Fatou’s Theorem and minimal graphs

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Abstract

In this paper we extend a recent result of Collin-Rosenberg (a solution to the minimal surface equation in the Euclidean disc has radial limits almost everywhere) to a large class of differential operators in Divergence form. Moreover, we construct an example (in the spirit of [3]) of a minimal graph in $M^2 \times \mathbb{R}$, where $M^2$ is a Hadamard surface, over a geodesic disc which has finite radial limits in a measure zero set.

Résumé

Dans ce papier nous généralisons un résultat récent de Collin-Rosenberg (une solution de l’équation de surface minimale sur le disque euclidien admet une limite radiale presque partout) à une vaste classe d’opérateurs différentiels sous forme divergence. De plus, nous construisons un exemple (dans le même esprit que [3]) d’un graphe minimal dans $M^2 \times \mathbb{R}$, où $M^2$ est une surface de Hadamard, sur un disque géodésique, qui admet une limite radiale finie sur un ensemble de mesure nulle.

Keywords: Radial limits, Minimal graphs, Operators in Divergence Form.

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1 Introduction

It is well known that a bounded harmonic function $u$ defined on the Euclidean disc $D$ has radial limits almost everywhere (Fatou’s Theorem [4]). Moreover, the radial limits cannot be plus infinity for a positive measure set. For fixed $\theta \in \mathbb{S}^{n-1}$, the radial limit $u(\theta)$ (if it exists) is defined as

$$u(\theta) = \lim_{r \to 1} u(r, \theta),$$

where we parametrize the Euclidean disc in polar coordinates $(r, \theta) \in [0, 1) \times \mathbb{S}^1$.

In 1965, J. Nitsche [8] asked if a Fatou Theorem is valid for the minimal surface equation, i.e., does a solution for the minimal surface equation in the Euclidean disc have radial limits almost everywhere? This question has been solved recently by P. Collin and H. Rosenberg [3]. Moreover, in the same paper [8], J. Nitsche asked: what is the largest set of $\theta$ for which a minimal graph on $D$ may not have radial limits? Again, this question was solved in [3] if one allows infinite radial limits. That is, they construct an example of a minimal graph in the Euclidean disc with finite radial limits only on a set of measure zero. In this example, the $+\infty$ radial limits (resp. $-\infty$) are taken on a set of measure $\pi$ (resp. $\pi$).

The aim of this paper is to extend both results. In Section 2, we extend Collin-Rosenberg’s Theorem to a large class of differential operators in divergence form (see Theorem 2.1). We show this applies to minimal graph sections of Heisenberg space. In Section 3, we construct an example of a minimal graph in $\mathbb{M}^2 \times \mathbb{R}$ over a geodesic disk $D \subset \mathbb{M}^2$ ( $\mathbb{M}^2$ is a Hadamard surface) for which the finite radial limits are of measure zero. Also, the $+\infty$ radial limits (resp. $-\infty$) are taken on a set of measure $\pi$ (resp. $\pi$).

2 Fatou’s Theorem

Henceforth $(\mathbb{B}, g)$ denotes the $n-$dimensional unit open ball, i.e.,

$$\mathbb{B} = \{(r, \theta) : 0 \leq r < 1, \theta \in \mathbb{S}^{n-1}\},$$

in polar coordinates with respect to $g$, $g$ a $C^2-$Riemannian metric on $\mathbb{B}$. Define $G := G(r, \theta) = \sqrt{\det(g)}$. Moreover, we denote by $\nabla$ the Levi-Civita connection associated to $g$ and by $\text{div}_g$ its associated divergence operator. Also, $L^1(\mathbb{B})$ denotes the set of integrable functions on $(\mathbb{B}, g)$.

Set $u \in C^2(\mathbb{B})-$function and $X_u$ be a $C^1(\mathbb{B})-$vector field so that its coordinates depend on $u$, its first derivatives and $C^1(\mathbb{B})-$functions.

For fixed $\theta \in \mathbb{S}^{n-1}$, the radial limit ( if it exists) $u(\theta)$ is defined as

$$u(\theta) = \lim_{r \to 1} u(r, \theta).$$

Theorem 2.1. Let $(\mathbb{B}, g, G, u, X_u)$ be as above. Assume that
Let $|f| \in L^1(B)$. If $u$ is a solution of
\[
\text{div}_g(X_u) \geq (\text{ or } \leq) f \text{ on } B,
\]
then $u$ has radial limits almost everywhere.

