Minimal surfaces and harmonic diffeomorphisms
from the complex plane onto certain Hadamard surfaces.

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Abstract. We construct harmonic diffeomorphisms from the complex plane \( \mathbb{C} \) onto any Hadamard surface \( \mathbb{M} \) whose curvature is bounded above by a negative constant. For that, we prove a Jenkins-Serrin type theorem for minimal graphs in \( \mathbb{M} \times \mathbb{R} \) over domains of \( \mathbb{M} \) bounded by ideal geodesic polygons and show the existence of a sequence of minimal graphs over polygonal domains converging to an entire minimal graph in \( \mathbb{M} \times \mathbb{R} \) with the conformal structure of \( \mathbb{C} \).

1 Introduction.

There are many harmonic diffeomorphisms from the complex plane \( \mathbb{C} \) onto the hyperbolic plane \( \mathbb{H} \). They were constructed by finding entire minimal graphs in \( \mathbb{H} \times \mathbb{R} \) whose conformal type is \( \mathbb{C} \) [CR]. The vertical projection of such a graph onto \( \mathbb{H} \) is such a harmonic diffeomorphism. It was conjectured that there was no such map [SY].

In this paper we will show there are harmonic diffeomorphisms from \( \mathbb{C} \) onto any Hadamard surface whose curvature is bounded above by a negative constant. The question of their existence was posed by R. Schoen.

We proceed as in [CR] by constructing entire minimal graphs in \( \mathbb{M} \times \mathbb{R} \), of conformal type \( \mathbb{C} \); \( \mathbb{M} \) a complete simply connected Riemannian surface with curvature \( K_{\mathbb{M}} \leq a < 0 \). The construction of these graphs in \( \mathbb{H} \times \mathbb{R} \) can be done in \( \mathbb{M} \times \mathbb{R} \); the geometry of the asymptotic boundary of \( \mathbb{M} \) is sufficiently close to that of \( \mathbb{H} \).

We are thus able to prove a Jenkins-Serrin type theorem for minimal graphs in \( \mathbb{M} \times \mathbb{R} \), over domains of \( \mathbb{M} \) bounded by ideal geodesic polygons. There are several constructions in our paper, which we believe will be useful for future research.

An interesting question is whether our theorems hold when \( K_{\mathbb{M}} < 0 \).

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2 Preliminaries.

We will devote this Section to present some basic properties of Hadamard manifolds, which will be necessary for our study (see for instance [E1, E2, E0] for details).

Let $\mathcal{M}$ be a Hadamard manifold, that is, a complete simply connected Riemannian manifold with non positive sectional curvature. It is classically known that there is a unique geodesic joining two points of $\mathcal{M}$. Thus, the concept of (geodesic) convexity is naturally defined for sets in $\mathcal{M}$.

We say that two geodesics $\gamma_1(t), \gamma_2(t)$ of $\mathcal{M}$, parametrized by arc length, are asymptotic if there exists a constant $c > 0$ such that the distance $d(\gamma_1(t), \gamma_2(t))$ is less than $c$ for all $t \geq 0$. Analogously, two unit vectors $v_1, v_2$ are said to be asymptotic if the corresponding geodesics $\gamma_{v_1}(t), \gamma_{v_2}(t)$ have this property; where, $\gamma_{v_i}(t)$ denotes the geodesic of $\mathcal{M}$ such that $\gamma'_{v_i}(0) = v_i$.

To be asymptotic is an equivalence relation on the oriented unit speed geodesics or on the set of unit vectors of $\mathcal{M}$. Each one of these equivalence classes will be called a point at infinity, and $\mathcal{M}(\infty)$ will denote the set of points at infinity.

We will denote by $\gamma(+\infty)$ or $v(\infty)$ the equivalence class of the corresponding geodesic $\gamma(t)$ or unit vector $v$.

Henceforth we shall always assume a Hadamard manifold $\mathcal{M}$ has sectional curvature bounded from above by a negative constant. Then any two asymptotic geodesics $\gamma_1, \gamma_2$ satisfy that the distance between the curves $\gamma_1|_{[0, +\infty)}, \gamma_2|_{[0, +\infty)}$ is zero for any $t_0 \in \mathbb{R}$. In addition, under this curvature hypothesis, given $x, y \in \mathcal{M}(\infty)$ there exists a unique oriented unit speed geodesic $\gamma$ such that $\gamma(+\infty) = x$ and $\gamma(-\infty) = y$, where $\gamma(-\infty)$ is the corresponding point at infinity when we change the orientation of $\gamma$.

For any point $p$ of a general Hadamard manifold, there is a bijective correspondence between the set of unit vectors at $p$ and $\mathcal{M}(\infty)$, where a unit vector $v$ is mapped to the point at infinity $v(\infty)$. Equivalently, given a point $p \in \mathcal{M}$ and a point $x \in \mathcal{M}(\infty)$, there exists a unique oriented unit speed geodesic $\gamma$ such that $\gamma(0) = p$ and $\gamma(+\infty) = x$. In particular, $\mathcal{M}(\infty)$ is bijective to a sphere.

In fact, there exists a topology on $\mathcal{M}^* = \mathcal{M} \cup \mathcal{M}(\infty)$ satisfying

1. the restriction to $\mathcal{M}$ agrees with the topology induced by the Riemannian distance,

2. for any $p \in \mathcal{M}$ and any homeomorphism $h : [0, 1] \rightarrow [0, \infty]$ the function $\varphi$, from the closed unit ball of $T_p\mathcal{M}$ onto $\mathcal{M}^*$, given by $\varphi(v) = \exp(h(\|v\|)v)$ is a homeomorphism. Moreover, $\varphi$ identifies $\mathcal{M}(\infty)$ with the unit sphere,

3. the map $v \rightarrow v(\infty)$ is a homeomorphism from the unit sphere of the tangent plane at a fixed point $p$ onto $\mathcal{M}(\infty)$.

This topology is called the cone topology of $\mathcal{M}^*$ and can be obtained as follows. Let $p \in \mathcal{M}$ and $\mathcal{U}$ an open set in the unit sphere of its tangent plane. Define for any $r > 0$

$$T(\mathcal{U}, r) = \{ \gamma_v(t) \in \mathcal{M}^* : v \in \mathcal{U}, r < t \leq +\infty \}.$$
The cone topology is the unique one such that its restriction to $\mathbb{M}$ is the topology induced by the Riemannian distance and such that the sets $T(\mathbb{U}, r)$ containing a point $x \in \mathbb{M}(\infty)$ form a neighborhood basis at $x$.

Given a set $A \subseteq \mathbb{M}$, we denote by $\partial_\infty A$ the set $\partial A \cap \mathbb{M}(\infty)$, where $\partial A$ is the boundary of $A$ for the cone topology.

Horospheres are defined in terms of Busemann functions. Given a unit vector $v$, the Busemann function $B_v : \mathbb{M} \rightarrow \mathbb{R}$, associated to $v$, is

$$B_v(p) = \lim_{t \to +\infty} d(p, \gamma_v(t)) - t.$$

This function verifies some important properties

1. $B_v$ is a $C^2$ convex function on $\mathbb{M}$,
2. the gradient $\nabla B_v(p)$ is the unique unit vector $w$ at $p$ such that $v(\infty) = -w(\infty)$,
3. if $w$ is a unit vector such that $v(\infty) = w(\infty)$ then $B_v - B_w$ is a constant function on $\mathbb{M}$.

Given a point $x \in \mathbb{M}(\infty)$ and a unit vector $v$ such that $v(\infty) = x$ we define the horospheres at $x$ as the level sets of the Busemann function $B_v$. By property 3, the horospheres at $x$ do not depend on the choice of $v$. The horospheres at a point $x \in \mathbb{M}(\infty)$ give a foliation of $\mathbb{M}$ and, from property one, each one bounds a convex domain in $\mathbb{M}$ called a horoball. Moreover, the intersection between a geodesic $\gamma$ and a horosphere at $\gamma(+\infty)$ is always orthogonal from property two.

With respect to distance from horospheres we present the following facts (see [EO]).

1. Let $p \in \mathbb{M}$, $H_x$ a horosphere at $x$ and $\gamma$ the geodesic passing through $p$ having $x$ as a point at infinity, then $H_x \cap \gamma$ is the closest point on $H_x$ to $p$.
2. If $\gamma$ is a geodesic with points at infinity $x, y$, and $H_x, H_y$ are disjoint horospheres at these points then the distance between $H_x$ and $H_y$ agrees with the distance between the points $H_x \cap \gamma$ and $H_y \cap \gamma$.
3. The function $D : \mathbb{M} \times \mathbb{M}^* \times \mathbb{M} \rightarrow \mathbb{R}$ given by

$$D(a, b, c) = \begin{cases} d(c, b) - d(a, b) & \text{if } b \in \mathbb{M} \\ B_v(c) & \text{if } b \in \mathbb{M}(\infty) \end{cases}$$

is continuous, where $v$ is the unique unit tangent vector at $a$ such that $v(\infty) = b$. $D(a, b, c)$ measures the difference between the oriented distance from $a$ and $c$ to any horosphere at $b \in \mathbb{M}(\infty)$. In particular, $D(a, b, c) < 0$ means that $c$ is in the horoball whose boundary is the horosphere at $b$ passing across $a$. 
3 A Jenkins-Serrin type theorem for ideal polygons.

