BOOTSTRAP PERCOLATION, AND OTHER AUTOMATA

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Abstract. Many fundamental and important questions from statistical physics lead to beautiful problems in extremal and probabilistic combinatorics. One particular example of this phenomenon is the study of bootstrap percolation, which is motivated by a variety of 'real-world' cellular automata, such as the Glauber dynamics of the Ising model of ferromagnetism, and kinetically constrained spin models of the liquid–glass transition.

In this review article, we will describe some dramatic recent developments in the theory of bootstrap percolation (and, more generally, of monotone cellular automata with random initial conditions), and discuss some potential extensions of these methods and results to other automata. In particular, we will state numerous conjectures and open problems.

1. Introduction

Cellular automata are interacting particle systems whose update rules are 'local' and homogeneous. In recent years, a great deal of progress has been made in understanding the behaviour of a particular class of *monotone* cellular automata, commonly known as 'bootstrap percolation'. In particular, if one considers only two-dimensional automata, then we now have a fairly precise understanding of the typical evolution of these processes, starting from random initial conditions. The aim of this article is to describe some of these developments, and discuss a number of open problems and conjectures about related cellular automata. In particular, we will focus our attention on kinetically constrained spin models, the Glauber dynamics of the zero-temperature Ising model, and the abelian sandpile.[1]

Let us begin by defining a large class of d-dimensional monotone cellular automata, which were recently introduced by Bollobás, Smith and Uzzell [13].

Definition 1.1. Let $\mathcal{U} = \{X_1, \ldots, X_m\}$ be an arbitrary finite collection of finite subsets of $\mathbb{Z}^d \setminus \{\mathbf{0}\}$. The *$\mathcal{U}$-bootstrap process* on the d-dimensional torus $\mathbb{Z}_n^d$ is defined as follows: given a set $A \subset \mathbb{Z}_n^d$ of initially *infected* sites, set $A_0 = A$, and define for each $t \geqslant 0$,

$$A_{t+1} = A_t \cup \big\{ v \in \mathbb{Z}_n^d \,:\, v + X \subset A_t \text{ for some } X \in \mathcal{U} \big\}.$$

We write $[A]_{\mathcal{U}} = \bigcup_{t \geqslant 0} A_t$ for the *closure* of A under the $\mathcal{U}$-bootstrap process.

Thus, a vertex v becomes infected at time $t+1$ if the translate by v of one of the sets in $\mathcal{U}$ (which we refer to as the *update family*) is already entirely infected at time t, and infected vertices remain infected forever. For example, if we take $\mathcal{U}$ to be $\mathcal{N}_r^d$, the family of

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[1] We would like to reassure the reader that the definitions of these models are all purely combinatorial, non-technical, and easy to understand, and that no knowledge of physics will be required in this paper.

r-subsets of the $2d$ nearest neighbours in $\mathbb{Z}^d$ of the origin, we obtain the classical r-neighbour bootstrap process, which was first introduced in 1979 by Chalupa, Leath and Reich [18].

We are interested in the typical global behaviour of the $\mathcal{U}$-bootstrap process acting on random initial sets. One of the key insights of Bollobás, Smith and Uzzell [13] was that, at least in two dimensions, this behaviour should be determined by the action of the process on discrete half-planes. To be more precise, for each $u \in S^1$, let $\mathbb{H}_u := \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}$ be the discrete half-plane whose boundary is perpendicular to u. We say that u is a *stable direction* if $[\mathbb{H}_u]_{\mathcal{U}} = \mathbb{H}_u$ and we denote by $\mathcal{S} = \mathcal{S}(\mathcal{U}) \subset S^1$ the collection of stable directions. Let us say that a two-dimensional update family $\mathcal{U}$ is:

- *supercritical* if there exists an open semicircle in S^1 that is disjoint from $\mathcal{S}$,
- *critical* if there exists a semicircle in S^1 that has finite intersection with $\mathcal{S}$, and if every open semicircle in S^1 has non-empty intersection with $\mathcal{S}$,
- *subcritical* if every semicircle in S^1 has infinite intersection with $\mathcal{S}$.

To justify this trichotomy, we need a couple more simple definitions. Let us say that a set $A \subset \mathbb{Z}_n^d$ is *p-random* if each of the vertices of $\mathbb{Z}_n^d$ is included in A independently with probability p, and define the *critical probability* of the $\mathcal{U}$-bootstrap process on $\mathbb{Z}_n^d$ to be

$$p_c(\mathbb{Z}_n^d, \mathcal{U}) := \inf \Big\{ p \,:\, \mathbb{P}_p\big([A]_{\mathcal{U}} = \mathbb{Z}_n^d\big) \geqslant 1/2 \Big\},$$

where $\mathbb{P}_p$ denotes the product probability measure on $\mathbb{Z}_n^d$ with density p.[2] The following theorem was proved by Bollobás, Smith and Uzzell [13] (parts (a) and (b)) and by Balister, Bollobás, Przykucki and Smith [4] (part (c)).

Theorem 1.2. *Let $\mathcal{U}$ be a two-dimensional update family.*

(a) *if $\mathcal{U}$ is supercritical then* $p_c(\mathbb{Z}_n^2, \mathcal{U}) = n^{-\Theta(1)}$.

(b) *if $\mathcal{U}$ is critical then* $p_c(\mathbb{Z}_n^2, \mathcal{U}) = (\log n)^{-\Theta(1)}$.

(c) *if $\mathcal{U}$ is subcritical then* $\liminf_{n\to\infty} p_c(\mathbb{Z}_n^2, \mathcal{U}) > 0$.

It is perhaps difficult to convey to the reader how surprising it is that such a simple and beautiful characterization could be proved in such extraordinary generality. Nevertheless, this is not the end of the story: to state the main result of [11], which determines $p_c(\mathbb{Z}_n^2, \mathcal{U})$ for critical families up to a *constant* factor, we will need a couple more definitions.

