Removable singularities for sections of Riemannian submersions of prescribed mean curvature.

Claudemir Leandro and Harold Rosenberg

Abstract

We will prove that isolated singularities of sections with prescribed mean curvature of a Riemannian submersion fibered by geodesics of a vertical Killing field, are removable. Also we obtain information on the growth of the difference of two sections $u, v : \Omega \rightarrow \bar{M}$, having the same prescribed mean curvature and $u = v$ on $\partial \Omega$. This generalizes theorem 2 of [2].

Keywords: removable singularities for prescribed mean curvature, sections of Riemannian submersions.

Mathematics Subject Classification (2000) 35J60-53C42

1 Introduction

Let $\pi : \bar{M}^{n+1} \rightarrow M^n$ be a Riemannian submersion. Assume the fibers are the integral curves of a unit Killing vector field $\xi$, and each orbit of $\xi$ is a geodesic of $\bar{M}$. Let $\Omega \subset M$ be a domain and $H : \Omega \rightarrow \mathbb{R}$ continuous. We will prove that a section $u : \Omega \rightarrow \bar{M}$ of mean curvature $H$, can not have isolated singularities. This was first proved for minimal graphs in $\mathbb{R}^3$ by L. Bers [1] and for graphs of prescribed mean curvature in $\mathbb{R}^3$, by R. Finn [4]. We refer the reader to the paper of Nitsche [5] for some history.

We remark that our theorem applies to graphs in $M^n \times \mathbb{R}$ and sections of the bundles: Heisenberg space $\rightarrow \mathbb{R}^2$ and $PSL(2, \mathbb{R}) \rightarrow \mathbb{H}^2$.

2 Killing Graphs

Let $\pi : \bar{M}^{n+1} \rightarrow M^n$ be Riemannian submersion such that the orbits of the vertical fibers are geodesics of a nonsingular unit Killing field denoted by $\xi \in \mathcal{X}(\bar{M})$. Let $\Omega \subset M$ be a domain. We assume that the integral curves $\phi_s$ of $\xi$ in $\bar{M}_0 = \pi^{-1}(\Omega)$ are complete non compact.

We first derive a formula for the mean curvature of a section of $\bar{M}$.

Lemma 2.1. Let $\Sigma$ be a hypersurface of $\bar{M}$, transverse to the fibers of $\xi$. Let $N$ be a unit normal vector field to $\Sigma$. Then

$$\text{div}_M(\pi_* N) = n \left\langle \bar{H}, N \right\rangle_{\bar{M}} = nH$$
Proof.
Let $\bar{x} \in \Sigma$, $\pi(\bar{x}) = x \in \Omega$, and let $X_1, \ldots, X_n$ be an orthonormal frame of $M$ in a neighborhood of $x$. Let $\bar{X}_1, \ldots, \bar{X}_n$ be their horizontal lifts to $\bar{M}$.

Extend a neighborhood of $\bar{x}$ in $\Sigma$ to a foliation of a neighborhood of $\bar{x}$ in $\bar{M}$, by the flow of $\xi$. Also extend $N$ to this neighborhood as well (by $\xi$). It is well known that

$$\text{div}_{\bar{M}}(N) = n \left( \langle N, \bar{H} \rangle \right) = nH,$$

where $\bar{H}$ is the mean curvature vector of the leaves of the local foliation.

Write $N = N_h + \langle N, \xi \rangle \xi$. Then

$$\text{div}_{\bar{M}}(N) = \sum_{i=1}^{n} \langle \bar{\nabla}_{\bar{X}_i}N, \bar{X}_i \rangle + \langle \bar{\nabla}_{\xi}N, \xi \rangle.$$

Since $\langle \xi, N \rangle = 0$, $\langle \bar{\nabla}_{\xi}N, \xi \rangle = -\langle N, \bar{\nabla}_{\xi}\xi \rangle = 0$.
We have $N = N_h + \langle N, \xi \rangle \xi$, so

$$\langle \bar{\nabla}_{\bar{X}_i}N, \bar{X}_i \rangle = \langle \nabla_{X_i}N_h, X_i \rangle + \langle N, \xi \rangle \langle \bar{\nabla}_{\bar{X}_i}\xi, \bar{X}_i \rangle$$

By O’Neil’s formula [6], $\bar{\nabla}_{\bar{X}_i}\bar{X}_i$ is horizontal, so differentiating $\langle \xi, X_i \rangle = 0$,

$$\langle \bar{\nabla}_{\bar{X}_i}\xi, \bar{X}_i \rangle = -\langle \xi, \bar{\nabla}_{\bar{X}_i}\bar{X}_i \rangle = 0$$

Again, by O’Neil’s formula,

$$\langle \bar{\nabla}_{\bar{X}_i}N_h, \bar{X}_i \rangle_{\bar{M}} = \langle \nabla_{X_i}\pi(N_h), X_i \rangle_{M},$$

and this proves the lemma.

