APPENDIX TO "THE TRIANGLE-FREE PROCESS AND THE RAMSEY NUMBER $R(3,k)$"

GONZALO FIZ PONTIVEROS, SIMON GRIFFITHS, AND ROBERT MORRIS

Abstract. This file contains some of the straightforward but technical calculations from the paper "The triangle-free process and the Ramsey number $R(3,k)$" which were omitted from the proof in order not to distract from the main argument. We have gathered these calculations here in order to save the interested reader the trouble of reproving them herself.

1. Introduction

This short file is an Appendix to the paper [2]. In that paper we follow the triangle-free process to its asymptotic end, and show that it is an excellent 'Ramsey graph', in the sense that it gives very strong lower bounds on the Ramsey numbers $R(3,k)$.

Recall from [2] that we denote by $G_{n,\triangle}$ the (random) maximal triangle-free graph on $\{1,\dots,n\}$ obtained via the triangle-free process. The main results of [2] were as follows:[1]

Theorem 1.1.

$$e\big(G_{n,\triangle}\big) \;=\; \left(\frac{1}{2\sqrt{2}} + o(1)\right) n^{3/2}\sqrt{\log n},$$

with high probability as $n \to \infty$.

The Ramsey number $R(3,k)$ is the smallest integer n such that every red-blue colouring of the edges of the complete graph K_n contains either a red K_k or a blue triangle.

Theorem 1.2.

$$R(3,k) \;\geqslant\; \left(\frac{1}{4} - o(1)\right)\frac{k^2}{\log k}$$

as $k \to \infty$.

The basic heuristic behind Theorems 1.1 and 1.2 is that, with high probability, the graph G_m obtained after m steps of the triangle-free process approximates (in a certain sense) the Erdős-Rényi random graph $G_{n,m}$, except in the fact that it contains no triangles. More precisely, there exists a (large) collection $\mathcal{S}$ of variables all of which take (approximately) the values one would expect in $G_{n,m}$, and all of whose derivatives at time $t = m \cdot n^{-3/2}$ may be bounded by functions which depend only on the values of variables in $\mathcal{S}$ at time t. We showed that moreover almost all of these variables exhibit a certain 'self-correction', and were thus able to control their evolution with a fairly high degree of precision.

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[1] We remark that when we restate results from [2] we shall use the numbering of that paper, whereas new statements will be given the prefix 'A'.

In this file we shall give the details of various straightforward but technical calculations which were omitted from [2] due to considerations of space and aesthetics. More precisely:

- In Section 2 we shall give an extended version of [2, Section 3.4], prove a generalized version of Lemma 3.9, and prove Lemma 4.23 and Proposition 5.5.
- In Sections 3 and 4 we shall do the same for [2, Section 4]; in particular, we shall prove Propositions 4.55 and 4.56 in Section 4.
- In Section 5 we shall derive the equations which govern the 'whirlpool' of [2, Section 6], and prove Lemma 6.7.
- Finally, in Section 6, we shall (for completeness) adapt the proof (from [3]) of Lemma 3.1 to our setting.

2. Section 3.2: Tracking the variables X_e

Recall that $C = C(\varepsilon) > 0$ is chosen sufficiently large, and that

$$g_y(t) = e^{2t^2} n^{-1/4} (\log n)^4 \qquad \text{and} \qquad g_x(t) = C g_y(t).$$

Set $a = \omega \cdot n^{3/2}$ and define, for each $m \in [m^*]$,

$$\mathcal{K}^{\mathcal{X}}(m) = \mathcal{E}(m) \cap \mathcal{X}(a) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m).$$

In this section we shall prove Lemmas 3.9 and 3.11 of [2], which were used in [2, Section 3] to prove the following proposition.

Proposition 3.8. *Let $\omega \cdot n^{3/2} < m \leqslant m^*$. With probability at least $1 - n^{-C \log n}$ either $\mathcal{K}^{\mathcal{X}}(m-1)^c$ holds, or*

$$X_e(m) \in \Big(1 \pm C e^{2t^2} n^{-1/4} (\log n)^4\Big) \cdot 2e^{-8t^2} n \;=\; \big(1 \pm g_x(t)\big) \tilde{X}(m) \qquad (1)$$

for every open edge $e \in O(G_m)$.

Let (W, A) denote the graph structure pair with $v(W) = 4$, $v_A(W) = 1$, $e(W) = 1$ and $o(W) = 2$. We shall use the following two immediate consequences of the event $\mathcal{E}(m)$: that

$$X_e(m) \leqslant 2 \cdot \tilde{X}(m)$$

for every open edge $e \in O(G_m)$, and that

$$N_\phi(W)(m) \;\leqslant\; \max\Big\{4te^{-8t^2}\sqrt{n}, (\log n)^\omega\Big\}$$

for every $\omega \cdot n^{3/2} < m \leqslant m^*$ and every faithful map ϕ.

The first step is to show that the variables X_e are self-correcting as long as the event $\mathcal{K}(m) = \mathcal{K}^X(m) \cap \mathcal{X}(m)$ holds. Define

$$X_e^*(m) \;=\; \frac{X_e(m) - \tilde{X}(m)}{g_x(t)\tilde{X}(m)},$$

the normalized error. We shall prove the following lemma.

Lemma 3.9. *Let $\omega \cdot n^{3/2} \leqslant m \leqslant m^*$. If $\mathcal{E}(m) \cap \mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds, then*

$$\mathbb{E}\big[\Delta X_e^*(m)\big] \,\in\, \frac{4t}{n^{3/2}}\Big(- X_e^*(m) \pm \varepsilon\Big)$$

for every $e \in O(G_m)$.

It is easy to see that

$$\mathbb{E}\big[\Delta X_e(m)\big] \,=\, -\frac{2}{Q(m)} \sum_{f \in X_e(m)} \Big(Y_f(m) + 1\Big), \tag{2}$$

for every $e \in O(G_m)$. Indeed, for each open triangle T in G_m containing e, the probability that one of the open edges (f and h, say) of T other than e is closed (or chosen) in step $m+1$ is equal to

$$\frac{\big|Y_f(m) \cup Y_h(m)\big| + 2}{Q(m)} \,=\, \frac{Y_f(m) + Y_h(m) + 2}{Q(m)}.$$

To see this, simply note that if $Y_f(m) \cap Y_h(m) \neq \emptyset$, then the endpoints of e have a common neighbour in G_m, which means that $e \notin O(G_m)$, a contradiction. Since X_e decreases by two for each open triangle which is destroyed, (2) follows.

In order to deduce Lemma 3.9 from (2), we shall prove the following, more general statement. We also use this more general version in [2, Section 7.4].

Lemma A.2.1. *Let $A(m)$ be a random variable which denotes both a collection of open edges of G_m, and the size of that collection. Suppose that*

$$\mathbb{E}\big[\Delta A(m)\big] \,\in\, -\frac{\ell \pm o(1)}{Q(m)} \sum_{f \in A(m)} Y_f(m) \tag{3}$$

for some $\ell \in \mathbb{N}$ and every $\omega \cdot n^{3/2} < m \leqslant m^*$, and set $\tilde{A}(m) = e^{-4\ell t^2} A(0)$ and

$$A^*(m) \,=\, \frac{A(m) - \tilde{A}(m)}{g(t)\tilde{A}(m)},$$

for some $g\colon (0,t^*) \to \mathbb{R}^+$ which satisfies $g(t) \geqslant g_x(t)$ and $g \sim g_x$.[2] *Then*

$$\mathbb{E}\big[\Delta A^*(m)\big] \,\in\, \frac{4t}{n^{3/2}}\Big(- A^*(m) \pm \varepsilon\Big).$$

for every $\omega \cdot n^{3/2} \leqslant m \leqslant m^*$ such that $\mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds and $|A^*(m)| \leqslant 1$.

We emphasize that ℓ is an absolute fixed constant; in fact, in our applications we shall need to consider only the cases $\ell = 1$ and $\ell = 2$. In the proof of Lemma A.2.1, we shall use the product rule, which was stated in [2, Section 4]

[2]We write $g \sim g_x$ to indicate that $g(t) = \lambda(n) \cdot g_x(t)$ for some function $\lambda(n)$. Since $g(t) \geqslant g_x(t)$, we have $\lambda(n) \geqslant 1$ for every $n \in \mathbb{N}$.

The product rule. *For any random variables $a(m)$ and $b(m)$,*

$$\mathbb{E}\big[\Delta\big(a(m)b(m)\big)\big] \;=\; a(m)\mathbb{E}\big[\Delta b(m)\big] + b(m)\mathbb{E}\big[\Delta a(m)\big] + \mathbb{E}\big[\big(\Delta a(m)\big)\big(\Delta b(m)\big)\big].$$

In particular, if $a(m)$ is deterministic, then

$$\mathbb{E}\big[\Delta\big(a(m)b(m)\big)\big] \;=\; a(m)\mathbb{E}\big[\Delta b(m)\big] + \Delta a(m)\Big(b(m) + \mathbb{E}\big[\Delta b(m)\big]\Big).$$

To simplify the calculations below, we shall write $a \approx b$ to denote that the inequalities $a/b \in 1 \pm O\big(1/n\big)$ hold.

Proof of Lemma A.2.1. Observe first that, differentiating with respect to t, we have

$$\Delta\tilde{A}(m) \approx -\frac{8\ell t}{n^{3/2}} \cdot \tilde{A}(m) \quad \text{and} \quad \Delta\big(g(t)\tilde{A}(m)\big) \approx -\frac{(8\ell - 4)t}{n^{3/2}} \cdot g(t)\tilde{A}(m). \tag{4}$$

We claim that if $\mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds and $|A^*(m)| \leqslant 1$, then

$$\begin{aligned}
\mathbb{E}\big[\Delta A(m)\big] - \Delta\tilde{A}(m) \;&\in\; -\frac{\ell \pm o(1)}{Q(m)} \sum_{f \in A(m)} Y_f(m) \;+\; \frac{8\ell t}{n^{3/2}} \cdot \tilde{A}(m) \\
&\in\; \left(\tilde{A}(m) \;-\; \frac{1 \pm g_y(t)}{1 \pm g_q(t)} \cdot A(m)\right) \cdot \big(\ell \pm o(1)\big) \cdot \frac{\tilde{Y}(m)}{\tilde{Q}(m)} \\
&\subseteq\; \Big(1 \;-\; \big(1 \pm 2g_y(t)\big)\big(1 + g(t)A^*(m)\big)\Big) \cdot \big(\ell \pm o(1)\big) \cdot \frac{\tilde{Y}(m) \cdot \tilde{A}(m)}{\tilde{Q}(m)} \\
&\subseteq\; \Big(-\,\ell \cdot A^*(m) \pm \varepsilon^2\Big) \cdot \frac{g(t) \cdot \tilde{Y}(m) \cdot \tilde{A}(m)}{\tilde{Q}(m)}.
\end{aligned}$$

Indeed, this follows since $g_q(t) \ll g_y(t) \leqslant \varepsilon^3 g_x(t) \leqslant \varepsilon^3 g(t)$ and $\omega < t \leqslant t^*$. Thus

$$\frac{\mathbb{E}\big[\Delta A(m)\big] - \Delta\tilde{A}(m)}{g(t)\tilde{A}(m)} \;\in\; -\,\frac{8\ell t}{n^{3/2}} \cdot A^*(m) \;\pm\; \frac{\varepsilon t}{n^{3/2}}. \tag{5}$$

Now, since $A(m) - \tilde{A}(m) = g(t)\tilde{A}(m) \cdot A^*(m)$, by the product rule we have

$$\frac{\mathbb{E}\big[\Delta A(m)\big] - \Delta\tilde{A}(m)}{g(t)\tilde{A}(m)} \;=\; \mathbb{E}\big[\Delta A^*(m)\big] \;+\; \frac{\Delta\big(g(t)\tilde{A}(m)\big)}{g(t)\tilde{A}(m)}\Big(A^*(m) \;+\; \mathbb{E}\big[\Delta A^*(m)\big]\Big). \tag{6}$$

Combining (4), (5) and (6), we obtain

$$\begin{aligned}
\mathbb{E}\big[\Delta A^*(m)\big] \;&\in\; -\,\frac{8\ell t}{n^{3/2}} \cdot A^*(m) + \frac{(8\ell - 4)t}{n^{3/2}} \cdot A^*(m) \;\pm\; \frac{2\varepsilon t}{n^{3/2}} \\
&\subseteq\; \frac{4t}{n^{3/2}} \cdot \Big(-\,A^*(m) \pm \varepsilon\Big),
\end{aligned}$$

as required. □

Recall next the following bound on $|\Delta X_e(m)|$, which was proved in [2].

Lemma 3.10. *Let $\omega \cdot n^{3/2} < m \leqslant m^*$. If $\mathcal{E}(m)$ holds, then*

$$|\Delta X_e(m)| \;\leqslant\; 2 \cdot \max\big\{4te^{-8t^2}\sqrt{n}, (\log n)^\omega\big\}$$

for every $e \in O(G_m)$.

Using Lemmas 3.9 and 3.10, we can bound $|\Delta X_e^*(m)|$ and $\mathbb{E}\big[|\Delta X_e^*(m)|\big]$.

Lemma 3.11. *Let $\omega \cdot n^{3/2} < m \leqslant m^*$. If $\mathcal{E}(m) \cap \mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds, then*

$$|\Delta X_e^*(m)| \;\leqslant\; \frac{C}{g_x(t)} \cdot \frac{e^{8t^2}}{n} \max\big\{te^{-8t^2}\sqrt{n}, (\log n)^\omega\big\} \quad \text{and} \quad \mathbb{E}\big[|\Delta X_e^*(m)|\big] \;\leqslant\; \frac{C}{g_x(t)} \cdot \frac{\log n}{n^{3/2}}$$

for every $e \in O(G_m)$.

In order to deduce Lemma 3.11 from Lemma 3.10, and also in Section 5, below, we shall use the following result, which was stated (but not proved) in [2].

Lemma 4.23. *Let $A(m)$ be a random variable, let $\tilde{A}(m)$ and $g(t)$ be functions, and set*

$$A^*(m) \;=\; \frac{A(m) - \tilde{A}(m)}{g(t)\tilde{A}(m)}.$$

If $|A(m)| \leqslant \big(1 + g(t)\big)\tilde{A}(m)$,

$$|\Delta \tilde{A}(m)| \ll \frac{\log n}{n^{3/2}} \cdot \tilde{A}(m) \quad \text{and} \quad |\Delta\big(g(t)\tilde{A}(m)\big)| \ll \frac{\log n}{n^{3/2}} \cdot g(t)\tilde{A}(m), \tag{7}$$

then

$$|\Delta A^*(m)| \;\leqslant\; 2 \cdot \left(\frac{|\Delta A(m)|}{g(t)\tilde{A}(m)} \;+\; \frac{1 + g(t)}{g(t)} \cdot \frac{\log n}{n^{3/2}} \right).$$

Proof. Observe first that, for arbitrary functions $a, b, c \colon \mathbb{N} \to \mathbb{R}^+$, if $a^*(m) = \frac{a(m) - b(m)}{c(m)}$ then

$$\Delta a(m) - \Delta b(m) \;=\; \Delta\big(a^*(m)c(m)\big) \;=\; c(m+1)\Delta a^*(m) + a^*(m)\Delta c(m),$$

and hence

$$\Delta a^*(m) \;\in\; \frac{\Delta a(m)}{c(m+1)} \;\pm\; \frac{|\Delta b(m)| \cdot c(m) + \big(a(m) + b(m)\big)|\Delta c(m)|}{c(m) \cdot c(m+1)}. \tag{8}$$

Applying (8) to the functions $A(m)$, $\tilde{A}(m)$ and $g(t)\tilde{A}(m)$, and using the assumptions that $A(m) \leqslant \big(1 + g(t)\big)\tilde{A}(m)$ and

$$|\Delta \tilde{A}(m)| \ll \frac{\log n}{n^{3/2}} \cdot \tilde{A}(m) \quad \text{and} \quad |\Delta\big(g(t)\tilde{A}(m)\big)| \ll \frac{\log n}{n^{3/2}} \cdot g(t)\tilde{A}(m),$$

we obtain

$$|\Delta A^*(m)| \;\leqslant\; 2 \cdot \left(\frac{|\Delta A(m)|}{g(t)\tilde{A}(m)} \;+\; \frac{1 + g(t)}{g(t)} \cdot \frac{\log n}{n^{3/2}} \right),$$

as claimed. $\square$

Specializing to the case X_e, we obtain the following easy corollary.

Lemma A.2.2. *For every $\omega \cdot n^{3/2} < m \leqslant m^*$, if $\mathcal{X}(m)$ holds, then*

$$|\Delta X_e^*(m)| \leqslant \frac{3}{g_x(t)} \cdot \left(\frac{|\Delta X_e(m)|}{\tilde{X}(m)} + \frac{\log n}{n^{3/2}} \right)$$

for every $e \in O(G_m)$.