Let $\mathbb{Q}_1 \subset S^1$ denote the set of rational directions on the circle, and for each $u \in \mathbb{Q}_1$, let ℓ_u^+ be the (infinite) subset of the line $\ell_u := \{x \in \mathbb{Z}^2 : \langle x, u \rangle = 0\}$ consisting of the origin and the sites to the right of the origin as one looks in the direction of u. Similarly, let $\ell_u^- := (\ell_u \setminus \ell_u^+) \cup \{\mathbf{0}\}$ consist of the origin and the sites to the left of the origin.

Now, given a two-dimensional bootstrap percolation update family $\mathcal{U}$, let $\alpha_{\mathcal{U}}^+(u)$ be the minimum (possibly infinite) cardinality of a set $Z \subset \mathbb{Z}^2$ such that $[\mathbb{H}_u \cup Z]_{\mathcal{U}}$ contains infinitely many sites of ℓ_u^+, and define $\alpha_{\mathcal{U}}^-(u)$ similarly (using ℓ_u^- in place of ℓ_u^+).

[2] Thus a p-random set is one chosen according to the distribution $\mathbb{P}_p$.

Definition 1.3. Given $u \in \mathbb{Q}_1$, the *difficulty* of u (with respect to $\mathcal{U}$) is[3]

$$\alpha(u) := \begin{cases} \min\left\{\alpha^+_{\mathcal{U}}(u), \alpha^-_{\mathcal{U}}(u)\right\} & \text{if } \alpha^+_{\mathcal{U}}(u) < \infty \text{ and } \alpha^-_{\mathcal{U}}(u) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}$ denote the collection of open semicircles of S^1. We define the *difficulty* of $\mathcal{U}$ to be

$$\alpha := \min_{C \in \mathcal{C}} \max_{u \in C} \alpha(u). \tag{1}$$

Roughly speaking, this says that a direction u has finite difficulty if there exists a finite set of sites that, together with the half-plane $\mathbb{H}_u$, infect the entire line ℓ_u. Moreover, the difficulty is at least k if it is necessary (in order to infect ℓ_u) to find at least k infected sites that are 'close' to one another. If the open semicircle centred at u contains no direction of difficulty greater than k, then it is possible for a 'critical droplet' of infected sites to grow in direction u without ever finding more than k infected sites close together.

The final definition we need is as follows.

Definition 1.4. A critical update family $\mathcal{U}$ is *balanced* if there exists a closed semicircle C such that $\alpha(u) \leqslant \alpha$ for all $u \in C$. It is said to be *unbalanced* otherwise.

It turns out that growth under the action of balanced critical families is completely two-dimensional, while that for unbalanced critical families is asymptotically one-dimensional. Despite this fact, it turns out that analyzing the $\mathcal{U}$-bootstrap process when $\mathcal{U}$ is unbalanced is significantly more difficult than when it is balanced. The following theorem was proved by Bollobás, Duminil-Copin, Smith and the author [11].

Theorem 1.5. *Let $\mathcal{U}$ be a critical two-dimensional update family.*

(*a*) *If $\mathcal{U}$ is balanced, then*

$$p_c(\mathbb{Z}_n^2, \mathcal{U}) = \Theta\bigg(\frac{1}{\log n}\bigg)^{1/\alpha}.$$

(*b*) *If $\mathcal{U}$ is unbalanced, then*

$$p_c(\mathbb{Z}_n^2, \mathcal{U}) = \Theta\bigg(\frac{(\log\log n)^2}{\log n}\bigg)^{1/\alpha}.$$

This theorem generalizes a number of earlier results about specific bootstrap processes, and the proof uses some of the techniques developed in those earlier works, especially those of Aizenman and Lebowitz [1] and Holroyd [36], in addition to those introduced by Bollobás, Smith and Uzzell [13]. In order to help the reader to understand Theorem 1.5, we will next briefly discuss a few of these models.

[3]In order to slightly simplify the notation, and since the update family $\mathcal{U}$ will always be clear from the context, we will not emphasize the dependence of the difficulty on $\mathcal{U}$.

1.1. **Two-neighbour bootstrap percolation.** The most extensively-studied bootstrap process in two-dimensions is the so-called two-neighbour model, in which a vertex is infected if at least two of its four nearest neighbours are already infected. As noted above, this corresponds to the update family $\mathcal{N}_2^2$ consisting of all subsets of $\{(1,0),(0,1),(-1,0),(0,-1)\}$ of size 2. The conclusion of Theorem 1.5 in this setting[4] was first proved by Aizenman and Lebowitz [1] in 1988 (and later, independently, by Balogh and Pete [8]), and the following sharp threshold was determined by Holroyd [36] in 2003:

$$p_c\big(\mathbb{Z}_n^2, \mathcal{N}_2^2\big) = \left(\frac{\pi^2}{18} + o(1)\right)\frac{1}{\log n}.$$

The key insight of Aizenman and Lebowitz was that if A percolates, then 'internally spanned droplets' must exist at all scales, and there exists a 'bottleneck scale' at which this is unlikely. The key technical breakthrough of Holroyd was the introduction of so-called 'hierarchies', which correspond (loosely speaking) to 'ways' in which a droplet can be internally spanned. The probability of each individual hierarchy occurring can be bounded, and one can then use the union bound over the (relatively small) set of all hierarchies. Generalizations of both of these ideas play a crucial role the proof of Theorem 1.5.

We remark that certain families of balanced two-dimensional update families were also considered earlier by Gravner and Griffeaths [34] and by Duminil-Copin and Holroyd [22]. In particular, in [22] a sharp threshold was determined for update rules formed by the r-element subsets of a centrally symmetric star-set[5] $N \subset \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, in the cases where such models are critical and balanced. Determining the sharp threshold for critical update families in general is an important (and likely very difficult) open problem.