Now consider two sections $u, v : \Omega \to \bar{M}$, transverse to $\xi$, such that the surfaces $\Sigma_u = u(\Omega)$ and $\Sigma_v = v(\Omega)$ have the same prescribed mean curvature at each $x \in \Omega$. We assume the mean curvature function $H$ is continuous on $\Omega$. Let $X_u, X_v$ be the vector fields on $\Omega$, the projection of the unit normals $N_u, N_v$ to the sections. Let $\varphi = u - v$ be the function on $\Omega$, the signed distance along the $\xi$ orbits from $v(x)$ to $u(x)$, $x \in \Omega$.

It is not hard to see that

$$\langle \nabla \varphi, X_u - X_v \rangle_{M} = \langle \bar{\nabla} \varphi, N_u - N_v \rangle_{\bar{M}} \geq 0,$$

and one has equality precisely when $\varphi$ is constant. It is useful to have an explicit formula for the quantities involved, so we will prove the above statement using formulas in local coordinates derived in [3]. We will prove:

$$\langle \nabla \varphi, X_u - X_v \rangle_{M} = \left( \frac{W_u + W_v}{2} \right) \|N_u - N_v\|^2,$$

where $W_u = \frac{1}{\langle N_u, \xi \rangle}, W_v = \frac{1}{\langle N_v, \xi \rangle}$.

Consider a smooth embedding $i : \Omega \to \bar{M}$, which is a section of the fibration, and assume $\Sigma_0 = i(\Omega)$ is transverse to $\xi$.

Thus, the hypersurfaces $\Sigma_s = \phi_s(\Sigma_0)$ foliate $\bar{M}_0$ by isometric hypersurfaces.
Definition 2.1. *The Killing graph* \( \Sigma = \Sigma_u \) of a function \( u \in C^2(\Omega) \) is the hypersurface
\[
\Sigma = \{ \phi (u(p), p) ; p \in \Sigma_0 \}
\]
where \( u \) is seen as a function on \( \Sigma_0 \) by taking \( u(p) = u(x) \) when \( \pi(p) = x \).

### 3 The Mean Curvature Equation

Let \( X_1, ..., X_n \) be a frame on \( \Omega \) and \( \sigma_{ij} = \langle X_i, X_j \rangle_M \). Let \( \bar{Y}_1, ..., \bar{Y}_n \) be the corresponding local frame on \( \Sigma_0 \), i.e., \( \bar{Y}_i(p) = \mathfrak{z}_* X_i(x) \), where \( x \in \Omega \) and \( p = \mathfrak{z}(x) \).

We extend \( \bar{Y}_i \) by the flow:
\[
\bar{Y}_i(\phi(s, p)) = (\phi)_*(\bar{Y}_i(p)),
\]
\( p \in \Sigma_0 \).

Let \( \bar{X}_1, ..., \bar{X}_n \) in \( \bar{M} \) denote the horizontal lifts of \( X_1, ..., X_n \). If \( q = \phi(s, p) \) for \( p \in \Sigma_0 \), then \( \pi(q) = \pi \circ \phi(s, p) = \pi(p) \). Therefore
\[
\bar{X}_i(q) = \phi_*(s, p)\bar{X}_i(p)
\]
since \( \phi_*(s, p)\bar{X}_i(p) \) is horizontal and
\[
\pi_*(q)\phi_*(s, p)\bar{X}_i(p) = (\pi \circ \phi)_*(s, p)\bar{X}_i(p) = \pi_*(p)\bar{X}_i(p).
\]

Also
\[
\langle \bar{X}_i, \bar{X}_j \rangle_{\bar{M}} = \langle X_i, X_j \rangle_M = \sigma_{ij}.
\]

We denote by \( s \), the function on \( \bar{M}_0 \) which is distance from \( q \) to \( \Sigma_0 \), along the integral curve of \( \xi \) from \( q \) to \( \Sigma_0 \). We calculate \( \bar{X}_i(s) \).