Proof. We apply Lemma 4.23 with $A(m) = X_e(m)$ and $g(t) = g_x(t)$. Since $\tilde{X}(m)$ is equal to e^{-8t^2} times a function of n, and $g_x(t)\tilde{X}(m)$ is equal to e^{-6t^2} times a function of n, we have

$$\Delta \tilde{X}(m) \in \frac{-16t \pm o(1)}{n^{3/2}} \cdot \tilde{X}(m) \quad \text{and} \quad \Delta\big(g_x(t)\tilde{X}(m)\big) \in \frac{-12t \pm o(1)}{n^{3/2}} \cdot g_x(t)\tilde{X}(m),$$

and hence

$$|\Delta \tilde{X}(m)| \ll \frac{\log n}{n^{3/2}} \cdot \tilde{X}(m) \quad \text{and} \quad |\Delta\big(g_x(t)\tilde{X}(m)\big)| \ll \frac{\log n}{n^{3/2}} \cdot g_x(t)\tilde{X}(m).$$

Moreover, the event $\mathcal{X}(m)$ implies that $X_e(m) \leqslant \big(1 + g_x(t)\big)\tilde{X}(m)$ for every $e \in O(G_m)$, and so

$$|\Delta X_e^*(m)| \leqslant \frac{3}{g_x(t)} \cdot \left(\frac{|\Delta X_e(m)|}{\tilde{X}(m)} + \frac{\log n}{n^{3/2}} \right),$$

as claimed. $\square$

We can now easily deduce Lemma 3.11.

Proof of Lemma 3.11. By Lemmas 3.10 and A.2.2, we have

$$\begin{aligned} |\Delta X_e^*(m)| &\leqslant \frac{3}{g_x(t)} \cdot \left(\frac{|\Delta X_e(m)|}{\tilde{X}(m)} + \frac{\log n}{n^{3/2}} \right) \\ &\leqslant \frac{3}{g_x(t)\tilde{X}(m)} \cdot \left(2 \cdot \max\big\{4te^{-8t^2}\sqrt{n}, (\log n)^{\omega}\big\} + \frac{e^{-8t^2}\log n}{\sqrt{n}} \right) \\ &\leqslant \frac{C}{g_x(t)} \cdot \frac{e^{8t^2}}{n} \max\big\{te^{-8t^2}\sqrt{n}, (\log n)^{\omega}\big\}, \end{aligned}$$

as claimed. Moreover, observe that, by (2), and using the event $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ and the fact that X_e is decreasing, we have

$$\mathbb{E}\big[|\Delta X_e(m)|\big] = \frac{2}{Q(m)} \sum_{f \in X_e(m)} \Big(Y_f(m) + 1\Big) \leqslant \frac{3 \cdot \tilde{X}(m) \cdot \tilde{Y}(m)}{\tilde{Q}(m)} = \frac{12t}{n^{3/2}} \cdot \tilde{X}(m).$$

Thus, by Lemma A.2.2, we have

$$\begin{aligned} \mathbb{E}\big[|\Delta X_e^*(m)|\big] &\leqslant \frac{3}{g_x(t)} \cdot \left(\frac{\mathbb{E}\big[|\Delta X_e(m)|\big]}{\tilde{X}(m)} + \frac{\log n}{n^{3/2}} \right) \\ &\leqslant \frac{3}{g_x(t)} \cdot \left(\frac{12t}{n^{3/2}} + \frac{\log n}{n^{3/2}} \right) \leqslant \frac{C}{g_x(t)} \cdot \frac{\log n}{n^{3/2}}, \end{aligned}$$

as required. $\square$

2.2. **The proof of Proposition 5.5.** We take this opportunity to prove a similar proposition from [2, Section 5].

Proposition 5.5. *Let* $\omega \cdot n^{3/2} < m \leqslant m^*$. *Then, with probability at least* $1 - n^{-C \log n}$, *either* $\mathcal{K}^{\mathcal{Y}}(m-1)^c$ *holds, or*

$$Y_e^L(m) \, \in \, \big(1 \pm g_x(t)\big) \cdot \big(2te^{-4t^2}\sqrt{n}\big) \tag{9}$$

for every $e \in O(G_m)$.

The proof is almost identical to that of Proposition 3.8, above, so we shall skip some of the details. Recall first the following special case of [2, Lemma 5.20], which is obtained from the version stated there by setting $\sigma = L$ and noting that $U_e^L(m)V_e^L(m) = \sum_{f \in Y_e^L(m)} Y_f(m)$.

Lemma 5.20. *Let* $\omega \cdot n^{3/2} < m \leqslant m^*$. *If* $\mathcal{E}(m) \cap \mathcal{U}(m) \cap \mathcal{X}(m) \cap \mathcal{Z}(m)$ *holds, then*

$$\mathbb{E}\big[\Delta Y_e^L(m)\big] \, \in \, -\frac{1}{Q(m)} \sum_{f \in Y_e^L(m)} Y_f(m) \, + \, \left(\frac{1}{2} \pm g_x(t)\right) \frac{\tilde{X}(m)}{Q(m)},$$

for every $e \in O(G_m)$.

Define

$$(Y_e^L)^*(m) \, = \, \frac{2 \cdot Y_e^L(m) - \tilde{Y}(m)}{g_x(t)\tilde{Y}(m)},$$

the normalized error. Since $\tilde{X}(m) \ll \tilde{Y}(m)^2$ for $t > \omega$, it follows from Lemmas 5.20 and A.2.1 that the variables Y_e^L are self-correcting.

Lemma A.2.3. *Let* $\omega \cdot n^{3/2} \leqslant m \leqslant m^*$. *If* $\mathcal{E}(m) \cap \mathcal{U}(m) \cap \mathcal{X}(m) \cap \mathcal{Z}(m) \cap \mathcal{Q}(m)$ *holds, then*

$$\mathbb{E}\big[\Delta(Y_e^L)^*(m)\big] \, \in \, \frac{4t}{n^{3/2}}\Big(- (Y_e^L)^*(m) \pm \varepsilon \Big)$$

for every $e \in O(G_m)$.

Next, recall the following lemma from [2, Section 5.3].

Lemma 5.17. *Let* $\omega \cdot n^{3/2} < m \leqslant m^*$. *If* $\mathcal{E}(m) \cap \mathcal{Z}(m)$ *holds, then*

$$\big|\Delta Y_e^L(m)\big| \, \leqslant \, (\log n)^3$$

for every $e \in O(G_m)$.

Using Lemmas 5.17 and A.2.3, we can bound $|\Delta(Y_e^L)^*(m)|$ and $\mathbb{E}\big[|\Delta(Y_e^L)^*(m)|\big]$.

Lemma A.2.4. *Let* $\omega \cdot n^{3/2} < m \leqslant m^*$. *If* $\mathcal{E}(m) \cap \mathcal{U}(m) \cap \mathcal{X}(m) \cap \mathcal{Z}(m) \cap \mathcal{Q}(m)$ *holds, then*

$$|\Delta(Y_e^L)^*(m)| \, \leqslant \, \frac{C}{g_x(t)} \cdot \frac{\log n}{\sqrt{n}} \qquad \text{and} \qquad \mathbb{E}\big[|\Delta(Y_e^L)^*(m)|\big] \, \leqslant \, \frac{C}{g_x(t)} \cdot \frac{\log n}{n^{3/2}}$$

for every $e \in O(G_m)$.

In order to deduce Lemma A.2.4 from Lemma 5.17, we shall use the following easy consequence of Lemma 4.23.

Lemma A.2.5. *For every $\omega \cdot n^{3/2} < m \leqslant m^*$, if $\mathcal{U}(m)$ holds, then*

$$|\Delta(Y_e^L)^*(m)| \leqslant \frac{4}{g_x(t)} \cdot \left(\frac{|\Delta Y_e^L(m)|}{\tilde{Y}(m)} + \frac{\log n}{n^{3/2}} \right)$$

for every $e \in O(G_m)$.

Proof. We apply Lemma 4.23 with $A(m) = Y_e^L(m)$ and $g(t) = g_x(t)$. Since $\tilde{Y}(m)$ is equal to $t \cdot e^{-4t^2}$ times a function of n, and $g_x(t)\tilde{Y}(m)$ is equal to $t \cdot e^{-2t^2}$ times a function of n, we have

$$\Delta\tilde{Y}(m) \in \frac{-8t \pm o(1)}{n^{3/2}} \cdot \tilde{Y}(m) \quad \text{and} \quad \Delta\big(g_x(t)\tilde{Y}(m)\big) \in \frac{-4t \pm o(1)}{n^{3/2}} \cdot g_x(t)\tilde{Y}(m),$$

and hence

$$|\Delta\tilde{Y}(m)| \ll \frac{\log n}{n^{3/2}} \cdot \tilde{X}(m) \quad \text{and} \quad |\Delta\big(g_x(t)\tilde{Y}(m)\big)| \ll \frac{\log n}{n^{3/2}} \cdot g_x(t)\tilde{X}(m).$$

Moreover, the event $\mathcal{U}(m)$ implies that $2 \cdot Y_e^L(m) \leqslant \big(1 + g_x(t)\big)\tilde{Y}(m)$ for every $e \in O(G_m)$, and so

$$|\Delta(Y_e^L)^*(m)| \leqslant \frac{4}{g_x(t)} \cdot \left(\frac{|\Delta Y_e^L(m)|}{\tilde{Y}(m)} + \frac{\log n}{n^{3/2}} \right),$$

as claimed. □

We can now easily deduce Lemma A.2.4.

Proof of Lemma A.2.4. By Lemmas 5.17 and A.2.5, we have

$$|\Delta(Y_e^L)^*(m)| \leqslant \frac{4}{g_x(t)} \cdot \left(\frac{|\Delta Y_e^L(m)|}{\tilde{Y}(m)} + \frac{\log n}{n^{3/2}} \right) \leqslant \frac{C}{g_x(t)} \cdot \frac{(\log n)^3}{\sqrt{n}},$$

as claimed. Moreover, since

$$\frac{\mathbb{E}\big[|\Delta Y_e^L(m)|\big]}{\tilde{Y}(m)} \leqslant \frac{\tilde{Y}(m)}{\tilde{Q}(m)} \leqslant \frac{\log n}{n^{3/2}},$$

it follows from Lemma A.2.5 that

$$\mathbb{E}\big[|\Delta(Y_e^L)^*(m)|\big] \leqslant \frac{4}{g_x(t)} \cdot \left(\frac{\mathbb{E}\big[|\Delta Y_e^L(m)|\big]}{\tilde{Y}(m)} + \frac{\log n}{n^{3/2}} \right) \leqslant \frac{C}{g_x(t)} \cdot \frac{\log n}{n^{3/2}},$$

as required. □

Finally, let's deduce Proposition 5.5.

Proof of Proposition 5.5. We begin by choosing a family of parameters as in [2, Definition 3.4]. Set $\mathcal{K}(m) = \mathcal{K}^{\mathcal{Y}}(m) \cap \mathcal{U}(m) \cap \big\{|(Y_e^L)^*(a)| < 1/2\big\}$ and $I = [a, b] = [\omega \cdot n^{3/2}, m^*]$, and let

$$\alpha(t) = \frac{C}{g_x(t)} \cdot \frac{(\log n)^3}{\sqrt{n}} \qquad \text{and} \qquad \beta(t) = \frac{C}{g_x(t)} \cdot \frac{\log n}{n^{3/2}}.$$

Moreover, set $\lambda = C$, $\delta = \varepsilon$ and $h(t) = t \cdot n^{-3/2}$. We claim that $(\lambda, \delta; g_x, h; \alpha, \beta; \mathcal{K})$ is a reasonable collection, and that Y_e^L satisfies the conditions of [2, Lemma 3.2] if $e \in O(G_m)$.

To prove the first statement, we need to show that α and β are λ-slow, and that

$$\min\big\{\alpha(t),\, \beta(t),\, h(t)\big\} \,\geqslant\, \frac{\varepsilon t}{n^{3/2}}$$

and $\alpha(t) \leqslant \varepsilon^2$ for every $\omega < t \leqslant t^*$, each of which is obvious, since $g_x(t) \leqslant 1$ for all $t \leqslant t^*$. To prove the second, we need to show that Y_e^L is $(g_x, h; \mathcal{K})$-self-correcting, which follows from Lemma A.2.3, that, for every $\omega \cdot n^{3/2} < m \leqslant m^*$, if $\mathcal{K}(m)$ holds then

$$|\Delta(Y_e^L)^*(m)| \leqslant \alpha(t) \qquad \text{and} \qquad \mathbb{E}\big[|\Delta(Y_e^L)^*(m)|\big] \leqslant \beta(t),$$

which follows from Lemma A.2.4, and that $|(Y_e^L)^*(a)| < 1/2$, which follows from $\mathcal{K}(m)$.

Observe that

$$\alpha(t)\beta(t)n^{3/2} \,\leqslant\, \frac{C^2}{g_x(t)^2} \cdot \frac{(\log n)^4}{\sqrt{n}} \,\leqslant\, \frac{1}{(\log n)^3}$$

for every $\omega < t \leqslant t^*$. By [2, Lemma 3.1], and summing over edges $e \in E(K_n)$ the probability that $e \in O(G_m)$ and $(Y_e^L)^*(m) > 1$, it follows that

$$\mathbb{P}\Big(\mathcal{U}(m)^c \cap \mathcal{K}(m-1) \text{ for some } m \in [a,b]\Big) \,\leqslant\, n^6 \exp\Big(- \delta'(\log n)^3\Big) \,\leqslant\, n^{-C\log n}.$$

Finally, we remark that

$$\mathbb{P}\Big(\mathcal{K}^{\mathcal{Y}}(a) \cap \big\{|(Y_e^L)^*(a)| \geqslant 1/2\big\}\Big) \,\leqslant\, n^{-C\log n},$$

by Proposition 4.56 (see Section 4, below), and so the proposition follows. □

3. Section 4: Everything Else

In this section we shall give the details omitted from [2, Section 4], recall from some of the lemmas from that section which we shall need below, and prove some simple variants. Recall the following important definitions from [2].

Definition 2.10. Define

$$t_A^*(F) \,=\, \inf\Big\{t > 0 \,:\, \tilde{N}_A(F)(m) \leqslant (2t)^{e(F)}\Big\} \,\in\, [0,\infty] \tag{10}$$

and

$$t_A(F) \,=\, \min\Big\{\min\big\{t_A^*(H) : A \subsetneq H \subseteq F\big\},\, t^*\Big\}.$$

We call $t_A(F)$ the *tracking time* of the pair (F, A).

Moreover, for each graph structure pair (F, A) with $t_A(F) > 0$, define

$$c \,=\, c(F,A) \,:=\, \max\bigg\{\max_{A\subsetneq H\subseteq F}\bigg\{\frac{2o(H)}{2v_A(H) - e(H)}\bigg\},\, 2\bigg\}, \tag{11}$$

so in particular $e^{ct^2} = n^{1/4}$ when $t = t_A(F)$ if $t_A(F) < t^*$.

We begin with the following crucial remark.

Remark 4.2. Let (F, A) be a graph structure pair with $v(F) = n^{o(1)}$, and let $m \in [m^*]$. If the event $\mathcal{E}(m)$ holds, then $N_\phi(F)(m)$ satisfies the conclusion (i.e., either (a), (b) or (c)) of [2, Theorem 4.1], for every faithful injection $\phi\colon A \to V(G_m)$.

Proof of Remark 4.2. Let us denote by (F', A', ϕ') the graph structure obtained by removing the isolated vertices from F, so $A' = V(F') \cap A$ and $\phi' = \phi|_{A'}$. Note that $t_{A'}(F') = t_A(F)$, since $t^*_A(H' \cup A) = t^*_{A'}(H')$ for every $A' \subsetneq H' \subseteq F'$. Assume that the event $\mathcal{E}(m)$ holds and that ϕ is faithful at time t. We shall consider in turn the three cases corresponding to parts (a), (b) and (c) of the theorem.

Suppose first that $0 < t \leqslant \omega < t_A(F)$, and note that

$$N_{\phi'}(F')(m) \,\in\, \tilde{N}_{A'}(F')(m) \pm f_{F',A'}(t)\tilde{N}_{A'}(F')(n^{3/2}),$$

since the event $\mathcal{E}(m)$ holds, and $t_{A'}(F') = t_A(F)$. Now, each copy of F' in G_m, rooted at $\phi'(A')$, extends to between $(n - v(F))^k$ and n^k copies of F rooted at $\phi(A)$. Set $k = v_A(F) - v_{A'}(F')$, and note that $\tilde{N}_A(F) = n^k \cdot \tilde{N}_{A'}(F')$ and that $\gamma(F, A) > \gamma(F', A')$ if $k \neq 0$. It follows that

$$N_\phi(F)(m) \,\in\, \tilde{N}_A(F)(m) \pm f_{F,A}(t)\tilde{N}_A(F)(n^{3/2}),$$

as required.