**Remark 2.1.** This Theorem 2.1 does generalize the theorem of Collin and the second author [3]. Consider the prescribed mean curvature equation for graphs in Euclidean space, i.e., let $|H| \in L^1(B)$ and $u \in C^2(B)$ satisfy
\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \right) = H,
\]
the graph of $u$ over $B$ is a graph with mean curvature $H$. We consider the flat metric, $\langle , \rangle$, in $B$, and so $\nabla$ and div are taken in the flat metric.

Since we are considering the flat metric, $G(r, \theta) = r^{n-1}$, so Item a) is easy to check. Item b) follows from
\[
\|X_u\| = \left\| \frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \right\| \leq 1.
\]

To check Item c), we use
\[
\langle \nabla u, X_u \rangle = \langle \nabla u, \frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \rangle = \frac{\|\nabla u\|^2}{\sqrt{1 + \|\nabla u\|^2}}
\]
\[
= \frac{1 + \|\nabla u\|^2}{\sqrt{1 + \|\nabla u\|^2}} - \frac{1}{\|\nabla u\|^2} = \sqrt{1 + \|\nabla u\|^2} - \frac{1}{\sqrt{1 + \|\nabla u\|^2}}
\]
\[
\geq \|\nabla u\| - 1,
\]
so, we obtain Item c) for $h \equiv -1$ whose absolute value is integrable in $B$ with the flat metric.

**Proof of Theorem 2.1.** First, let us prove the case
\[
\text{div}_g(X_u) \geq f.
\]
For \( r < 1 \) fixed, set \( \mathbb{B}(r) \) the \( n \)-dimensional open ball of radius \( r \). Let \( \eta : \mathbb{R} \rightarrow (0, +1) \) be a smooth function so that \( 0 < \eta'(x) < 1 \) for all \( x \in \mathbb{R} \). Define \( \psi := \eta \circ u \).

On the one hand, by direct computations and item \( c) \), we have

\[
\text{div}_g(\psi X_u) = \psi \text{div}_g(X_u) + g(\nabla \psi, X_u) \geq \psi f + \eta' \cdot g(\nabla u, X_u)
\]

\[
\geq \psi f + \eta' (\delta |\nabla u| + h) = \delta \eta' |\nabla u| + (\psi f + \eta' h)
\]

\[
= \delta |\nabla \psi| + (\psi f + \eta' h),
\]

thus

\[
\int_{\mathbb{B}(r)} \text{div}_g(\psi X_u) \geq \delta \int_{\mathbb{B}(r)} |\nabla \psi| + C
\]

(2.1)

where \( C \) is some constant. This follows since \( |h| \) and \( |f| \) are \( L^1 \)-functions on \( \mathbb{B} \).

On the other hand, by Stokes’ Theorem and items \( a) \) and \( b) \), we obtain for \( r < 1 \) fixed

\[
\int_{\mathbb{B}(r)} \text{div}_g(\psi X_u) = \int_{\partial \mathbb{B}(r)} \psi g(X_u,v) \leq \int_{\partial \mathbb{B}(r)} M
\]

\[
= M \int_{\theta \in \mathbb{S}^{n-1}} G(r,\theta) d\theta \leq M \beta \int_{\theta \in \mathbb{S}^{n-1}}
\]

(2.2)

where \( v \) is the outer conormal to \( \partial \mathbb{B}(r) \) and \( \omega_{n-1} \) is the volume of \( \mathbb{S}^{n-1} \).

So, from (2.1), (2.2) and letting \( r \) go to one, we conclude that \( |\nabla \psi| \) is integrable in \( \mathbb{B} \), i.e.,

\[
\int_{\mathbb{B}} |\nabla \psi| < +\infty
\]

(2.3)

Since \( \frac{\partial \psi}{\partial r} \leq |\nabla \psi| \), we have from Fubini’s Theorem and (2.3)

\[
\int_{\mathbb{B}} \frac{\partial \psi}{\partial r} = \int_{\theta \in \mathbb{S}^{n-1}} \left( \int_{0}^{1} \frac{\partial \psi}{\partial r} G(r,\theta) dr \right) d\theta < \infty.
\]

Thus, as \( G(r,\theta) \) is bounded below by a positive constant, for \( r > 1/2 \) and almost all \( \theta \in \mathbb{S}^{n-1} \),

\[
\lim_{r \rightarrow 1} \psi(r,\theta) - \psi(0,0) = \int_{0}^{1} \frac{\partial \psi}{\partial r}(r,\theta) dr < \infty,
\]

that is, \( \psi \) has radial limits almost everywhere. Since \( \psi = \eta \circ u \), we conclude \( u \) has radial limits almost everywhere (which may be \( \pm \infty \)).