1.2. **Unbalanced models.** As noted above, the proof of Theorem 1.5 for unbalanced families was significantly more difficult than that for balanced families, and a sharp threshold has been determined for only two[6] specific examples. These are the 'anisotropic' model, with update rule $\mathcal{A}$ consisting of the 3-subsets of $\big\{(-2,0),(-1,0),(0,1),(0,-1),(1,0),(2,0)\big\}$, and the Duarte model, with update rule $\mathcal{D}$ consisting of the 2-subsets of $\big\{(-1,0),(0,1),(0,-1)\big\}$. The conclusion of Theorem 1.5 for these two models was first proved by van Enter and Hulshof [24] and by Mountford [42], respectively. The sharp threshold

$$p_c\big(\mathbb{Z}_n^2, \mathcal{A}\big) = \left(\frac{1}{12} + o(1)\right)\frac{(\log\log n)^2}{\log n}$$

was proved by Duminil-Copin and van Enter [23], and the sharp threshold

$$p_c\big(\mathbb{Z}_n^2, \mathcal{D}\big) = \left(\frac{1}{8} + o(1)\right)\frac{(\log\log n)^2}{\log n} \qquad (2)$$

[4] The reader can easily check that $\alpha = 1$ for $\mathcal{N}_2^2$ and also the update families $\mathcal{A}$ and $\mathcal{D}$ mentioned below, and that $\mathcal{N}_2^2$ is balanced, while $\mathcal{A}$ and $\mathcal{D}$ are unbalanced.

[5] This means that if $x \in N$, then every vertex of $\mathbb{Z}^2$ on the straight line between x and $-x$ is in N.

[6] Not counting a small number of minor variants, for which sharp thresholds can be proved by simple modifications of the proofs in [12, 23].

was proved recently by Bollobás, Duminil-Copin, Smith and the author [12]. The key property that makes these models (and those studied in [22]) easier to deal with is symmetry: there exist four pairwise-opposite stable directions, and (with the exception of the family $\mathcal{D}$) growth is equally difficult in opposite directions. Nevertheless, even in this relatively simple setting, to prove (2) it was necessary to adapt a number of the innovations of [11] to the setting of non-polygonal droplets, a task involving substantial technical challenges.

1.3. **Higher dimensions.** Fix an integer $d \geqslant 2$ and let $\mathcal{U}$ be a d-dimensional update family. Let $\mathbb{H}_u^d := \{ x \in \mathbb{Z}^d : \langle x, u \rangle < 0 \}$ denote the discrete half-space with normal $u \in S^{d-1}$, define the stable set $\mathcal{S} = \mathcal{S}(\mathcal{U})$ to be the set of $u \in S^{d-1}$ such that $[\mathbb{H}_u^d]_{\mathcal{U}} = \mathbb{H}_u^d$, and let μ denote Lebesgue measure. We say that a d-dimensional update family is:

- *supercritical* if there exists an open hemisphere in S^{d-1} that is disjoint from $\mathcal{S}$,
- *critical* if there exists a hemisphere in S^{d-1} such that $\mu(C \cap \mathcal{S}) = 0$, and every open hemisphere in S^{d-1} has non-empty intersection with $\mathcal{S}$,
- *subcritical* if $\mu(C \cap \mathcal{S}) > 0$ for every hemisphere $C \subset S^{d-1}$.

Note that, as in two dimensions, the subcritical/critical/supercritical trichotomy depends only on the stable set $\mathcal{S}$. The following conjecture was made in [11].

Conjecture 1.6. *Let $\mathcal{U}$ be a d-dimensional update family.*

(a) *If $\mathcal{U}$ is supercritical then $p_c(\mathbb{Z}_n^d, \mathcal{U}) = n^{-\Theta(1)}$.*

(b) *If $\mathcal{U}$ is critical then there exists $r = r(\mathcal{U}) \in \{2, \dots, d\}$ such that*

$$p_c(\mathbb{Z}_n^d, \mathcal{U}) = \left(\frac{1}{\log_{(r-1)} n} \right)^{\Theta(1)}, \tag{3}$$

where $\log_{(r-1)}$ denotes an $(r-1)$-times iterated logarithm.

(c) *If $\mathcal{U}$ is subcritical then $\liminf_{n \to \infty} p_c(\mathbb{Z}_n^d, \mathcal{U}) > 0$.*

If $\mathcal{U}$ satisfies (3) (i.e., if $r(\mathcal{U}) = r$) then we will say that $\mathcal{U}$ is *r-critical.* For the r-nearest neighbour model $\mathcal{N}_r^d$, this conjecture was proved[7] (with $r(\mathcal{N}_r^d) = r$) in the case $d = r = 3$ by Cerf and Cirillo [16], and for all $d \geqslant r \geqslant 3$ by Cerf and Manzo [17]. The sharp threshold in these cases was determined by Balogh, Bollobás, Duminil-Copin and the author [5, 7].

1.4. **The structure of the paper.** In the sections below we will introduce or recall variants of the $\mathcal{U}$-bootstrap process in the context of some cellular automata that have been well-studied by the mathematical physics and probability theory communities. More precisely, in Section 2 we will discuss kinetically constrained spin models, in Section 3 we will consider the zero-temperature Glauber dynamics of the Ising model, and in Section 4 we will discuss the abelian sandpile. We will state several open problems and conjectures about each of these models; these problem vary widely in difficulty, and we have attempted in the text to guide the reader towards those that seem most vulnerable to attack. We strongly believe that ideas and techniques from extremal and probabilistic combinatorics have an important role to play in the continuing development of our understanding of cellular automata.

[7] More precisely, the papers [16, 17] determined $p_c(\mathbb{Z}_n^d, \mathcal{N}_r^d)$ up to a constant factor.

2. Kinetically constrained spin models

In this section we will discuss, from a combinatorial perspective, a model of the liquid-glass transition that has been extensively studied in the statistical physics literature, see for example [33, 44]. The model is simply a biased random walk on the family of percolating sets[8] in the $\mathcal{U}$-bootstrap process; the following definition is a special case of an even more general class of kinetically constrained spin models introduced in [15].