The proof in case (b) is almost identical, so we skip the details and proceed to case (c). Here we need the observation that the minimal $A \subsetneq H \subseteq F$ such that $t < t_H(F)$ contains no isolated vertices. Indeed, this is obvious, since moving an isolated vertex from H to F can only increase $t_H(F)$, and $t > t_A(F)$. It follows that the minimal $A \subsetneq H \subseteq F$ such that $t < t_H(F)$, is equal to A union the minimal $A' \subsetneq H' \subseteq F'$ such that $t < t_{H'}(F')$. Since $\Delta(F', H', A') < \Delta(F, H, A)$ unless $k = 0$, it follows that

$$N_\phi(F)(m) \,\leqslant\, n^k \cdot (\log n)^{\Delta(F',H',A')} \tilde{N}_{H'}(F')(m^+) \,\leqslant\, (\log n)^{\Delta(F,H,A)} \tilde{N}_H(F)(m^+),$$

by the event $\mathcal{E}(m)$, as required. □

We remark that the same argument implies that the building sequences

$$A \subseteq H_0 \subsetneq \cdots \subsetneq H_\ell = F \qquad \text{and} \qquad A' \subseteq H'_0 \subsetneq \cdots \subsetneq H'_\ell = F'$$

of (F, A) and (F', A') respectively are identical, in the sense that $H_j = H'_j \cup A$ for every $0 \leqslant j \leqslant \ell - 1$. That is, all of the isolated vertices of F lie in $V(F) \setminus H_{\ell-1}$.

We next recall three simple properties of the collection $\mathcal{F}^o_F$, see [2, Section 4.2].

Observation 4.20. *Let (F, A) be a graph structure pair. Then $|\mathcal{F}^o_F| = e(F)$, and*

$$2te^{4t^2} \cdot \tilde{N}_A(F^o)(m) \,=\, \sqrt{n} \cdot \tilde{N}_A(F)(m)$$

for every $F^o \in \mathcal{F}^o_F$.

Observation 4.24. *If (F, A) is a graph structure pair and $F^o \in \mathcal{F}^o_F$, then $t_A(F) \leqslant t_A(F^o)$.*

Proof. If $t_A(F^o) = t^*$ there is nothing to prove, so suppose that $A \subsetneq H^o \subseteq F^o$ satisfies $t^*_A(H^o) = t_A(F^o) < t^*$. Then either $H^o \subseteq F$, in which case $t_A(F) \leqslant t^*_A(H^o) = t_A(F^o)$, as required, or $H^o \in \mathcal{F}^o_H$ for some $A \subsetneq H \subseteq F$.

We claim that, in the latter case, we have $t_A(F) \leqslant t^*_A(H) \leqslant t^*_A(H^o) = t_A(F^o)$. Note that $e(H) = e(H^o) - 1$, and therefore, by Observation 4.20,

$$\tilde{N}_A(H^o)(m) \,=\, \frac{\sqrt{n}}{2te^{4t^2}} \cdot \tilde{N}_A(H)(m) \,\geqslant\, \frac{n^\varepsilon}{2t} \cdot \tilde{N}_A(H)(m) \,>\, (2t)^{e(H)-1} \,=\, (2t)^{e(H^o)}$$

for every $t \leqslant t^*_A(H) < t^*$. By (10), it follows that $t^*_A(H) \leqslant t^*_A(H^o)$, as required. □

Observation 4.25. *Let (F, A) be a graph structure pair with $v_A(F) \geqslant 1$, and let $F^o \in \mathcal{F}^o_F$. Then*

$$o(F) g_y(t) \ll g_{F,A}(t) \qquad \text{and} \qquad g_{F^o,A}(t) \ll g_{F,A}(t)$$

as $n \to \infty$.

Proof. Observe first that $c(F, A) \geqslant c(F^o, A) \geqslant 2$ for every $F^o \in \mathcal{F}^o_F$, by Observation 4.24. Indeed, $c = c(F, A)$ was chosen (see (11)) so that $e^{ct^2} = n^{1/4}$ at $t = t_A(F)$, and $t_A(F) \leqslant t_A(F^o)$. Now, noting that $v_A(F) = v_A(F^o)$, $e(F) = e(F^o) + 1$ and $o(F) = o(F^o) - 1$, we have

$$\gamma(F, A) \,\geqslant\, \gamma(F^o, A) + \sqrt{\gamma(F, A)},$$

by the definition of $\gamma(F, A)$. □

Recall next that

$$f_{F,A}(t) \,=\, e^{C(o(F)+1)(t^2+1)} n^{-1/4} (\log n)^{\Delta(F,A) - \sqrt{\Delta(F,A)}} \tag{12}$$

for each graph structure pair (F, A), and that

$$f_y(t) \,=\, e^{Ct^2} n^{-1/4} (\log n)^{5/2} \qquad \text{and} \qquad f_x(t) \,=\, e^{-4t^2} f_y(t).$$

The following variant of [2, Observation 4.25] follows easily from the definitions.

Observation A.3.1. *Let (F, A) be a graph structure pair with $v_A(F) \geqslant 1$, and let $F^o \in \mathcal{F}^o_F$. Then*

$$(\log n)^{e(F)+o(F)} \big(f_y(t) + f_{F^o,A}(t) \big) \ll f_{F,A}(t)$$

as $n \to \infty$.

Proof. Since $e(F^o) = e(F) - 1$ and $o(F^o) = o(F) + 1$, we have

$$\Delta(F, A) \,=\, \big(\delta(F^o, A) + 1 \big)^C \,\geqslant\, \Delta(F^o, A) + \sqrt{\Delta(F, A)}.$$

The claimed bound now follows immediately. □

Our next few results give useful properties of the collection of graph structures $\mathcal{F}^-_{F,A}$ which was defined in [2, Section 4.3]. The first four were also proved there.

Observation 4.32. *$\mathcal{F}^+_{F,A} \subseteq \mathcal{F}^-_{F,A}$ for every graph structure pair (F, A).*

Note that, by this observation, the following results all also hold for each $(F', A') \in \mathcal{F}^+_{F,A}$.

Observation 4.34. *Let (F, A) be a graph structure pair, and let $(F', A') \in \mathcal{F}^-_{F,A}$. For every $A' \subseteq H' \subseteq F'$, we have $\tilde{N}_{H'}(F') \leqslant \tilde{N}_{H' \cap F}(F)$.*

Observation 4.37. *Let (F, A) be a graph structure pair, and let $(F', A') \in \mathcal{F}^-_{F,A}$. Then*

$$\Delta(F', H', A') \,\leqslant\, \Delta(F' - v, A) \,\leqslant\, \Delta(F, A) - 3\sqrt{\Delta(F, A)}$$

for every $A' \subsetneq H' \subsetneq F'$. Moreover, the same bounds holds if $H' = F'$ and $A' \cap F \neq A$.

Observation 4.39. *Let (F,A) be a graph structure pair, and let $(F',A') \in \mathcal{F}^-_{F,A}$. Then*

$$\Delta(F',A') \leqslant (1+\varepsilon)\Delta(F,A).$$

The following lemma was stated but not proved in [2].

Lemma 4.41. *Let (F,A) be a graph structure pair with $t_A(F) > 0$, and let $(F',A') \in \mathcal{F}^-_{F,A}$. If $(\log n)^{\gamma(F,A)} \leqslant n^{v_A(F)+e(F)+1}$, then*

$$\max\left\{\Delta(F',A') - \Delta(F,A), \sqrt{\Delta(F,A)}\right\} \leqslant \frac{\varepsilon^2}{v_A(F)} \cdot \frac{\log n}{\log\log n}.$$

Proof. We have, as in the proof above,

$$\begin{aligned}\big|\Delta(F',A') - \Delta(F,A)\big| + \sqrt{\Delta(F,A)} &\leqslant \big(\delta(F,A)+2\big)^C - \delta(F,A)^C + \delta(F,A)^{C/2} \\ &\leqslant 4C \cdot \delta(F,A)^{C-1} \leqslant \frac{4C \cdot \Delta(F,A)}{\delta(F,A)} \leqslant \frac{4}{C} \cdot \frac{\Delta(F,A)}{v_A(F)^2}.\end{aligned}$$

Note that $e(F) \leqslant 2v_A(F) - 1$, since $t_A(F) > 0$. Since $(\log n)^{\gamma(F,A)} \leqslant n^{v_A(F)+e(F)+1}$, it follows that

$$v_A(F) \geqslant \frac{v_A(F)+e(F)+1}{3} \geqslant \frac{\gamma(F,A)}{3} \cdot \frac{\log\log n}{\log n} \geqslant \frac{\Delta(F,A)}{4} \cdot \frac{\log\log n}{\log n}.$$

It follows that

$$\big|\Delta(F',A') - \Delta(F,A)\big| + \sqrt{\Delta(F,A)} \leqslant \frac{16}{C \cdot v_A(F)} \cdot \frac{\log n}{\log\log n} \leqslant \frac{\varepsilon}{v_A(F)} \cdot \frac{\log n}{\log\log n},$$

as claimed. ∎

We next prove three new lemmas about the collection $\mathcal{F}^-_{F,A}$. Recall Table 4.1.

	(a)	(b)	(c)	(d)	(e)	(f)
$v_{A^-}(F^-) - v_A(F)$	0	-1	0	-2	-1	-1
$e(F^-) - e(F)$	1	$\leqslant 0$	1	$\leqslant 0$	$\leqslant 0$	$\leqslant 1$
$o(F^-) - o(F)$	0	$\leqslant 0$	0	$\leqslant 0$	$\leqslant 0$	$\leqslant 0$

Table 4.1

Lemma A.3.2. *Let (F,A) be a graph structure pair with $t_A(F) > 0$, and let $(F',A') \in \mathcal{F}^-_{F,A}$. If $(\log n)^{\gamma(F,A)} \leqslant n^{v_A(F)+e(F)+1}$ and $t \leqslant \omega$, then*

$$f_{F',A'}(t) \leqslant \min\left\{n^{\varepsilon}, (\log n)^{\varepsilon\Delta(F,A)}\right\} \cdot (\log n)^{\Delta(F,A) - 3\sqrt{\Delta(F,A)}}.$$

Proof. This follows from the definition (12), using Observation 4.39 and Lemma 4.41. ∎

Lemma A.3.3. *Let (F, A) be a graph structure pair with $t_A(F) > 0$. If $(F', A') \in \mathcal{F}_{F,A}^-$, then*

$$\tilde{N}_{A'}(F')(n^{3/2}) \;\leqslant\; \frac{2 \cdot e^{4o(F)}}{\sqrt{n}} \cdot \tilde{N}_A(F)(n^{3/2}). \tag{13}$$

Proof. Note that $\tilde{N}_A(F)(n^{3/2}) = 2^{e(F)} e^{-4o(F)} n^{v_A(F) - e(F)/2}$ for every pair (F, A), since $t = 1$ when $m = n^{3/2}$, and that $e(F') \leqslant e(F) + 1$. We are thus required to prove that

$$v_{A'}(F') - v_A(F) \;\leqslant\; \frac{e(F') - e(F) - 1}{2}. \tag{14}$$

In cases (a) and (c), this follows immediately from Table 4.1. In cases (b), (d), (e) and (f) we need the following extra observation: $t_A(F) > 0$ implies that $t_A^*(A' \cap F) > 0$. This gives

$$e(F) - e(F') \;\leqslant\; e(A' \cap F) \;\leqslant\; 2v_A(A' \cap F) - 1 \;=\; 2\big(v_A(F) - v_{A'}(F')\big) - 1,$$

as required. □

Lemma A.3.4. *Let (F, A, ϕ) be a graph structure triple, and let $(F', A') \in \mathcal{F}_{F,A}^-$. Suppose that $t_{A'}(F') < t \leqslant \omega < t_A(F)$, and that $\phi' \colon A' \to V(G_m)$ is faithful at time t. If $\mathcal{E}(m)$ holds and $(\log n)^{\Delta(F,A)} \leqslant n^{2e(F)+2}$, then*

$$N_{\phi'}(F')(m) \;\leqslant\; \frac{(\log n)^{\Delta(F,A) - 2\sqrt{\Delta(F,A)}}}{\sqrt{n}} \cdot \tilde{N}_A(F)(n^{3/2}). \tag{15}$$

Proof. Note that since $t_{A'}(F') < t \leqslant \omega$, it follows that $t_{A'}(F') = 0$. There are two cases: either (F', A') is balanced, or it is not. In the former case, since $\mathcal{M}(m)$ holds we have

$$N_{\phi'}(F')(m) \;\leqslant\; (\log n)^{\Delta(F' - v, A')} \;\leqslant\; (\log n)^{\Delta(F,A) - 3\sqrt{\Delta(F,A)}},$$

and so in this case we are done. In the latter case, since $\mathcal{E}(m)$ holds we have

$$N_{\phi'}(F')(m) \;\leqslant\; (\log n)^{\Delta(F', H', A')} \tilde{N}_{H'}(F')(m^+), \tag{16}$$

where $A' \subsetneq H' \subsetneq F'$ is minimal such that $t < t_{H'}(F')$, and $m^+ = \max\{m, n^{3/2}\}$. Now

$$(\log n)^{\Delta(F', H', A')} \tilde{N}_{H'}(F')(m^+) \;\leqslant\; (\log n)^{\Delta(F,A) - 3\sqrt{\Delta(F,A)}} \tilde{N}_H(F)(m^+),$$

where $H = H' \cap F$, by Observations 4.34 and 4.37. Next, note that, since $t_A(F) > 0$, we have $t_A^*(H) > 0$, and thus $e(H) \leqslant 2v_A(H) - 1$. Hence

$$\tilde{N}_A(H)(n^{3/2}) \;\geqslant\; 2^{e(F)} e^{-4o(H)} \sqrt{n}.$$

Since $0 < t \leqslant \omega$, it follows that

$$\tilde{N}_H(F)(m^+) \;\leqslant\; \omega^{e(F)} e^{4o(F)} \cdot \tilde{N}_H(F)(n^{3/2}) \;\leqslant\; \frac{\omega^{e(F)} e^{8o(F)}}{\sqrt{n}} \cdot \tilde{N}_A(F)(n^{3/2}),$$

and thus

$$N_{\phi'}(F')(m) \;\leqslant\; \frac{(\log n)^{\Delta(F,A) - 2\sqrt{\Delta(F,A)}}}{\sqrt{n}} \cdot \tilde{N}_A(F)(n^{3/2}),$$

as claimed. □

4. Section 4.6: the land before time $t = \omega$

In this section we shall use the method of Bohman [1] to control the variables $N_\phi(F)$ up to time $t = \omega$, assuming that $t_A(F) > 0$. Recall once again that

$$f_{F,A}(t) \,=\, e^{C(o(F)+1)(t^2+1)} n^{-1/4} (\log n)^{\Delta(F,A) - \sqrt{\Delta(F,A)}}$$

for each graph structure pair (F, A), that

$$f_y(t) \,=\, e^{Ct^2} n^{-1/4} (\log n)^{5/2} \qquad \text{and} \qquad f_x(t) \,=\, e^{-4t^2} f_y(t),$$

and that $\mathcal{K}^{\mathcal{E}}(m) = \mathcal{Y}(m) \cap \mathcal{Z}(m) \cap \mathcal{Q}(m)$. We shall prove the following proposition.

Proposition 4.55. *Let (F, A) be a graph structure pair, and let $0 < t \leqslant \omega < t_A(F)$. Then, with probability at least $1 - n^{-3 \log n}$, either $\big(\mathcal{E}(m-1) \cap \mathcal{M}(m-1) \cap \mathcal{K}^{\mathcal{E}}(m-1)\big)^c$ holds, or*

$$N_\phi(F)(m) \,\in\, \tilde{N}_A(F)(m) \,\pm\, f_{F,A}(t) \cdot \tilde{N}_A(F)(n^{3/2}) \tag{17}$$

for every $\phi\colon A \to V(G_m)$ which is faithful at time t.

We remark, for future reference, that (17) holds trivially if $(\log n)^{\Delta(F,A)} > n^{e(F)+2}$, since then $f_{F,A}(t) \cdot \tilde{N}_A(F)(n^{3/2}) > n^{v_A(F)}$, and that we have $e(F) < 2v_A(F)$, since $t_A(F) > 0$.

The proof of Proposition 4.55 relies heavily on the fact that the event $\mathcal{Y}(m)$ gives us stronger bounds (in the range $t \leqslant \omega$) on the variables Y_e than those given by the event $\mathcal{E}(m)$. The bounds we need are essentially due to Bohman [1], although he stated a slightly weaker version of the following proposition. For completeness, we shall sketch the proof.

Recall from [2, Section 5] the definition of the variables $Y_e^L(m)$. We shall need the following slight strengthening of [2, Proposition 4.56] in [2, Section 5] in order to show that the event $\mathcal{U}(a)$ holds, where $a = \omega \cdot n^{3/2}$.

