MEAN FIELD CONDITIONS FOR COALESCING RANDOM WALKS

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WHAT IS THE VOTER MODEL?
WHAT IS THE VOTER MODEL?

Arrival process (Poisson)
WHAT IS THE VOTER MODEL?
WHAT IS THE VOTER MODEL?

Neighbor chosen at random
WHAT IS THE VOTER MODEL?
The original motivation for studying coalescing random walks is the voter model.
DUALITY
DUALITY

[Diagram showing a series of points connected by arrows, possibly illustrating a concept of duality over time.]
DUALITY
DUALITY
Duality

The upshot is that the voter model is dual to

Coalescing random walks,

the main subject of this talk. Will discuss:

C:= full coalescence time.

Results extend to voters with i.i.d. initial opinions.
The case of the complete graph is easy. Other cases turn out to be similar.
THE COMPLETE GRAPH

System of $n$ coalescing random walks on a complete graph $K_n$. (ie. all transition rates equal to 1).
THE COMPLETE GRAPH

System of $n$ coalescing random walks on a complete graph $K_n$.

(ie. all transition rates equal to 2).

Time to move from $k$ to $k-1$ particles:

$$P(Z_k \geq t) = e^{-\left(\frac{k}{2}\right) t}$$
THE COMPLETE GRAPH

System of $n$ coalescing random walks on a complete graph $K_n$.

(i.e. all transition rates equal to 2).

Time to move from $k$ to $k-1$ particles:

$$P(Z_k \geq t) = e^{-\left(\frac{k}{2}\right)t}$$

i.e. exponential with mean $\left(\frac{k}{2}\right)^{-1}$. 
THE COMPLETE GRAPH

System of $n$ coalescing random walks on a complete graph $K_n$.

(i.e. all transition rates equal to 2).

$$C = \sum_{k=2}^{n} \mathbb{E}_k$$

$\mathbb{E}_k$ independent, $\mathbb{E}_k = d \exp \left( \frac{1}{\sqrt{k}} \right)$. 
THE COMPLETE GRAPH

System of $n$ coalescing random walks on a complete graph $K_n$. (ie. all transition rates equal to 2).

$$C \sim \sum_{k=2}^{\infty} z_k$$

$$E(C) \sim 2 = 2 \ E(\text{Meet of 2})$$
A RESULT BY COX

**Cox’91**: Consider CRW based on simple random walk in \((\mathbb{Z}_n)^d\)

where \(d\) is at least 2. Then:
A RESULT BY COX

**Cox'91:** Consider CRW based on simple random walk in 

$$(Z_n)^d$$

where $d$ is at least 2. Then:

$$\sum_{k \geq 2} z_k$$
A RESULT BY COX

**Cox’91**: Consider CRW based on simple random walk in \((Z_n)^d\) where \(d\) is at least 2. Then:

\[
\frac{C}{M_{Z_n^d}} \xrightarrow{n \to \infty} \sum_{k \geq 2} Z_k = (1/2)^{-1}
\]

\(\mathbb{E}(\text{Meeting time of } Z)\)
A PROBLEM FROM ALDOUS AND FILL

Prove that this is universal over large transitive graphs
with relaxation time small.
A PROBLEM FROM ALDOUS AND FILL

**PROBLEM:**

\[ \frac{C}{M G_n} \xrightarrow{n \to +\infty} \sum_{k \geq 2} \mathbb{Z}_k \]

\( \implies \) independent exponentials mean \( (k/2)^{-1} \).

\( E(\text{Meeting time of } Z) \gg t_{rel} \)

**ASSUMPTION**
A PROBLEM FROM ALDOUS AND FILL

Prove that this is universal over large transitive graphs with relaxation time smaller than expected meeting time.

Some assumption is needed: no mean field behavior for star graphs or one-dimensional cycles.
A MORE GENERAL PROBLEM BY DURRETT

In *Random Graph Dynamics* Durrett studies the same kind of problem over certain *random graphs*.

Those have *power law degrees* and are “very non transitive” in many ways.

Nevertheless, D. obtains some partial results in the direction of *universality of mean field behavior*. 
Mean field behavior is indeed very general. We give two results.
A THEOREM FOR TRANSITIVE GRAPHS

\( \mathcal{Q}_n \): sequence of reversible, transitive chains on finite spaces.

\( \mathcal{M}_n \): expected meeting time of 2 indep. \( Q_n \)-walks
A THEOREM FOR TRANSITIVE GRAPHS

$\Omega_{n^3}$: sequence of reversible, transitive chains on finite spaces. $[t_{\text{mix}} = \text{mixing time}]

\mu_{n^3}$: expected meeting time of 2 indep. $Q_n$-walks
A Theorem for Transitive Chains

\[ \frac{C_m}{M_m} \quad \Rightarrow \quad m \to +\infty \quad \sum_{k \geq 2} \epsilon_k \]

Whenever

\[ \lim_{m \to +\infty} \frac{t_{\text{mix}, m}}{M_m} = 0 \]
A THEOREM FOR TRANSITIVE CHAINS

\[ \frac{C_m}{M_m} \rightarrow \sum_{k \geq 2} z_k \]

whenever \( t_{\text{mix}, n} \xrightarrow{n \to +\infty} 0 \) 

[Aldo/Fill: \( t_{\text{rel}, n} / M_n \rightarrow 0 \).]
A THEOREM FOR GENERAL CHAINS

We also have a theorem not requiring transitivity or reversibility, with messier assumptions.