Let \( \bar{\nabla}s \) be the gradient of the function \( s \). Using
\[
\pi_*(q)\bar{Y}_i = \pi_*(p)\mathfrak{z}_* X_i(x) = X_i(x) = \pi_*(q)\bar{X}_i
\]
and
\[
1 = \xi(s) = \langle \bar{\nabla}s, \xi \rangle_{\bar{M}}
\]
we have that two frames in \( \bar{M} \) are related by
\[
\begin{cases}
\bar{\nabla}s = \xi + \sigma^{ij}\bar{X}_j(s)\bar{X}_i, \\
\bar{Y}_i = \delta_i \xi + \bar{X}_i.
\end{cases}
\]

where \( \sigma^{ij} \) is the inverse of \( \sigma_{ij} \). Then
\[
\delta_i = \langle \bar{Y}_i(q), \xi(q) \rangle_{\bar{M}} = \langle (\phi_*)_*(p)\bar{Y}_i(p), (\phi_*)_*(p)\xi(p) \rangle_{\bar{M}} = \langle \bar{Y}_i(p), \xi(p) \rangle_{\bar{M}}
\]
and
\[
0 = \bar{Y}_j(s) = \langle \bar{\nabla}s, \bar{Y}_j \rangle_{\bar{M}} = \delta_j + \bar{X}_j(s).
\]

3
Let $\Sigma$ be a Killing graph. Consider $\Sigma$ as given by the immersion
\[ I_u : x \in \Omega \subset M \mapsto \phi(u(x), \xi(x)). \]
Its tangent bundle is spanned by the vector fields
\[ (I_u)_* X_i = X_i(u) \phi_s + (\phi \circ \xi)_* X_i \]
\[ = X_i(u) \xi + \bar{Y}_i \]
\[ = X_i(u) \xi + \bar{Y}_i \]
\[ = \bar{X}_i(u) + \delta_i \xi(u) = \bar{Y}_i(u) = X_i(u). \]

We may regard $u$ as a function in $\bar{M}_0$ by means of the extension $u(q) = u(x)$ if $\pi(q) = x$. Thus $\xi(u) = 0$ and hence
\[ \bar{X}_i(u) = \bar{Y}_i(u) - \delta_i \xi(u) = \bar{Y}_i(u) \]
Therefore, we have using (1) that
\[ (I_u)_* X_i = \bar{X}_i(u) \xi + \bar{Y}_i \]
\[ = \bar{X}_i(u) \xi + (\delta_i \xi + \bar{X}_i) \]
\[ = \bar{X}_i(u) \xi - \bar{X}_i(s) \xi + \bar{X}_i \]
\[ = \bar{X}_i(u - s) \xi + \bar{X}_i \]

Thus it is easy see that the unit normal vector field to $\Sigma$ pointing upwards is
\[ N = \frac{1}{W}(\xi - \hat{\varpi}^j \bar{X}_j) \]
where $\hat{\varpi}^j = \sigma^{ij} \bar{X}_i(u - s)$ and $W^2 = 1 + \sigma_{ij} \hat{\varpi}^i \hat{\varpi}^j = 1 + \hat{\varpi}^i \hat{\varpi}_i$ for $\hat{\varpi}_i = \sigma_{ij} \hat{\varpi}^j$. We extend $N$ to $\bar{M}_0$ by the flow $\xi$.

Observe that if we define
\[ Gu = \hat{\varpi}^j \bar{X}_j = \sigma^{ij} \bar{X}_i(u - s) \bar{X}_j \]
then $\frac{Gu}{W} = N^h$ and Lemma 2.1 gives $\text{div}_M \left( \pi \left( \frac{Gu}{W} \right) \right) = nH$.

Also
\[ Gu = \sigma^{ij} \bar{X}_i(u) \bar{X}_j - \sigma^{ij} \bar{X}_i(s) \bar{X}_j \]
\[ = \sigma^{ij} \bar{X}_i(u) \bar{X}_j - \sigma^{ij} \bar{X}_i(s) \bar{X}_j \]
\[ = \nabla u - \sigma^{ij} \bar{X}_i(s) \bar{X}_j. \]

Notice that $\sigma^{ij} \bar{X}_i(s) \bar{X}_j$ are defined on $M$ since they are independent of $s$. Hence for two sections $u, v : G(u) - G(v) = \nabla u - \nabla v$. 

