min\{0, \text{sec}(N)\}|}$, then $\partial M \neq \emptyset$.

In [12] it is proved that an $H$-hypersurface in $\mathbb{R}^{n+1}$, with finite total curvature, is minimal, so, if it is stable, it is a hyperplane (cf. [4]). For $n = 3, 4$, we are able to generalize this result in the following sense. We do not need the finite total curvature hypothesis on $M$ and the ambient space can be any manifold with uniformly bounded sectional curvature, provided the mean curvature $|H|$ is large enough (See Corollary 1).

As a consequence of the diameter estimate in Theorem 1, we have the Maximum Principle at Infinity.

**Theorem 2** Let $N^{n+1}$ have uniformly bounded sectional curvature, $n = 3, 4$. If $|H| > 2\sqrt{|\min\{0, \text{sec}(N)\}|}$ and $M_1, M_2$ are properly embedded $H$-hypersurfaces in $N^{n+1}$, which bound a connected domain $W$, then the mean curvature vector points out of $W$ along the boundary of $W$. 

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The proof of Theorem 2 is the same as in [8], where the result is proved for \( n = 2 \).

After this paper was submitted for publication, we received a preprint of Xu Cheng, where she also establishes our Theorem 1 [3].

2 Proofs

Let \( M \) be a \( H \)-hypersurface in a manifold \( \mathcal{N} \) and let \( N \) be a unit vector field normal to \( M \) in \( \mathcal{N} \). The stability operator of \( M \) is \( L = \Delta + |A|^2 + \text{Ric}(N) \) where \( \text{Ric}(N) \) is the Ricci curvature of the ambient manifold \( \mathcal{N} \) in the direction of \( N \) and \( A \) is the shape operator of the immersion. We say that \( M \) is stable if

\[
- \int_M uLu \geq 0,
\]

for any smooth function \( u \) with compact support on \( M \). Our definition of stability is usually known as strong stability. The usual definition of stability (weak stability) also requires the test function \( u \) to satisfy \( \int_M u = 0 \). Geodesic spheres in a space form are weakly stable but they are not stable (cf. [2]). We remark that the solutions of the Plateau problem are stable hypersurfaces in our sense as well as any \( H \)-hypersurface transverse to some Killing vector field of the ambient manifold. The proof of the latter is standard (cf. for example [7]). For further relations between the two notions of stability see [1] and [2].

**Proof of Theorem 1.** Consider the traceless operator \( \Phi = A - HI \). One can write the stability operator of \( M \) in terms of \( \Phi \), namely, \( L = \Delta + |\Phi|^2 + nH^2 + \text{Ric}(N) \). Since \( M \) is stable, there exists a function \( u > 0 \) on \( M \) such that \( Lu = 0 \) on \( M \) (cf. [6]).

Denote by \( ds^2 \) the metric on \( M \) induced by the immersion in \( \mathcal{N} \) and and let \( d\bar{s}^2 = u^{2k}ds^2 \), with \( \frac{3(n-1)}{4n} \leq k < \frac{4}{n-1} \). This choice of \( k \) will be justified later. Notice that, in order to have some \( k \) satisfying the previous inequality, one needs \( n = 3, 4 \).

Consider \( p \in M \) and let \( r > 0 \) be such that the intrinsic ball \( B_r \) of \( M \), centered at \( p \) of \( ds \)-radius \( r \), is contained in the interior of \( M \). Let \( \gamma \) be a \( d\bar{s} \)-minimizing geodesic in \( B_r \) joining \( p \) to \( \partial B_r \). Let \( a \) be the \( ds \)-length of \( \gamma \). Then \( a \geq r \) and it is enough to prove that there exists a constant \( c(n, H, \sec(\mathcal{N})) \) such that \( a \leq c \).