It covers the random graphs of Durrett + many other examples (eg. supercritical percolation in 3 or more dimensions).

There certainly is room for improvement here.
Exponential hitting times, with good error bounds + quantiles + control of big bang phase.
THE THEOREM FOR TRANSITIVE CHAINS

\[ \frac{C_m}{M_m} \rightarrow \sum_{k \geq 2} \zeta_k \]

whenever \[ \frac{t_{\text{mix}, m}}{M_m} \rightarrow 0 \]
The random variables $Z_k$ have a clearly defined meaning in the complete graph case.

Time to move from $k$ to $k-1$ particles:

$\mathbb{P}(Z_k \geq t) = e^{-\left(\frac{1}{2}\right)t}$
COMPARE WITH COMPLETE GRAPH

The random variables $Z_k$ have a clearly defined meaning in the complete graph case.

Time to move from $k$ to $k-1$ particles:

$$P(Z_k \geq t) = e^{-\left(\frac{1}{2}\right) t}$$

**BASIC IDEA:** prove similar result for general chains.
CONNECTION WITH HITTING TIMES

$k$ random walks on $V$\Rightarrow$one random walk on $V^k$ (product)
**CONNECTION WITH HITTING TIMES**

$k$ random walks on $V$ \(\Rightarrow\) one random walk on \(V^k\) (product)

\[ T_k = \text{1st coalescence among } k \quad \Rightarrow \quad \Delta_k = \exists (x_i)_{i=1}^k : \exists 1 \leq i < j \leq k, x_i = x_j \]
EXPONENTIAL HITTING TIMES

Well-known “metatheorem” (Aldous, Aldous/Brown,...).

\[ P = \text{chain on } S, \quad A \subset S \quad \text{with} \]
\[ \forall \text{ stationary, } \mathbb{E}_\pi (\tau_A) \gg t_{\text{mix}}. \]
EXPONENTIAL HITTING TIMES

Well-known “metatheorem” (Aldous, Aldous/Brown,...).

\[ P = \text{chain on } \mathbb{R}, \quad C < \mathbb{R} \text{ with stationary, } E_\pi(\tau_A) \gg t_{\text{mix}}. \]

\[ P_\pi \left( \tau_A \vee E_\pi(\tau_A) > t \right) \sim e^{-t} \]
EXPONENTIAL HITTING TIMES

\[ T_k = 1^{st} \text{roalescence among } k \Rightarrow \text{Hitting time of } \Delta_k = \{ (x_i)_i = 0 : \exists 1 \leq i < j \leq k \} \]

\[ k = 2 \Rightarrow E_\pi(T_2) = M_n \gg t_{\text{mix}} \]
**EXPONENTIAL HITTING TIMES**

\[ T_k = 1^{st} \text{ coalescence among } k \Rightarrow \text{Hitting time of } \Delta_k = \{ (x_i)_{i=1}^n : \exists 1 \leq i < j \leq n, x_i = x_j \} \]

\[(k=2) \Rightarrow E^{\pi} (T_2) = M_n \gg t_{mix} \]

\[ P^{\pi} \left( T_A / E^{\pi} (T_A) > t \right) \sim e^{-t} \]
EXPONENTIAL HITTING TIMES

\[ T_k = \text{1st coalescence among } k \]

Hitting time of \( \Delta_k = \{ x: i = 0: \exists 1 \leq i < j \leq k \} \)

(larger \( k \))

\[ \Pr_{\pi} \left( T_k / \frac{E\pi(T_k)}{} > t \right) \sim e^{-t} \]
WHAT IS MISSING?

\[ \frac{C}{M} = \sum_{h \geq 2} \frac{T_k}{M} \approx \sum_{h \geq 2} 2^k \]

EXPO-
NEN-
TAILS
WHAT IS MISSING?

\[ \frac{C}{\mathcal{M}} = \sum_{k \geq 2} \frac{T_k}{\mathcal{M}} \]

**Expectation of** \( T_2 \)

**Hitting of** \( \Delta_k = \{(x,i) : i \leq k, \exists i < j \text{ s.t. } x_i = x_j\} \)

**Exponentials**
WHAT IS MISSING?

\[ \frac{C}{M} = \sum_{k \geq 2} \frac{T_k}{M} = \sum_{k \geq 2} 2^k \]

**Expected Value of \( T_2 \)**

**Problem #1**

Better estimates for tail of \( T_k \).
WHAT IS MISSING?

\[ \frac{C}{M} = \sum_{k \geq 2} \frac{T_k}{M} \]

\[ \approx ? \sum_{k \geq 2} 2^k \]

Expectation of \( T_2 \)

Exponentials

Hitting of

\[ \Delta_h = \{ (x_i) : i \leq h \} : \exists i < j : x_j = x_i \]