From now on we will assume $\mathbb{M}$ is a simply connected, complete surface with Gauss curvature bounded from above by a negative constant.

We say that $\Gamma$ is an ideal polygon if $\Gamma$ is a Jordan curve in $\mathbb{M}^*$ which is a geodesic polygon with an even number of sides and all the vertices in $\mathbb{M}(\infty)$. As usual, we will denote by $A_1, B_1, \ldots, A_k, B_k$ the sides of $\Gamma$, which are oriented counter-clockwise.

Now, we study the Dirichlet problem for the minimal surface equation in the domain $D$ bounded by an ideal polygon $\Gamma$. That is, we look for a solution $u : D \rightarrow \mathbb{R}$ to the equation

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$  \hspace{1cm} (3.1)

Here, we prescribe the $+\infty$ data on each side $A_i$ and $-\infty$ on each side $B_i$.

For relatively compact domains $D \subseteq \mathbb{M}$, it is well-known that there are necessary and sufficient conditions on the lengths of the sides of polygons inscribed in $\Gamma$ in order to solve this Dirichlet problem (see [JS], [NR], [P]).

When $\Gamma$ is an ideal polygon the length of each side is infinity and the previous conditions make no sense. However, in [CR], the authors devise a manner to compare the “lengths” of sides.

Fix an ideal polygon $\Gamma$ and consider pairwise disjoint horocycles $H_i$ at each vertex $a_i$ of $\Gamma$.

For each side $A_i$, let us denote by $\tilde{A}_i$ the compact geodesic arc between the horocycles at the vertices of $A_i$, and by $|A_i|$ the length of $\tilde{A}_i$, that is, the distance between the horocycles. Analogously, one defines $\tilde{B}_i$ and $|B_i|$ for each side $B_i$, (cf. Figure 1).

![Figure 1.](image)

Observe that if we define

$$a(\Gamma) = \sum_{i=1}^{k} |A_i|, \quad b(\Gamma) = \sum_{i=1}^{k} |B_i|,$$

then...
then $a(\Gamma) - b(\Gamma)$ does not depend on the choice of horocycles. For that, it is sufficient to observe that given two horocycles $H_1, H_2$ at a point $x \in \mathbb{M}(\infty)$ and any geodesic $\gamma$ with $x$ as a point at infinity, then the distance between $H_1$ and $H_2$ agrees with the distance between the points $\gamma \cap H_1$ and $\gamma \cap H_2$. Hence, if we change a horocycle at a vertex of $\Gamma$ then $a(\Gamma)$ and $b(\Gamma)$ increase or decrease in the same quantity.

Let $D$ be the domain bounded by an ideal polygon $\Gamma$. We say that a simple closed geodesic polygon $P$ is *inscribed* in $D$ if each vertex of $P$ is a vertex of $\Gamma$.

Each side of $P$ is one side $A_i$ or $B_i$ of $\Gamma$, or a geodesic contained in $D$ (cf. Figure 2). Thus, the definition of $a(\Gamma)$ and $b(\Gamma)$ extends to $P$ as follows. Consider pairwise disjoint horocycles $H_i$ at each vertex $a_i$ of $P$. For each side $A_i \subseteq P$, denote by $\widetilde{A}_i$ the compact geodesic arc between the horocycles at the vertices of $A_i$, and by $|A_i|$ the length of $\widetilde{A}_i$, that is, the distance between the horocycles. Analogously, one defines $\widetilde{B}_i$ and $|B_i|$ for each side $B_i \subseteq P$. Then, we consider

$$a(P) = \sum_{A_i \subseteq P} |A_i|, \quad b(\Gamma) = \sum_{B_i \subseteq P} |B_i|.$$  

In addition, we define the truncated length of the inscribed polygon $|P|$ as the sum of the lengths of the compact arcs of each side of $P$ bounded by the horocycles at its vertices.

![Figure 2.](image)

Now, we can state a Jenkins-Serrin type theorem on domains of $\mathbb{M}$ bounded by an ideal polygon $\Gamma$.

**Theorem 3.1.** There is a solution to the Dirichlet problem for the minimal surface equation in the domain $D$ bounded by $\Gamma$ with prescribed data $+\infty$ at $A_i$ and $-\infty$ at $B_i$ if, and only if, the following two conditions are satisfied

1. $a(\Gamma) - b(\Gamma) = 0$,
2. for all inscribed polygons $P$ in $D$ different from $\Gamma$ there exist horocycles at the vertices such that

$$2a(P) < |P| \quad \text{and} \quad 2b(P) < |P|.$$
Moreover, the solution is unique up to additive constants.

Remark 3.1. Notice that \( a(\mathcal{P}) \) and \( b(\mathcal{P}) \) depend on the chosen horocycles at the vertices. However, if condition 2 is satisfied for a particular choice of horocycles then it is also satisfied for all smaller horocycles at the vertices.

In addition, let \( A_i, B_j \) be the two sides of \( \Gamma \) with a common vertex of \( \mathcal{P} \). If the side \( A_i \) does not belong to \( \mathcal{P} \) then \( 2a(\mathcal{P}) < \left| \mathcal{P} \right| \) is satisfied for the choice of a small horocycle at the vertex. Thus, if \( \mathcal{P} \) is an inscribed polygon in \( D \) such that there exists a vertex of \( \mathcal{P} \) not containing the adjacent side \( A_i \) of \( \Gamma \) and another vertex of \( \mathcal{P} \) not containing the adjacent side \( B_j \) of \( \Gamma \), then condition 2 is satisfied for the polygon \( \mathcal{P} \).

Proof of Theorem 3.1. This Theorem was proved in [CR] when \( M \) is the hyperbolic plane \( \mathbb{H} \). The reader can check their proof works for a general surface \( M \) once the existence of a Scherk type surface on each halfspace of \( M \) is established.

This Scherk type surface in \( \mathbb{H} \) is unique up to isometries of the ambient space and was explicitly computed by U. Abresch and R. Sa Earp [Sa]. For a general \( M \) we now show its existence.

Proposition 3.1. Let \( \gamma \) be a complete geodesic in \( M \) and \( \Omega \) a connected component of \( M - \gamma \). There exists a positive solution \( u \) to the Dirichlet problem for the minimal surface equation in \( \Omega \) with prescribed data \(+\infty\) at \( \gamma \) and such that

\[
\lim\{u(p_n)\} = 0
\]

for each sequence \( \{p_n\} \) of points in \( \Omega \) with distance to \( \gamma \) going to infinity.

Proof. Since \( M \) is a Hadamard surface then

\[
\varphi(s, t) = \exp_{\gamma(t)}(s J\gamma'(t)), \quad (s, t) \in \mathbb{R}^2
\]

is a global parametrization of \( M \), where the geodesic \( \gamma(t) \) is parametrized by arc length, \( \exp \) is the usual exponential map and \( J \) stands for the rotation in \( M \) by \( \pi/2 \). In addition, we can assume that \( \Omega \) is parametrized for \( s > 0 \).

We observe that

\[
\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \rangle = 1, \quad \langle \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \rangle = 0
\]

where \( \langle , \rangle \) is the induced metric in \( M \). Moreover, if we denote by \( G(s, t) \) the function \( \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle \) then

\[
G(0, t) = 1, \quad G_s(0, t) = 0, \quad t \in \mathbb{R}
\]

since \( \gamma(t) \) is a geodesic. Here, \( G_s \) denotes the derivative of \( G \) with respect to \( s \).

Now, we consider a graph \( \psi(s, t) = (\varphi(s, t), h(s)) \) on \( \Omega \) which has constant height for equidistant points to \( \gamma \), that is, when \( s \) is constant.
For the unit normal of the graph pointing down, the mean curvature of the immersion is positive if and only if
\[ G_s h_s (1 + h_s^2) + 2G h_{ss} < 0, \quad s > 0, t \in \mathbb{R}. \] (3.3)

On the other hand, the Gauss curvature of \( M \) is given by
\[ K(s, t) = -\frac{1}{4} \left( \frac{G_s}{G} \right)^2 - \frac{1}{2} \left( \frac{G_s}{G} \right)_s. \] (3.4)

We notice that, for any constant \( d < 0 \), the function \( \tilde{G}(s) = \cosh^2(\sqrt{-d}s) \) verifies
\[ d = -\frac{1}{4} \left( \frac{\tilde{G}_s}{\tilde{G}} \right)^2 - \frac{1}{2} \left( \frac{\tilde{G}_s}{\tilde{G}} \right)_s. \] (3.5)

Observe that \( ds^2 + \tilde{G}(s) \, dt^2 \) is the hyperbolic metric of curvature \( d \). Moreover, the function
\[ \tilde{h}(s) = -\frac{1}{\sqrt{-d}} \log \left( \tanh \left( \frac{1}{2} \sqrt{-d}s \right) \right), \quad s > 0, \] (3.6)
is decreasing and satisfies
\[ \tilde{G}_s \tilde{h}_s (1 + \tilde{h}_s^2) + 2\tilde{G} \tilde{h}_{ss} = 0. \] (3.7)
That is, \( \tilde{h}(s) \) is the minimal graph in the hyperbolic space found by Abresch and Sa Earp.