Let \( R \) be the curvature tensor of \( M \) in the metric \( ds \) and \( d\bar{s} \), respectively. Choose a basis \( \{ \tilde{e}_1 = \frac{\partial}{\partial s}, \tilde{e}_2, \ldots, \tilde{e}_n \} \) orthonormal for the metric \( d\bar{s} \), such that \( \tilde{e}_2, \ldots, \tilde{e}_n \) are parallel along \( \gamma \) and let \( \tilde{e}_{n+1} = N \). The basis \( \{ e_1 = \frac{\partial}{\partial s} = u^k\tilde{e}_1, e_2 = u^k\tilde{e}_2, \ldots, e_n = u^k\tilde{e}_n \} \) is orthonormal for the metric \( ds \). Denote by \( R_{11} \) and \( \bar{R}_{11} \) the Ricci curvature in the direction of \( e_1 \) for the metric \( ds \) and \( d\bar{s} \) respectively. Let \( \bar{R} \) be the curvature tensor of the ambient manifold \( \mathcal{N} \) and write \( \text{Ric}(N) = \bar{R}_{n+1,n+1} \).

Let \( \tilde{r} \) be the length of \( \gamma \) in the \( d\bar{s} \) metric. Since \( \gamma \) is \( d\bar{s} \) minimizing, by the second variation formula, one has

\[
\int_0^{\tilde{r}} \left[ (n-1) \left( \frac{d\varphi}{d\bar{s}} \right)^2 - \bar{R}_{11} \varphi^2 \right] d\bar{s} \geq 0, \tag{1}
\]

for any smooth function \( \varphi \) such that \( \varphi(0) = \varphi(\tilde{r}) = 0 \).

As it is proved in the Appendix.
\[ \tilde{R}_{11} = u^{-2k} \left\{ R_{11} - k(n-2)(\ln u)_{ss} - k \frac{\Delta u}{u} + k \frac{\nabla u^2}{u^2} \right\}. \]  

(2)

Now use that \( Lu = (\Delta + |\Phi|^2 + nH^2 + \tilde{R}_{n+1,n+1})u = 0 \) to obtain

\[ \tilde{R}_{11} = u^{-2k} \left\{ R_{11} - k(n-2)(\ln u)_{ss} + k(|\Phi|^2 + nH^2 + \tilde{R}_{n+1,n+1}) + k \frac{\nabla u^2}{u^2} \right\}. \]  

(3)

From the Gauss equation one has

\[ R_{ijij} = \tilde{R}_{ijij} + h_{ii} h_{jj} - h_{ij}^2, \]  

(4)

which can be rewritten as

\[ R_{ijij} = \tilde{R}_{ijij} + (\Phi_{ii} + H)(\Phi_{jj} + H) - (\Phi_{ij} + H\delta_{ij})^2. \]

Taking \( i = 1 \) and summing up in \( j = 2, \ldots, n \) we obtain

\[ R_{11} = \sum_{j=2}^{n} \tilde{R}_{1j1j} + \sum_{j=2}^{n} \Phi_{1j} \Phi_{jj} + (n-2)H \Phi_{11} + \sum_{j=1}^{n} \Phi_{jj} H + (n-1)H^2 - \sum_{j=2}^{n} \Phi_{1j}^2. \]

Since \( \sum_{j=1}^{n} \Phi_{jj} = 0 \), we have

\[ R_{11} = \sum_{j=2}^{n} \tilde{R}_{1j1j} - \Phi_{11}^2 + (n-2)H \Phi_{11} + (n-1)H^2 - \sum_{j=2}^{n} \Phi_{1j}^2. \]

Replacing the last relation in equation (3), yields

\[ \tilde{R}_{11} = u^{-2k} \left( \sum_{j=2}^{n} \tilde{R}_{1j1j} + k\tilde{R}_{n+1,n+1} + (kn + n - 1)H^2 + (n-2)H \Phi_{11} \right) \]

\[ + u^{-2k} \left[ k|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^{n} \Phi_{1j}^2 - k(n-2)(\ln u)_{ss} + k \frac{\nabla u^2}{u^2} \right]. \]