Proposition 4.56 (Bohman [1])**.** *Let $m \leqslant \omega \cdot n^{3/2}$. With probability at least $1 - n^{-C \log n}$, either $\big(\mathcal{Z}(m-1) \cap \mathcal{Q}(m-1)\big)^c$ holds, or*

$$X_e(m) \in \tilde{X}(m) \pm f_x(t) \tilde{X}(n^{3/2}) \qquad \text{and} \qquad 2 \cdot Y_e^L(m) \in \tilde{Y}(m) \pm f_y(t) \tilde{Y}(n^{3/2}) \tag{18}$$

for every $e \in O(G_m)$.

Recall the following martingale lemma from [2, Section 3].

Lemma 3.2. *Let M be a super-martingale, defined on $[0, s]$, such that*

$$-\beta \leqslant \Delta M(m) \leqslant \alpha$$

for every $m \in [0, s-1]$. Then, for every $0 \leqslant x \leqslant \min\{\alpha, \beta\} \cdot s$,

$$\mathbb{P}\big(M(s) > M(0) + x\big) \,\leqslant\, \exp\left(-\frac{x^2}{4\alpha\beta s}\right).$$

Proof of Proposition 4.56. Consider the event that $m_0 \leqslant \omega \cdot n^{3/2}$ is the minimal value of m such that the event $\mathcal{Z}(m) \cap \mathcal{Q}(m)$ holds, and moreover either

$$\mathcal{X}(m)^c \text{ holds}, \qquad \text{or} \qquad 2 \cdot Y_e^L(m) \notin \tilde{Y}(m) \pm f_y(t) \tilde{Y}(n^{3/2}).$$

We shall consider the two subcases: $\mathcal{X}(m_0)$ holds, or it does not, separately.

Suppose first that $\mathcal{X}(m_0)^c$ holds, and note that $\tilde{X}(m) = \Theta\big(e^{-8t^2} \cdot \tilde{X}(n^{3/2})\big)$ and $\tilde{Y}(m) = \Theta\big(t \cdot e^{-4t^2} \cdot \tilde{Y}(n^{3/2})\big)$. By (2), and using the event $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$, which holds for all $m < m_0$, we have

$$\begin{aligned}\mathbb{E}\big[\Delta X_e(m)\big] \; &= \; -\frac{2}{Q(m)} \sum_{f \in X_e(m)} Y_f(m) \\ &\in \; -\frac{2}{\tilde{Q}(m)} \Big(1 \pm \varepsilon \cdot f_y(t) e^{4t^2}\Big)\Big(\tilde{X}(m) \pm f_x(t)\tilde{X}(n^{3/2})\Big)\Big(\tilde{Y}(m) \pm f_y(t)\tilde{Y}(n^{3/2})\Big).\end{aligned}$$

Multiplying out the brackets, we obtain

$$\tilde{X}(m)\tilde{Y}(m) \; \pm \; O\Big(\varepsilon t e^{-8t^2} f_y(t) + t e^{-4t^2} f_x(t) + e^{-8t^2} f_y(t) + f_x(t) f_y(t)\Big)\tilde{X}(n^{3/2})\tilde{Y}(n^{3/2}),$$

and hence, recalling that $f_y(t) = e^{4t^2} f_x(t) \ll e^{-8\omega^2}$, we obtain

$$\mathbb{E}\big[\Delta X_e(m)\big] \; \in \; -\frac{16t}{n^{3/2}} \cdot \tilde{X}(m) \; \pm \; \sqrt{C} \cdot \left(\frac{t+1}{n^{3/2}}\right) \cdot f_x(t)\tilde{X}(n^{3/2}).$$

It follows that

$$M^{\pm}_{X_e}(m') \; = \; \sum_{m=0}^{m'-1} \left[\Delta X_e(m) + \frac{16t}{n^{3/2}} \cdot \tilde{X}(m) \; \pm \; \sqrt{C} \cdot \left(\frac{t+1}{n^{3/2}}\right) \cdot f_x(t)\tilde{X}(n^{3/2})\right].$$

is a super-/sub-martingale pair on $0 \leqslant m' < m_0$. Moreover, the number of open triangles which are destroyed in step $m+1$ of the triangle-free process is at most

$$\max_{e \in O(G_m)} Y_e(m) \; \leqslant \; \sqrt{n}$$

since $\mathcal{Y}(m)$ holds for every $m < m_0$. It follows that $-\sqrt{n} \leqslant \Delta X_e(m) \leqslant 0$, and hence

$$-\,2\sqrt{n} \; \leqslant \; \Delta M^{\pm}_{X_e}(m) \; \leqslant \; \frac{C}{\sqrt{n}}, \tag{19}$$

for every $e \in O(G_m)$, while $\mathcal{Y}(m)$ holds. Now, set

$$\alpha \; = \; \frac{C}{\sqrt{n}} + \frac{f_x(\omega)\tilde{X}(n^{3/2})}{m_0} \qquad \text{and} \qquad \beta \; = \; 2\sqrt{n} + \frac{f_x(\omega)\tilde{X}(n^{3/2})}{m_0},$$

and observe that, since $f_x(t)\tilde{X}(n^{3/2}) \geqslant n^{3/4}(\log n)^{5/2}$ and we may assume[3] that $m_0 \geqslant n^{1/4}$,

$$\alpha \cdot \beta \cdot m_0 \; \leqslant \; \frac{\big(f_x(t)\tilde{X}(n^{3/2})\big)^2}{(\log n)^4}.$$

Hence, applying Lemma 3.2, we obtain

$$\mathbb{P}\bigg(\Big(M^{-}_{X_e}(m_0) > \frac{1}{2} f_x(t)\tilde{X}(n^{3/2})\Big) \cap \mathcal{K}(m-1)\bigg) \; \leqslant \; e^{-(\log n)^3},$$

[3]If $m_0 \leqslant n^{1/4}$, then (19) implies that $X_e(m_0) > n - O(n^{3/4})$, and hence $\mathcal{X}(m_0) \cup \mathcal{Y}(m_0 - 1)^c$ holds.

and similarly for $M^+_{X_e}$. Finally, noting that

$$\frac{1}{n^{3/2}} \sum_{m=0}^{m'-1} 16t \cdot \tilde{X}(m) \in \big(1 - e^{-8t'^2}\big) n \pm 1 \quad \text{and} \quad \frac{1}{n^{3/2}} \sum_{m=0}^{m'-1} (t+1) f_x(t) \leqslant \frac{1}{C} \cdot f_x(t'),$$

it follows that, with probability at least $1 - n^{-\log n}$, we have

$$\begin{aligned} X_e(m') \,&\in\, n \,-\, \sum_{m=0}^{m'-1} \left[\frac{16t}{n^{3/2}} \cdot \tilde{X}(m) \pm \sqrt{C} \cdot \left(\frac{t+1}{n^{3/2}} \right) \cdot f_x(t) \tilde{X}(n^{3/2}) \right] \,\pm\, \frac{1}{2} f_x(t') \tilde{X}(n^{3/2}) \\ &\in\, \tilde{X}(m') \,\pm\, f_x(t') \tilde{X}(n^{3/2}) \end{aligned}$$

for every $e \in O(G_m)$, as required.

The proof in the case that $\mathcal{X}(m_0)$ holds is similar. The first step is to break $\Delta Y^L_e(m)$ into two pieces, $C^Y_e(m)$ and $D^Y_e(m)$, where $C^Y_e(m)$ denotes the number of edges of $Y^L_e(m+1)$ which were created in step $m+1$ of the triangle-free process, and $D^Y_e(m)$ denotes the number of edges of $Y^L_e(m)$ which were closed in that step. It follows immediately from the definition that

$$Y^L_e(m') \,=\, \sum_{m=0}^{m'-1} \Big(C^Y_e(m) - D^Y_e(m) \Big). \tag{20}$$

We claim first that

$$\begin{aligned} \mathbb{E}\big[C^Y_e(m) \,|\, G_m \big] \,&=\, \frac{1}{2} \cdot \frac{X_e(m)}{Q(m)} \,\in\, \frac{\big(1 \pm O(\varepsilon) e^{4t^2} f_y(t)\big)\big(\tilde{X}(m) \pm f_x(t) \tilde{X}(n^{3/2})\big)}{2 \cdot \tilde{Q}(m)} \\ &\subseteq\, \frac{2e^{-4t^2} n \pm \big(e^{4t^2} f_x(t) + O(\varepsilon) f_y(t)\big) \tilde{X}(n^{3/2})}{n^2} \,\subseteq\, \frac{2e^{-4t^2} \pm f_y(t)}{n}, \end{aligned}$$

where we used the event $\mathcal{X}(m) \cap \mathcal{Q}(m)$, and the facts that $f_y(t) = e^{4t^2} f_x(t)$, and that $\tilde{X}(m) = \Theta\big(e^{-8t^2} \cdot \tilde{X}(n^{3/2})\big) \gg f_x(t) \tilde{X}(n^{3/2})$ for $t \leqslant \omega$. Similarly, we have

$$\begin{aligned} \mathbb{E}\big[D^Y_e(m) \,|\, G_m \big] \,&=\, \frac{1}{Q(m)} \sum_{f \in Y^L_e(m)} Y_f(m) \,\in\, \frac{\big(1 \pm O(\varepsilon) e^{4t^2} f_y(t)\big)\big(\tilde{Y}(m) \pm f_y(t) \tilde{Y}(n^{3/2})\big)^2}{2 \cdot \tilde{Q}(m)} \\ &\subseteq\, \frac{\tilde{Y}(m)^2}{2 \cdot \tilde{Q}(m)} \,\pm\, O\Big(\varepsilon \cdot t^2 e^{-4t^2} f_y(t) + t e^{-4t^2} f_y(t) + f_y(t)^2 \Big) \frac{\tilde{Y}(n^{3/2})^2}{\tilde{Q}(m)} \\ &\subseteq\, \frac{16t^2 e^{-4t^2} \pm \big(t^2 + O(t)\big) f_y(t)}{n}, \end{aligned}$$

using the event $\mathcal{Y}(m) \cap \mathcal{Q}(m)$, and the fact that $\tilde{Y}(m) = \Theta\big(t \cdot e^{-4t^2} \cdot \tilde{Y}(n^{3/2})\big)$. Hence

$$M^{\pm}_{C^Y_e}(m') \,=\, \sum_{m=0}^{m'-1} \left[C^Y_e(m) - \frac{2e^{-4t^2} \pm f_y(t)}{n} \right].$$

and

$$M^{\pm}_{D^Y_e}(m') \;=\; \sum_{m=0}^{m'-1} \left[D_e^Y(m) - \frac{16t^2 e^{-4t^2} \pm \big(t^2 + O(t)\big) f_y(t)}{n} \right].$$

are both super-/sub-martingale pairs while the event $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds.

Next, observe that

$$0 \leqslant C_e^Y(m) \leqslant 1 \qquad \text{and} \qquad 0 \leqslant D_e^Y(m) \leqslant (\log n)^2,$$

while $\mathcal{Z}(m)$ holds, and hence

$$-\frac{C}{n} \;\leqslant\; \Delta M^{\pm}_{C^Y_e}(m) \;\leqslant\; 2 \qquad \text{and} \qquad -\frac{C}{n} \;\leqslant\; \Delta M^{\pm}_{D^Y_e}(m) \;\leqslant\; 2(\log n)^2.$$

Now, set

$$\alpha \;=\; 2(\log n)^2 + \frac{f_y(\omega)\tilde{Y}(n^{3/2})}{m_0} \qquad \text{and} \qquad \beta \;=\; \frac{C}{n} + \frac{f_y(\omega)\tilde{Y}(n^{3/2})}{m_0},$$

and observe that, since $f_y(t)\tilde{Y}(n^{3/2}) \geqslant n^{1/4}(\log n)^{5/2}$ and we may assume that $m_0 \geqslant n^{1/4}$,

$$\alpha \cdot \beta \cdot m_0 \;\leqslant\; \frac{\big(f_y(t)\tilde{Y}(n^{3/2})\big)^2}{\omega \cdot (\log n)^2}.$$

Hence, applying Lemma 3.2, we obtain

$$\mathbb{P}\bigg(\Big(M^{-}_{D^Y_e}(m) > \frac{1}{4} f_y(t)\tilde{Y}(n^{3/2}) \Big) \cap \mathcal{K}(m-1) \bigg) \;\leqslant\; n^{-2C\log n},$$

and similarly for $M^{+}_{C^Y_e}$, $M^{-}_{C^Y_e}$ and $M^{+}_{D^Y_e}$.

Finally, note that $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}(m) = \big(4 - 32t^2\big)e^{-4t^2}\sqrt{n}$ and $\frac{1}{n^{3/2}}\sum_{m=0}^{m'-1}(t^2+1)f_y(t) \leqslant \frac{1}{C} \cdot f_y(t')$. Hence, with probability at least $1 - n^{-2C\log n}$, we have

$$\begin{aligned}
2 \cdot Y_e(m') \;&=\; 2 \cdot \sum_{m=0}^{m'-1} \Big(C_e^Y(m) - D_e^Y(m) \Big) \\
&\in\; \sum_{m=0}^{m'-1} \left[\frac{(4 - 32t^2)e^{-4t^2}}{n} \pm \frac{O(t^2+1)f_y(t)}{n} \right] \;\pm\; \frac{1}{2} f_y(t')\tilde{Y}(n^{3/2}) \\
&\in\; \tilde{Y}(m') \;\pm\; f_y(t')\tilde{Y}(n^{3/2}),
\end{aligned}$$

as required. Putting the pieces together, it follows that

$$\mathbb{P}\bigg(\bigcup_{t \leqslant \omega} \big(\mathcal{X}(m) \cap \mathcal{Y}(m)\big)^c \cap \mathcal{Z}(m) \cap \mathcal{Q}(m) \bigg) \;\leqslant\; n^{-C\log n},$$

and moreover the same holds if we replace $\mathcal{Y}(m)$ by the event in (18). □

4.1. **Proof of Proposition 4.55.** Returning to the proof of Proposition 4.55, let us fix a graph structure triple (F, A, ϕ) with $t_A(F) > 0$. The first step in the proof is to break up $N_\phi(F)$ as follows:

$$N_\phi(F)(m') = \sum_{m=0}^{m'-1} \Big(C_\phi(F)(m) - D_\phi(F)(m) \Big), \tag{21}$$

where $C_\phi(F)(m)$ denotes the number of copies of F rooted at $\phi(A)$ which are created at step $m+1$ of the triangle-free process, and $D_\phi(F)(m)$ denotes the number of such copies which are destroyed in that step. Let's deal first with $C_\phi(F)$. It was proved in [2, Lemma 4.21] that

$$\mathbb{E}\big[C_\phi(F)(m) \,|\, G_m\big] \in \frac{1}{Q(m)} \sum_{F^o \in \mathcal{F}_F^o} N_\phi(F^o)(m) \tag{22}$$

for every ϕ which is faithful at time t. Using the events $\mathcal{E}(m)$ and $\mathcal{Q}(m)$, we easily obtain a super-/sub-martingale pair.