For

\[
\text{div}_g(X_u) \leq f,
\]

we just have to follow the above proof by changing \( \eta : \mathbb{R} \rightarrow (-1, 0) \) so that \( 0 < \eta'(x) < 1 \) for all \( x \in \mathbb{R} \). \( \square \)
2.1 Applications

Moreover, we will see now how Theorem 2.1 applies to get radial limits almost everywhere for minimal graphs in ambient spaces besides $\mathbb{R}^3$. We work here in Heisenberg space, but it is not hard to check that we could work with minimal graphs in a more general submersion (see [7]).

First, we need to recall some definitions in Heisenberg space (see [1]). The Heisenberg spaces are $\mathbb{R}^3$ endowed with a one parameter family of metrics indexed by bundle curvature by a real parameter $\tau \neq 0$. When we say the Heisenberg space, we mean $\tau = 1/2$, and we denote it by $\mathcal{H}$.

In global exponential coordinates, $\mathcal{H}$ is $\mathbb{R}^3$ endowed with the metric

$$g = (dx^2 + dy^2) + \left(\frac{1}{2}(ydx - xdy) + dz\right)^2.$$  

The Heisenberg space is a Riemannian submersion $\pi : \mathcal{H} \to \mathbb{R}$ over the standard flat Euclidean plane $\mathbb{R}^2$ whose fibers are the vertical lines, i.e., they are the trajectories of a unit Killing vector field and hence geodesics.

Let $S_0 \subset \mathcal{H}$ be the surface whose points satisfy $z = 0$. Let $D \subset \mathbb{R}^2$ be the unit disc. Henceforth, we identify domains in $\mathbb{R}^2$ with its lift to $S_0$. The Killing graph of a function $u \in C^2(D)$ is the surface

$$\Sigma = \{(x, y, u(x, y)) ; (x, y) \in D\}.$$  

Moreover, the minimal graph equation is

$$\text{div}_{\mathbb{R}^2}(X_u) = 0,$$

here $\text{div}_{\mathbb{R}^2}$ stands for the divergence operator in $\mathbb{R}^2$ with the Euclidean metric $\langle , \rangle$, and

$$X_u := \frac{\alpha}{W} \partial_x + \frac{\beta}{W} \partial_y,$$

where

$$\alpha := \frac{y}{2} + u_x, \quad \beta := \frac{-x}{2} + u_y,$$

and

$$W^2 = 1 + \alpha^2 + \beta^2.$$  

Thus, for verifying $u$ has radial limits almost everywhere (which may be $\pm \infty$), we have to check conditions $a)$, $b)$ and $c)$. Item $a)$ is immediate since we are working with the Euclidean metric.

Item $b)$ follows from

$$|X_u|^2 = \frac{\alpha^2 + \beta^2}{1 + \alpha^2 + \beta^2} \leq 1.$$
Now, we need to check Item c). On the one hand, using polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$, we have

\[
W^2 = 1 + \alpha^2 + \beta^2 = 1 + u_x^2 + u_y^2 + (yu_x - xu_y) + \frac{x^2 + y^2}{4}
\]

\[
= 1 + |\nabla u|^2 + \langle \nabla u, (-y, x) \rangle + \frac{x^2 + y^2}{4}
\]

\[
\geq 1 + |\nabla u|^2 - |\nabla u||(-y, x)| + \frac{x^2 + y^2}{4}
\]

\[
= 1 + |\nabla u|^2 - r|\nabla u| + \frac{r^2}{4}
\]

thus,

\[
W \geq \sqrt{1 + \left(\frac{|\nabla u| - \frac{r}{2}}{2}\right)^2} \geq \frac{1 + |\nabla u| - r/2}{2}.
\]