Definition 2.1. Let $\mathcal{U}$ be a finite collection of finite subsets of $\mathbb{Z}^d \setminus \{\mathbf{0}\}$, and let $p \in (0,1)$. The *$\mathcal{U}$-kinetically constrained spin model* on $\mathbb{Z}^d$ with density p is defined as follows:

- (a) Each vertex has an independent exponential clock which rings randomly at rate 1.
- (b) If the clock at vertex v rings at (continuous) time $t \geqslant 0$, and

$$v + X \subset A_t \text{ for some } X \in \mathcal{U},$$

  where $A_t \subset \mathbb{Z}^d$ is the set of infected vertices at time t, then v becomes infected with probability p, and healthy with probability $1-p$, independently of all other events.

In other words, the state (infected or healthy) of a vertex is resampled (independently) at rate 1 if it could be infected in the next step of the $\mathcal{U}$-bootstrap process, and remains in its current state otherwise. The infected sites at time zero are normally chosen to be p-random, in which case it follows easily that the distribution is also given by $\mathbb{P}_p$ at every later time t (though, of course, the distributions at different times are not independent of one another). In order to avoid additional technical definitions, we will focus on the following simple and natural parameter of the model:

$$\tau \,=\, \tau\big(\mathbb{Z}^d,\mathcal{U}\big) \,:=\, \inf\big\{t \geqslant 0 : \mathbf{0} \in A_t\big\},$$

that is, the time at which the origin is first infected. Our aim is to understand the rate at which it grows (either in expected value, or with high probability) as $p \to 0$.

2.1. A lower bound on τ via the $\mathcal{U}$-bootstrap process.

Let us construct a collection B of vertices that are infected at time zero, and are used to infect the origin. We do so recursively[9] by choosing a set $X \in \mathcal{U}$ such that $X \subset A_\tau$, and taking B to be the union of sets constructed in a similar way for each $v \in X$, each of which was first infected at some strictly earlier time. It follows from Definitions 1.1 and 2.1 that $\mathbf{0} \in [B]_{\mathcal{U}}$, since only sites that would have been infected in the $\mathcal{U}$-bootstrap process can be infected in the $\mathcal{U}$-kinetically constrained spin model. However, in order for a vertex v at distance k from the origin to be in B, there must have been a sequence of at least $\Omega(k)$ clock-rings which transferred the information that v was initially infected to the origin. A simple first moment calculation shows that this is likely to take time at least $\Omega(k)$, and in particular we can assume that B has diameter $O(\tau)$. This argument, which is due to Cancrini, Martinelli, Roberto and Toninelli [15], gives the following corollary to (the proof of) Theorem 1.5.

[8]To be precise, this holds as long as the initial set of infected sites percolates. Note that if the infected sites at time zero are chosen to be p-random, then this holds almost surely if $p_c(\mathbb{Z}^d,\mathcal{U}) = 0$. Note also that if Conjecture 1.6 is correct, then this holds if and only if $\mathcal{U}$ is not subcritical.

[9]If the origin is infected at time $t = 0$ then we set $B = \{\mathbf{0}\}$.

Corollary 2.2. *Let $\mathcal{U}$ be a critical two-dimensional bootstrap percolation update family. There exists a constant $c = c(\mathcal{U}) > 0$ such that, with high probability as $p \to 0$,*

(*a*) *if $\mathcal{U}$ is balanced, then*

$$\tau\big(\mathbb{Z}^2,\mathcal{U}\big) \geqslant \exp\Big(cp^{-\alpha}\Big).$$

(*b*) *if $\mathcal{U}$ is unbalanced, then*

$$\tau\big(\mathbb{Z}^2,\mathcal{U}\big) \geqslant \exp\Big(cp^{-\alpha}\big(\log(1/p)\big)^2\Big).$$

Cancrini, Martinelli, Roberto and Toninelli [15] also introduced a powerful and general analytic technique that allows one to prove upper bounds on $\tau\big(\mathbb{Z}^d,\mathcal{U}\big)$ for many specific update families. More recently, Martinelli and Toninelli [39] have refined this method to obtain much more precise bounds for the 2-neighbour model $\mathcal{N}_2^2$, and there is some hope that their ideas, combined with the techniques introduced in [11, 13], might be sufficient to prove an almost-matching upper bound for a large collection of update families.

Recall from Definition 1.3 the definitions of $\alpha(u)$ and $\alpha = \alpha(\mathcal{U})$, the difficulty of a direction, and an update family $\mathcal{U}$, respectively. We believe that the following definition will prove to be important in understanding the behaviour of $\tau\big(\mathbb{Z}^2,\mathcal{U}\big)$ for general critical update families.

Definition 2.3. A critical two-dimensional update family $\mathcal{U}$ is *α-rooted* if there are exist two non-opposite stable directions in S^1, each with difficulty strictly greater than $\alpha = \alpha(\mathcal{U})$. Otherwise, we will say that $\mathcal{U}$ is *α-unrooted*.

With Martinelli and Toninelli, we make the following conjecture.

Conjecture 2.4. *Let $\mathcal{U}$ be an α-unrooted critical two-dimensional update family. Then*

$$\tau\big(\mathbb{Z}^2,\mathcal{U}\big) = \exp\Big(p^{-\alpha}\big(\log(1/p)\big)^{O(1)}\Big) \tag{4}$$

with high probability as $p \to 0$.

The idea behind Conjecture 2.4 is that if $\mathcal{U}$ is α-unrooted then a 'critical droplet' of size $p^{-\alpha}\big(\log(1/p)\big)^{O(1)}$ will drift randomly until it arrives at the origin. On the other hand, we do not expect (4) to hold when $\mathcal{U}$ is α-rooted, since then a droplet of this size can typically only move in one direction. We are not sure what to expect in general, and as a first step it would be interesting to resolve the problem in some special cases, such as the Duarte model.

Conjecture 2.5. *Let $\mathcal{D}$ consist of the 2-subsets of $\big\{(-1,0),(0,1),(0,-1)\big\}$. Then*

$$\tau\big(\mathbb{Z}^2,\mathcal{D}\big) = \exp\Big(p^{-2}\big(\log(1/p)\big)^{O(1)}\Big),$$

with high probability as $p \to 0$.