Since \( K(s, t) \) is bounded from above by a negative constant \( c \), we can choose \( d \) such that \( c < d < 0 \). Then from (3.4) and (3.5)
\[ \left( \frac{G_s}{G} \right)^2 + 2 \left( \frac{G_s}{G} \right)_s > \left( \frac{\tilde{G}_s}{\tilde{G}} \right)^2 + 2 \left( \frac{\tilde{G}_s}{\tilde{G}} \right)_s. \] (3.8)

Now, we observe that given two real functions \( f(x), g(x) \) defined on an interval \( I \), with \( f(x_0) = g(x_0) \) and satisfying
\[ 2f'(x) + f(x)^2 > 2g'(x) + g(x)^2, \]
then \( f(x) > g(x) \), for all \( x > x_0 \) on \( I \).

Thus, from (3.2), (3.7) and (3.8),
\[ \frac{G_s}{G} > \frac{\tilde{G}_s}{\tilde{G}} = \frac{-2\tilde{h}_{ss}}{\tilde{h}_s(1 + \tilde{h}_s^2)}, \quad s > 0, t \in \mathbb{R} \]
or equivalently, \( h = \tilde{h} \) satisfies the inequality in (3.3). That is, the graph \( \psi(s, t) = (\varphi(s, t), \tilde{h}(s)) \) on \( \Omega \) has strictly positive mean curvature for its unit normal pointing down. This graph has value \(+\infty\) on \( \gamma \) and goes to zero when the distance to \( \gamma \) tends to infinity.

Finally, we obtain the minimal graph with the desired properties as follows. Let \( p \in \gamma \subseteq \mathbb{M} \), \( C(n) \) the geodesic circumference in \( \mathbb{M} \) centered at \( p \) and radius \( n \), \( A(n) = C(n) \cap \Omega \) and \( B(n) \) the segment \( \gamma([-n, n]) \). Now, consider the Jordan curve \( \Gamma(n) \) in \( \mathbb{M} \times \mathbb{R} \) obtained by the arcs \( A(n) \times \{0\}, B(n) \times \{n\} \) and the vertical segments joining their end points. Let \( \Sigma_n \) be the minimal disk which is a solution of the Plateau problem for \( \Gamma(n) \).

\( \Sigma_n \) is the graph of a function \( u_n \) on the domain bounded by \( A(n) \cup B(n) \) with \( u_n|A(n) = 0 \) and \( u_n|B(n) = n \). The sequence \( \{u_n\} \) is non decreasing and non negative. In addition, from the comparison principle, it is bounded from above by \( \tilde{h} \).

Thus, for any compact set \( K \subseteq \Omega \), we have a non decreasing and bounded sequence of minimal graphs \( \{u_n|K\} \), and therefore \( \{u_n\} \) must converge to a minimal graph \( u \) on \( \Omega \).

Now, consider a sequence \( \{p_n\} \) of points in \( \Omega \) with distance to \( \gamma \) going to infinity. Then, \( p_n = \varphi(s_n, t_n) \) with \( \{s_n\} \) going to infinity. Thus, since

\[
0 \leq u(p_n) \leq \tilde{h}(s_n) \quad \text{and} \quad \lim_{s \to \infty} \tilde{h}(s) = 0,
\]

we obtain \( \lim\{u(p_n)\} = 0 \).

On the other hand, let \( \{q_n\} \) be a sequence of points in \( \Omega \) converging to \( q \in \gamma \) and let \( R > 0 \). Given \( N \) large enough, \( u_N(q) > R \), and so there exists an open neighbourhood \( U \) of \( q \) such that \( u_N(p) > R \) for all \( p \in U \). Since \( \lim\{q_n\} = q \) there exists \( n_0 \) such that if \( n \geq n_0, p_n \in U \) and \( u(p_n) \geq u_N(p_n) > R \). Therefore, \( u \) is a minimal graph in \( \Omega \) with prescribed data \(+\infty\) at \( \gamma \).

This completes the proof of Proposition 3.1; hence, Theorem 3.1 as well. \( \square \)

As in [CR], we can extend Theorem 3.1 to more general domains. We say that a convex domain \( D \subseteq \mathbb{M} \) is admissible if

1. the (non empty) finite set \( \partial_{\infty} D \) are the vertices of an ideal polygon,
2. given two convex arcs \( C_1, C_2 \subseteq \partial D \) with a common vertex \( x \in \partial_{\infty} D \) there exist two sequences of points \( x_n \in C_1, y_n \in C_2 \) converging to \( x \), such that the distance between \( x_n \) and \( y_n \) tends to zero.

The second condition in the above definition is used in order to obtain a maximum principle for minimal graphs over the domain \( D \). Moreover, the domain bounded by an ideal polygon is admissible since the distance between two geodesics with a common point at infinity goes to zero.

Now, we present a Jenkins-Serrin type theorem when we fix continuous boundary data on some components of the boundary, whose proof can be shown as in [CR].

Let \( D \) be an admissible domain in \( \mathbb{M} \). We seek a solution to the Dirichlet problem for the minimal surface equation in \( D \) which is \(+\infty\) on geodesic sides \( A_1, \ldots, A_k \) of \( \partial D \), and equals
on other geodesic sides $B_1, \ldots, B_k'$ of $\partial D$, and equal to continuous functions $f_i : C_i \longrightarrow \mathbb{R}$ on the remaining (nonempty) convex arcs of $\partial D$.

**Theorem 3.2.** There exists a unique solution in $D$ to the above Dirichlet problem if, and only if,

$$2a(\mathcal{P}) < |\mathcal{P}| \quad \text{and} \quad 2b(\mathcal{P}) < |\mathcal{P}|$$

for all inscribed polygon $\mathcal{P}$ in $D$.

The uniqueness property in the two previous Theorems is guaranteed by a maximum principle over admissible domains. The proof when $\mathbb{M} = \mathbb{H}$ was given in [CR] and it also works in our situation.

**Theorem 3.3.** (Generalized Maximum Principle) Let $D \subseteq \mathbb{M}$ be an admissible domain. Let us consider a domain $\Omega \subseteq D$ and $u, v \in C^0(\bar{\Omega})$ two solutions to the minimal surface equation in $\Omega$ with $u \leq v$ on $\partial \Omega$. Then $u \leq v$ in $\Omega$.

In addition, as it was explained in [CR], the previous maximum principle also applies when the solutions $u, v$ to the minimal surface equation have infinite boundary values along some geodesic arcs of $\partial D$. When $\partial D$ is made of complete geodesics and $u, v$ take the same infinite value on the whole boundary of $D$, then $u$ and $v$ agree up to an additive constant.

### 4 Extending the domain of a solution.

The aim of this section is to show that a solution $u$ to the minimal surface equation on an admissible polygonal domain $D$ with infinity boundary data can be “extended” to a larger polygonal domain. By this we mean that a solution $v$ on a larger domain can be chosen arbitrarily close to $u$ on a fixed compact set $K$ of $D$.

To establish this result, we first need to show the existence of some special ideal quadrilaterals. We will call ideal Scherk surface a graph given by Theorem 3.1, and Scherk domain the domain where it is defined.

**Proposition 4.1.** Let $x, y, z$ be three points in $\mathbb{M}(\infty)$. Let $\gamma$ be the geodesic joining $x$ and $y$, and $\Omega$ the connected component of $\mathbb{M} - \gamma$ such that $z \notin \partial_{\infty} \Omega$. Then there exist a point $w \in \partial_{\infty} \Omega$ and an ideal Scherk surface over the domain bounded by the ideal quadrilateral with vertices $x, y, z, w$. (Cf. Figure 3.)
We start by establishing some previous Lemmas in order to prove the above Proposition.

**Lemma 4.1.** Let $H_1, H_2$ be two different horocycles in $\mathbb{M}$. Then they intersect at most at two points.

**Proof.** Let us assume that $H_1, H_2$ are horocycles at $x, y \in \mathbb{M}(\infty)$, respectively. If $x = y$ then they do not intersect. So, we can suppose $x \neq y$.

Let $p \in H_1 \cap H_2$. Then, the intersection between the two horocycles at $p$ is transversal, unless $p$ is in the geodesic $\gamma_{xy}$ joining $x$ and $y$. In the latter case, if $\gamma_p$ is the geodesic through $p$ tangent to $H_1$ and $H_2$, then each convex horodisk $B_1, B_2$, with respective boundaries $H_1, H_2$, must be on different sides of $\gamma_p$. Therefore, each horocycle is in the concave part of the other and the intersection is only $p$.