We need a lower bound for $W$ in terms of $|\nabla u|$. To do so, we distinguish two cases:

**Case** $|\nabla u| \leq 5/4$: Since

\[
1 - r|\nabla u| + \frac{r^2}{4} \geq 1 - \frac{5r}{4} + \frac{r^2}{4} \geq 0 \text{ for all } r \leq 1,
\]

we obtain

\[
W \geq \sqrt{|\nabla u|^2 + 1 - r|\nabla u| + \frac{r^2}{4}} \geq |\nabla u|.
\]

**Case** $|\nabla u| > 5/4$: We already know that

\[
W \geq |\nabla u| - \frac{r}{2},
\]

thus, for $|\nabla u| > 5/4$, it is easy to see that

\[
|\nabla u| - \frac{r}{2} \geq \frac{3}{10} |\nabla u| \text{ for all } r \leq 1.
\]

So, in any case, for $\delta = 3/10 > 0$

\[
W \geq \delta |\nabla u|.
\]

(2.4)

On the other hand,

\[
\langle \nabla u, X_u \rangle = \frac{u_x^2 + u_y^2 + \frac{1}{2}(yu_x - xu_y)}{W}
\]

\[
= \frac{1 + u_x^2 + u_y^2 + (yu_x - xu_y) + \frac{x^2 + y^2}{4}}{W} - \frac{1 + \frac{1}{2}(yu_x - xu_y) + \frac{x^2 + y^2}{4}}{W}
\]

\[
= \frac{W^2}{W} + h = W + h \geq \delta |\nabla u| + h,
\]
where we have used (2.4) and $h$ denotes the bounded function

$$h = -\frac{1 + \frac{1}{2}(yu_x - xu_y) + \frac{x^2 + y^2}{4}}{\sqrt{1 + u_x^2 + u_y^2 + (yu_x - xu_y) + \frac{x^2 + y^2}{4}}}.$$ 

that is, Item $c$) is satisfied. So,

**Corollary 2.1.** A solution for the minimal surface equation in the Heisenberg space defined over a disc has radial limits almost everywhere (which may be $\pm\infty$).

### 3 An example in a Hadamard surface

The aim of this Section is to construct an example of a minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ over a geodesic disk $D \subset \mathbb{H}^2$ ($\mathbb{H}^2$ is a Hadamard surface) for which the finite radial limits are of measure zero.

We need to recall preliminary facts about graphs over a Hadamard surface (see [5] for details). Henceforth, $\mathbb{H}^2$ denotes a simply connected with Gauss curvature bounded above by a negative constant, i.e., $K_{\mathbb{H}^2} \leq c < 0$.

Let $p_0 \in \mathbb{H}^2$ and $D$ be the geodesic disk in $\mathbb{H}^2$ centered at $p_0$ of radius one. Re-scaling the metric, we can assume that

$$\max \{K_{\mathbb{H}^2}(p); p \in \partial D\} = -1.$$ 

From the Hessian Comparison Theorem (see e.g. [6]), $\partial D$ bounds a strictly convex domain. We assume that $\partial D$ is smooth, otherwise we can work in a smaller disc. We identify $\partial D = S^1$ and orient it counter-clockwise.

We say that $\Gamma$ is an admissible polygon in $D$ if $\Gamma$ is a Jordan curve in $\partial D$ which is a geodesic polygon with an even number of sides and all the vertices in $\partial D$. We denote by $A_1, B_1, \ldots, A_k, B_k$ the sides of $\Gamma$ which are oriented counter-clockwise. Recall that any two sides can not intersect in $D$. Set $D$ the domain in $D$ bounded by $\Gamma$. By $|A_i|$ (resp. $|B_j|$), we denote the length of such a geodesic arc.

**Theorem 3.1 ([9]).** Let $\Gamma \subset \mathbb{H}^2$ be a compact polygon with an even number of geodesic sides $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$, in that order, and denote by $D$ the domain with $\partial D = \Gamma$. The necessary and sufficient conditions for the existence of a minimal graph $u$ on $D$, taking values $+\infty$ on each $A_i$, and $-\infty$ on each $B_j$, are the two following conditions:

1. $\sum_{i=1}^n |A_i| = \sum_{i=1}^n |B_i|$.