Combining the last equation with inequality (1) gives (by abuse of notation we denote again by \( \varphi \) the composition \( \varphi \circ \tilde{s} \), hence \( \varphi(0) = \varphi(a) = 0 \))

\[ (n-1) \int_{0}^{a} (\varphi s)^2 u^{-k} ds \geq \int_{0}^{a} \varphi^2 u^{-k} \left( \sum_{j=2}^{n} \tilde{R}_{1j1j} + k\tilde{R}_{n+1,n+1} \right) ds \]

\[ + \int_{0}^{a} \varphi^2 u^{-k} \left[ (kn + n - 1)H^2 + (n-2)H \Phi_{11} + k|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^{n} \Phi_{1j}^2 \right] ds \]

\[ - \int_{0}^{a} \varphi^2 u^{-k} \left[ k(n-2)(\ln u)_{ss} + k \frac{\nabla u^2}{u^2} \right] ds. \]
Replace \( \varphi \) by \( \varphi u^k \) to get rid of \( u^k \) in the denominator. The last relation becomes

\[
(n - 1) \int_0^a (\varphi_s)^2 ds + k(n - 1) \int_0^a \varphi \varphi_s u_s u^{-1} ds + \frac{k^2(n - 1)}{4} \int_0^a \varphi^2 u_s^2 u^{-2} ds \\
\geq \int_0^a \varphi^2 \left[ \sum_{j=2}^n \hat{R}_{1j} + k \hat{R}_{n+1,n+1} \right] ds \\
+ \int_0^a \varphi^2 \left[ (kn + n - 1)H^2 + (n - 2)H \Phi_{11} + k|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 \right] ds \\
- \int_0^a \varphi^2 \left[ (n - 2)(\ln u)_{ss} + k \frac{|\nabla u|^2}{u^2} \right] ds.
\]

Integration by parts gives

\[
\int \varphi^2 (\ln u)_{ss} ds = -2 \int \varphi \varphi_s u_s^{-1} ds.
\]

Then, replacing in inequality (5), we obtain

\[
(n - 1) \int_0^a (\varphi_s)^2 ds \geq k(n - 3) \int_0^a \varphi \varphi_s u_s u^{-1} ds - \frac{(n - 1)}{4} \int_0^a \varphi^2 (\ln u^k)^2 ds \\
+ k \int_0^a \varphi \frac{|\nabla u|^2}{u^2} ds + \int_0^a \varphi^2 \left[ k \hat{R}_{n+1,n+1} + \sum_{j=2}^n \hat{R}_{1j} \right] ds \\
+ \int_0^a \varphi^2 \left[ (kn + n - 1)H^2 + (n - 2)H \Phi_{11} + k|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 \right] ds,
\]

that is

\[
(n - 1) \int_0^a (\varphi_s)^2 ds \geq k(n - 3) \int_0^a \varphi \varphi_s u_s u^{-1} ds + \left[ \frac{1}{k} - \frac{(n - 1)}{4} \right] \int_0^a \varphi^2 (\ln u^k)^2 ds \\
+ \int_0^a \varphi^2 \left[ k \hat{R}_{n+1,n+1} + \sum_{j=2}^n \hat{R}_{1j} \right] ds \\
+ \int_0^a \varphi^2 \left[ (kn + n - 1)H^2 + (n - 2)H \Phi_{11} + k|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 \right] ds.
\]

We now use that \( a^2 + b^2 \geq -2ab \) with \( a = (n - 2)H \) and \( b = \frac{\Phi_{11}}{2} \), to obtain

\[
(n - 2)^2 H^2 + \frac{\Phi_{11}^2}{4} \geq -(n - 2)H \Phi_{11}.
\]

Replacing in inequality (6) yields
(n - 1) \int_0^a (\varphi_s)^2 ds \geq k(n - 3) \int_0^a \varphi \varphi_s u^{-1} ds + \left[ \frac{1}{k} - \frac{(n - 1)}{4} \right] \int_0^a \varphi^2 (\ln u^k) s ds \\
+ \int_0^a \varphi^2 \left[ k \tilde{R}_{n+1,n+1} + \sum_{j=2}^n \tilde{R}_{1j1j} + (kn - n^2 + 5n - 5)H^2 \right] ds.