Thus, if $H_1 \cap H_2$ has more than one point then each intersection is transversal. Let us consider $p_1, p_2 \in H_1 \cap H_2$ and $I$ the compact arc in $H_1$ joining $p_1$ and $p_2$. Let $p_0$ be a point in $I$ at the largest distance from $H_2$. Then, the horocycle $H_2$ at $y$ passing trough $p_0$ intersects $H_1$ in a tangent way. In particular, $p_0$ is the point $\gamma_{xy} \cap H_1$.

So, for any two points $p_1, p_2 \in H_1 \cap H_2$, we have that $p_0$ is in the interior of the compact arc of $H_1$ joining $p_1$ and $p_2$. Therefore, there are at most two points in the intersection between $H_1$ and $H_2$. \hfill $\square$

**Lemma 4.2.** Let $x, y \in \mathbb{M}(\infty)$, and $H_x, H_y$ two disjoint horocycles at $x, y$, respectively. Consider the open set $I$ of points in $\mathbb{M}(\infty)$ between $x$ and $y$, where we assume $\mathbb{M}(\infty)$ ordered counter-clockwise. For any point $z$, we define $L : I \longrightarrow \mathbb{R}$ as

$$L(z) = d(H_y, H_z) - d(H_z, H_x),$$

where $H_z$ is any horocycle at $z$ disjoint from the previous ones and $d(,)$ denotes distance in $\mathbb{M}$. Then, $L$ is a homeomorphism from $I$ onto $\mathbb{R}$. 

Proof. We observe that the definition of the function $L$ does not depend on the chosen horocycle $H_z$ at $z$, and also makes sense when $H_z$ intersects $H_x$ or $H_y$ in one point.

Let $h_1 : \mathbb{M}(\infty) - \{x\} \to H_x$ be the map defined as follows. For each point $z \in \mathbb{M}(\infty) - \{x\}$, we consider $h_1(z)$ as the unique point given by the intersection between $H_x$ and the geodesic joining $x$ and $z$. It is well known that $h_1$ is a homeomorphism. Analogously, we consider the homeomorphism $h_2 : \mathbb{M}(\infty) - \{y\} \to H_y$.

Then $L(z) = d(h_2(z), H_z) - d(H_z, h_1(z)) = \mathcal{D}(h_1(z), z, h_2(z))$ is a continuous function, where $\mathcal{D}$ is given by (2.1).

Now, we see that $L$ is injective. Let $p, q$ be two points in $\mathcal{I}$ oriented such that $x < p < q < y$. Let $H_p, H_q$ be the smallest horocycles (i.e. bounding the smallest convex closed domain) at $p, q$, respectively, such that they intersect $H_x \cup H_y$. We distinguish three cases:

1. $H_q$ only intersects $H_y$ and $H_p$ only intersects $H_x$,
2. $H_q$ intersects $H_x$,
3. $H_p$ intersects $H_y$.

The case 1 is trivial because $L(q) < 0 < L(p)$. And, since the cases 2 and 3 are symmetric, we only need to study case 2.

First, we observe that $H_p$ does not intersects $H_y$. Otherwise, the intersection holds at $h_2(p)$ and each connected component of $H_p - \{h_2(p)\}$ intersects twice to $H_q$, which contradicts Lemma 4.1 (cf. Figure 4).

![Figure 4](image.png)

Figure 4.

Thus, $H_p$ and $H_q$ intersect $H_x$ and we only need to show that $d(H_q, H_y) < d(H_p, H_y)$ in order to obtain that $L(q) < L(p)$. For that, enlarge $H_y$ to the first horocycle $\tilde{H}_y$ that intersects $H_p \cup H_q$. From the previous discussion, replacing $H_y$ and $\tilde{H}_y$, $H_p$ does not intersects $\tilde{H}_y$ (cf. Figure 5). Hence, $d(H_q, H_y) < d(H_p, H_y)$ and the case 2 is proved.
Now, let us see that there exists a sequence of points \( p_n \in I \) such that \( L(p_n) \) goes to \(-\infty\).

Let \( \gamma \) be the geodesic joining \( x \) and \( y \) and \( \Omega \) the connected component of \( \mathbb{M} - \gamma \) which contains \( I \) in its ideal boundary. Consider \( q \in H_y \cap \Omega \) and \( \gamma_q \) the unique geodesic joining \( q \) and \( x \). Let us denote by \( p \) the other point of \( \gamma_q \) at the ideal boundary.

The smallest horocycle \( H_p \) intersecting \( H_x \cup H_y \) does not intersects \( H_x \). Otherwise, \( \gamma_q \) would be contained in the horodisks bounded by \( H_x \) and \( H_q \). But, since \( q \in \gamma_q \cap H_y \) then \( q \) would be in the horodisk bounded by \( H_p \), which is a contradiction (cf. Figure 6).

Thus \( L(p) = -d(H_p, H_x) = -(d(H_p, q) + d(q, H_x)) \leq -d(q, H_x) \). Therefore, taking a sequence \( q_n \in H_y \cap \Omega \) converging to \( y \), we obtain the corresponding sequence \( p_n \) such that \( \lim L(p_n) \leq -\lim d(q_n, H_x) = -\infty \).

Analogously, it can be proved that there exists a sequence of points \( p_n \in I \) such that \( L(p_n) \) goes to \(+\infty\).

Hence, since \( I \) has the topology of an interval, \( L \) is a strictly monotonous continuous function, and there exist sequences in \( I \) whose image tend to \(-\infty \) and \(+\infty \), then \( L \) is a homeomorphism from \( I \) onto \( \mathbb{R} \).
Now, let us denote by $|xy|$ the distance between two points in $\mathbb{M}^* = \mathbb{M} \cup \mathbb{M}(\infty)$, where we indicate distance between horocycles if $x$ or $y$ are in $\mathbb{M}(\infty)$.

**Lemma 4.3. (Generalized Triangle Inequality.)** Consider a triangle with vertices $x_1, x_2$ in $\mathbb{M}^* = \mathbb{M} \cup \mathbb{M}(\infty)$ and another point $x_3 \in \mathbb{M}$. Then,

$$|x_1x_2| \leq |x_1x_3| + |x_3x_2|.$$  

Moreover, if $x_1, x_2, x_3 \in \mathbb{M}(\infty)$ then there exist horocycles at these points such that the following three inequalities are simultaneously satisfied

$$|x_ix_j| < |x_ix_k| + |x_kx_j|, \quad \{i, j, k\} = \{1, 2, 3\}.$$

**Remark 4.1.** When $x_3 \in \mathbb{M}$, the quantity $|x_1x_3| + |x_3x_2| - |x_1x_2|$ does not depend on the chosen disjoint horocycles, if any. However, it is important to bear in mind that $|x_ix_k| + |x_kx_j| - |x_ix_j|$ depends on the chosen horocycles if $x_1, x_2, x_3 \in \mathbb{M}(\infty)$.

**Proof of Lemma 4.3.** First, we consider the case $x_3 \in \mathbb{M}$.

If $x_1, x_2 \in \mathbb{M}$ then the inequality is clear. Thus, let us assume $x_1 \in \mathbb{M}(\infty)$. Then, if $x_2 \in \mathbb{M}$, enlarge the horocycle $H_{x_1}$ to another horocycle $\tilde{H}_{x_1}$ which intersects $x_2$ or $x_3$ for the first time.

Otherwise, if $x_2 \in \mathbb{M}(\infty)$, enlarge the horocycle $H_{x_1}$ to another horocycle $\tilde{H}_{x_1}$ which intersects $H_{x_2}$ or $x_3$ for the first time.

If $x_2$ (or $H_{x_2}$) intersects $\tilde{H}_{x_1}$, the inequality is clear. Otherwise, $x_3 \in \tilde{H}_{x_1}$, and so the distance from $x_2$ (or $H_{x_2}$) to $\tilde{H}_{x_1}$ is less than or equal to the distance to $x_3$ and the inequality also holds.

Finally, we consider $x_1, x_2, x_3 \in \mathbb{M}(\infty)$ and three pairwise disjoint horocycles $H_{x_1}, H_{x_2}, H_{x_3}$. Now, fix $H_{x_1}, H_{x_2}$ and consider a small enough horocycle $\tilde{H}_{x_3}$ such that $|x_1x_2| < |x_1x_3| + |x_3x_2|$.

To obtain the second inequality, we consider a smaller horocycle $\tilde{H}_{x_2}$ at $x_2$, if necessary, such that $|x_1x_3| < |x_1x_2| + |x_2x_3|$. And now we observe that the first inequality remains unchanged for the horocycles $H_{x_1}, \tilde{H}_{x_2}, \tilde{H}_{x_3}$.

