On the Optimality of Spectral Compression of Meshes

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Spectral compression of triangle meshes has shown good results in practice, but there has been little or no theoretical support for the optimality of this compression. We show that for certain classes of geometric mesh models, spectral decomposition using the eigenvectors of the symmetric Laplacian of the connectivity graph is equivalent to principal component analysis. The key component of the proof is that the Laplacian is identical, up to a constant factor, to the inverse covariance matrix of the distribution of valid mesh geometries. Hence spectral compression is optimal in the MSE sense.

Additional Key Words: triangle mesh, spectral decomposition, Laplacian

1. INTRODUCTION

Triangle meshes are a popular way of representing 3D shape models. As the size and detail of the models grow, compression of the models becomes more and more important. The size of the mesh data files can be reduced by compressing either the geometry or the connectivity of the mesh, or both.

There has been much research into both geometry and connectivity coding. Most connectivity coding schemes, for example the Edgebreaker [Rossignac 1999] and the TG methods [Touma & Gotsman 1998], are based on traversing the mesh and adding symbols representing new vertices or faces to the code as they are traversed. The quality of the compression results from the entropy of the symbol sequence.

Typical geometry coding schemes are based on the fact that the coordinates of a mesh are not independent, and specifically, the coordinates of neighboring vertices are highly correlated, especially in smooth meshes. This correlation can be used exploited by using "prediction rules" – the coordinates of a vertex are predicted from the coordinates of neighbouring vertices, and only the prediction error vector is coded [Taubin & Rossignac 1998, Touma & Gotsman 1998]. The better the prediction rule is, the smaller the errors are, and the smaller the entropy of the code will be.

Spectral compression of mesh geometry [Karni & Gotsman 2000] also exploits the correlation among neighboring vertices, and implicitly applies a prediction rule that every vertex is the simple average of all its immediate neighbours. Inspired by traditional signal coding, spectral decomposition has been proposed for lossy transform coding of the geometry of a mesh with irregular connectivity. Although the method yields good results, there is little theoretical support for the optimality, or even quality, of such compression.

Motivated by the optimality of DCT in signal processing, we wish to prove a similar result, that spectral compression is optimal for certain classes of irregular meshes. Our proof covers meshes in one and two dimensions, and certain 3D extensions.

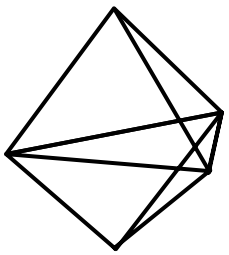
1.1 Previous Work

Let G be a graph, $G = (V,E)$, where V is the vertex set, and E the edge set. A *k-dimensional mesh* M is $M = (G, X)$ where X is a mapping $X: V(G) \rightarrow R^k$. X assigns *coordinate values* to the mesh vertices. We sometimes say that X is an *embedding* of G in R^k, or simply, the *geometry* of the mesh M.

Given a mesh $M(G(V,E),X)$ the *symmetric Laplacian* of M is the matrix L:

$$L_{i,j} = \begin{cases} d_i & i = j \\ 0 & (i,j) \notin E \\ -1 & (i,j) \in E \end{cases} \quad (1)$$

where d_i is the number of neighbours (*valence*) of the *i*'th vertex. A mesh with constant valences is called a *regular* mesh, otherwise it is called an *irregular* mesh. See Fig. 1 for an example of an irregular mesh, and its corresponding Laplacian.