Lemma 4.58. *Let (F, A, ϕ) be a graph structure triple, and suppose that ϕ is faithful at time t, where $0 < t \leqslant \omega < t_A(F)$. If $\mathcal{E}(m) \cap \mathcal{Z}(m) \cap \mathcal{Q}(m)$ holds, then*

$$\mathbb{E}\big[C_\phi(F)(m) \,|\, G_m\big] - \frac{e(F) \cdot \tilde{N}_A(F)(m)}{t \cdot n^{3/2}} \in \pm \frac{f_{F,A}(t) \cdot \tilde{N}_A(F)(n^{3/2})}{n^{3/2}}. \tag{23}$$

Proof. This follows from (22) via a straightforward calculation. Note first that the lemma holds trivially if $e(F) = 0$, since then $C_\phi(F)(m) = 0$ for every $m \in \mathbb{N}$. So assume that $e(F) > 0$, and recall that the event $\mathcal{Q}(m)$ implies that $Q(m) \in \big(1 \pm \varepsilon \cdot e^{4t^2} f_y(t)\big)\tilde{Q}(m)$, and that the event $\mathcal{E}(m)$ (and the fact that $t \leqslant \omega < t_A(F) \leqslant t_A(F^o)$, by Observation 4.24) implies that

$$N_\phi(F^o)(m) \in \tilde{N}_A(F^o)(m) \pm f_{F^o,A}(t) \cdot \tilde{N}_A(F^o)(n^{3/2}).$$

Moreover $2te^{4t^2} \cdot \tilde{N}_A(F^o)(m) = \sqrt{n} \cdot \tilde{N}_A(F)(m)$ and $|\mathcal{F}_F^o| = e(F)$, by Observation 4.20, and hence, by (22),

$$\mathbb{E}\big[C_\phi(F)(m) \,|\, G_m\big] \in \frac{1 \pm e^{4t^2} f_y(t)}{\tilde{Q}(m)} \left(\tilde{N}_A(F)(m) \frac{e(F)\sqrt{n}}{2te^{4t^2}} \pm \sum_{F^o \in \mathcal{F}_F^o} f_{F^o,A}(t) \tilde{N}_A(F)(n^{3/2}) \cdot \frac{\sqrt{n}}{2e^4} \right).$$

Thus the left-hand side of (23) lies in the interval

$$\pm \frac{1}{\tilde{Q}(m)} \left(\frac{e(F)\sqrt{n}}{2t} \cdot f_y(t) \tilde{N}_A(F)(m) + \sum_{F^o \in \mathcal{F}_F^o} f_{F^o,A}(t) \tilde{N}_A(F)(n^{3/2}) \cdot \frac{\sqrt{n}}{e^4} \right).$$

Since $\tilde{N}_A(F)(m) = t^{e(F)} e^{-4o(F)(t^2-1)} \tilde{N}_A(F)(n^{3/2})$, this is in turn contained in

$$\pm \frac{\tilde{N}_A(F)(n^{3/2})}{n^{3/2}} \Big(e(F) \cdot t^{e(F)-1} e^{4o(F)+4t^2} f_y(t) + e^{4t^2} f_{F^o,A}(t) \Big) \subseteq \pm \frac{f_{F,A}(t) \cdot \tilde{N}_A(F)(n^{3/2})}{n^{3/2}},$$

as required, where the final inequality follows since $e(F) > 0$ and

$$(\log n)^{e(F)+o(F)} \big(f_y(t) + f_{F^o,A}(t) \big) \ll f_{F,A}(t)$$

by Observation A.3.1. □

We shall also need a corresponding lemma for the variables $D_\phi(F)$. It was proved in [2, Lemma 4.21] that, if $\mathcal{Z}(m)$ holds and ϕ is faithful at time t, then

$$\mathbb{E}\big[D_\phi(F)(m)\,|\,G_m\big] \in \frac{1}{Q(m)}\Bigg(\sum_{F^*\in N_\phi(F)} \sum_{f\in O(F^*)} Y_f(m) \pm o(F)^2(\log n)^2 N_\phi(F)\Bigg). \tag{24}$$

We shall use the events $\mathcal{E}(m)$, $\mathcal{Y}(m)$ and $\mathcal{Q}(m)$ to obtain a super-/sub-martingale pair.

Lemma 4.59. *Let (F, A, ϕ) be a graph structure triple, and suppose that $0 < t \leqslant \omega < t_A(F)$, and that ϕ is faithful at time t. If $\mathcal{E}(m)\cap\mathcal{Y}(m)\cap\mathcal{Z}(m)\cap\mathcal{Q}(m)$ holds, then*

$$\mathbb{E}\big[D_\phi(F)(m)\,|\,G_m\big] - \frac{8t\cdot o(F)\cdot \tilde{N}_A(F)(m)}{n^{3/2}} \in \pm\frac{C\cdot o(F)\cdot(t+1)}{n^{3/2}}\cdot f_{F,A}(t)\tilde{N}_A(F)(n^{3/2}). \tag{25}$$

Proof. Note first that if $o(F) = 0$ then the result holds trivially, since in that case copies of F cannot be destroyed, and so $D_\phi(F)(m) = 0$ for every $m \in \mathbb{N}$. So assume that $o(F) > 0$ and recall that, since $\mathcal{E}(m)\cap\mathcal{Y}(m)\cap\mathcal{Q}(m)$ holds and $t \leqslant \omega < t_A(F)$, we have

$$Q(m) \in \big(1 \pm \varepsilon\cdot e^{4t^2} f_y(t)\big)\tilde{Q}(m), \qquad Y_f(m) \in \tilde{Y}(m) \pm f_y(t)\tilde{Y}(n^{3/2})$$

and

$$N_\phi(F)(m) \in \tilde{N}_A(F)(m) \pm f_{F,A}(t)\cdot\tilde{N}_A(F)(n^{3/2}).$$

By (24), it follows that $\mathbb{E}\big[D_\phi(F)(m)\,|\,G_m\big]$ is contained in the interval

$$\frac{1\pm\varepsilon\cdot e^{4t^2}f_y(t)}{\tilde{Q}(m)}\cdot o(F)\cdot\Big(\tilde{N}_A(F)(m) \pm f_{F,A}(t)\tilde{N}_A(F)(n^{3/2})\Big)\Big(\tilde{Y}(m) \pm 2f_y(t)\tilde{Y}(n^{3/2})\Big),$$

where we used our assumption[4] that $o(F) \leqslant n^{o(1)} \leqslant n^{1/4} \leqslant f_y(t)\tilde{Y}(n^{3/2})$ to absorb the final error term in (24). It follows that the left-hand side of (25) is contained in

$$\pm\frac{o(F)\cdot\tilde{Y}(n^{3/2})}{\tilde{Q}(m)}\bigg(\Big(t\cdot f_y(t) + 2f_y(t)\Big)\tilde{N}_A(F)(m) + \Big(O(t)\cdot e^{-4t^2} + 2f_y(t)\Big) f_{F,A}(t)\tilde{N}_A(F)(n^{3/2})\bigg),$$

since $\tilde{Y}(m) = O(t)\cdot e^{-4t^2}\cdot\tilde{Y}(n^{3/2})$. This in turn is a subset of

$$\pm\frac{C\cdot o(F)}{n^{3/2}}\cdot\Big(t^{e(F)}(t+1)e^{4o(F)}f_y(t) + \big(t + e^{4t^2}f_y(t)\big)f_{F,A}(t)\Big)\cdot\tilde{N}_A(F)(n^{3/2}).$$

since $\tilde{N}_A(F)(m) = \tilde{N}_A(F)(n^{3/2})\cdot t^{e(F)}e^{-4o(F)(t^2-1)}$ and $o(F) > 0$.

Finally, note that $e^{4t^2}f_y(t) \ll 1$ for every $t \leqslant \omega$, and recall that $(\log n)^{e(F)+o(F)}f_y(t) \ll f_{F,A}(t)$, by Observation A.3.1. It follows that

$$\mathbb{E}\big[D_\phi(F)(m)\,|\,G_m\big] - \frac{8t\cdot o(F)\cdot\tilde{N}_A(F)(m)}{n^{3/2}} \in \pm\frac{C\cdot o(F)\cdot(t+1)}{n^{3/2}}\cdot f_{F,A}(t)\tilde{N}_A(F)(n^{3/2}),$$

as required. ∎

In order to use Lemma 3.2, we shall need bounds on $C_\phi(F)(m)$ and $D_\phi(F)(m)$ which hold deterministically for all $0 < t \leqslant \omega$. We shall prove the following bounds.

---

[4]Recall that Proposition 4.55 is trivial if $o(F) \geqslant (\log n)^{1/5}$.

Lemma 4.60. *Let (F, A, ϕ) be a graph structure triple, and suppose that $0 < t \leqslant \omega < t_A(F)$, and that ϕ is faithful at time t. If $\mathcal{E}(m) \cap \mathcal{M}(m)$ holds, then*

$$0 \leqslant C_\phi(F)(m) \leqslant \min\Big\{ n^\varepsilon, (\log n)^{\varepsilon \Delta(F,A)} \Big\} \cdot \frac{(\log n)^{\Delta(F,A) - 2\sqrt{\Delta(F,A)}}}{\sqrt{n}} \cdot \tilde{N}_A(F)(n^{3/2}). \qquad (26)$$

Moreover, the same bounds also hold for $D_\phi(F)(m)$.

The required bounds on $C_\phi(F)(m)$ follow easily from Lemmas A.3.2, A.3.3 and A.3.4.

Proof of Lemma 4.60. Recall first that, by [2, Lemma 4.28], we have

$$C_\phi(F)(m) \leqslant \sum_{(F',A') \in \mathcal{F}^+_{F,A}} \max_{\phi' \colon A' \to V(G_m)} N_{\phi'}(F')(m).$$

There are two cases to consider: $t \leqslant \omega < t_{A'}(F')$ and $t > t_{A'}(F') = 0$. Set

$$\Upsilon(F, A) := \min\Big\{ n, (\log n)^{\Delta(F,A)} \Big\},$$

and recall that, by Lemmas A.3.2 and A.3.3, we have

$$f_{F',A'}(t) \leqslant \Upsilon(F,A)^\varepsilon \cdot (\log n)^{\Delta(F,A) - 3\sqrt{\Delta(F,A)}} \quad \text{and} \quad \tilde{N}_{A'}(F')(n^{3/2}) \leqslant \frac{e^{4o(F)+1}}{\sqrt{n}} \cdot \tilde{N}_A(F)(n^{3/2}).$$

Together with the event $\mathcal{E}(m)$, this implies that

$$\begin{aligned} N_{\phi'}(F')(m) &\leqslant \tilde{N}_{A'}(F')(m) + f_{F',A'}(t) \cdot \tilde{N}_{A'}(F')(n^{3/2}) \\ &\leqslant \Upsilon(F,A)^\varepsilon \cdot \frac{(\log n)^{\Delta(F,A) - 2\sqrt{\Delta(F,A)}}}{\sqrt{n}} \cdot \tilde{N}_A(F)(n^{3/2}). \end{aligned}$$

In the latter case, Lemma A.3.4 gives us

$$N_{\phi'}(F')(m) \leqslant \frac{(\log n)^{\Delta(F,A) - 2\sqrt{\Delta(F,A)}}}{\sqrt{n}} \cdot \tilde{N}_A(F)(n^{3/2}).$$

Since $|\mathcal{F}^+_{F,A}| \leqslant v_A(F)^2 \leqslant (\log n)^{2/5}$, the upper bound in (26) follows. The proof of the bounds on $D_\phi(F)(m)$ is identical, using [2, Lemma 4.30]. $\square$

We can now apply Lemma 3.2 to the variables $C_\phi(F)$ and $D_\phi(F)$.

Proof of Proposition 4.55. For each $m \in [m^*]$, set $\mathcal{K}(m) = \mathcal{E}(m) \cap \mathcal{M}(m) \cap \mathcal{K}^{\mathcal{E}}(m)$. We shall bound, for each $m_0 \leqslant \omega \cdot n^{3/2}$, the probability that m_0 is the minimal $m \in \mathbb{N}$ such that $\mathcal{K}(m-1)$ holds, and

$$N_\phi(F)(m) \notin \tilde{N}_A(F)(m) \pm f_{F,A}(t) \cdot \tilde{N}_A(F)(n^{3/2})$$

for some ϕ which is faithful at time $t = m \cdot n^{-3/2}$. Note that the event in the statement of the proposition implies that this event holds for some $m \leqslant \omega \cdot n^{3/2}$.

Fix $m_0 \leqslant \omega \cdot n^{3/2}$, and for each $m' \leqslant m_0$, define random variables

$$M_C^{\pm}(m') = \sum_{m=0}^{m'-1} \left[C_\phi(F)(m) - \frac{e(F) \cdot \tilde{N}_A(F)(m)}{t \cdot n^{3/2}} \pm \frac{f_{F,A}(t) \cdot \tilde{N}_A(F)(n^{3/2})}{n^{3/2}} \right]$$

and

$$M_D^{\pm}(m') = \sum_{m=0}^{m'-1} \left[D_\phi(F)(m) - \frac{8t \cdot o(F) \cdot \tilde{N}_A(F)(m)}{n^{3/2}} \pm \frac{C \cdot o(F) \cdot (t+1)}{n^{3/2}} f_{F,A}(t) \tilde{N}_A(F)(n^{3/2}) \right].$$

It follows from Lemmas 4.58 and 4.59 that, while the event $\mathcal{E}(m) \cap \mathcal{Y}(m) \cap \mathcal{Z}(m) \cap \mathcal{Q}(m)$ holds, $M_C^{\pm}$ and $M_D^{\pm}$ are both super-/sub-martingale pairs. Now, set

$$\alpha \; = \; \left(\Upsilon(F,A)^{\varepsilon} \cdot \frac{(\log n)^{\Delta(F,A) - 2\sqrt{\Delta(F,A)}}}{\sqrt{n}} + \frac{f_{F,A}(\omega)}{m_0} \right) \cdot \tilde{N}_A(F)(n^{3/2})$$

where $\Upsilon(F,A) = \min\big\{ n, \, (\log n)^{\Delta(F,A)} \big\}$, and

$$\beta \; = \; \left(\frac{(\log n)^{e(F)+o(F)}}{n^{3/2}} + \frac{f_{F,A}(\omega)}{m_0} \right) \cdot \tilde{N}_A(F)(n^{3/2}).$$

By Lemma 4.60, we have

$$-\beta \; \leqslant \; \Delta M_C^{\pm}(m) + \Delta M_D^{\pm}(m) \; \leqslant \; \alpha$$

while $\mathcal{E}(m) \cap \mathcal{M}(m)$ holds. Moreover, since $f_{F,A}(t_0) \geqslant n^{-1/4} (\log n)^{\Delta(F,A) - \sqrt{\Delta(F,A)}}$, and we may assume that $m_0 \geqslant n^{\varepsilon}$, we have

$$\frac{\alpha \cdot \beta \cdot m_0}{\tilde{N}_A(F)(n^{3/2})^2} \; \leqslant \; \frac{f_{F,A}(t_0)^2}{(\log n)^4}.$$

Hence, by Lemma 3.2, we obtain

$$\mathbb{P}\Bigg(\bigg(M_C^{-}(m_0) > \frac{1}{4} f_{F,A}(t_0) \tilde{N}_A(F)(n^{3/2}) \bigg) \cap \mathcal{K}(m_0 - 1) \Bigg) \; \leqslant \; e^{-(\log n)^3},$$

and similarly for M_C^{+}, M_D^{-} and M_D^{+}.

To complete the proof, note that

$$\sum_{m=0}^{m'-1} C(t+1)(o(F)+1) \cdot f_{F,A}(t) \; \leqslant \; \frac{n^{3/2}}{C} \cdot f_{F,A}(t').$$

Therefore, if

$$\max\Big\{ \min\big\{ |M_C^{+}(m)|, |M_C^{-}(m)| \big\}, \min\big\{ |M_D^{+}(m)|, |M_D^{-}(m)| \big\} \Big\} \; \leqslant \; \frac{1}{4} f_{F,A}(t) \tilde{N}_A(F)(n^{3/2})$$

then

$$\sum_{m=0}^{m'-1} \Big(C_\phi(F)(m) - D_\phi(F)(m) \Big) \in \sum_{m=0}^{m'-1} \left(\frac{e(F)}{t} - 8t \cdot o(F) \right) \frac{\tilde{N}_A(F)(m)}{n^{3/2}} \pm f_{F,A}(t') \tilde{N}_A(F)(n^{3/2}).$$

Finally, note that $\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{N}_A(F)(m) = \big(\frac{e(F)}{t} - 8t\cdot o(F)\big)\tilde{N}_A(F)(m)$, and that the number of choices for ϕ is negligible, since $|A| \leqslant (\log n)^{1/5}$. Hence, with probability at least $1 - n^{-3\log n}$, we have

$$N_\phi(F)(m') \,=\, \sum_{m=0}^{m'-1} \Big(C_\phi(F)(m) - D_\phi(F)(m) \Big) \,\in\, \tilde{N}_A(F)(m') \,\pm\, f_{F,A}(t')\tilde{N}_A(F)(n^{3/2}),$$

as required. □

4.2. **The number of open edges in a set before time $t = \omega$.** Let us finish this section by using Bohman's method to prove [2, Lemma 7.12]. The proof is almost identical to that of the bounds on $X_e(m)$ in Proposition 4.56.

Lemma 7.12. *Let $S \subseteq V(G_m)$, and let $\mathcal{N} = (A_1, \ldots, A_k)$ be a collection of subsets of S. If $|S|\sqrt{n} \leqslant o_{\mathcal{N}}(S,0) \leqslant n^{5/4}$, then*

$$\mathbb{P}\Big(\big\{ |o^*_{\mathcal{N}}(S,m)| > 1 \big\} \cap \tilde{\mathcal{N}}(S,m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m) \Big) \,\leqslant\, n^{-|S|n^{4\delta}}$$

for every $m \leqslant \omega \cdot n^{3/2}$.

In fact we shall prove the following, slightly stronger statement. Set

$$f_o(t) \,=\, n^{3\delta} \cdot f_y(t),$$

and note that we have $f_o(t) \ll g_o(t)e^{-4t^2}$ for every $m \leqslant \omega \cdot n^{3/2}$, and so the following lemma trivially implies Lemma 7.12.