Following the same process, we take a small horocycle $\tilde{H}_{x_1}$ at $x_1$ such that $|x_2x_3| < |x_2x_1| + |x_1x_3|$. Since the previous two inequalities do not change, the Lemma follows. $\square$

**Proof of Proposition 4.1.** Let $H_z$ be a horocycle at $z$. Take $H_x, H_y$ disjoint horocycles at $x, y$, respectively, at the same distance from $H_z$. Then, using Lemma 4.2, there exists a point $w \in \partial_\infty \Omega$ such that $d(H_w, H_x) = d(H_w, H_y)$ for any horocycle $H_w$ at $w$ disjoint from $H_x$ and $H_y$. That is, $a(\Gamma) - b(\Gamma) = 0$ for the ideal quadrilateral $\Gamma$ with vertices $x, y, z, w$.

In addition, if $\mathcal{P}$ is an inscribed polygon in the domain bounded by the ideal quadrilateral $\Gamma$, and different from $\Gamma$, then $\mathcal{P}$ must be an ideal triangle. So, $\mathcal{P}$ has a vertex not containing an adjacent side $A_i$ and another vertex not containing an adjacent side $B_i$. Hence, from Remark
3.1, condition 2 in Theorem 3.1 is satisfied and, so, there exists an ideal Scherk surface over the domain bounded by the ideal quadrilateral $\Gamma$. □

Now, we establish some notation.

Given an even set of points $a_0, a_1, \ldots, a_{2n-1}$ in $\mathbb{M}(\infty)$, which we will assume ordered counter-clockwise, we denote by $\mathcal{P}(a_0, a_1, \ldots, a_{2n-1})$ the ideal polygon in $\mathbb{M}$ whose vertices are these points.

In order to obtain an ideal Scherk surface on the domain bounded by $\mathcal{P}(a_0, a_1, \ldots, a_{2n-1})$, we will fix $+\infty$ boundary data on the sides $[a_{2k}, a_{2k+1}]$ and $-\infty$ boundary data on the sides $[a_{2k+1}, a_{2k+2}]$. Here, $[x, y]$ denotes the complete geodesic joining the points $x, y \in \mathbb{M}(\infty)$, and we identify $a_{2n} = a_0$.

**Proposition 4.2.** Let $u$ be an ideal Scherk graph on the domain $D$ bounded by an ideal polygon $\mathcal{P}(a_0, a_1, a_2, \ldots, a_{2n-1})$. Now, we attach to $D$ two Scherk domains bounded by $\mathcal{P}(a_0, b_1, b_2, a_1)$ and $\mathcal{P}(a_1, b_3, b_4, a_2)$. Then, given a compact set $K \subseteq D$ and $\varepsilon > 0$, there exists an ideal Scherk graph $v$ on the domain bounded by the ideal polygon $\mathcal{P}(a_0, b_1', b_2', a_1, b_3', b_4, a_2, \ldots, a_{2n-1})$ such that

$$\|v - u\|_{C^2(K)} \leq \varepsilon$$

where $b_2', b_3'$ can be chosen in any punctured neighborhood of $b_2, b_3$ in $\mathbb{M}(\infty)$, respectively. (Cf. Figure 7.)

![Figure 7](image.png)

**Remark 4.2.** We observe that the existence of the Scherk domains bounded by $\mathcal{P}(a_0, b_1, b_2, a_1)$ and $\mathcal{P}(a_1, b_3, b_4, a_2)$ is guaranteed by Proposition 4.1.

The proof of Proposition 4.2 proceeds as in [CR]; we will first prove three lemmas. Following the notation in Proposition 4.2, we denote by $E_1$ the domain bounded by the polygon $\mathcal{P}(a_0, b_1, b_2, a_1)$, by $E_2$ the domain bounded by $\mathcal{P}(a_1, b_3, b_4, a_2)$ and by $D_0$ the global domain bounded by $\Gamma = \mathcal{P}(a_0, b_1, b_2, a_1, b_3, b_4, a_2, \ldots, a_{2n-1})$. Then, it is clear that $\Gamma$ satisfies Condition 1 in Theorem 3.1 and, in addition, one obtains
**Lemma 4.4.** Condition 2 in Theorem 3.1 is satisfied by every inscribed polygon in $D_0$, except the boundaries of $E_1, E_2, D_0 - E_1$ and $D_0 - E_2$.

**Proof.** It is clear that the boundaries of $E_1, E_2, D_0 - E_1$ and $D_0 - E_2$ do not satisfy Condition 2 in Theorem 3.1. Therefore, we start with a inscribed polygon $P$ in $D_0$ different from them.

From Remark 3.1, we can assume that $P$ has the adjacent side of $\partial D_0$ with $+\infty$ data, at any vertex of $P$. And we only have to prove that $2a(P) < |P|$, for some choise of horocycles at the vertices.

Let $D_P$ be the domain bounded by $P$ and $P'$ the boundary of $D_P - E_2$. $P'$ is a polygon with some possible vertices in $\mathbb{M}$ (cf. Figure 8). We are going to show that if $2a(P') < |P'|$ then $2a(P) < |P|$. And, so, we will only need to prove that the inequality is true for $P'$.

![Figure 8.](image-url)

If the geodesic $[b_3, b_4]$ does not belong to $P$ then $P = P'$ and the result is obvious. Let $d_1$ be the vertex of $P$ previous to $b_3$, and $d_2$ the vertex following $b_4$. Consider $q_1 = [d_1, b_3] \cap [a_1, a_2]$ and $q_2 = [b_4, d_2] \cap [a_1, a_2]$ (cf. Figure 8). Observe that $q_i$ could be $a_i$; in that case we have $|a_i q_i| = 0$. We have,

$$a(P) = a(P') + |b_3 b_4|,$$

$$|P| = |P'| - |q_1 q_2| + |q_1 b_3| + |b_3 b_4| + |b_4 q_2|.$$

On the other hand, since $E_2$ is a Scherk domain we can assume $d_0 = |a_1 b_3| = |b_3 b_4| = |b_4 a_2| = |a_1 a_2|$. Thus, from the Generalized Triangle Inequality, $|a_1 b_3| \leq |a_1 q_1| + |q_1 b_3|$, $|a_2 b_4| \leq |a_2 q_2| + |q_2 b_4|$, and using that $2a(P') < |P'|$, we have

$$|P| - 2a(P) > |q_1 b_3| + |b_4 q_2| - |q_1 q_2| - |b_3 b_4|$$

$$= |q_1 b_3| + |b_4 q_2| - 2|b_3 b_4| + |a_1 q_1| + |a_2 q_2|$$

$$= (|q_1 b_3| + |a_1 q_1| - |a_1 b_3|) + (|b_4 q_2| + |a_2 q_2| - |a_2 b_4|) \geq 0.$$

Therefore, in order to finish the proof of Lemma 4.4 we only need to see that $2a(P') < |P'|$.  

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Let $D_{\mathcal{P}'}$ be the domain bounded by $\mathcal{P}'$, and $\mathcal{P}''$ the boundary of $D_{\mathcal{P}'} - E_1$. We use a flux inequality, for the initial minimal graph $u$, over the domain bounded by $\mathcal{P}''$ to obtain the desired inequality.

From the minimal graph equation (3.1), the field $X_u = (\nabla u)/W$, with $W = \sqrt{1 + |\nabla u|^2}$, is divergence free. We state the Flux theorem for $X = X_u$ on the domain $D$ of $u$ (see [CR], [P]). For any domain $Q$ in $D$ with $\partial Q$ compact, the flux of $X_u$ along $\partial Q$ is zero. By this we mean $F(\partial Q) = \int_{\partial Q} (X, \nu) = 0$, where $\nu$ is the outer unit normal to $\partial Q$. If $\beta$ is an arc of $\partial D$ and $u = \infty$ on $\beta$ (i.e. $u(p) \to \infty$ as $p \to \beta$, $p \in D$) then $F(\beta) = |\beta|$. If $u = -\infty$ on $\beta$ then $F(\beta) = -|\beta|$. And if $u$ extends continuously to $\beta$ then $|F(\beta)| < |\beta|$.

We write $\mathcal{P}''$ as the union of three sets: the union of all geodesic arcs of $\mathcal{P}''$ with boundary data $+\infty$ and disjoint from $[a_0, a_1], I_1 = \mathcal{P}'' \cap [a_0, a_1]$ and $J$ the union of the remaining arcs.

If $\nu$ is the unit outward normal to $\mathcal{P}''$ then it is easy to see that $X = \nu$ on the sides with $+\infty$ boundary data (see [CR, P]), and so

$$0 = F_u(\partial \mathcal{P}'') = a(\mathcal{P}'') + |I_1| + F_u(J) + \rho,$$

where, for instance, $F_u(\partial \mathcal{P}'') = \int_{\partial \mathcal{P}''} (X, \nu)$ denotes the flux of $u$ along $\partial \mathcal{P}''$.

Here, the flux $F_u(J)$ is taken on the compact arcs of $J$ outside the horocycles, and the number $\rho$ corresponds to the remaining flux of $X$ along some parts of horocycles.