4
4 Some Results

We will now prove the removable singularities theorem. First, a lemma.

**Lemma 4.1.** Let \( u \) and \( v \) be functions in \( C^2(\Omega) \). Then

\[
\left\langle Gu - Gv, \frac{Gu}{W_u} - \frac{Gv}{W_v} \right\rangle_M = \left( \frac{W_u + W_v}{2} \right) \|N_u - N_v\|_M^2 \geq 0
\]  \hspace{1cm} (2)

where \( W_u^2 = 1 + \|Gu\|_M^2 \) and \( W_v^2 = 1 + \|Gv\|_M^2 \), with equality at a point if and only if, \( \nabla u = \nabla v \).

**Proof.**

We know that \( Gu = \hat{u}^i X_i \), \( Gv = \hat{v}^j X_j \), \( W_u^2 = 1 + \hat{u}^i \hat{u}^i \), \( W_v^2 = 1 + \hat{v}^j \hat{v}^j \), \( N_u = \frac{1}{W_u} (\xi - \hat{u}^i Y_i) \), \( N_v = \frac{1}{W_v} (\xi - \hat{v}^j Y_j) \) and

\[
\left\langle \bar{X}_i, \bar{X}_j \right\rangle_M = \left\langle X_i, X_j \right\rangle_M .
\]

Thus

\[
\begin{align*}
\left\langle N_u - N_v, N_u - N_v \right\rangle_M &= \left\langle N_u, N_u \right\rangle_M + \left\langle N_v, N_v \right\rangle_M - 2 \left\langle N_u, N_v \right\rangle_M \\
&= 2 - 2 \left\langle N_u, N_v \right\rangle_M \\
&= 2 - \frac{2}{W_u W_v} \left( 1 + \hat{u}^i \hat{v}^i \right).
\end{align*}
\]

Then

\[
\hat{u}^i \hat{v}^i = \left( \frac{W_u W_v}{2} \left\| N_u - N_v \right\|_M^2 \right) - 1.
\]  \hspace{1cm} (3)

Further

\[
\begin{align*}
\left\langle Gu - Gv, \frac{Gu}{W_u} - \frac{Gv}{W_v} \right\rangle_M &= \frac{\hat{u}^i \hat{u}^i}{W_u} - \frac{\hat{v}^i \hat{v}^i}{W_v} - \frac{\hat{u}^i \hat{v}^i}{W_u} - \frac{\hat{v}^i \hat{v}^i}{W_v} \\
&= \frac{W_u^2 - 1}{W_u} - \hat{u}^i \hat{v}^i \left( \frac{1}{W_u} + \frac{1}{W_v} \right) + \frac{W_v^2 - 1}{W_v} \\
&= \frac{W_u + W_v}{2} \left\| N_u - N_v \right\|_M^2 \geq 0
\end{align*}
\]

by (3). Thus equality yields \( N_u = N_v \) and this implies that \( \nabla u = \nabla v \) since \( Gu - Gv = \nabla u - \nabla v \).

---

**Theorem 4.1.** Let \( u : \Omega - \{p\} \rightarrow \mathbb{R}, \Omega \subset M \), be a function whose Killing graph has prescribed mean curvature \( H \). Then \( u \) extends smoothly to a solution at \( p \).
Proof.

Let $R$ be small so that there exists a smooth function $v$ defined on $B_R(p)$, with:

$$\begin{align*}
\begin{cases}
div_M \frac{Gv}{W} = nH, & \text{in } B_R(p), \\
v = u, & \text{in } \partial B_R(p),
\end{cases}
\end{align*}$$

This exists by [3].

Let $C$ be a positive constant. Define

$$\varphi = \begin{cases} 
u - v, & \text{if } |u - v| < C, \\
C, & \text{if } |u - v| \geq C,
\end{cases}$$

Then, $\varphi$ is Lipschitz and $\nabla \varphi = \nabla u - \nabla v = Gu - Gv$ in the set $|u - v| < C$ and $\nabla \varphi = 0$ in the complement of this set. We have for $0 < r < R$

$$\int_{\partial A (r,R)} \varphi \left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, \nu \right\rangle \leq 2C \text{vol}(S_r),$$

where $W_u = \sqrt{1 + ||Gu||^2}$, $W_v = \sqrt{1 + ||Gv||^2}$ and $\text{vol}(S_r)$ is the volume of $S_r = \partial B_r(p)$.