We expect that the rate at which the $\mathcal{D}$-kinetically constrained spin model relaxes is determined by an 'energy barrier', as in the East model, see the discussion below. Determining the limit of $\log\log\tau\big(\mathbb{Z}^2,\mathcal{U}\big)/\log p$ as $p \to 0$ for general critical two-dimensional update families seems to be an extremely challenging and important open problem.

2.2. **Supercritical models.** One supercritical model that has been extensively studied and whose behaviour is understood in great detail is the so-called 'East model' (see e.g. [2, 20, 26, 32]), which is given by the update family $\mathcal{E}_d = \big\{\{-e_1\}, \dots, \{-e_d\}\big\}$, where e_i denotes the neighbour of the origin in direction i. For example, it was proved in [15, 19] that

$$\tau\big(\mathbb{Z}^d, \mathcal{E}_d\big) = \exp\left(\frac{1+o(1)}{2d\log 2}\left(\log\frac{1}{p}\right)^2\right)$$

with high probability as $p \to 0$. In order to gain some intuition for why $\tau(\mathbb{Z}^d, \mathcal{E}_d)$ grows at rate $p^{-\Theta(\log(1/p))}$, let us consider the case $d = 1$, and assume that $-n$ is the nearest infected site to the left of the origin at time zero, where $n = \Theta(1/p)$. Chung, Diaconis and Graham [20] proved two combinatorial lemmas about this setting, which imply that there exists a collection $\mathcal{A}(k)$ of $e^{O(k)}2^{\binom{k}{2}}k!$ subsets of $[-n, 0]$ of size $k = \lceil \log_2 n \rceil$, at least one of which must be entirely infected at some time $t \leqslant \tau(\mathbb{Z}, \mathcal{E}_1)$. Since, as noted above, the distribution of infected sites at time t is given by $\mathbb{P}_p$ for every $t \geqslant 0$, it follows easily by Markov's inequality that $\tau(\mathbb{Z}, \mathcal{E}_1) \geqslant p^{-\Omega(\log(1/p))}$.

The reader should think of the set $\mathcal{A}(k)$ as forming an 'energy barrier' that must be crossed by the process in order to reach a configuration in which the origin is infected. As noted above, we expect some α-rooted critical update families (such as the Duarte model) to exhibit similar behaviour, and combinatorial arguments analogous to those used in [20] are likely to play an important role in their analysis.

It would be very interesting to obtain similar results for general supercritical update families; analogously to Definition 2.3 we suggest the following partition.

Definition 2.6. A supercritical two-dimensional update family $\mathcal{U}$ is *rooted* if there are exist two non-opposite stable directions in S^1, and otherwise it is *unrooted*.

We propose the following conjecture.

Conjecture 2.7. *Let $\mathcal{U}$ be a supercritical two-dimensional update family. Then, with high probability as $p \to 0$,*

(*a*) *if $\mathcal{U}$ is unrooted, then*

$$\tau\big(\mathbb{Z}^2, \mathcal{U}\big) = p^{-\Theta(1)}.$$

(*b*) *if $\mathcal{U}$ is rooted, then*

$$\tau\big(\mathbb{Z}^2, \mathcal{U}\big) = p^{-\Theta(\log(1/p))}.$$

The idea behind this conjecture is that if $\mathcal{U}$ is rooted then the two non-opposite stable directions restrict the growth of a droplet to a cone, which it cannot escape until it is 'saved' by the growth of another droplet. We expect that, for such an update family, the behaviour of the $\mathcal{U}$-kinetically constrained spin model should thus resemble (in broad outline) that of the East model in two dimensions. On the other hand, if there are two disjoint open semicircles that contain no stable directions, then a droplet can drift randomly in both directions along a 1-dimensional line. Similarly, in d dimensions one might expect $\tau\big(\mathbb{Z}^d, \mathcal{U}\big)$ to be polynomial if and only if there exist two disjoint open hemispheres that contain no stable directions.

3. The Glauber dynamics of the zero-temperature Ising model

In this section we will discuss another celebrated cellular automaton, the stochastic Ising model. This is an extremely well-studied model of the dynamics of the ions in a ferromagnet, and provided the original motivation for the introduction of bootstrap percolation in [18]. In this model, unlike in the previous sections, we have symmetry between the two states, and to emphasize this we will denote them by $+$ and $-$.

Definition 3.1. Let $\mathcal{U}$ be an arbitrary finite collection of finite subsets of $\mathbb{Z}^d \setminus \{\mathbf{0}\}$. The *$\mathcal{U}$-Ising dynamics* on $\mathbb{Z}^d$ are defined as follows:

(a) Each vertex has two independent exponential clocks which ring randomly at rate 1.
(b) If the $+$ clock at vertex v rings at (continuous) time $t \geqslant 0$, and there exists $X \in \mathcal{U}$ such that the set $v + X$ is entirely in state $+$, then the state of v becomes $+$.
(c) If the $-$ clock at vertex v rings at (continuous) time $t \geqslant 0$, and there exists $X \in \mathcal{U}$ such that the set $v + X$ is entirely in state $-$, then the state of v becomes $-$.

For example, if $\mathcal{U} = \mathcal{N}_d^d$ is the d-nearest neighbour model, then the state of a vertex updates (at rate 1) to agree with the majority of its neighbours, and switches randomly at rate 1 if it has an equal number of neighbours in each state. This process is usually referred to as the zero-temperature Glauber dynamics of the Ising model, and has been extensively studied for many years (see e.g. [38] and the references therein).

Let us say that the $\mathcal{U}$-Ising dynamics *fixate* at $+$ if the state of each vertex is eventually[10] $+$. As usual, let the initial states be chosen p-randomly (that is, independently at random, with probability p of being $+$). Define $p_c^{\mathrm{Is}}\big(\mathbb{Z}^d,\mathcal{U}\big)$, the critical probability of the $\mathcal{U}$-Ising dynamics on $\mathbb{Z}^d$, to be the infimum over p such that the $\mathcal{U}$-Ising dynamics almost surely fixate at $+$.