Since $|\mathcal{P}''| = a(\mathcal{P}'') + |I_1| + |J|$, we have

$$|\mathcal{P}''| - 2(a(\mathcal{P}'') + |I_1|) = |J| + F_u(J) + \rho. \quad (4.1)$$

We are assuming that $\mathcal{P}$ has the adjacent side of $\partial D_0$ with $+\infty$ data, for any vertex of $\mathcal{P}$. Therefore, the estimation of $|\mathcal{P}'| - 2a(\mathcal{P}')$ does not depend on the chosen horocycles. Moreover, $\mathcal{P}''$ is not empty and it is different from the boundary of $D$. To see that, observe that if $\mathcal{P}''$ is empty then the previous condition on $\mathcal{P}$ implies that $\mathcal{P}$ is the boundary of $E_1$, and if $\mathcal{P}''$ is the boundary of $D$ then $\mathcal{P}$ is the boundary of $D \cup E_2$ or $D_0$.

Hence, $\mathcal{P}''$ has interior arcs in $D$, and $F_u(\alpha) + |\alpha|$ is positive on each interior arc $\alpha$. Thus, $|J| + F_u(J)$ is positive and non decreasing when we choose smaller horocycles. In addition, we can select these horocycles so that $|\rho|$ is as small as desired. Therefore, from (4.1), we can assume

$$|\mathcal{P}''| - 2(a(\mathcal{P}'') + |I_1|) > 0 \quad (4.2)$$

for suitable horocycles at the vertices.

Now, we show that $2a(\mathcal{P}') < |\mathcal{P}'|$. For that, we distinguish three cases

1. $a_0$ and $a_1$ are vertices of $\mathcal{P}$,
2. only one of the points $a_0, a_1$ is a vertex of $\mathcal{P}$,
3. neither $a_0$ nor $a_1$ are vertices of $\mathcal{P}$.
In the case 1, \([a_0, b_1]\) and \([b_2, a_1]\) must be sides of \(P\). So, \([b_1, b_2]\) is also a side of \(P\) and \(E_1\) is contained in the domain bounded by \(P\). Thus,

\[
a(P') = a(P'') + 2d_1, \quad |P'| = |P''| + 2d_1 \quad \text{and} \quad |I_1| = d_1,
\]

where \(d_1 = |a_0 b_1| = |b_1 b_2| = |b_2 a_1| = |a_1 a_0|\).

And (4.2) gives us \(2a(P') < |P'|\).

For the case 2, we assume for instance that \(a_0\) is a vertex of \(P\), but not \(a_1\). Then \(b_2\) is not a vertex of \(P\) and we consider the point \(q\) which is the intersection between the geodesic \([a_0, a_1]\) and the geodesic joining \(b_1\) and the following vertex of \(P\) (cf. Figure 9). Then \(I_1 = [a_0, q]\) and

\[
a(P') = a(P'') + d_1, \quad |P'| = |P''| - |I_1| + d_1 + |b_1 q|,
\]

where \(d_1 = |a_0 b_1|\).

Figure 9.

And using (4.2)

\[
0 < (|P'| + |I_1| - d_1 - |b_1 q|) - 2(a(P') - d_1 + |I_1|)
= |P'| - 2a(P') + d_1 - |I_1| - |b_1 q|.
\]

Therefore, using the Generalized Triangle Inequality for the triangle with vertices \(a_0, b_1, q\) we have \(d_1 - |I_1| - |b_1 q| \leq 0\) and the inequality \(2a(P') < |P'|\) is proved.

Case 3 is clear because the domain bounded by \(P'\) lies on \(D\) and the flux formula (4.2) gives the desired inequality.

Now we will perturb \(E_1\) and \(E_2\) to obtain a Scherk domain. We will do this so that \(E_1, E_2\) and their complements are inscribed polygons satisfying Condition 2 of Theorem 3.1. By the previous Lemma 4.4, the other inscribed polygons in \(D_0\) also satisfy Condition 2. Hence, if we make the perturbation of \(E_1, E_2\) small enough, the strict inequalities (Condition 2) satisfied by these inscribed polygons will remain strict inequalities. We now make this precise.
Lemma 4.5. For any punctured neighborhoods of $b_2, b_3$ in $\mathbb{M}(\infty)$, we can choose, respectively, two points $b'_2, b'_3$ such that the domain bounded by $\mathcal{P}(a_0, b_1, b'_2, a_1, b'_3, a_2, \ldots, a_{2n-1})$ is, in fact, a Scherk domain.

Proof. We first observe that

$$
|a_0 a_1| - |a_1 b_2| + |b_2 b_1| - |b_1 a_0| = 0
$$

since $E_1$ and $E_2$ are Scherk domains.

Now, using Lemma 4.2, there exist unique $b_2(t), b_3(t)$ such that

$$
t = |a_0 a_1| - |a_1 b_2(t)| + |b_2(t) b_1| - |b_1 a_0|
= |a_2 a_1| - |a_1 b_3(t)| + |b_3(t) b_4| - |b_4 a_2|, \tag{4.3}
$$

where $b_2(t)$ varies in the open interval between $b_1$ and $a_1$ at infinity, $b_3(t)$ between $a_1$ and $b_4$, and $t \in \mathbb{R}$. In addition, the functions $b_i(t)$ are homeomorphisms.

Let $\Gamma(t) = \mathcal{P}(a_0, b_1, b_2(t), a_1, b_3(t), b_4, a_2, \ldots, a_{2n-1})$. From (4.3), Condition 1 in Theorem 3.1 is satisfied for any $t \in \mathbb{R}$. In addition, if $t > 0$ then the domains $E_1(t)$ and $E_2(t)$ bounded by $\mathcal{P}(a_0, b_1, b_2(t), a_1), \mathcal{P}(a_1, b_3(t), b_4, a_2)$ and their complements also satisfy Condition 2. For that, observe that if we denote $\mathcal{P}_1(t) = \mathcal{P}(a_0, b_1, b_2(t), a_1)$ and $\overline{\mathcal{P}}_1(t) = \mathcal{P}(a_0, a_1, b_3(t), b_4, a_2, a_3, \ldots, a_{2n-1})$, then from (4.3)

$$
2a(\mathcal{P}_1(t)) - |\mathcal{P}_1(t)| = |b_1 a_0| + |a_1 b_2(t)| - |a_0 a_1| - |b_2(t) b_1| = -t < 0,
$$

$$
2b(\overline{\mathcal{P}}_1(t)) - |\overline{\mathcal{P}}_1(t)| = -|a_2 a_1| + |a_1 b_3(t)| - |b_3(t) b_4| + |b_4 a_2| = -t < 0.
$$

Thus, from Remark 3.1, Condition 2 in Theorem 3.1 is satisfied for the domain $E_1(t)$ and its complementary. Analogously, it can be proved for $E_2(t)$.

In order to obtain Condition 2 for the other inscribed polygons we argue as follows.

From Lemma 4.4, every polygon inscribed in the domain bounded by $\Gamma(0)$ satisfies Condition 2, except $E_1(0), E_2(0)$ and their complements. Observe that the inequalities in Condition 2 are strict, and the number of inscribed polygons is finite. From Lemma 4.2, these inequalities depend continuously on $b_2(t)$ and $b_3(t)$, so one has that there exists $t_0 > 0$ such that Condition 2 is also satisfied for any domain bounded by $\Gamma(t)$, with $0 < t < t_0$. \hfill \Box

Proof of Proposition 4.2. The proof is a verification that the arguments in [CR] work in our context. Let us denote by $D_t$ the Scherk domain bounded by $\Gamma(t)$. Consider the graph of a Scherk surface $u_t$ defined on $D_t$ with the corresponding infinite boundary data. First, we show that $\nabla u$ is the limit of $\nabla u_t|D$ when $t$ goes to zero.

Let us consider the divergence free fields $X_t = (\nabla u_t)/W_t$ and $X = (\nabla u)/W$, with $W_t = \sqrt{1 + |\nabla u_t|^2}, W = \sqrt{1 + |\nabla u|^2}$, associated to $u_t$ and $u$, respectively. We now see that $X_t$ converges to $X$ on $D$ when $t$ tends to zero.
Consider the outer pointing normal $\nu$ along the boundary of $D$. We have fixed the same infinite boundary data on $\partial D - ([a_0, a_1] \cup [a_1, a_2])$, so $X_t = X = \pm \nu$ on this set.

On the boundary of $E_1(t)$ truncated by the horocycles, the flux of $X_t$ is zero. Hence,

$$0 = |a_0 b_1| - |b_1 b_2(t)| + |b_2(t) a_1| + \int_{[a_0', a_1']} \langle X_t, -\nu \rangle + F_{u_t}(I_t),$$

where $[a_0', a_1']$ is the compact geodesic arc between the horocycles at $a_0$ and $a_1$, and $I_t$ is the set of arcs included in the four horodisks. Then, from (4.3),

$$t = \int_{[a_0', a_1']} (1 - \langle X_t, \nu \rangle) + F_{u_t}(I_t),$$

and taking limits for smaller horocycles at the vertices one has the convergence of the integral on the whole geodesic and

$$t = \int_{[a_0, a_1]} (1 - \langle X_t, \nu \rangle) = \int_{[a_0, a_1]} \langle X - X_t, \nu \rangle,$$

since $X = \nu$ on $[a_0, a_1]$.