2. for each inscribed polygon $P$ in $D$ (the vertices of $P$ are among the vertices of $\Gamma$) $P \neq D$, one has the two inequalities:

$$2a(P) < |P| \text{ and } 2b(P) < |P|.$$
Here \( a(P) = \sum_{A_j \in P} |A_j| \), \( b(P) = \sum_{B_j \in P} |B_j| \) and \( |P| \) is the perimeter of \( P \).

The construction of this example follows the steps in [3, Section III], but here we have to be more careful in the choice of the first inscribed square and the trapezoids. We need to choose them as symmetric as possible.

Let us first explain how we take the inscribed square: Let \( L = \text{length}(\partial D) \) and \( \gamma(x_0, x_1) \), be the geodesic arc in \( D \) joining \( x_0, x_1 \in \partial D \). Fix \( x_0 \in \partial D \) and let \( \alpha : \mathbb{R}/[0, L) \rightarrow \partial D \) an arc-length parametrization of \( \partial D \) (oriented count-clockwise). Set \( x_1 = \alpha(L/2) \). Consider \( x_0^\pm(s) = \alpha(\pm s) \) and \( x_1^\pm(s) = \alpha(L/2 \pm s) \) for \( 0 \leq s \leq L/2 \) (c.f. Figure 1), and denote

\[
\begin{align*}
B_1(s) &= \gamma(x_0^+(s), x_1^-(s)) \\
A_1(s) &= \gamma(x_1^-(s), x_1^+(s)) \\
B_2(s) &= \gamma(x_1^+(s), x_0^-(s)) \\
A_2(s) &= \gamma(x_0^-(s), x_1^+(s)).
\end{align*}
\]

![Figure 1: We move the points along \( \partial D \)](image)
Hence (c.f. Figure 2),

\[
|A_1(s)| + |A_2(s)| > |B_1(s)| + |B_2(s)| \quad \text{for } s \text{ close to } 0.
\]

\[
|A_1(s)| + |A_2(s)| < |B_1(s)| + |B_2(s)| \quad \text{for } s \text{ close to } L/2.
\]

Figure 2: How does the length change?

Thus, there exist \(s_0 \in (0, L/2)\) so that

\[
|A_1(s_0)| + |A_2(s_0)| = |B_1(s_0)| + |B_2(s_0)|.
\]

So, given a fixed point \(x_0 \in \partial D\), we have the existence of four distinct points \(p_1 = \alpha(s_0), p_2 = \alpha(L/2 - s_0), p_3 = \alpha(L/2 + s_0)\) and \(p_4 = \alpha(-s_0)\) ordered counter-clockwise so that

\[
|A_1| + |A_2| = |B_1| + |B_2|,
\]

where

\[
B_1 = \gamma(p_1, p_2),
A_1 = \gamma(p_2, p_3),
B_2 = \gamma(p_3, p_4),
A_2 = \gamma(p_4, p_1).
\]
In analogy with the Euclidean case [3],

**Definition 3.1.** Fix a point $x_0 \in \partial D$, let $p_i$, $i = 1, \ldots, 4$ be the points constructed above associated to $x_0 \in D$, then $\Gamma_{x_0} = A_1 \cup B_1 \cup A_2 \cup A_3$ is called the quadrilateral associated to $x_0 \in D$ and it satisfies

$$|A_1| + |A_2| = |B_1| + |B_2|,$$

where

$$
\begin{align*}
B_1 &= \gamma(p_1, p_2) \\
A_1 &= \gamma(p_2, p_3) \\
B_2 &= \gamma(p_3, p_4) \\
A_2 &= \gamma(p_4, p_1).
\end{align*}
$$

Moreover, the interior domain $D_{x_0}$ bounded by $\Gamma_{x_0}$ is the square inscribed associated to $x_0 \in D$ (note that $D_{x_0}$ is a topological disc), and $B_1$ is called the bottom side (c.f. Figure 3).