Arratia [3] proved[11] over 30 years ago that $p_c^{\mathrm{Is}}\big(\mathbb{Z},\mathcal{N}_1^1\big) = 1$, and a well-known conjecture states that $p_c^{\mathrm{Is}}\big(\mathbb{Z}^d,\mathcal{N}_d^d\big) = 1/2$ for every $d \geqslant 2$. Motivated by this conjecture, the following problem seems to us to be the central question about the $\mathcal{U}$-Ising dynamics on $\mathbb{Z}^d$.

Problem 3.2. *Characterize the d-dimensional update families $\mathcal{U}$ such that*

$$p_c^{\mathrm{Is}}\big(\mathbb{Z}^d,\mathcal{U}\big) = \frac{1}{2}.$$

For critical update families, we conjecture that the following upper bound always holds.

Conjecture 3.3. *If $\mathcal{U}$ is a critical d-dimensional bootstrap percolation update family, then*

$$p_c^{\mathrm{Is}}\big(\mathbb{Z}^d,\mathcal{U}\big) < 1.$$

This was proved for $\mathcal{U} = \mathcal{N}_d^d$ by Fontes, Schonmann and Sidoravicius [31]. It seems plausible that Conjecture 3.3 could be proved by combining the techniques of [31] and [11,

[10] Note that this does not imply that there exists a finite time after which every vertex is in state $+$, but rather that for each vertex v there exists a time, depending on v, after which the state of v is $+$.

[11] More precisely, this is an easy consequence of the main result of [3], which is a generalization of a conjecture of Erdős and Ney [25] about annihilating random walks on $\mathbb{Z}$.

13]. We expect that $p_c^{\rm Is}\big(\mathbb{Z}^d,\mathcal{U}\big) = 1$ whenever $\mathcal{U}$ is subcritical, but it seems likely that there exist supercritical families with $p_c^{\rm Is}\big(\mathbb{Z}^d,\mathcal{U}\big) < 1$ whenever $d \geqslant 2$ (see the discussion after Conjecture 3.7, below). For $d = 1$, on the other hand, we suspect that the following conjecture could be proved using the theorem of Arratia [3] mentioned above.

Conjecture 3.4. *If $\mathcal{U}$ is a 1-dimensional bootstrap percolation update family, then*

$$p_c^{\rm Is}\big(\mathbb{Z},\mathcal{U}\big) = 1.$$

To see why this conjecture should follow from Arratia's theorem, note first that when $d = 1$ every update family $\mathcal{U}$ is either supercritical or subcritical. Let $p < 1$ and couple the (p-random) initial state σ with one in which all sites are $+$ except for sufficiently long $-$ intervals in σ. If $\mathcal{U}$ is supercritical then one might expect the endpoints of these intervals to perform annihilating random walks as in [3], which suggests that each vertex almost surely spends an infinite amount of time in state $-$, even in the coupled process. On the other hand, if $\mathcal{U}$ is subcritical then a long interval of $+$s (or $-$s) cannot be invaded from outside, so the system almost surely fixates with both states present.

Understanding for which families this holds in higher dimensions is an important open problem, and (one hopes) is likely to be significantly easier than Problem 3.2.

Problem 3.5. *Characterize the d-dimensional update families $\mathcal{U}$ such that*

$$p_c^{\rm Is}\big(\mathbb{Z}^d,\mathcal{U}\big) = 1.$$

Another interesting open question is to understand the behaviour of the $\mathcal{U}$-Ising dynamics for the r-nearest neighbour model, i.e., for the update family $\mathcal{U} = \mathcal{N}_r^d$. We make the following conjecture in high dimensions.

Conjecture 3.6. *Let $r = r(d) \in \mathbb{N}$ be such that $r \leqslant d/2$. Then*

$$p_c^{\rm Is}\big(\mathbb{Z}^d,\mathcal{N}_r^d\big) + \frac{r}{2d} \to 1 \qquad (5)$$

as $d \to \infty$.

The corresponding statement for $r = d$ was proved by the author [41], using techniques from bootstrap percolation [6] together with the results of [31], and the techniques of [41] could potentially be extended to prove the upper bound in (5) for all $r \leqslant d$. We expect a matching lower bound to hold when $r \leqslant d/2$, since in that case the corners of a cube that is entirely $+$ are vulnerable to attack from neighbours outside with (uniform) random states. For larger values of r it is not clear (at least to us) whether or not such a droplet would be likely to grow or shrink: in the latter case (5) should still hold; otherwise one might instead expect that $p_c^{\rm Is}\big(\mathbb{Z}^d,\mathcal{N}_r^d\big) = 1/2$.

The technique of [41] does not seem flexible enough to deal with general update families; in particular, the proof strongly uses the fact that at time zero, a typical vertex is vulnerable to their $+$ clock ringing, but not to their $-$ clock. On the other hand, it might be possible to adapt them to prove the following conjecture, which would imply that Conjecture 3.3 is (in a certain sense) best possible. Recall from Section 1.3 that we say that a d-dimensional update family $\mathcal{U}$ is r-critical if $p_c(\mathbb{Z}_n^d,\mathcal{U}) = \big(\log_{(r-1)} n\big)^{-\Theta(1)}$.

Conjecture 3.7. *For every $d \geqslant r \geqslant 2$, $1/2 \leqslant \alpha \leqslant 1$ and $\varepsilon > 0$, there exists an r-critical d-dimensional update family $\mathcal{U}$ such that*

$$\alpha - \varepsilon \leqslant p_c^{\mathrm{Is}}\big(\mathbb{Z}^d, \mathcal{U}\big) \leqslant \alpha + \varepsilon.$$

In other words, the critical probabilities of r-critical update families are dense in $[1/2, 1]$.