Lemma A.4.1. *Let $S \subseteq V(G_m)$, and let $\mathcal{N} = (A_1, \ldots, A_k)$ be a collection of subsets of S. Suppose that $|S|\sqrt{n} \leqslant o_{\mathcal{N}}(S,0) \leqslant n^{5/4}$, and let $m \leqslant \omega \cdot n^{3/2}$. Then, with probability at least $1 - n^{-|S|n^{4\delta}}$ either $\big(\tilde{\mathcal{N}}(S,m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)\big)^c$ holds, or*

$$o_{\mathcal{N}}(S,m) \in e^{-4t^2} o_{\mathcal{N}}(S,0) \pm f_o(t) o_{\mathcal{N}}(S,0). \tag{27}$$

Proof. Let $m_0 \leqslant \omega \cdot n^{3/2}$ be minimal such that the event $\tilde{\mathcal{N}}(S,m_0) \cap \mathcal{Y}(m_0) \cap \mathcal{Q}(m_0)$ holds, and

$$o_{\mathcal{N}}(S,m_0) \notin e^{-4t_0^2} o_{\mathcal{N}}(S,0) \pm f_o(t_0) o_{\mathcal{N}}(S,0),$$

where $t_0 = m_0 \cdot n^{-3/2}$. It follows that, if $m < m_0$, then

$$\begin{aligned}
\mathbb{E}\big[\Delta o_{\mathcal{N}}(S,m)\big] \,&=\, -\frac{1}{Q(m)} \sum_{f \in O_{\mathcal{N}}(S,m)} Y_f(m) \\
&\in\, -\frac{e^{-4t^2} o_{\mathcal{N}}(S,0)}{\tilde{Q}(m)} \big(1 \pm e^{4t^2} f_o(t)\big)^2 \Big(\tilde{Y}(m) \pm f_y(t) \tilde{Y}(n^{3/2}) \Big) \\
&\subset\, -\frac{8t}{n^{3/2}} \cdot e^{-4t^2} o_{\mathcal{N}}(S,0) \,\pm\, \sqrt{C} \cdot \left(\frac{t+1}{n^{3/2}} \right) \cdot f_o(t) o_{\mathcal{N}}(S,0),
\end{aligned}$$

since $\tilde{Y}(m) = \Theta\big(t \cdot e^{-4t^2} \cdot \tilde{Y}(n^{3/2})\big)$, and using the event $\mathcal{Y}(m) \cap \mathcal{Q}(m)$ and the bounds (27), which hold for all $m < m_0$. It follows that

$$M^{\pm}_{S,\mathcal{N}}(m') \;=\; \sum_{m=0}^{m'-1} \left[\Delta o_{\mathcal{N}}(S,m) + \frac{8t}{n^{3/2}} \cdot e^{-4t^2} o_{\mathcal{N}}(S,0) \;\pm\; \sqrt{C} \cdot \left(\frac{t+1}{n^{3/2}} \right) \cdot f_o(t) o_{\mathcal{N}}(S,0) \right].$$

is a super-/sub-martingale pair on $0 \leqslant m' < m_0$. Moreover, we have

$$-\, n^{\delta} \;\leqslant\; \Delta o_{\mathcal{N}}(S,m) \;\leqslant\; 0$$

for every $m < m_0$, since $\tilde{\mathcal{N}}(S,m)$ holds (see [2, Section 7.4]), and so

$$-\, 2 \cdot n^{\delta} \;\leqslant\; \Delta M^{\pm}_{S,\mathcal{N}}(m) \;\leqslant\; \frac{C \cdot o_{\mathcal{N}}(S,0)}{n^{3/2}},$$

for every $m < m_0$. Since $m_0 \geqslant \dfrac{f_o(t) o_{\mathcal{N}}(S,0)}{4 \cdot n^{\delta}} \gg \dfrac{f_o(t) o_{\mathcal{N}}(S,0)^2}{n^{3/2}}$ and $f_o(t)^2 o_{\mathcal{N}}(S,0) \geqslant n^{6\delta}|S|$, it follows by Lemma 3.2 that

$$\mathbb{P}\left(M^{-}_{X_e}(m_0) > \frac{1}{2} f_o(t) o_{\mathcal{N}}(S,0) \right) \;\leqslant\; \exp\left(- \frac{f_o(t)^2 o_{\mathcal{N}}(S,0)}{C^2 \cdot n^{\delta}} \right) \;\ll\; n^{-|S| n^{4\delta}},$$

and similarly for $M^{+}_{X_e}$. It follows that, with probability at least $1 - n^{-|S|n^{4\delta}}$, we have

$$\begin{aligned} o_{\mathcal{N}}(S,m') &\in o_{\mathcal{N}}(S,0) - \sum_{m=0}^{m'-1} \left[\frac{8t}{n^{3/2}} \cdot o_{\mathcal{N}}(S,m) \pm \sqrt{C} \left(\frac{t+1}{n^{3/2}} \right) f_o(t) o_{\mathcal{N}}(S,0) \right] \pm \frac{1}{2} f_o(t') o_{\mathcal{N}}(S,0) \\ &\subset\; e^{-4t'^2} o_{\mathcal{N}}(S,0) \pm f_o(t') o_{\mathcal{N}}(S,0) \end{aligned}$$

as required. □

5. Section 6: Whirlpools

In this section, we shall prove that the variables $\overline{X}$, $\overline{Y}$ and Q follow (in expected value) a three-dimensional dynamical system which looks like a whirlpool. Set

$$\overline{X}^*(m) \;=\; \frac{\overline{X}(m) - \tilde{X}(m)}{g_q(t)\tilde{X}(m)}, \quad \overline{Y}^*(m) \;=\; \frac{\overline{Y}(m) - \tilde{Y}(m)}{g_q(t)\tilde{Y}(m)} \quad \text{and} \quad Q^*(m) \;=\; \frac{Q(m) - \tilde{Q}(m)}{g_q(t)\tilde{Q}(m)}$$

for each $m \in \mathbb{N}$. We shall prove the following lemma.

Lemma 6.2. *Let $\omega \cdot n^{3/2} < m \leqslant m^*$, and suppose that $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds. Then*

(a) $\mathbb{E}\big[\Delta Q^*(m)\big] \;\in\; \dfrac{4t}{n^{3/2}}\Big(-2\overline{Y}^*(m) + Q^*(m) \pm o(1)\Big)$.

(b) $\mathbb{E}\big[\Delta \overline{Y}^*(m)\big] \;\in\; \dfrac{4t}{n^{3/2}}\Big(-3\overline{Y}^*(m) + 2Q^*(m) \pm o(1)\Big)$.

(c) $\mathbb{E}\big[\Delta \overline{X}^*(m)\big] \;\in\; \dfrac{4t}{n^{3/2}}\Big(-\overline{X}^*(m) - 4\overline{Y}^*(m) + 4Q^*(m) \pm o(1)\Big)$.

We begin by calculating the expected step-change in the variables $\overline{X}(m)$, $\overline{Y}(m)$ and $Q(m)$.

Lemma 6.3. *For every $m \in \mathbb{N}$,*

$$\mathbb{E}\big[\Delta Q(m)\big] \; = \; -\overline{Y}(m) - 1.$$

Proof. This is trivial, since if edge e is chosen in step $m+1$, then $\Delta Q(m) = -Y_e(m) - 1$. □

Lemma 6.4. *Let $\omega \cdot n^{3/2} < m \leqslant m^*$. If $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds, then*

$$\mathbb{E}\big[\Delta \overline{Y}(m)\big] \; \in \; \frac{1}{Q(m)}\Big(- \overline{Y}(m)^2 + \overline{X}(m) - 2 \cdot \mathrm{Var}\big(Y_e(m)\big) \pm O\big(\tilde{Y}(m)\big)\Big).$$

Lemma 6.5. *Let $\omega \cdot n^{3/2} < m \leqslant m^*$. If $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds, then*

$$\mathbb{E}\big[\Delta \overline{X}(m)\big] \; \in \; \frac{1}{Q(m)}\bigg(- 2 \cdot \overline{X}(m)\overline{Y}(m) - 3 \cdot \mathrm{Cov}(X,Y) \pm O\Big(\tilde{X}(m) + \tilde{Y}(m)^2\Big)\bigg).$$

Recall first that the Y-graph has vertex set $O(G_m)$, and an edge between each pair $\{f, f'\}$ such that $f' \in Y_f(m)$. We begin with a simple observation about this graph.

Observation A.5.1. *The Y-graph is triangle-free, i.e., for every $m \in \mathbb{N}$, there do not exist three distinct open edges e, f and g of G_m with $e \in Y_f(m)$, $f \in Y_g(m)$ and $g \in Y_e(m)$.*

Proof. This follows from the fact that G_m is triangle-free. Indeed, let $\{e, f, g\} \subset O(G_m)$ be open edges which form a triangle in the Y-graph, and suppose first that they have a common endpoint. Then the other endpoints of e, f and g form a triangle in G_m, which is a contradiction. On the other hand, if e, f and g do not share a common endpoint, then they must form a triangle (since they are pairwise intersecting), which contradicts (e.g.) the assumption that $e \in Y_f(m)$. It follows that no such triple exists, as claimed. □

In order to prove Lemmas 6.4 and 6.5, we shall use the variables

$$\mathbb{Y}(m) = \sum_{e \in Q(m)} Y_e(m) \qquad \text{and} \qquad \mathbb{X}(m) = \sum_{e \in Q(m)} X_e(m),$$

which are exactly twice the number of edges in the Y-graph, and six times the number of open triangles in G_m, respectively.

We first bound the expected change in $\mathbb{Y}(m)$.

Lemma A.5.2.

$$\mathbb{E}\big[\Delta \mathbb{Y}(m)\big] \; = \; \frac{1}{Q(m)}\bigg(\mathbb{X}(m) \; - \; 2 \sum_{e \in Q(m)} Y_e(m)^2\bigg).$$

Proof. Suppose we add edge $e \in O(G_m)$ in step $m+1$. We claim that

$$\Delta \mathbb{Y}(m) \; = \; X_e(m) - 2 \sum_{f \in Y_e(m)} Y_f(m). \tag{28}$$

To see this, observe that in step $m+1$ we remove from the Y-graph the vertices corresponding to each $f \in Y_e(m)$, and add a matching between the vertices corresponding to $X_e(m)$. Since the Y-graph is triangle-free, by Observation A.5.1, and each $f \in Y_e(m)$ is (by definition) a Y-neighbour of e, it follows that the number of edges of the Y-graph which are removed is exactly $\sum_{f\in Y_e(m)} Y_f(m)$. Since $\mathbb{Y}(m)$ is equal to twice the number of edges in the Y-graph, (28) follows.

Summing over edges $e \in Q(m)$, it follows that

$$\mathbb{E}\big[\mathbb{Y}(m+1) - \mathbb{Y}(m) \,\big|\, G_m\big] \;=\; \frac{1}{Q(m)} \sum_{e\in Q(m)} \bigg[X_e(m) - 2 \sum_{f\in Y_e(m)} Y_f(m)\bigg].$$

Now recall that $f \in Y_e(m)$ if and only if $e \in Y_f(m)$, and thus

$$\sum_{e\in Q(m)} \sum_{f\in Y_e(m)} Y_f(m) \;=\; \sum_{f\in Q(m)} Y_f(m)^2.$$

(Indeed, this is simply the number of walks of length two in the Y-graph.) Hence

$$\mathbb{E}\big[\Delta\mathbb{Y}(m)\big] \;=\; \frac{\mathbb{X}(m)}{Q(m)} \;-\; \frac{2}{Q(m)} \sum_{e\in Q(m)} Y_e(m)^2,$$

as required. □

Recall the following simple lemma, which was stated (but not proved) in [2, Section 5].

Lemma 5.28.

$$\mathbb{E}\bigg[\Delta\bigg(\frac{A(m)}{B(m)}\bigg)\bigg] \;=\; \frac{\mathbb{E}\big[\Delta A(m)\big]}{B(m)} \;-\; \frac{1}{B(m)}\mathbb{E}\bigg[\frac{A(m+1)\Delta B(m)}{B(m+1)} \,\bigg|\, G_m\bigg]$$

Proof. We have

$$\mathbb{E}\bigg[\frac{A(m+1)}{B(m+1)} - \frac{A(m)}{B(m)} \,\bigg|\, G_m\bigg] = \mathbb{E}\bigg[\frac{B(m)A(m+1) - B(m+1)A(m)}{B(m)(B(m+1)} \,\bigg|\, G_m\bigg]$$
$$= \frac{1}{B(m)}\mathbb{E}\bigg[\frac{B(m+1)\big(A(m+1) - A(m)\big) + \big(B(m) - B(m+1)\big)A(m+1)}{B(m+1)} \,\bigg|\, G_m\bigg]$$
$$= \frac{\mathbb{E}\big[\Delta A(m)\big]}{B(m)} \;+\; \frac{1}{B(m)}\mathbb{E}\bigg[\frac{\big(B(m) - B(m+1)\big)A(m+1)}{B(m+1)} \,\bigg|\, G_m\bigg],$$

as claimed. □

Recall that $\mathrm{Var}\big(Y_e(m)\big) = \mathbb{E}\big[Y_e(m)^2\big] - \overline{Y}(m)^2$, where the expectation is over the choice of the edge $e \in O(G_m)$, and that the event $\mathcal{X}(m) \cap \mathcal{Y}(m)$ implies that

$$X_e(m) \in \big(2 \pm o(1)\big)e^{-8t^2} n \quad \text{and} \quad Y_e(m) \in \big(4 \pm o(1)\big)te^{-4t^2}\sqrt{n} \tag{29}$$

for every $e \in O(G_m)$. We can now prove Lemma 6.4.

Proof of Lemma 6.4. By Lemma 5.28, we have

$$\mathbb{E}\big[\Delta\overline{Y}(m)\big] \;=\; \frac{\mathbb{E}\big[\Delta\mathbb{Y}(m)\big]}{Q(m)} \;-\; \frac{1}{Q(m)}\mathbb{E}\bigg[\frac{\mathbb{Y}(m+1)\Delta Q(m)}{Q(m+1)}\bigg]. \tag{30}$$

Now, we claim that

$$\frac{\mathbb{Y}(m+1)}{Q(m+1)} \in \left(1 \pm \frac{1}{\tilde{Y}(m)^2}\right) \frac{\mathbb{Y}(m)}{Q(m)} \subset \overline{Y}(m) \pm 1.$$

To see this, note that by (28) and (29), since $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds we have

$$\frac{|\Delta \mathbb{Y}(m)|}{\mathbb{Y}(m)} \leqslant 3 \cdot \frac{\tilde{X}(m) + \tilde{Y}(m)^2}{\tilde{Y}(m) \cdot \tilde{Q}(m)} \ll \frac{1}{\tilde{Y}(m)^2} \quad \text{and} \quad \frac{|\Delta Q(m)|}{Q(m)} \leqslant 2 \cdot \frac{\tilde{Y}(m)}{\tilde{Q}(m)} \ll \frac{1}{\tilde{Y}(m)^2}.$$

Since $\mathbb{E}\big[\Delta Q(m)\big] = -\overline{Y}(m) - 1$, and using (30), it follows that

$$\mathbb{E}\big[\Delta \overline{Y}(m)\big] \in \frac{\mathbb{E}\big[\Delta \mathbb{Y}(m)\big]}{Q(m)} + \frac{\overline{Y}(m)^2 \pm O\big(\tilde{Y}(m)\big)}{Q(m)}.$$

By Lemma A.5.2, and since $\mathbb{E}\big[Y_e(m)^2\big] = \mathrm{Var}\big(Y_e(m)\big) + \overline{Y}(m)^2$, this implies that

$$\begin{aligned}\mathbb{E}\big[\Delta \overline{Y}(m)\big] &\in \frac{1}{Q(m)} \left(\frac{\mathbb{X}(m)}{Q(m)} - 2\Big(\overline{Y}(m)^2 + \mathrm{Var}\big(Y_e(m)\big)\Big)\right) + \frac{\overline{Y}(m)^2 \pm O\big(\tilde{Y}(m)\big)}{Q(m)} \\ &= \frac{1}{Q(m)}\Big(\overline{X}(m) - \overline{Y}(m)^2 - 2 \cdot \mathrm{Var}\big(Y_e(m)\big) \pm O\big(\tilde{Y}(m)\big)\Big),\end{aligned}$$

as required. □

The proof for $\overline{X}(m)$ is similar. Recall that $\mathrm{Cov}(X,Y) = \mathbb{E}\big[X_e \cdot Y_e\big] - \overline{X} \cdot \overline{Y}$, where the expectation is over the (uniformly random) choice of the edge $e \in O(G_m)$.