We will now prove that the last term in inequality (7) is greater or equal than zero. We know that

|\Phi|^2 \geq \Phi_{11}^2 + \Phi_{22}^2 + \cdots + \Phi_{nn}^2 + 2 \sum_{j=2}^n \Phi_{1j}^2

and since \sum_{j=1}^n \Phi_{jj} = 0, we have

|\Phi|^2 \geq \frac{n}{n-1} \Phi_{11}^2 + 2 \sum_{j=2}^n \Phi_{1j}^2,

(8)

Since \( k \geq \frac{5(n-1)}{4n} \), using inequality (8), we obtain

\[ k|\Phi|^2 - \frac{5}{4} \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 \]
\[ \geq \frac{5}{4} \Phi_{11}^2 + \frac{5(n-1)}{2n} \sum_{j=2}^n \Phi_{1j}^2 - \frac{5}{4} \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 = \frac{3n - 5}{2n} \sum_{j=2}^n \Phi_{1j}^2 \geq 0. \]

Then, inequality (7) yields

\[ (n - 1) \int_0^a (\varphi_s)^2 ds \geq (n - 3) \int_0^a \varphi \varphi_s (\ln u^k)_s ds + \left[ \frac{1}{k} - \frac{(n - 1)}{4} \right] \int_0^a \varphi^2 (\ln u^k)_s ds \\
+ \int_0^a \varphi^2 \left[ k \tilde{R}_{n+1,n+1} + \sum_{j=2}^n \tilde{R}_{1j1j} + (kn - n^2 + 5n - 5)H^2 \right] ds. \]

We now use that \( a^2 + b^2 \geq -2ab \) with \( a = \left( \frac{1}{k} - \frac{(n-1)}{4} \right)^{\frac{1}{2}} \varphi (\ln u^k)_s \) and \( b = \frac{(n-3)}{2} \left( \frac{1}{k} - \frac{(n-1)}{4} \right)^{-\frac{1}{2}} \varphi_s \), to obtain

\[ \left( \frac{1}{k} - \frac{(n-1)}{4} \right) \varphi^2 (\ln u^k)_s + \frac{(n-3)^2}{4} \left( \frac{1}{k} - \frac{(n-1)}{4} \right)^{-1} \varphi_s^2 \geq -(n-3) \varphi \varphi_s (\ln u^k)_s. \]
The last inequality together with inequality (9) gives

\[(n-1) \int_0^a (\varphi_s)^2 ds \geq -\frac{(n-3)^2}{4} \left( \frac{1}{k} - \frac{(n-1)}{4} \right)^{-1} \int_0^a (\varphi_s)^2 ds + \int_0^a \varphi^2 \left[ (kn - n^2 + 5n - 5)H^2 + k\hat{R}_{n+1,n+1} + \sum_{j=2}^n \hat{R}_{1j1j} \right] ds.\]

Then setting \( A = \frac{4k(2-n)+(n-1)}{4-k[n-1]} \) and making a suitable choice of a positive constant \( B \), we can rewrite the last inequality as

\[A \int_0^a (\varphi_s)^2 ds \geq B \int_0^a \varphi^2 ds. \tag{10}\]

We remark that \( A \) is positive as soon as \( k < \frac{4}{n-1} \leq \frac{n-1}{n-2} \). We now want to choose \( B \) such that

\[0 < B \leq (kn - n^2 + 5n - 5)H^2 + \left( k\hat{R}_{n+1,n+1} + \sum_{j=2}^n \hat{R}_{1j1j} \right).\]

When the curvature of the ambient manifold is non-negative, we set \( B = (kn - n^2 + 5n - 5)H^2 \), which is positive if \( H \neq 0 \) (remember that \( k > \frac{5(n-1)}{4n} \) and that \( n = 3, 4 \)). In this case we can set \( c_1 = 0 \).

Otherwise, we proceed as follows. By a straightforward computation one has

\[k\hat{R}_{n+1,n+1} + \sum_{j=2}^n \hat{R}_{1j1j} \geq (kn + n - 1) \inf \{\text{sectional curvatures of } N\} = \text{sec}(N),\]

then we set \( B = (kn - n^2 + 5n - 5)H^2 + (kn + n - 1)\text{sec}(N) \). If

\[H^2 > \frac{kn + n - 1}{kn - n^2 + 5n - 5} \text{sec}(N), \tag{11}\]

then \( B \) is positive. In this case, one can set \( c_1 = 2\sqrt{|\text{sec}(N)|} \) (using the restrictions on \( k \) one can prove that \( \frac{kn + n - 1}{kn - n^2 + 5n - 5} < 4 \)).