Since the Killing graphs of $u$ and $v$ have the same mean curvature, we have

$$\begin{align*}
div_M \varphi \left( \frac{Gu}{W_u} - \frac{Gv}{W_v} \right) &= \left\langle \nabla \varphi, \frac{Gu}{W_u} - \frac{Gv}{W_v} \right\rangle + \varphi \text{div}_M \left( \frac{Gu}{W_u} - \frac{Gv}{W_v} \right) \\
&= \left\langle \nabla \varphi, \frac{Gu}{W_u} - \frac{Gv}{W_v} \right\rangle \\
&= \left\langle \nabla u - \nabla v, \frac{Gu}{W_u} - \frac{Gv}{W_v} \right\rangle \\
&= \left\langle Gu - Gv, \frac{Gu}{W_u} - \frac{Gv}{W_v} \right\rangle.
\end{align*}$$

on $|u - v| < C$. By Stokes Theorem, we have

$$\int_{A (r,R)} \text{div}_M \varphi \left( \frac{Gu}{W_u} - \frac{Gv}{W_v} \right) = \int_{\partial A (r,R)} \varphi \left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, \nu \right\rangle \leq 2C \text{vol}(S_r). \quad (4)$$

Furthermore, by Lemma 4.1, we get

$$\begin{align*}
div_M \varphi \left( \frac{Gu}{W_u} - \frac{Gv}{W_v} \right) &= \left\langle Gu - Gv, \frac{Gu}{W_u} - \frac{Gv}{W_v} \right\rangle_M \\
&= \frac{W_u + W_v}{2} ||N_u - N_v||_M^2,
\end{align*}$$

when $|u - v| < C$ and $\text{div}_M \varphi \left( \frac{Gu}{W_u} - \frac{Gv}{W_v} \right) = 0$ when $|u - v| \geq C$. Thus we have, by (4) and (5) that
As $r$ decreases to zero we get that $N_u = N_v$ on the set $|u - v| < C$. Hence $Gu = Gv$ in the set $|u - v| < C$.

Since $C$ was arbitrary, we have that $Gu = Gv$ in $A(0, R)$ and $u = v$ in $B_R(p) - \{p\}$. Thus $u = v$ in $B_R(p)$. 

\section{Asymptotic Properties of Sections $u, v : \Omega \to \bar{M}$ with the same prescribed mean curvature and equal on $\partial\Omega$.}

In [2], Pascal Collin and Romain Krust studied graphs $u, v$ over non compact domains $\Omega \subset \mathbb{R}^2$, that have the same mean curvature and with $u = v$ on $\partial\Omega$. They proved that when $u \neq v$, then $|u - v|$ must grow at least like $\log(r)$, $r$ radial distance in $\mathbb{R}^2$.

This theorem of Collin and Krust, and its technique of proof, have had many applications and generalizations.

We now show that their techniques apply to sections $u, v : \Omega \subset M \to \bar{M}$, provided the volume of $\Omega$ intersected with the geodesic spheres of $M$, grows at most linearly in the radius of the sphere. For example, when $\bar{M} = \text{Heisenberg Space}$ and $M = \text{the flat } \mathbb{R}^2$, this is always the case.

\textbf{Theorem 5.1.} Let $\Omega \subset M^n$ be a domain such that $\Omega$ intersects the boundary of each geodesic ball centered at a fixed point in a region whose volume is bounded by a constant times the radius, and $u, v$ two $C^2(\Omega)$ functions such that their Killing graphs have the same mean curvature, $H(u) = H(v)$ in $\Omega$ and $u|\partial \Omega$ and $v|\partial \Omega$ are piecewise differentiable and coincide in the points of continuity. Let $M(r) = \sup_{\Lambda_r} |u - v|$ where $\Lambda_r = \Omega \cap \{x \in M; \text{dist}(x, a) = r\}$. Then \( \liminf_{r \to \infty} \frac{M(r)}{\log r} \geq 0 \) if $u \neq v$. If the volume of $\Lambda_r$ is uniformly bounded then \( \liminf_{r \to \infty} \frac{M(r)}{r} > 0 \).