Problem # 1

Better estimates for tail of \( T_k \).
WHAT IS MISSING?

\[ \frac{C}{m} = \sum_{k \geq 2} \frac{T_k}{m} \]

Expectation of \( T_2 \)

Exponentials

Hitting of \( \Delta_k = \{ x : i \leq k, \exists j : x_i < x_j \} \)

Problem #2

Need to consider non-stationary starts.
WHAT IS MISSING?

\[ \frac{C}{M} = \sum_{k \geq 2} \frac{T_k}{M} \]

\[ = \sum_{k \geq 2} \frac{2k}{M} \]

Hitting of
\[ \Delta_h = \{ (x_1, \dotsc, x_i) : x_i < x_{i+1} \} \]

Problem #3

Must show
\[ E(T_n) \sim \mathcal{N}(\frac{1}{2}) \]
WHAT IS MISSING?

\[ \frac{C}{m} = \sum_{k \geq 2} \frac{T_k}{m} \]

EXPECTION OF \( T_2 \)

HITTING OF \( \Delta_h = \{ (x,i) : i < h \} \quad \text{and} \quad X_i = X_j \)

Problem #4

\( \lambda \) "big" \( \Rightarrow \) not exp. (Big Bang)
EXPONENTIAL HITTING TIMES (NEW)

Sharper theorem.

\[ \Pr = \text{chain on } \mathcal{X}, \ A < \mathcal{X} \text{ with stationary } \Pi; \ E = \Theta \left( \frac{t_{\text{mix}}}{E(\Pi)} \right)^{2/3} \]

\[ \Pr \left( \frac{Z_A}{E(\Pi)} > t \right) \leq \left( 3 + t \right) e^{-\frac{t}{3+2}} \]
EXPONENTIAL HITTING TIMES (NEW)

Sharper theorem.

\[ P = \text{chain on } \mathcal{X}, \quad A < \infty \text{ with stationary; } 3 = \Omega \left( \frac{\text{mix}_{\text{min}}}{\mathbb{E}_{\pi}(T_A)} \right)^{1/2} \] 

\[ 2) \quad \mathbb{P} \left( \frac{\text{mix}_{\text{min}}}{\mathbb{E}_{\pi}(T_A)} > t \right) \geq (3-1) e^{-\frac{t}{3-1}} \]
EXPONENTIAL HITTING TIMES (NEW)

Sharper theorem.

\[ P = \text{chain on } \mathbb{S}, \ A < \mathbb{S} \text{ with } \quad \Pi \text{ stationary; } 3 = S(L(\frac{\text{max}}{E_{\Pi}(\tau_{A})})) < < 1 \]

2) \[ P_{x} \left( \frac{\tau_{A}}{E_{\Pi}(\tau_{A})} > t \right) \geq (1 - e^{-\frac{t}{3-L}}) \]
EXPONENTIAL HITTING TIMES (NEW)

Sharper theorem.

\[ P = \text{chain on } \mathbb{R}, \quad A < \mathbb{R} \text{ with stationary } \Pi \quad \exists \quad 3 = \Omega \left( \left( \frac{\text{time mix}}{\mathbb{E}_\Pi (\tau_A)} \right)^{\frac{1}{3}} \right) \gg 1 \]

3) $3$-quantile of $\tau_A \approx 3 \mathbb{E}_\Pi (\tau_A)$
APPLICATION TO $E(T_k)$

Recall $T_k =$ Hitting time of

$$\Delta_k = \left\{(x_i)_{i=1}^k : \exists i < j \quad x_i = x_j \right\}$$
Recall $T_k$ = Hitting time of

$$\Delta_\mathbb{L} = \left\{ (x_i)_{i=1}^k : \exists i \neq j \quad x_i = x_j \right\}$$

$$\Rightarrow P(T_k \leq 3 \mathbb{E}(T_2)) \leq \left( \frac{3}{2} \right)^k$$
APPLICATION TO $E(T_k)$

Recall $T_k = \text{Hitting time of}$

$$\Delta_L = \{(x_i)_{i=1}^k : \exists i < j, \exists x_i = x_j\}$$

$$\Rightarrow P_T(T_k \leq 3 \cdot \mathbb{E}(T_2)) \leq \left(\frac{\theta}{2}\right)^3$$

Also need reverse inequality!
APPLICATION TO $E(T_k)$

$$P_{\Pi} \left( T_k \leq \varepsilon E(T_2) \right) \geq \left( \frac{1}{2} \right) \varepsilon$$

$$- O(k^4) P_{\Pi} \left( T_2^{1,2} \leq \varepsilon E(T_2), \quad T_2^{2,3} \leq \varepsilon E(T_2) \right)$$
BOUNDING CORRELATIONS (TRANSITIVE)

\[ P_{\pi \otimes^3} (T^{1,2} \leq t, T^{2,3} \leq t) \leq 2 \left( P_{\pi \otimes^2} (T^{1,2} \leq t) \right)^2 \]

(on blackboard!)
Thanks for your attention. Here is a [link to the paper].