Analogously, one has

$$t = -\int_{[a_1, a_2]} \langle X - X_t, \nu \rangle.$$

Thus, for any family $\alpha$ of disjoint arcs of $\partial D$

$$\left| \int_{\alpha} \langle X - X_t, \nu \rangle \right| \leq \int_{[a_0, a_1] \cup [a_1, a_2]} \|\langle X - X_t, \nu \rangle\| = 2t. \tag{4.4}$$

Now, we study the behavior of the field $X - X_t$ on the interior of $D$. Let $\Sigma$ be the graph of $u$ and $\Sigma_t$ the graph of $u_t$. These graphs are stable, complete and satisfy uniform curvature estimates by Schoen’s curvature estimates [Sc]. Thus,

$$\forall \varepsilon > 0 \ \exists \rho > 0 \text{ such that } \forall p \in D \ \forall q \in \Sigma_t \cap B((p, u_t(p)), \rho) \text{ one has } \|N_t(p) - N_t(q)\| \leq \varepsilon.$$

Here, $\rho$ does not depend on $t$, and $N_t$ denotes the normal to $\Sigma_t$ pointing down and $B((p, v_t(p)), \rho)$ the ball of radius $\rho$, centered at $(p, v_t(p)) \in \mathbb{M} \times \mathbb{R}$. These estimates remain true for $\Sigma$.

One may think of these curvature estimates as follows. For $\varepsilon > 0$, there is a $\delta > 0$ such that for $\Sigma$ a stable minimal surface in a Riemannian 3-manifold $M$, and for $p \in \Sigma - \partial \Sigma$, $\Sigma$ is a graph (in exponential coordinates) over the disk of radius $\delta$ in the tangent plane to $\Sigma$ at $p$. This graph and its first and second derivatives are bounded on the $\delta-$disk. This bound and $\delta$ depend on the geometry of $M$ and the distance of $p$ to $\partial \Sigma$ in $M$, and not on $\Sigma$. We refer to reader to Schoen [Sc], and Rosenberg, Souam and Toubiana [RST].
Therefore, one obtains that fixed $\varepsilon > 0$ and $p \in D$ there exists $\rho_1 \leq \rho/2$, which depends continuously on $p$ but does not depend on $t$, such that for every $q$ in the disk $B(p, \rho_1)$ in $\mathbb{M}$ with center $p$ and radius $\rho_1$, we have $|u(q) - u(p)| \leq \rho/2$.

Let us assume now that $\|N_t(p) - N(p)\| \geq 3 \varepsilon$. Consider the connected component $\Omega_t(p)$ of $\{q \in D : u(q) - u_t(q) > u(p) - u_t(p)\}$ with $p$ in its boundary, and $\Lambda_t$ the component of $\partial \Omega_t(p)$ containing $p$. Since $\Lambda_t$ is a level curve of $u - u_t$ then it is piecewise smooth. Let $\sigma \subseteq \Sigma$, $\sigma_t \subseteq \Sigma_t$ be the two curves which project on $\Lambda_t \cap B(p, \rho_1)$.

For the points of $\sigma$, we have that if $q \in \Lambda_t \cap B(p, \rho_1)$ then for the product distance $|(q, u(q)) - (p, u(p))| \leq \rho_1 + \rho/2 \leq \rho$ and so $\|N(q) - N(p)\| \leq \varepsilon$. The same is also true on the curve $\sigma_t$, that is, $\|N_t(q) - N_t(p)\| \leq \varepsilon$ for all $q \in \Lambda_t \cap B(p, \rho_1)$.

Thus, using these inequalities and the assumption on the normals at $p$, we obtain for all $q \in \Lambda_t \cap B(p, \rho_1)$ that $\|N(q) - N_t(q)\| \geq \|N_t(p) - N_t(q)\| - 2 \varepsilon \geq \varepsilon$.

From [P, Assertion 3.1] or [CR, Lemma A.1],

$$\langle X - X_t, \eta \rangle \geq \frac{\|N - N_t\|^2}{4}$$

(4.5)

with $\eta = \nabla(u - u_t)/|\nabla(u - u_t)|$ orienting the level curve $\Lambda_t$ at its regular points (see also [CR]). Thus, one has

$$\int_{\Lambda_t \cap B(p, \rho_1)} \langle X - X_t, \eta \rangle \geq \frac{\rho_1 \varepsilon^2}{2}.$$ 

In addition, from (4.5), $\langle X - X_t, \eta \rangle$ is non negative outside the isolated points where $\nabla(u - u_t) = 0$, and so, for every compact arc $\beta \subseteq \Lambda_t$ containing $\Lambda_t \cap D(p, \rho_1)$ we have

$$\int_{\beta} \langle X - X_t, \eta \rangle \geq \frac{\rho_1 \varepsilon^2}{2}.$$ 

(4.6)

By the maximum principle, $\Lambda_t$ is not compact in $D$. And, since $\Lambda_t$ is proper on $D$, its two infinite branches go close to $\partial D$. Then there exists a connected compact part $\beta$ of $\Lambda_t$, containing $\Lambda_t \cap B(p, \rho_1)$, and two arcs $\delta$ in $D$ small enough and joining the extremities of $\beta$ to $\partial D$. Eventually truncating by a family of horocycles, the flux formula for $X - X_t$ yields

$$0 = \int_{\beta} \langle X - X_t, -\eta \rangle + \int_{\alpha} \langle X - X_t, \nu \rangle + F_{u - u_t}(\delta \cup \delta'),$$

where $\alpha$ is contained in $\partial D$ and $\delta'$ is contained in the horocycles and correctly oriented. Using (4.4) and (4.6) we obtain

$$\frac{\rho_1 \varepsilon^2}{2} \leq 2t + F_{u - u_t}(\delta \cup \delta').$$

When the length of $\delta \cup \delta'$ goes to zero, one has

$$\frac{\rho_1 \varepsilon^2}{4} \leq t.$$
Hence, if \( t \leq (\rho_1 \varepsilon^2)/4 \) then \( \|X(p) - X_t(p)\| \leq \|N(p) - N_t(p)\| \leq 3 \varepsilon \). Since \( \rho_1 \) only depends continuously on \( p \), this gives us the desired convergence of \( X_t \) to \( X \) when \( t \) goes to zero.

With all of this, we have proved that \( \nabla u_t|D \) converges uniformly to \( \nabla u \) when \( t \) goes to zero. Therefore, after the normalization \( u_t(p_0) = u(p_0) \) for a fixed \( p_0 \in D \), we have that \( u_t|D \) converges uniformly to \( u \) in relatively compact domains \( \tilde{D} \) of \( D \) when \( t \) tends to zero, and the convergence is \( C^\infty \) on compact sets of \( \tilde{D} \).

Hence, given a compact set \( K \subseteq D \) and \( \varepsilon > 0 \), there exists a \( t \) small enough such that \( \|u_t - u\|_{C^2(K)} \leq \varepsilon \). \( \square \)

5 Entire minimal graphs.

We now establish our main result.

**Theorem 5.1.** Let \( \mathbb{M} \) be a Hadamard surface with Gauss curvature bounded from above by a negative constant. Then, there exist harmonic diffeomorphisms from the complex plane onto \( \mathbb{M} \).

**Proof.** The vertical projection from a minimal surface \( \Sigma \subseteq \mathbb{M} \times \mathbb{R} \) into \( \mathbb{M} \) is a harmonic map. Therefore, in order to prove the Theorem, we only need to show that there exist entire minimal graphs in \( \mathbb{M} \times \mathbb{R} \) with the conformal structure of the complex plane.

Let us fix a point \( p_0 \) in a Scherk domain \( D_1 \subseteq \mathbb{M} \) and also a compact disk \( K_1 \subseteq D_1 \).

Observe that the existence of \( D_1 \) is guaranteed by Theorem 3.1 and Proposition 4.1.

As it was explained in Section 2, we can consider the homeomorphism \( h \) from the set \( S^1_{p_0} \) of unit tangent vectors at \( p_0 \) onto \( \mathbb{M}(\infty) \), which maps a vector \( v \in S^1_{p_0} \) to the point in \( \mathbb{M}(\infty) \) given by \( \gamma_v(+\infty) \). Here, \( \gamma_v(0) \) is the unique geodesic in \( \mathbb{M} \) with initial conditions \( \gamma_v(0) = p_0 \) and \( \gamma_v'(0) = v \). Using the continuous function \( h^{-1} \), we will measure the angle between two points \( x, y \in \mathbb{M}(\infty) \) as the angle between the vectors \( h^{-1}(x), h^{-1}(y) \in S^1_{p_0} \).