One possible way to construct a family $\mathcal{U}$ with $p_c^{\mathrm{Is}}\big(\mathbb{Z}^d, \mathcal{U}\big) \approx \alpha \in (1/2, 1)$ might be to choose k disjoint sets of size s, where $(1-\alpha)^s k \approx 1$ and $k, s \gg 1$. If $p = \alpha + \varepsilon$ then a typical vertex will only be vulnerable to a ring of its + clock, whereas if $p = \alpha - \varepsilon$ then it will be vulnerable to either clock. Moreover, a large 'droplet' of vertices in state + will be vulnerable to attack from vertices with (uniformly) random states outside. It seems likely that a similar construction could give a supercritical update family with $p_c^{\mathrm{Is}}\big(\mathbb{Z}^d, \mathcal{U}\big) \approx \alpha$.

Finally, we make the following conjecture about the symmetric case, $p = 1/2$, in which the initial states are chosen independently and uniformly at random.

Conjecture 3.8. *If $\mathcal{U}$ is a critical two-dimensional update family and $p = 1/2$, then almost surely every vertex changes state an infinite number of times.*

For $\mathcal{U} = \mathcal{N}_2^2$ this was proved by Nanda, Newman and Stein [43], but determining whether or not it holds in higher dimensions (i.e., for $\mathcal{U} = \mathcal{N}_d^d$) is a well-known open problem. In particular, we expect the following problem to be hard.

Problem 3.9. *Characterize the d-dimensional bootstrap percolation update families such that if $p = 1/2$, then almost surely every vertex changes state an infinite number of times.*

Finally, let us note that there are a number of interesting and natural variants of the $\mathcal{U}$-Ising dynamics, some of which may also be amenable to techniques from bootstrap percolation. For example, one could define the *$\mathcal{U}$-voter dynamics* as follows:

(a) Each vertex has an independent exponential clock which rings randomly at rate 1.
(b) When the clock at vertex v rings at (continuous) time $t \geqslant 0$, the vertex chooses a set $X \in \mathcal{U}$ uniformly at random. If all sites in $v + X$ agree, then the state of v updates to agree with them, otherwise it does not change.

In this model there is an extra bias in favour of the dominant state, and as a result for some of the examples mentioned above the behaviour of the dynamics is likely to be very different. In particular, it seems quite possible that the critical probability could be 1/2 in this model for *every* critical update family.

Another (less dramatic) way to change the dynamics would be to give the + and − clocks different rates; for example, the + clock could ring at rate p, and the − clock at rate $1-p$. This would likely change the critical probability in many cases: for example, for the r-nearest neighbour rule $\mathcal{N}_r^d$ one might expect the critical probability to decrease from $1 - r/2d + o(1)$ to $1 - r/d + o(1)$, which would now be the point at which a droplet becomes stable.

4. The abelian sandpile

In this final section, we will discuss a deterministic cellular automaton which was first introduced almost 30 years ago by Bak, Tang and Wiesenfeld [10] and has since attracted a great deal of attention from both mathematicians [14, 27, 29, 37] and physicists [9, 48].

Definition 4.1. Given a finite set $U \subset \mathbb{Z}^d \setminus \{\mathbf{0}\}$ and a function $A\colon \mathbb{Z}^d \to \mathbb{Z}$, the *$U$-abelian sandpile* on $\mathbb{Z}^d$ with initial state A is defined as follows:

(a) At time zero, place $A(v)$ balls at vertex v for each $v \in \mathbb{Z}^d$.

(b) For each (integer) $t \geqslant 0$, do the following: for each $v \in \mathbb{Z}^d$ with at least $|U|$ balls at time t, remove $|U|$ balls from v and place one at each element of $v + U$.

When v sends $|U|$ balls to $v + U$, we say that it 'topples'. The property we are interested in understanding is as follows: given a random starting configuration, does the U-abelian sandpile stop after some finite time, or does it continue toppling forever? More precisely, if Y is an integer-valued random variable, then let us say that A is *Y-random* if each $A(v)$ is an independent copy of Y. The following problem seems to be extremely difficult.

Problem 4.2. *Determine, for each $d \in \mathbb{N}$ and each finite set $U \subset \mathbb{Z}^d \setminus \{\mathbf{0}\}$, the collection of random variables Y with the following property: in the U-abelian sandpile on $\mathbb{Z}^d$ with a Y-random initial state, almost surely every site topples infinitely many times.*

This problem was introduced by Dickman, Muñoz, Vespagnani and Zapperi [21] and has subsequently been investigated in, for example, [27, 29, 30, 40]. In particular, it follows easily from the techniques developed in those papers that:

(a) if $\mathbb{E}[Y] > |U| - 1$, then almost surely every site topples infinitely many times,

whereas

(b) if $\mathbb{E}[Y] < |U|/2$ and the set U is centrally symmetric (i.e., $U = -U$), then almost surely every site topples only a finite number of times.[12]

Moreover, the deterministic random variable $Y = |U| - 1$ shows that the first bound is sharp and, assuming Conjecture 1.6, the second bound is also sharp, cf. [27, Proposition 1.4]. Indeed, this is an easy consequence of the following observation.

Observation 4.3. *Let $U \subset \mathbb{Z}^d \setminus \{\mathbf{0}\}$ be a finite, centrally symmetric set, and let $\mathcal{U}$ be the collection of subsets of U of size $|U|/2$. Then $\mathcal{U}$ is not a subcritical update family.*

Proof. Note that a direction is stable if, and only if, it is perpendicular to an element of U. This implies immediately that the set of stable directions $\mathcal{S}(\mathcal{U})$ has measure zero. □

Now, given a finite, centrally symmetric set $U \subset \mathbb{Z}^d \setminus \{\mathbf{0}\}$, and $\varepsilon > 0$, define a random variable Y by

$$\mathbb{P}\big(Y = |U|/2\big) = 1 - \varepsilon \qquad \text{and} \qquad \mathbb{P}\big(Y = |U|\big) = \varepsilon.$$

[12] Note that the condition $U = -U$ allows one to define a graph G with edge set $\{uv : u - v \in U\}$. One can now prove the latter statement by considering the last time a ball passed along each edge of G.