Lemma A.5.3. *If $\mathcal{X}(m) \cap \mathcal{Y}(m)$ holds, then*

$$\mathbb{E}\big[\Delta \mathbb{X}(m)\big] \in -\frac{3}{Q(m)} \sum_{f \in Q(m)} X_f(m) \cdot Y_f(m) \pm O\Big(\tilde{X}(m) + \tilde{Y}(m)^2\Big).$$

Proof. Observe first that $\mathbb{X}(m)$ is simply the number of labelled open triangles in G_m (i.e., six times the number of unlabelled open triangles). We claim that if edge e is chosen in step $m+1$ of the triangle-free process, then

$$-3 \cdot X_e(m) \leqslant \Delta \mathbb{X}(m) + 3 \sum_{f \in Y_e(m)} X_f(m) \leqslant 6 \cdot Y_e(m)^2 \tag{31}$$

To see the lower bound, note that no new open triangles are created, and each edge $f \in Y_e(m) \cup \{e\}$ which is closed in step $m+1$ destroys at most $X_f(m)/2$ unlabelled open triangles. On the other hand, an open triangle is destroyed in two different ways only if it has two edges in $Y_e(m) \cup \{e\}$, and hence we have double-counted at most $Y_e(m)^2$ unlabelled open triangles, which gives the upper bound in (31).

Now, summing over edges $e \in Q(m)$, it follows that

$$\mathbb{E}\big[\Delta \mathbb{X}(m)\big] \in -\frac{3}{Q(m)} \sum_{e \in Q(m)} \sum_{f \in Y_e(m)} X_f(m) \pm O\Big(\tilde{X}(m) + \tilde{Y}(m)^2\Big).$$

Since $f \in Y_e(m)$ if and only if $e \in Y_f(m)$, it follows that

$$\sum_{e \in Q(m)} \sum_{f \in Y_e(m)} X_f(m) \,=\, \sum_{f \in Q(m)} X_f(m) \cdot Y_f(m),$$

and hence

$$\mathbb{E}\big[\Delta \mathbb{X}(m)\big] \,\in\, -\frac{3}{Q(m)} \sum_{f \in Q(m)} X_f(m) \cdot Y_f(m) \,\pm\, O\Big(\tilde{X}(m) + \tilde{Y}(m)^2\Big),$$

as required. □

We can now prove Lemma 6.5.

Proof of Lemma 6.5. By Lemma 5.28 we have

$$\mathbb{E}\big[\Delta \overline{X}(m)\big] \,=\, \frac{\mathbb{E}\big[\Delta \mathbb{X}(m)\big]}{Q(m)} \,-\, \frac{1}{Q(m)} \mathbb{E}\bigg[\frac{\mathbb{X}(m+1)\Delta Q(m)}{Q(m+1)}\bigg],$$

and we have

$$\frac{\mathbb{X}(m+1)}{Q(m+1)} \,\in\, \bigg(1 \pm \frac{1}{\tilde{Y}(m)^2}\bigg) \frac{\mathbb{X}(m)}{Q(m)} \,\subseteq\, \overline{X}(m) \,\pm\, 1.$$

To see this, note that by (29) and (31), since $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds we have

$$\frac{|\Delta \mathbb{X}(m)|}{\mathbb{X}(m)} \,\leqslant\, 4 \cdot \frac{\tilde{X}(m) \cdot \tilde{Y}(m)}{\tilde{X}(m) \cdot \tilde{Q}(m)} \,\ll\, \frac{1}{\tilde{Y}(m)^2} \quad \text{and} \quad \frac{|\Delta Q(m)|}{Q(m)} \,\leqslant\, 2 \cdot \frac{\tilde{Y}(m)}{\tilde{Q}(m)} \,\ll\, \frac{1}{\tilde{Y}(m)^2}.$$

Since $\mathbb{E}\big[\Delta Q(m)\big] = -\overline{Y}(m) - 1$, it follows that

$$\mathbb{E}\big[\Delta \overline{X}(m)\big] \,\in\, \frac{\mathbb{E}\big[\Delta \mathbb{X}(m)\big]}{Q(m)} \,+\, \frac{\overline{X}(m)\overline{Y}(m) \pm O\big(\overline{X}(m) + \overline{Y}(m)\big)}{Q(m)}.$$

By Lemma A.5.3, and since $\mathbb{E}\big[X_e \cdot Y_e\big] = \overline{X} \cdot \overline{Y} + \mathrm{Cov}(X,Y)$, this implies that

$$\mathbb{E}\big[\Delta \overline{X}(m)\big] \,\in\, \frac{1}{Q(m)}\bigg(- 2 \cdot \overline{X}(m)\overline{Y}(m) - 3 \cdot \mathrm{Cov}(X,Y) \pm O\Big(\tilde{X}(m) + \tilde{Y}(m)^2\Big)\bigg),$$

as required. □

We can now deduce Lemma 6.2 via a rather tedious calculation.

Proof of Lemma 6.2. Let $\omega \cdot n^{3/2} < m \leqslant m^*$, and suppose that $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds. By Lemma 5.28 we have

$$\mathbb{E}\big[\Delta Q^*(m)\big] \,=\, \frac{\mathbb{E}\big[\Delta Q(m) - \Delta \tilde{Q}(m)\big]}{g_q(t)\tilde{Q}(m)} \,-\, \frac{\Delta\big(g_q(t)\tilde{Q}(m)\big)}{g_q(t)\tilde{Q}(m)} \cdot \mathbb{E}\Big[Q^*(m+1) \,\big|\, G_m\Big].$$

Recall that $\mathbb{E}\big[\Delta Q(m)\big] = -\overline{Y}(m) - 1$ and $\Delta \tilde{Q}(m) \in -\tilde{Y}(m) \pm 1$, and note that

$$\Delta\big(g_q(t)\tilde{Q}(m)\big) \,\in\, -\frac{4t}{n^{3/2}} \cdot g_q(t)\tilde{Q}(m) \pm 1,$$

since $g_q(t)\tilde{Q}(m)$ is equal to e^{-2t^2} times some function of n. It follows that

$$\left(1 - \frac{\big(4+o(1)\big)t}{n^{3/2}}\right)\mathbb{E}\big[\Delta Q^*(m)\big] \,\in\, \frac{\tilde{Y}(m) - \overline{Y}(m)}{g_q(t)\tilde{Q}(m)} + \frac{4t}{n^{3/2}} \cdot Q^*(m) \pm \frac{3}{g_q(t)\tilde{Q}(m)},$$

since the event $\mathcal{Q}(m)$ implies that $|\overline{Y}^*(m)| + |Q^*(m)| \leqslant 2$, and hence

$$\mathbb{E}\big[\Delta Q^*(m)\big] \,\in\, -\frac{\overline{Y}^*(m)\cdot\tilde{Y}(m)}{\tilde{Q}(m)} + \frac{4t}{n^{3/2}}\cdot Q^*(m) \pm \frac{1}{n^{3/2}} \,\subset\, \frac{4t}{n^{3/2}}\Big(-2\overline{Y}^*(m) + Q^*(m) \pm o(1)\Big),$$

as required.

We turn next to $\overline{Y}^*(m)$. Observe that, by Lemma 5.28, we have

$$\mathbb{E}\big[\Delta\overline{Y}^*(m)\big] \,=\, \frac{\mathbb{E}\big[\Delta\overline{Y}(m) - \Delta\tilde{Y}(m)\big]}{g_q(t)\tilde{Y}(m)} \,-\, \frac{\Delta\big(g_q(t)\tilde{Y}(m)\big)}{g_q(t)\tilde{Y}(m)}\cdot\mathbb{E}\Big[\overline{Y}^*(m+1)\,\big|\,G_m\Big], \tag{32}$$

and recall that, by Lemma 6.4, if $\mathcal{X}(m)\cap\mathcal{Y}(m)\cap\mathcal{Q}(m)$ holds then

$$\mathbb{E}\big[\Delta\overline{Y}(m)\big] \,\in\, \frac{1}{Q(m)}\Big(-\overline{Y}(m)^2 + \overline{X}(m) - 2\cdot\mathrm{Var}\big(Y_e(m)\big) \pm O\big(\tilde{Y}(m)\big)\Big). \tag{33}$$

Now, since $\tilde{Y}(m) = 4te^{-4t^2}\sqrt{n}$, a simple calculation gives

$$\Delta\tilde{Y}(m) \,\in\, -\left(\frac{8t^2-1}{t\cdot n^{3/2}}\right)\tilde{Y}(m) \pm \frac{1}{n^2} \,\subseteq\, \frac{-\tilde{Y}(m)^2 + \tilde{X}(m)}{\tilde{Q}(m)} \pm \frac{1}{n^2}, \tag{34}$$

and similarly, since $g_q(t)\tilde{Y}(m) = 4te^{-2t^2}n^{1/4}(\log n)^3$ and $t \geqslant \omega$, we obtain

$$\Delta\big(g_q(t)\tilde{Y}(m)\big) \in -\left(\frac{4t^2-1}{t\cdot n^{3/2}}\right) g_q(t)\tilde{Y}(m) \pm \frac{1}{n^2} \,\subseteq\, -\big(1\pm o(1)\big)\cdot\frac{4t}{n^{3/2}}\cdot g_q(t)\tilde{Y}(m). \tag{35}$$

Moreover, since $\mathcal{Y}(m)$ holds, we have

$$\mathrm{Var}\big(Y_e(m)\big) \,\leqslant\, g_y(t)^2\tilde{Y}(m)^2 \,\ll\, g_q(t)\tilde{Y}(m)^2. \tag{36}$$

Recalling that $\overline{Y}(m) = \big(1+g_q(t)\overline{Y}^*(m)\big)\tilde{Y}(m)$ and $Q(m) = \big(1+g_q(t)Q^*(m)\big)\tilde{Q}(m)$, it follows from (33), (34) and (36) that

$$\begin{aligned}\mathbb{E}\big[\Delta\overline{Y}(m)\big] - \Delta\tilde{Y}(m) \,\in\, &-\frac{\tilde{Y}(m)^2}{Q(m)}\Big(\big(1+g_q(t)\overline{Y}^*(m)\big)^2 - \big(1+g_q(t)Q^*(m)\big)\Big)\\ &+\frac{\tilde{X}(m)}{Q(m)}\Big(\big(1+g_q(t)\overline{X}^*(m)\big) - \big(1+g_q(t)Q^*(m)\big)\Big) \,\pm\, \frac{o(1)\cdot g_q(t)\tilde{Y}(m)^2}{\tilde{Q}(m)}.\end{aligned}$$

Observing that $\tilde{X}(m) \ll \tilde{Y}(m)^2$, which holds since $t \geqslant \omega$, we deduce that

$$\mathbb{E}\big[\Delta\overline{Y}(m)\big] - \Delta\tilde{Y}(m) \,\in\, \frac{g_q(t)\tilde{Y}(m)^2}{Q(m)}\Big(-2\overline{Y}^*(m) + Q^*(m) \pm o(1)\Big) \tag{37}$$

if $|\overline{X}^*(m)| + |\overline{Y}^*(m)| + |Q^*(m)| = O(1)$, which follows from the event $\mathcal{Q}(m)$.

Now, combining (32), (35) and (37), we obtain

$$\mathbb{E}\big[\Delta\overline{Y}^*(m)\big] \in \frac{\tilde{Y}(m)}{Q(m)}\Big(-2\overline{Y}^*(m) + Q^*(m) \pm o(1)\Big) + \big(1 \pm o(1)\big)\frac{4t}{n^{3/2}}\Big(\overline{Y}^*(m) + \mathbb{E}\big[\Delta\overline{Y}^*(m)\big]\Big),$$

which easily implies that

$$\mathbb{E}\big[\Delta\overline{Y}^*(m)\big] \,\in\, \frac{4t}{n^{3/2}}\Big(-3\overline{Y}^*(m) + 2Q^*(m) \pm o(1)\Big),$$

as required.

Finally, we turn to $\overline{X}^*(m)$. By Lemma 5.28, we have

$$\mathbb{E}\big[\Delta\overline{X}^*(m)\big] \,=\, \frac{\mathbb{E}\big[\Delta\overline{X}(m) - \Delta\tilde{X}(m)\big]}{g_q(t)\tilde{X}(m)} \,-\, \frac{\Delta\big(g_q(t)\tilde{X}(m)\big)}{g_q(t)\tilde{X}(m)} \cdot \mathbb{E}\Big[\overline{X}^*(m+1) \,\big|\, G_m\Big], \qquad (38)$$

and by Lemma 6.5, if $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds then

$$\mathbb{E}\big[\Delta\overline{X}(m)\big] \,\in\, \frac{1}{Q(m)}\bigg(-2 \cdot \overline{X}(m)\overline{Y}(m) - 3 \cdot \mathrm{Cov}(X,Y) \pm O\Big(\tilde{X}(m) + \tilde{Y}(m)^2\Big)\bigg). \qquad (39)$$

Now, since $\tilde{X}(m) = 2e^{-8t^2}n$, a simple calculation gives

$$\Delta\tilde{X}(m) \,\in\, -\frac{16t}{n^{3/2}} \cdot \tilde{X}(m) \pm \frac{1}{n^2} \,\subseteq\, -\frac{2 \cdot \tilde{X}(m) \cdot \tilde{Y}(m)}{\tilde{Q}(m)} \pm \frac{o(1)}{n^{3/2}}, \qquad (40)$$

and similarly, since $g_q(t)\tilde{X}(m) = 2e^{-6t^2}n^{3/4}(\log n)^3$ and $t \geqslant \omega$, we obtain

$$\Delta\big(g_q(t)\tilde{X}(m)\big) \in -\frac{12t}{n^{3/2}} \cdot g_q(t)\tilde{X}(m) \pm \frac{1}{n^2} \,\subseteq\, -\big(1 \pm o(1)\big) \cdot \frac{12t}{n^{3/2}} \cdot g_q(t)\tilde{X}(m). \qquad (41)$$

Moreover, since $\mathcal{X}(m) \cap \mathcal{Y}(m)$ holds, we have

$$\mathrm{Cov}\big(X_e(m), Y_e(m)\big) \,\leqslant\, g_x(t)\tilde{X}(m) \cdot g_y(t)\tilde{Y}(m) \,\ll\, g_q(t)\tilde{X}(m)\tilde{Y}(m), \qquad (42)$$

and note also that $\tilde{X}(m) + \tilde{Y}(m)^2 \ll g_q(t)\tilde{X}(m)\tilde{Y}(m)$ for every $m \leqslant m^*$.

Combining (39), (40) and (42), it follows that $\mathbb{E}\big[\Delta\overline{X}(m)\big] - \Delta\tilde{X}(m)$ is contained in

$$\frac{2 \cdot \tilde{X}(m)\tilde{Y}(m)}{Q(m)}\Big(-\big(1 + g_q(t)\overline{X}^*(m)\big)\big(1 + g_q(t)\overline{Y}^*(m)\big) + \big(1 + g_q(t)Q^*(m)\big) \pm o(1) \cdot g_q(t)\Big),$$

and hence

$$\mathbb{E}\big[\Delta\overline{X}(m)\big] - \Delta\tilde{X}(m) \,\in\, \frac{2 \cdot g_q(t)\tilde{X}(m)\tilde{Y}(m)}{Q(m)}\Big(-\overline{X}^*(m) - \overline{Y}^*(m) + Q^*(m) \pm o(1)\Big). \qquad (43)$$

Finally, it follows from (38), (41) and (43) that

$$\mathbb{E}\big[\Delta\overline{X}^*(m)\big] \in \frac{2 \cdot \tilde{Y}(m)}{Q(m)}\Big(-\overline{X}^*(m) - \overline{Y}^*(m) + Q^*(m) \pm o(1)\Big)$$
$$+ \,\big(1 \pm o(1)\big) \cdot \frac{12t}{n^{3/2}} \cdot \Big(\overline{X}^*(m) + \mathbb{E}\big[\Delta\overline{X}^*(m)\big]\Big),$$

which, since $|\overline{X}^*(m)| + |\overline{Y}^*(m)| + |Q^*(m)| = O(1)$, by $\mathcal{Q}(m)$, easily implies that

$$\mathbb{E}\big[\Delta\overline{X}^*(m)\big] \,\in\, \frac{4t}{n^{3/2}}\Big(-\overline{X}^*(m) - 4\overline{Y}^*(m) + 4Q^*(m) \pm o(1)\Big),$$

as required. □

Finally, let's prove Lemma 6.7. Recall that

$$\begin{pmatrix} \overline{Y}^* \\ Q^* \end{pmatrix} = \varepsilon \begin{pmatrix} 4 & 5 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$$

and $\Lambda(m) \,=\, \lambda(m)^2 + \mu(m)^2$.

Lemma 6.7. *Let $\omega \cdot n^{3/2} < m \leqslant m^*$, and suppose that $\mathcal{X}(m) \cap \mathcal{Y}(m) \cap \mathcal{Q}(m)$ holds. Then*

$$|\Delta\overline{X}^*(m)| + |\Delta\overline{Y}^*(m)| + |\Delta Q^*(m)| \,\leqslant\, \frac{(\log n)^3}{g_q(t) \cdot n^{3/2}},$$

and hence

$$|\Delta\Lambda(m)| \,\leqslant\, \frac{(\log n)^4}{g_q(t) \cdot n^{3/2}} \qquad \text{and} \qquad \mathbb{E}\big[|\Delta\Lambda(m)|\big] \,\leqslant\, \frac{(\log n)^4}{g_q(t) \cdot n^{3/2}}.$$

We shall use the following bound, which follows easily from Lemma 4.23.