![Figure 3: Scherk domain](image)

Second, let us explain how to take the regular trapezoids: As above, fix $x_0 \in \partial D$ (from now on, $x_0$ will be fixed and we will omit it) and parametrize $\partial D$ as $\alpha : \mathbb{R}/[0, L) \to \partial D$. Let
\[ 0 \leq s_1 < s_2 < L, \] or equivalently, two distinct and ordered points \( p_i = \alpha(s_i) \in \partial D, i = 1, 2. \)

The aim is to construct a trapezoid in the region bounded by \( \gamma(p_1, p_2) \) and \( \alpha([s_1, s_2]) \). To do so, set \( \bar{s} = \frac{s_1 + s_2}{2} \), i.e., \( \bar{p} = \alpha(\bar{s}) \) is the mid-point. Define \( \bar{p}^\pm(s) = \alpha(\bar{s} \pm s) \) for \( 0 \leq s \leq \bar{s} \).

Set

\[
\begin{align*}
    l_1(s) &= \text{Length} \left( \gamma(p_1, \bar{p}^-(s)) \right) \\
    l_2(s) &= \text{Length} \left( \gamma(\bar{p}^-(s), \bar{p}^+(s)) \right) \\
    l_3(s) &= \text{Length} \left( \gamma(\bar{p}^+(s), p_2) \right) \\
    l_4(s) &= \text{Length} \left( \gamma(p_2, p_1) \right).
\end{align*}
\]

Hence, for \( s \) close to zero

\[ l_1(s) + l_3(s) > l_2(s) + l_4(s) \]

by the Triangle Inequality, and for \( s \) close to \( \bar{s} \)

\[ l_1(s) + l_3(s) < l_2(s) + l_4(s), \]

since \( l_1 \) and \( l_3 \) go to zero and \( l_4 \) has positive length (c.f. Figure 4).

Figure 4: How does the trapezoid vary?
Thus, there exists $s_0 \in (0, \bar{s})$ so that

$$l_1(s_0) + l_3(s_0) = l_2(s_0) + l_4(s_0).$$

So, given a fixed point $x_0 \in \partial D$ and a geodesic arc $A := \gamma(p_1, p_2)$ joining two (distinct and oriented) points in $\partial D$, we have the existence of two distinct points $p^- = \alpha(\bar{s} - s_0)$ and $p^+ = \alpha(\bar{s} + s_0)$ ordered count-clockwise so that

$$l_1 + l_3 = l_2 + l_4,$$

where

\begin{align*}
l_1 &= \text{Length} \left( \gamma(p_1, p^-) \right) \\
l_2 &= \text{Length} \left( \gamma(p^- , p^+) \right) \\
l_3 &= \text{Length} \left( \gamma(p^+, p_2) \right) \\
l_4 &= \text{Length} \left( \gamma(p_2, p_1) \right).
\end{align*}

Moreover, the domain bounded by $\gamma(p_1, p^-) \cup \gamma(p^- , p^+) \cup \gamma(p^+, p_2) \cup \gamma(p_2, p_1)$ is a topological disc.

![Figure 5: (Left) Regular Trapezoid](image1)

![Figure 6: (Right) First Scherk domain](image2)

Again, in analogy with the Euclidean case,
**Definition 3.2.** $E = \gamma(p_1, p^-) \cup \gamma(p^-, p^+) \cup \gamma(p^+, p_2) \cup \gamma(p_1, p_2)$ is called the regular trapezoid associated to the side $A$, here $A = \gamma(p_1, p_2)$ (and, of course, once we have fixed a point $x_0 \in \partial D$), and $p^\pm$ are given by the above construction (c.f. Figure 5).

Now, we can begin the example. We only highlight the main steps in the construction since, in essence, it is as in [3, Section III].

Fix $x_0 \in \partial D$ and let $D_1$ the inscribed quadrilateral associated to $x_0$ and $\Gamma_1 = \partial D_1$ (see Definition 3.1). We label $A_1, B_1, A_2, B_2$ the sides of $\Gamma_1$ ordered count-clockwise, with $B_1$ the bottom side. By construction, $D_1$ is a Scherk domain. One can check this fact using the Triangle Inequality. From Theorem 3.1, there is a minimal graph $u_1$ in $D_1$ which is $+\infty$ on the $A_i$'s sides and equals $-\infty$ on the $B_i$'s sides (c.f. Figure 6).

![Figure 7: Attaching trapezoids](image)

Henceforth, we will attach regular trapezoids (see Definition 3.2) to the sides of the quadrilateral $\Gamma_1$ in the following way. Let $E_1$ the regular trapezoid associated to the side $A_1$, and $E'_1$ the regular trapezoid associated to the side $B_1$.