Integration by parts in inequality (10) yields

\[\int_0^a (\varphi_{ss} A + B\varphi)\varphi ds \leq 0.\]

Choosing \( \varphi = \sin(\pi sa^{-1}) \), \( s \in [0, a] \) one has

\[\int_0^a \left[ B - \frac{A\pi^2}{a^2} \right] \sin^2(\pi sa^{-1}) ds \leq 0.\]

Finally

\[B - \frac{A\pi^2}{a^2} \leq 0,\]

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and this gives the desired inequality, if we choose
\[ c = \frac{2\pi \sqrt{k(2 - n) + (n - 1)}}{\sqrt{(4 - k(n - 1))((kn - n^2 + 5n - 5)H^2 + (kn + n - 1)\min\{0, \sec(N)\})}}. \]

\[ \square \]

**Proof of Corollary 1.** Assume that such an \( M \) exists. In the proof of Theorem 1, we showed that the radius of an intrinsic disc of \( M \), that does not touch \( \partial M \), is at most \( c \). Hence, when \( \partial M = \emptyset \), the diameter of \( M \) is at most \( c \) and then \( M \) is compact. As \( M \) is stable, there exists a positive function \( f \) on \( M \) such that \( L(f) = 0 \) (cf. [6]). Let \( p \in M \) be a minimum of the function \( f \). At \( p \), one has:
\[ 0 \leq \Delta f(p) = -|\Phi|^2(p) + nH^2 + \hat{R}_{n+1,n+1}(p)f(p). \]

By our choice of \( H \), the potential \( |\Phi|^2 + nH^2 + \hat{R}_{n+1,n+1} \) is strictly positive on \( M \), hence the previuos inequality yields a contradiction.

\[ \square \]

### 3 Appendix

The transformation law of the curvature under the conformal change of the metric \( d\tilde{s}^2 = u^{2k}ds^2 \) is the following (cf. [10] page 184 and [11] formula (4))

\[ \tilde{R}_{11} = \tilde{Ric}(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial s}) = \left\{ Ric(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial s}) - k(n - 2)Hess(ln u) \left( \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial s} \right) \right\} \]
\[ + k^2(n - 2)\left| \frac{\partial\gamma}{\partial s}(\ln u) \right|^2 - \left[ k\Delta(\ln u) + k^2(n - 2)|\nabla\ln u|^2 \right] u^{-2k} \]

In order to simplify this equation we need to compute \( \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial s} \).

Using the relation between the connections of conformal metrics we obtain
\[ \tilde{\nabla}_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial s} = \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial s} + 2k < \nabla\ln u, \frac{\partial\gamma}{\partial s} > \frac{\partial\gamma}{\partial s} - k\nabla\ln u. \]

Since \( \gamma \) is geodesic in the \( d\tilde{s}^2 \) metric we have that \( \tilde{\nabla}_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial s} = 0 \)

and thus
\[ \tilde{\nabla}_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial s} = k < \nabla\ln u, \frac{\partial\gamma}{\partial s} > \frac{\partial\gamma}{\partial s}. \]

The last two equations yield
\[ \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial s} = k(\nabla\ln u)^\perp, \]
where \( (\nabla\ln u)^\perp \) means the component of \( \nabla\ln u \) perpendicular to \( \frac{\partial\gamma}{\partial s} \). Now we observe that
\[\text{Hess}(\ln u)\left( \frac{\partial \gamma}{\partial \tilde{s}}, \frac{\partial \gamma}{\partial \tilde{s}} \right) = u^{-2k}(\ln u)_{ss} - \left( \nabla_{\frac{\partial \gamma}{\partial \tilde{s}}} \ln u \right)\]

where in the last equality we use (13). Replacing this last equation in (12) one obtains

\[\tilde{R}_{11} = u^{-2k}\left\{ R_{11} - k(n - 2)(\ln u)_{ss} + k^2(n - 2)(\nabla \ln u)^2 \right\},\]

which can be rewritten as

\[\tilde{R}_{11} = u^{-2k}\left\{ R_{11} - k(n - 2)(\ln u)_{ss} - k\Delta(\ln u) \right\}\]

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