Lemma A.5.4. *Let $A \in \{\overline{X}, \overline{Y}, Q\}$ and let $\tilde{A}$ be the corresponding member of $\{\tilde{X}, \tilde{Y}, \tilde{Q}\}$. For every $\omega \cdot n^{3/2} < m \leqslant m^*$, if $\mathcal{Q}(m)$ holds, then*

$$|\Delta A^*(m)| \,\leqslant\, \frac{3}{g_q(t)} \cdot \left(\frac{|\Delta A(m)|}{\tilde{A}(m)} \,+\, \frac{\log n}{n^{3/2}} \right).$$

Proof. Since $\tilde{A}(m)$ is equal to either te^{-kt^2} or e^{-kt^2} times some function of n, where $k \in \{4, 8\}$, $g_q(t)\tilde{A}(m)$ is equal to either $te^{-(k-2)t^2}$ or $e^{-(k-2)t^2}$ times some function of n, and $t > \omega$, we have

$$\Delta\tilde{A}(m) \in \frac{-2kt \pm o(1)}{n^{3/2}} \cdot \tilde{A}(m) \quad \text{and} \quad \Delta\big(g_q(t)\tilde{A}(m)\big) \in \frac{-(2k-4)t \pm o(1)}{n^{3/2}} \cdot g_q(t)\tilde{A}(m),$$

and hence

$$|\Delta\tilde{A}(m)| \,\ll\, \frac{\log n}{n^{3/2}} \cdot \tilde{A}(m) \quad \text{and} \quad |\Delta\big(g_q(t)\tilde{A}(m)\big)| \ll \frac{\log n}{n^{3/2}} \cdot g_q(t)\tilde{A}(m).$$

Moreover, the event $\mathcal{Q}(m)$ implies that $A(m) \leqslant \big(1 + g_q(t)\big)\tilde{A}(m)$, and $g_q(t) \ll 1$. Thus, applying Lemma 4.23, we obtain

$$|\Delta A^*(m)| \,\leqslant\, \frac{3}{g_q(t)} \left(\frac{|\Delta A(m)|}{\tilde{A}(m)} \,+\, \frac{\log n}{n^{3/2}} \right),$$

as claimed. □

Proof of Lemma 6.7. By Lemma A.5.4, we have

$$|\Delta\overline{X}^*(m)| + |\Delta\overline{Y}^*(m)| + |\Delta Q^*(m)| \leqslant \frac{6}{g_q(t)}\left(\frac{|\Delta\overline{X}(m)|}{\tilde{X}(m)} + \frac{|\Delta\overline{Y}(m)|}{\tilde{Y}(m)} + \frac{|\Delta Q(m)|}{\tilde{Q}(m)} + \frac{\log n}{n^{3/2}}\right),$$

so it will suffice to prove that

$$\max\left\{\frac{|\Delta\overline{X}(m)|}{\tilde{X}(m)}, \frac{|\Delta\overline{Y}(m)|}{\tilde{Y}(m)}, \frac{|\Delta Q(m)|}{\tilde{Q}(m)}\right\} \leqslant \frac{\log n}{n^{3/2}}.$$

For $Q(m)$, the bound is trivial, since if $\mathcal{Y}(m)$ holds then $\Delta Q(m) \in -(1 \pm \varepsilon)\tilde{Y}(m)$. To prove the bound for $\overline{Y}(m)$, recall first (see (28)) that

$$\Delta\mathbb{Y}(m) = X_e(m) - 2\sum_{f \in Y_e(m)} Y_f(m),$$

and observe that therefore, since $\mathcal{X}(m) \cap \mathcal{Y}(m)$ holds,

$$\Delta\mathbb{Y}(m) \in (1 \pm \varepsilon)\Big(\tilde{X}(m) - 2 \cdot \tilde{Y}(m)^2\Big).$$

Since $\tilde{X}(m) \ll \tilde{Y}(m)^2$ for $t \geqslant \omega$, and $\Delta Q(m) \in -(1 \pm \varepsilon)\tilde{Y}(m)$, it follows that

$$\overline{Y}(m+1) = \frac{\mathbb{Y}(m+1)}{Q(m+1)} \in \left(1 \pm \frac{\tilde{Y}(m)}{\tilde{Q}(m)}\right)\left(\frac{\mathbb{Y}(m) + \tilde{X}(m) - 2 \cdot \tilde{Y}(m)^2}{Q(m) - \tilde{Y}(m)}\right),$$

and hence

$$|\Delta\overline{Y}(m)| \leqslant \left(1 + \frac{\tilde{Y}(m)}{\tilde{Q}(m)}\right)\left(\frac{\mathbb{Y}(m) + \tilde{X}(m) - 2 \cdot \tilde{Y}(m)^2}{Q(m) - \tilde{Y}(m)}\right) - \overline{Y}(m) \leqslant \frac{3 \cdot \tilde{Y}(m)^2}{\tilde{Q}(m)},$$

as required.

Finally, to prove the bound for $\overline{X}(m)$, recall from (31) that

$$-3 \cdot X_e(m) \leqslant \Delta\mathbb{X}(m) + 3\sum_{f \in Y_e(m)} X_f(m) \leqslant 6 \cdot Y_e(m)^2$$

and observe that therefore, since $\mathcal{X}(m) \cap \mathcal{Y}(m)$ holds,

$$\Delta\mathbb{X}(m) \in -(3 \pm \varepsilon) \cdot \tilde{X}(m)\tilde{Y}(m).$$

As before, it follows that

$$\overline{X}(m+1) = \frac{\mathbb{X}(m+1)}{Q(m+1)} \in \left(1 \pm \frac{\tilde{Y}(m)}{\tilde{Q}(m)}\right)\left(\frac{\mathbb{X}(m) - 3 \cdot \tilde{X}(m)\tilde{Y}(m)}{Q(m) - \tilde{Y}(m)}\right),$$

and hence

$$|\Delta\overline{X}(m)| \leqslant \left(1 + \frac{\tilde{Y}(m)}{\tilde{Q}(m)}\right)\left(\frac{\mathbb{X}(m) - 3 \cdot \tilde{X}(m)\tilde{Y}(m)}{Q(m) - \tilde{Y}(m)}\right) - \overline{X}(m) \leqslant \frac{4 \cdot \tilde{X}(m)\tilde{Y}(m)}{\tilde{Q}(m)},$$

as required. As noted above, it follows from Lemma A.5.4 and our bounds on $|\Delta Q(m)|$, $|\Delta \overline{Y}(m)|$ and $|\Delta \overline{X}(m)|$ that

$$|\Delta \overline{X}^*(m)| + |\Delta \overline{Y}^*(m)| + |\Delta Q^*(m)| \leqslant \frac{(\log n)^3}{g_q(t) \cdot n^{3/2}}.$$

Finally, in order to deduce the claimed bounds on $|\Delta \Lambda(m)|$ and $\mathbb{E}\big[|\Delta \Lambda(m)|\big]$, simply note that

$$|\Delta \Lambda(m)| \leqslant 2\Big(\big|\lambda(m) \cdot \Delta\lambda(m)\big| + \big|\mu(m) \cdot \Delta\mu(m)\big| \Big) + |\Delta\lambda(m)|^2 + |\Delta\mu(m)|^2,$$

and that

$$|\Delta\lambda(m)| + |\Delta\mu(m)| = O\big(|\Delta \overline{Y}^*(m)| + |\Delta Q^*(m)|\big) \quad \text{and} \quad |\lambda(m)| + |\mu(m)| = O(1),$$

since the event $\mathcal{Q}(m)$ holds. It follows immediately that

$$|\Delta \Lambda(m)| \leqslant \frac{(\log n)^4}{g_q(t) \cdot n^{3/2}},$$

and therefore that the same bound holds for $\mathbb{E}\big[|\Delta \Lambda(m)|\big]$, as required. □

6. The martingale inequalities

In this section, for completeness, we shall give the proof our main martingale lemma [2, Lemma 3.1]. The proof below is taken from the survey of McDiarmid [3]. Recall that we assume throughout that for each martingale M we consider, $M(m)$ depends only on the graph G_{m+r} for each $m \in [0, s]$, for some (fixed) $r \in \mathbb{N}$ depending on M.

Lemma 3.1 (Theorem 3.15 of [3]). *Let M be a super-martingale, defined on $[0, s]$, such that*

$$|\Delta M(m)| \leqslant \alpha \qquad \text{and} \qquad \mathbb{E}\big[|\Delta M(m)|\big] \leqslant \beta \tag{44}$$

for every $m \in [0, s-1]$. Then, for every $0 \leqslant a \leqslant \beta s$,

$$\mathbb{P}\big(M(s) > M(0) + a\big) \leqslant \exp\left(-\frac{a^2}{4\alpha\beta s} \right).$$

We begin with a straightforward observation.

Lemma A.6.1 (Lemma 2.8 of [3]). *Let X be a random variable with $\mathbb{E}[X] \leqslant 0$ and $X \leqslant b$, and for each $x \in \mathbb{R} \setminus \{0\}$, set*

$$g(x) = \frac{e^x - x - 1}{x^2}.$$

The function g is increasing on $\mathbb{R} \setminus \{0\}$, and

$$\mathbb{E}\big[e^X\big] \leqslant \exp\Big(g(b)\mathbb{E}\big[X^2\big] \Big). \tag{45}$$

Proof. The fact that g is increasing follows from simple calculus. Indeed, for each $x \in \mathbb{R}\backslash\{0\}$,

$$g'(x) \;=\; \frac{(x-2)e^x + x + 2}{x^3} \;\geqslant\; 0$$

since $h(x) = (x-2)e^x + x + 2$ satisfies $h(0) = 0$ and $h'(x) = (x-1)e^x + 1 \geqslant 0$ for every $x \in \mathbb{R}$. To see the latter bound, simply note that $h'(0) = 0$ and $h''(x) = xe^x$.

To deduce (45), simply note that

$$e^x \;=\; 1 + x + x^2 g(x) \;\leqslant\; 1 + x + x^2 g(b)$$

for every $x \leqslant b$ (setting $g(0) = 0$). Since $\mathbb{E}[X] \leqslant 0$ and $X \leqslant b$, it follows immediately that

$$\mathbb{E}\big[e^X\big] \;\leqslant\; 1 + g(b)\mathbb{E}[X^2] \;\leqslant\; \exp\Big(g(b)\mathbb{E}[X^2]\Big),$$

as required. □

We next prove another relatively straightforward preliminary lemma.

Lemma A.6.2 (Lemma 3.16 of [3])**.** *Let M be a super-martingale defined on $[0,s]$, and let $r \in \mathbb{N}$. Suppose that $M(m)$ depends only on G_{m+r} for each $m \in [0,s]$. Then, for any $h \in \mathbb{R}$,*

$$\mathbb{E}\Big[e^{h(M(s)-M(0))} \,\Big|\, G_r\Big] \;\leqslant\; \sup\left(\prod_{m=0}^{s-1} \mathbb{E}\Big[e^{h\Delta M(m)} \,\big|\, G_{m+r}\Big] \,\Big|\, G_r\right). \tag{46}$$

Proof. The proof is by induction on s. When $s = 0$ the result is trivial, since (46) reduces to $1 \leqslant 1$. So let $s \geqslant 1$, and suppose the result holds for $s-1$. Set $A = e^{h(M(s)-M(1))}$ and

$$B \;=\; \prod_{m=1}^{s-1} \mathbb{E}\Big[e^{h\Delta M(m)} \,\big|\, G_{m+r}\Big].$$

By the induction hypothesis, we have $\mathbb{E}\big[A \,|\, G_{r+1}\big] \;\leqslant\; \mathbb{E}\big[B \,|\, G_{r+1}\big]$, and note that trivially $\sup\big(B \,|\, G_r\big) \leqslant \sup\big(B \,|\, G_{r+1}\big)$. It follows that

$$\begin{aligned}
\mathbb{E}\Big[e^{h(M(s)-M(0))} \,\Big|\, G_r\Big] \;&=\; \mathbb{E}\Big[e^{h\Delta M(0)}\mathbb{E}\big[A \,\big|\, G_{r+1}\big] \,\Big|\, G_r\Big] \\
&\leqslant\; \mathbb{E}\Big[e^{h\Delta M(0)} \sup\big(B \,\big|\, G_{r+1}\big) \,\Big|\, G_r\Big] \;\leqslant\; \mathbb{E}\Big[e^{h\Delta M(0)} \sup\big(B \,\big|\, G_r\big) \,\Big|\, G_r\Big] \\
&=\; \sup\big(B \,\big|\, G_r\big)\mathbb{E}\big[e^{h\Delta M(0)} \,\big|\, G_r\big] \;=\; \sup\left(\prod_{m=0}^{s-1} \mathbb{E}\Big[e^{h\Delta M(m)} \,\big|\, G_{m+r}\Big] \,\Big|\, G_r\right),
\end{aligned}$$

as required. □

We shall use one more trivial observation, which is an immediate consequence of [3, Lemma 2.4]. (Here log denotes the natural logarithm.)

Observation A.6.3. *For every $0 \leqslant x \leqslant 1$,*

$$(1+x)\log(1+x) - x \;\geqslant\; \frac{x^2}{4}.$$

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1. Note that $\mathbb{E}\big[\Delta M(m)\big] \leqslant 0$, since M is a super-martingale, and recall that $|\Delta M(m)| \leqslant \alpha$. Applying Lemma A.6.1 to the random variable $X = h \cdot \Delta M(m)$, it follows that

$$\mathbb{E}\Big[e^{h\Delta M(m)} \,\big|\, G_{m+r}\Big] \;\leqslant\; \exp\Big(h^2 g(h\alpha)\mathbb{E}\big[\big(\Delta M(m)\big)^2\big]\Big). \tag{47}$$

Next, note that

$$\mathbb{E}\big[\big(\Delta M(m)\big)^2\big] \;\leqslant\; \alpha\beta, \tag{48}$$

since $|\Delta M(m)| \leqslant \alpha$ and $\mathbb{E}\big[|\Delta M(m)|\big] \leqslant \beta$.

Now, combining Lemma A.6.2 with (47) and (48), we obtain

$$\begin{aligned}
\mathbb{E}\Big[e^{h(M(s)-M(0))} \,\Big|\, G_r\Big] \;&\leqslant\; \sup\left(\prod_{m=0}^{s-1} \mathbb{E}\Big[e^{h\Delta M(m)} \,\big|\, G_{m+r}\Big] \,\Big|\, G_r\right)\\
&\leqslant\; \sup\left(\prod_{m=0}^{s-1} \mathbb{E}\Big[\exp\Big(h^2 g(h\alpha)\mathbb{E}\big[\big(\Delta M(m)\big)^2\big]\Big) \,\Big|\, G_{m+r}\Big] \,\Big|\, G_r\right)\\
&\leqslant\; \exp\Big(h^2 g(h\alpha)\alpha\beta s\Big).
\end{aligned}$$

Thus, for any $h > 0$, by Markov's inequality,

$$\begin{aligned}
\mathbb{P}\big(M(s) > M(0) + a\big) \;&=\; \mathbb{P}\Big(e^{h(M(s)-M(0))} \geqslant e^{ha}\Big)\\
&\leqslant\; e^{-ha}\mathbb{E}\Big[e^{h(M(s)-M(0))} \,\Big|\, G_r\Big] \;\leqslant\; \exp\Big(-ha + h^2 g(h\alpha)\alpha\beta s\Big).
\end{aligned}$$

Setting $h = \frac{1}{\alpha}\log\big(\frac{a+\beta s}{\beta s}\big)$, and noting that

$$\begin{aligned}
-ha + h^2 g(h\alpha)\alpha\beta s \;&=\; -\frac{a}{\alpha}\log\left(\frac{a+\beta s}{\beta s}\right) + \frac{\beta s}{\alpha}\left(\frac{a+\beta s}{\beta s} - \log\left(\frac{a+\beta s}{\beta s}\right) - 1\right)\\
&=\; \frac{\beta s}{\alpha}\left(-\left(1 + \frac{a}{\beta s}\right)\log\left(1 + \frac{a}{\beta s}\right) + \frac{a}{\beta s}\right).
\end{aligned}$$

Finally, applying Observation A.6.3 with $x = a/\beta s$, it follows that

$$\mathbb{P}\big(M(s) > M(0) + a\big) \;\leqslant\; \exp\left(-\frac{\beta s}{\alpha}\cdot\frac{1}{4}\left(\frac{a}{\beta s}\right)^2\right) \;=\; \exp\left(-\frac{a^2}{4\alpha\beta s}\right),$$

since $a \leqslant \beta s$, as required. □

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Gonzalo Fiz Pontiveros, Simon Griffiths, Robert Morris
IMPA, Estrada Dona Castorina 110, Jardim Botânico, Rio de Janeiro, RJ, Brasil
E-mail address: gf232|sgriff|rob@impa.br