Now, fix a sequence of positive numbers \( \varepsilon_n \) such that \( \sum_{n \geq 1} \varepsilon_n < \infty \). We show the existence of an exhaustion of \( \mathbb{M} \) by Scherk domains \( D_n \) and by compact disks \( K_n \subseteq D_n \) such that each \( K_n \) is contained in the interior of \( K_{n+1} \) and a sequence of minimal graphs \( u_n \) on \( D_n \) satisfying

1. \( \|u_{n+1} - u_n\|_{C^2(K_n)} < \varepsilon_n \),

2. the conformal modulus of the minimal graph of \( u_n \) on the annulus \( K_{i+1} - \text{int}(K_i) \), with respect to the family of curves that separate its boundary components, is greater than one for each \( 1 \leq i \leq n - 1 \) (i.e. it is conformally equivalent to an annulus in \( \mathbb{C} \) of radii \( r < R \) with \( \frac{1}{2\pi} \log(R/r) > 1 \) for each \( i \)). Here \( \text{int}(K_i) \) denotes the interior of \( K_i \),

3. the angle between two consecutive vertices of the ideal polygon \( \partial D_n \) is less than \( \pi/2^{n-1} \).
The third condition is clear for \( n = 1 \) since \( p_0 \in D_1 \). Thus, we assume that there exists
the sequence \( (D_i, u_i, K_i) \) satisfying the three previous conditions for \( 1 \leq i \leq n \) and we obtain
\( (D_{n+1}, u_{n+1}, K_{n+1}) \).

Let \( x, y \) be the vertices of a side of \( \partial D_n \), and \( \mathcal{I} \) the arc between \( x \) and \( y \) that contains no
other vertex of \( \partial D_n \). We choose the unique point \( z \in \mathcal{I} \) such that the angle between \( x \) and \( z \)
agrees with the angle between \( y \) and \( z \). Thus, the angle between \( x \) and \( z \) is less than \( \pi/2 \).
Now, from Proposition 4.1, there exists \( w \in \mathcal{I} \) such that the domain bounded by the quadrilateral with
vertices \( x, y, z, w \) is a Scherk domain. Moreover, the angle between two consecutive vertices is
less than \( \pi/2 \).

We attach to each side of \( \partial D_n \) an ideal quadrilateral constructed as above. Then we use
Proposition 4.2 and perturb all the pairs of sides of \( \partial D_n \) to obtain an ideal Scherk graph
\( u_{n+1} \) on a larger domain \( D_{n+1} \). This perturbation of the vertices can be done as small as necessary so
that Conditions 1, 3 are satisfied, and also Condition 2 for \( 1 \leq i < n \).

Now, we use the following Lemma. We refer the reader to [CR] for its proof.

**Lemma 5.1.** Every ideal Scherk surface is conformally equivalent to the complex plane.

Hence, the minimal graph \( \Sigma \) of \( u_{n+1} \) is conformally the complex plane. Let \( \Sigma_0 \subseteq \Sigma \) be the
graph of \( u_{n+1} \) on the interior of \( K_n \). Thus, we can choose a closed disk \( \Sigma_1 \subseteq \Sigma \) containing \( \Sigma_0 \)
in its interior such that the conformal modulus of \( \Sigma_1 - \Sigma_0 \) is greater than one. Then, we take
\( K_{n+1} \) as the projection of \( \Sigma_1 \). In addition, we can enlarge \( K_{n+1} \), if necessary, in such a way that
\( K_{n+1} \) contains \( \hat{D}_{n+1} \cap B(p_0, n) \), where \( \hat{D}_{n+1} \) is the set of points in \( D_{n+1} \) a distance greater than
1 to its boundary and \( B(p_0, n) \) the geodesic disk centered at \( p_0 \) of radius \( n \). Thus, Condition 2
is also satisfied.

Observe now that \( M = \bigcup_{n \geq 1} D_n \). This is a straightforward consequence of Condition 3,
since the set of vertices of the domains \( D_n \) is dense in \( M(\infty) \). In addition, from the condition
between the distance of \( \partial K_n \) and \( \partial D_n \) one has that \( M = \bigcup_{n \geq 1} K_n \).

Once we have obtained the previous sequence, we can get the desired entire minimal graph.
Since \( u_n(p) \) is a Cauchy sequence for any \( p \in M \), we obtain an entire minimal graph \( u \). On the
other hand, on each compact set \( K_{i+1} - \text{int}(K_i) \) the sequence \( u_n \) converges uniformly to \( u \)
in the \( C^2 \)-topology. Hence, the conformal modulus of the minimal graph of \( u \) on \( K_{i+1} - \text{int}(K_i) \)
is at least one.

Now, we observe that the minimal annulus \( u|_{M - \text{int}K_1} \) must be conformal to an annulus of
rarii \( 1 = r < R \leq \infty \). But, from the first Grötzsch Lemma [V], \( R \) cannot be finite. Therefore,
the conformal type of the minimal graph \( u \) is the complex plane. \( \square \)

We also construct harmonic diffeomorphisms from the unit disk onto \( M \) by solving a Dirich-
let problem at infinity.

**Theorem 5.2.** Let \( \Upsilon \) be a continuous Jordan curve in the cylinder \( M(\infty) \times \mathbb{R} \), which is a vertical
graph. Then, there exists a unique entire minimal graph on \( M \) having \( \Upsilon \) as its asymptotic
boundary. Moreover, the conformal structure of this graph is that of the unit disk.
Proof. Let $\varphi : \mathbb{M}(\infty) \rightarrow \mathbb{R}$ be the continuous function whose graph is $\Upsilon$. Let us fix a point $p_0 \in \mathbb{M}$. Consider for any unit tangent vector $v$ at $p_0$, the unique geodesic $\gamma_v(t)$ satisfying $\gamma_v(0) = p_0$ and $\gamma_v'(0) = v$, and $h : S^1_{p_0} \rightarrow \mathbb{M}(\infty)$ the homeomorphism given by $h(v) = \gamma_v(+\infty)$.

For the continuous function $\varphi \circ h : S^1_{p_0} \rightarrow \mathbb{R}$, we consider a sequence of $C^2$–functions $\varphi_n : S^1_{p_0} \rightarrow \mathbb{R}$ converging uniformly to $\varphi \circ h$. Then, for any positive integer $n$ we consider the graph on the geodesic circle centered at $p_0$ of radius $n$ given by the curve $\Upsilon_n(v) = (\gamma_v(n), \varphi_n(v))$, $v \in S^1_{p_0}$.

Let $\Sigma_n$ be the minimal surface in $\mathbb{M} \times \mathbb{R}$ obtained as the Plateau solution with boundary $\Upsilon_n$. The surface $\Sigma_n$ can be seen as a graph $u_n$ on the geodesic disk centered at $p_0$ of radius $n$, by Rado’s theorem. Since the horizontal slices are minimal surfaces, from the maximum principle, the sequence $\{u_n\}$ is uniformly bounded on compact subsets of $\mathbb{M}$. Thus there is a subsequence converging to a entire minimal solution $u : \mathbb{M} \rightarrow \mathbb{R}$, uniformly on compact subsets of $\mathbb{M}$. Let $\Sigma$ be the entire minimal graph given by $u$.

We now prove that the asymptotic boundary of $\Sigma$ is $\Upsilon$. For that, observe that we only need to show that if $q$ is a point in $\mathbb{M}(\infty) \times \mathbb{R}$ such that $q \notin \Upsilon$ then $q$ does not belong to the asymptotic boundary of $\Sigma$.

Consider $q = (x_0, r) \in \mathbb{M}(\infty) \times \mathbb{R}$. We assume, for instance, $r > \varphi(x_0)$. Take $\varepsilon = (r - \varphi(x_0))/2 > 0$ and $\nu_0 = h^{-1}(x_0)$. Then, from the uniform convergence of $\varphi_n$ to $\varphi \circ h$ and the continuity of $\varphi_n$, we can assure the existence of $\delta > 0$ and $n_0$ such that for all $w \in S^1_{p_0}$ with $\|w - \nu_0\| \leq \delta$ and $n \geq n_0$

$$|\varphi_n(w) - \varphi(h(\nu_0))| \leq \varepsilon.$$

Let $w_1, w_2 \in S^1_{p_0}$ be the unit vectors at a distance $\delta$ from $\nu_0$. Let $\Omega \subseteq \mathbb{M}$ be the halfspace determined by the geodesic $\alpha$ joining the points at infinity $h(w_1)$ and $h(w_2)$ and having $x_0$ in $\partial_{\infty} \Omega$ (cf. Figure 10). From Proposition 3.1, there exists a Scherk type graph $v$ on the halfspace $\Omega$ with boundary data $+\infty$ on $\alpha$ and $(r + \varphi(x_0))/2$ on $\partial_{\infty} \Omega$.

![Figure 10](image)

From the maximum principle, $u_n \leq v$ on $\Omega$ for all $n \geq n_0$. In particular, $q = (x_0, r)$ does
not belong to the asymptotic boundary of the entire graph $\Sigma$. Thus, the asymptotic boundary of $\Sigma$ is $\Upsilon$.

The uniqueness part of the Theorem is a straightforward consequence of the maximum principle. In addition, since the height function is harmonic and bounded for the entire minimal graph $\Sigma$, then its conformal structure must be that of the unit disk. 

\[\square\]

**References**


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