We claim that in the U-abelian sandpile on $\mathbb{Z}^d$ with a Y-random initial state, almost surely every site topples infinitely many times. To see this, simply note that a site v topples at least once if at least $|U|/2$ elements of $v+U$ topple. Since the collection $\mathcal{U}$ of subsets of U of size $|U|/2$ is either a critical or a supercritical update family, by Observation 4.3, it follows from Conjecture 1.6 that $p_c(\mathbb{Z}_n^d,\mathcal{U}) = o(1)$, which implies that almost surely every site topples at least once. But if every site topples at least once then every site topples infinitely many times, as claimed. We remark that [27] was (as far as we are aware) the first paper to explore in detail the connection between the abelian sandpile and bootstrap percolation.

We remark that for non-symmetric sets U, we are not aware of *any* non-trivial analogue of statement (b), above. More precisely, it seems likely that for every set U there exists $\rho = \rho(U) > 0$ such that if $\mathbb{E}[Y] < \rho$ then almost surely every site topples only a finite number of times, but we do not know how to prove this.

4.1. **Poisson initial conditions.** In order to simplify Problem 4.2 a little, let us restrict to the case in which Y is a Poisson random variable, and U is the set of $2d$ nearest neighbours of the origin in $\mathbb{Z}^d$. Let us say that A is λ-Poisson if each $A(v)$ is chosen according to an independent Poisson random variable of mean λ, and (abusing notation slightly) write $\mathbb{P}_\lambda$ to denote the corresponding probability measure. Let us also say that A *percolates on* $\mathbb{Z}^d$ if every site topples infinitely many times in the nearest-neighbour abelian sandpile on $\mathbb{Z}^d$ starting from A, and define

$$\lambda_c^S(\mathbb{Z}^d) := \inf\Big\{\lambda \,:\, \mathbb{P}_\lambda\big(A \text{ percolates on } \mathbb{Z}^d\big) = 1\Big\}.$$

The following problem was introduced almost 20 years ago by Dickman, Muñoz, Vespagnani and Zapperi [21, 48] (see [28]), but is still wide open for all $d \geqslant 2$.

Problem 4.4. *Determine* $\lambda_c^S(\mathbb{Z}^d)$ *for every* $d \in \mathbb{N}$.

Note that, by the discussion above, we have $d \leqslant \lambda_c^S(\mathbb{Z}^d) \leqslant 2d-1$ for every $d \in \mathbb{N}$, and extensive simulations for $d=2$ (see [28]) indicate that $\lambda_c^S(\mathbb{Z}^2)$ is a little larger than 17/8. We make the following conjecture.

Conjecture 4.5.

$$\frac{\lambda_c^S(\mathbb{Z}^d)}{d} \to 1$$

as $d \to \infty$.

4.2. **Polluted bootstrap percolation.** Conjecture 4.5 is motivated by another conjecture, on the critical probability of so-called 'polluted' bootstrap percolation. Let us write $\mathbb{Z}^d(q)$ for the random induced subgraph of $\mathbb{Z}^d$ obtained by removing each vertex independently with probability q, and define

$$p_c^\infty\big(\mathbb{Z}^d(q),\mathcal{U}\big) := \inf\bigg\{p \,:\, \mathbb{P}_p\Big(\text{the closure of } A \text{ under the } \mathcal{U}\text{-bootstrap process on } \mathbb{Z}^d(q) \text{ contains an infinite component}\Big) \geqslant 1/2\bigg\}$$

for each d-dimensional bootstrap percolation update family $\mathcal{U}$.

Conjecture 4.6. *For each $d > r \geqslant 1$, there exists $q_0(d,r) > 0$ such that*

$$p_c^\infty\big(\mathbb{Z}^d(q), \mathcal{N}_r^d\big) = 0$$

almost surely for every $0 < q < q_0(d,r)$.

Note that when $r = 1$ we can take $q_0(d,1) = 1 - p_c^{\text{site}}(\mathbb{Z}^d)$, but the conjecture is open (and seems to be very difficult) for every $d > r \geqslant 2$. We remark that $p_c^\infty\big(\mathbb{Z}^d(q), \mathcal{N}_d^d\big) > 0$ for every $d \in \mathbb{N}$ and $q > 0$, and it was proved by Gravner and McDonald [35] that moreover

$$p_c^\infty\big(\mathbb{Z}^2(q), \mathcal{N}_2^2\big) = \Theta\big(\sqrt{q}\big).$$

The connection between the previous two conjectures is as follows. Let us label a vertex v as 'infected' if $A(v) \geqslant 2d$, as 'vulnerable' if $d < A(v) < 2d$, and as 'removed' if $A(v) \leqslant d$. Let $\lambda = (1+\varepsilon)d$, and set $p = \mathbb{P}_\lambda\big(A(v) \geqslant 2d\big)$ and $q = \mathbb{P}_\lambda\big(A(v) \leqslant d\big)$. Then $q \to 0$ as $d \to \infty$ and $p > 0$, so Conjecture 4.6 (almost[13]) implies that an infinite component of sites topple at least once. This is not quite sufficient to deduce Conjecture 4.5, but nevertheless it seems likely that a proof of Conjecture 4.6 could be adapted to the setting of the sandpile.

4.3. **The stochastic sandpile.** Finally, let us mention a closely-related model, the so-called *stochastic sandpile*, which has also received a great deal of attention. When a vertex v topples in this model, instead of sending one ball to each element of $v + U$, each of the $|U|$ balls goes to an independently (and uniformly) chosen random element of $v + U$. For this model it was only recently proved, by Rolla and Sidoravicius [45] (on $\mathbb{Z}$) and by Sidoravicius and Teixeira [46] (on $\mathbb{Z}^d$) that the critical probability for the $2d$ nearest-neighbours model is non-zero.[14] It would be very interesting to extend their results to more general sets U, and to determine more precisely the behaviour of the critical probability in high dimensions.

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