Consider the domain $D_2 = D_1 \cup E_1 \cup E'_1$, $\Gamma_2 = \partial D_2$. This new domain does not satisfy the second condition of Theorem 3.1, we only have to consider the inscribed polygon $E$ (c.f. Figure 7).
So, the next step is to perturb $D_2$ in such a way that it becomes an admissible domain. Let $p$ be the common vertex of $E_1$ and $E'_1$. Let $a_1$ the closed vertex of $E_1$ to $p$, and $b_1$ the closed vertex of $E'_1$ to $p$ (c.f. Figure 8).

One moves the vertex $a_1$ towards $b_1$ to a nearby point $a_1(\tau)$ on $\partial D$ (using the parametrization $\alpha : \mathbb{R}/[0, L) \rightarrow \partial D$ as we have been done throughout this Section). And then one moves $b_1$ towards $a_1$ to a nearby point $b_1(\tau)$ on $\partial D$.

Let $\Gamma_2(\tau)$ the inscribed polygon obtained by this perturbation, $E_1(\tau)$ and $E'_1(\tau)$ the perturbed regular trapezoids (c.f. Figure 9). Thus, for $\tau > 0$ small, it is clear that:

- $\Gamma_2(\tau)$ satisfies Condition 1 in Theorem 3.1.
- $2a(E_1(\tau)) < |E_1(\tau)|$ and $2b(E'_1(\tau)) < |E'_1(\tau)|$.

Now, we state the following Lemma that establish how we extend the Scherk surface in general.

**Lemma 3.1.** Let $u$ be a Scherk graph on a polygonal domain $D_1 = P(A_1, B_1, \ldots, A_k, B_k)$, where the $A_i$'s and $B_i$'s are the (geodesic) sides of $\partial D_1$ on which $u$ takes values $+\infty$ and $-\infty$ respectively. Let $K$ be a compact set in the interior of $D_1$. Let $D_2 = P(E_1, E'_1, A_2, B_2, \ldots, A_k, B_k)$ be the polygonal domain $D_1$ to which we attach two regular trapezoids $E_1$ to the side $A_1$ and $E'_1$ to the side $B_1$. Let $E_1(\tau)$ and $E'_1(\tau)$ be the perturbed polygons as above. Then
for all $\epsilon > 0$ there exists $\bar{\tau} > 0$ so that, for all $0 < \tau \leq \bar{\tau}$, $v$ is a Scherk graph on $P(E_1(\tau), E'_1(\tau), A_2, B_2, \ldots, A_k, B_k)$ such that

$$\|u - v\|_{C^2(K)} \leq \epsilon.$$  (3.1)

**Proof.** The proof of this Lemma relies on [3, Section IV] with the obvious differences that we need to use the results for Scherk graphs over a domain in a Hadamard surface stated in [9] and [5].

Before we return to the construction, let us explain how we construct a compact domain associated to any Scherk domain: Let $D = P(A_1, B_1, \ldots, A_k, B_k)$ be a Scherk domain in $\mathcal{D}$ with vertex $\{v_1, \ldots, v_{2k}\} \in \partial \mathcal{D}$. Let $\beta_{v_i} : [0, 1] \longrightarrow \mathcal{D}$ denote the radial geodesic starting at $p_0 \in \mathcal{D}$ (the center of the disc $\mathcal{D}$) and ending at $v_i \in \partial \mathcal{D}$. Note that any $\beta_{v_i}$ can not touch neither a $A_i$ side nor a $B_i$ side expect at the vertex.

Set $r < 1$ and $p_i = \beta_{v_i}(r) \in \mathcal{D}$ for $i = 1, \ldots, 2k$. Consider the polygon

$$P = \bigcup_{i=1}^{2k-1} \gamma(p_i, p_{i+1}) \cup \gamma(p_{2k}, p_1) \subset D,$$
and let $K'$ be the closure of the domain bounded by $P$, here $\gamma(p_i, p_{i+1})$ is the geodesic arc joining $p_i$ and $p_{i+1}$ in $D$. Let $D(p_i, 1 - r)$ be geodesic disc centered at $p_i$ of radius $1 - r$ for each $i = 1, \ldots, 2k$. Then,