\textbf{Proof.}

First observe that if $\bar{\Omega}$ is compact then $u = v$ in $\bar{\Omega}$. For, in this case, one can move the graph $\Sigma_u$ of $u$, by the flow $\phi_t$ of $\xi$, and $\phi_t(\partial \Sigma_u) \cap \partial \Sigma_u = \emptyset$ for $t \neq 0$. Choose a largest $|t| \neq 0$ such that $\phi_{|t|}(\Sigma_u) \cap \Sigma_v \neq \emptyset$. Then $\phi_{|t|}(\Sigma_u)$ and $\Sigma_u$ touch at an interior point, hence they are equal by the maximum principle. This is a contradiction.

Recall that the unit normal to the graph $\Sigma_u$ of $u$ is written:

\[ N_u = \frac{-Gu}{W_u} + \frac{1}{W_u} \xi, \]
where \(-Gu\) is horizontal. Let \(X_u\) and \(X_v\) be the horizontal projections of \(\frac{Gu}{W_u}\) and \(\frac{Gv}{W_v}\) to \(M\); also we will think of \(Gu\) and \(Gv\) as tangent to \(M\).

We saw that

\[ Gu - Gv = \nabla u - \nabla v \]

and

\[ \left\langle Gu - Gv, \frac{Gu}{W_u} - \frac{Gv}{W_v} \right\rangle_M = \langle \nabla u - \nabla v, X_u - X_v \rangle_M = \frac{(W_u + W_v)}{2} \| N_u - N_v \|^2_M. \]

These equations are precisely what one needs to prove theorem 5.1 using the technique of Collin-Krust.

We begin the argument and we refer the reader to [2] for the completion of the proof.

Assuming \(u \neq v\), we can suppose

\[ A = \{x \in \Omega / u(x) > v(x)\} \]

is not bounded and connected.

Define \(A_r = \{x \in A, dist(x, p) < r\}\) and \(\Lambda_r = \{x \in A, dist(x, p) = r\}\). We will denote \(vol(\Lambda_r)\) the volume of the \(\Lambda_r\). Let \(r_0\) such that \(\mu = \int_{A_{r_0}} |X_u - X_v|^2 > 0\), where \(X_u = \frac{Gu}{W_u}, X_v = \frac{Gv}{W_v} (\mu \text{ exists since } u \neq v \text{ and } A_{r_0} \neq \emptyset)\).

By Stokes theorem we have

\[
\int_{\partial A_r} (u - v) \langle X_u - X_v, \nu \rangle = \int_{A_r} div((u - v)(X_u - X_v)) = \int_{A_r} \langle \nabla u - \nabla v, X_u - X_v \rangle = \int_{A_r} \langle Gu - Gv, X_u - X_v \rangle. \quad (6)
\]

By Lemma 4.1 we get

\[
\langle Gu - Gv, X_u - X_v \rangle_M = \frac{1}{2}(W_u + W_v) \| N_u - N_v \|^2_M. \quad (7)
\]

Since \(\frac{1}{2}(W_u + W_v) \geq 1\). We have by (6) and (7) that

\[
\int_{A_r} (u - v) \langle X_u - X_v, \nu \rangle = \int_{\partial A_r} (u - v) \langle X_u - X_v, \nu \rangle \geq \int_{A_r} |X_u - X_v|^2. \quad (8)
\]

By (8) we have

\[
\mu + \int_{A_r - A_{r_0}} |X_u - X_v|^2 \leq M(r) \eta(r), \quad (9)
\]
where
\[ \eta(r) = \int_{\Lambda_r} |X_u - X_v|. \]

By Schwartz's lemma, we have
\[ \eta(r)^2 \leq \text{vol}(\Lambda_r) \int_{\Lambda_r} |X_u - X_v|^2. \]  
(10)

Now the reader can read [2], for the completion of the argument.

**Remark.** For graphs of prescribed mean curvature in Heisenberg space, over domains \( \Omega \subset \mathbb{R}^2 \), we conclude there is at most one bounded solution of the mean curvature equation over \( \Omega \), with given boundary values. In particular, a bounded entire minimal graph is constant.

**References**


Claudemir Leandro
Instituto de Matemática Pura e Aplicada - IMPA, Estrada Dona Castorina 110, Rio de Janeiro, 22460-320 Brasil; e-mail: claudemi@impa.br

Harold Rosenberg
Institut de Mathématiques, Université Paris VII, 2 place Jussieu, 75005 Paris, France; e-mail: rosen@math.jussieu.fr