**Definition 3.3.** For $r < 1$ close to 1, the **compact domain associated to the Scherk domain $D$** is given by

$$K = K' \setminus \bigcup_{i=1}^{2k} D(p_i, 1 - r).$$

![Figure 10: (Left) Compact domain associated to the inscribed quadrilateral](image1)

![Figure 11: (Right) Attaching perturbed regular trapezoids](image2)

Next, we continue with the construction. Let $D_1 = P(A_1, B_1, A_2, B_2)$ be the inscribed square in $D$ (given in Definition 3.1), and the Scherk graph $u_1$ on $D_1$ which is $+\infty$ on the $A_i's$ sides and $-\infty$ on the $B_i's$ sides. Let $K_1$ be the compact domain associated to $D_1$ (see Definition 3.3). We choose $r_1 < 1$ close enough to one so that $u_1 > 1$ on the geodesic sides of $\partial K_1$ closer to the $A_i's$ sides and $u_1 < -1$ on the geodesic sides of $\partial K_1$ closer to the $B_i's$ sides (cf. Figure 10).

Next, we attach perturbed regular trapezoids to the sides $A_1$ and $B_1$, so from Lemma 3.1, for any $\epsilon_2 > 0$ there exists $\tau_2 > 0$ so that $D_2(\tau) = D_1 \cup E_1(\tau) \cup E_1'(\tau)$ is a Scherk domain and $u_2(\tau)$, the Scherk graph defined on $D_2(\tau)$, satisfy

$$\|u_1 - u_2(\tau)\|_{C^2(K_1)} \leq \epsilon_2,$$
for all $0 < \tau \leq \tau_2$. Moreover, we can choose $u_2(\tau)$ so that $u_1(p_0) = u_2(\tau)(p_0)$ (here $p_0$ is the center of $D$). Then, choose $\epsilon_2 > 0$ so that $u_2(\tau) > 1$ on the geodesic sides of $\partial K_1$ closer to the $A_i'$s sides and $u_2(\tau) < -1$ on the geodesic sides of $\partial K_1$ closer to the $B_i'$s sides.

Let $K_2(\tau)$ be the compact domain associated to the Scherk domain $D_2(\tau)$. Choose $r_2 < 1$ close enough to one (in the definition of $K_2(\tau)$ given by Definition 3.3) so that, for $0 < \tau \leq \tau_2$, $u_2(\tau) > 2$ on those geodesic sides of $\partial K_2(\tau)$ parallel to the sides of $D_2(\tau)$ where $u_2(\tau) = +\infty$, and $u_2(\tau) < -2$ on the sides of $\partial K_2(\tau)$ parallel to sides of $D_2(\tau)$ where $u_2(\tau) = -\infty$ (cf. Figure 12).

Continue by constructing the Scherk domain $D_3(\tau)$ by attaching perturbed regular trapezoids (as above) to the sides $A_2$ and $B_2$ of $D_1$. We know, for $\epsilon_3 > 0$, that there exist $\tau_3 > 0$ so that if $0 < \tau \leq \tau_3$ then the Scherk graph $u_3(\tau)$ exists, $u_3(\tau)(p_0) = u_1(p_0)$ and

$$\|u_3(\tau) - u_2(\tau)\|_{C^2(K_2(\tau))} \leq \epsilon_3.$$ 

Moreover, choose $\epsilon_3 > 0$ so that $u_3(\tau) > 3$ on the geodesic sides of $\partial K_2(\tau)$ closer to the $A_i'$s sides and $u_3(\tau) < -3$ on the geodesic sides of $\partial K_2(\tau)$ closer to the $B_i'$s sides (cf. Figure 13).

Now choose $\epsilon_n \to 0$, $\tau_n \to 0$, $K_n(\tau_n)$ so that $K_n(\tau_n) \subset K_{n+1}(\tau_{n+1})$, $\bigcup_n K_n(\tau_n) = D$. Then the $u_n(\tau_n)$ converge to a graph $u$ on $D$. 
To see \( u \) has the desired properties, we refer the reader to [3, pages 13 and 14] with the only difference that we need to use now Theorem 2.1.

**Remark 3.1.** The above construction can be carried out in a more general situation. Actually, if we ask that

- The geodesic disc \( D \) has strictly convex boundary.
- There is a unique minimizing geodesic joining any two points of the disc.

Then, we can extend the above example.

**References**