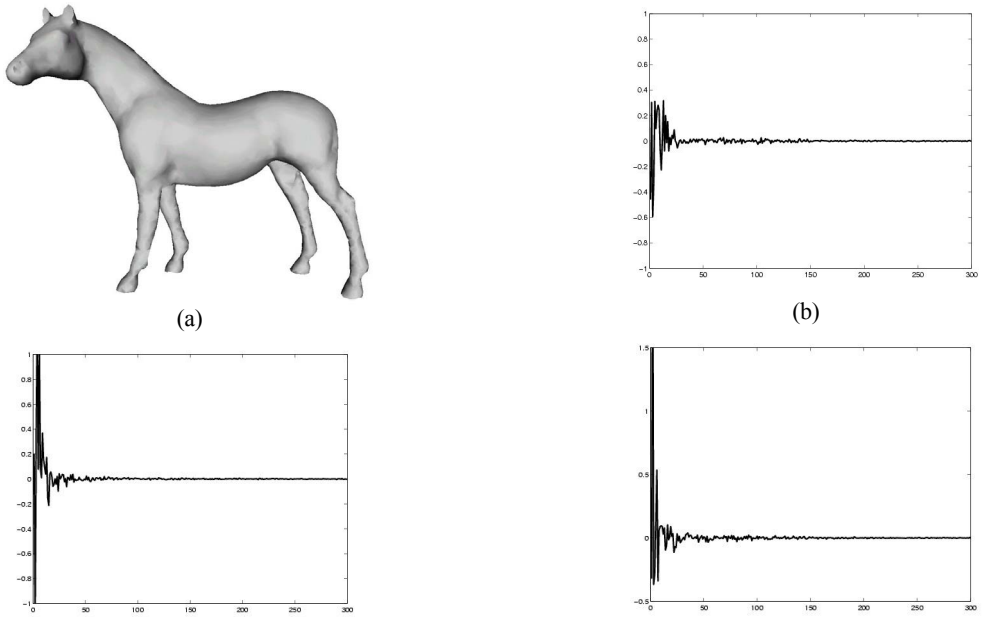
$$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{pmatrix}$$

Figure 1: A simple irregular mesh and its symmetric Laplacian

Spectral decomposition was first introduced by Karni and Gotsman [2000] as a tool for mesh geometry compression. In their work, the eigenvectors of the Laplacian matrix are used as the basis for the decomposition of all meshes having the same connectivity edge set E. Karni and Gotsman argued that since these vectors were a generalization of the Fourier basis vectors obtained for regular (grid-type) connectivities, they should work just as well for irregular connectivities. Indeed, they showed that the coefficients of such decomposition decay rapidly to zero (see Fig. 2), hence a small number of coefficients can be used to code the mesh geometry. The eigenvalues corresponding to the eigenvectors are analogous to the concept of frequency in harmonic analysis.

Eigenanalysis of the graph Laplacian is a standard tool in spectral graph theory [Chung 1997]. Several variants of the Laplacian have been studied. Eigenvectors of Laplacians are also used for graph drawing [Hall 1970, Koren 2003], graph partitioning [Alpert & Yao 1995], and parametrization [Gotsman, Gu & Sheffer 2003]. For a more extensive survey of the uses of Laplacians in digital mesh processing, see [Gotsman 2003].

The rest of the paper is organized as follows. Section 2 defines the terminology used throughout the paper. Section 3 reviews the concept of principal component analysis, a key tool in our proofs, which is the motivation for studying the eigenvectors of the covariance matrix. Sections 4 and 5 prove the optimality result for the 1D and 2D cases respectively. Section 6 introduces possible 3D extensions. To conclude, Section 7 discusses our model and explore future research directions.



(a) (b) (c) (d)

Figure 2: Spectral decomposition. (a) The horse model and (b,c,d) the spectral coefficients of its decomposition in the *X, Y* and *Z* dimensions.

2. DEFINITIONS

The *decomposition* of a vector X_{nx1}, by the orthonormal vector basis of R^n: $U_{\text{nxn}} = \{U_1,\ldots,U_n\}$, is:

$$X = \sum_{i=1}^{n} c_i U_i \,, \tag{1}$$

where $C = \{c_i\}$ are the *coefficients* of the decomposition.

The *reconstruction* of a vector X from its decomposition using the first k coefficients is:

$$X_{(U,k)} = \sum_{i=1}^{k} c_i U_i \quad . \tag{2}$$

Note that the order of the basis vectors is important, as only a prefix of them is used in the k-th reconstruction.

Because of the orthonormality of U, the Parseval identity maintains that $||c||=||X||$. A decomposition is useful when a small number of the coefficients contain a large portion of the norm (energy) of the vector. If this is the case, those coefficients alone may be used in the reconstruction, and the Euclidean distance between the reconstructed vector and the original X will be small. This is a useful feature for compression applications.

For a specific vector X, the best basis will always be $\{X,0,0,\ldots,0\}$, since then all the energy is contained in the first coefficient. Hence there is a meaningful answer to the question "what is the optimal basis for decomposition" only if we deal with an entire family of vectors (finite or infinite). In our context, the family will be all the geometries which are valid for a given mesh connectivity E. These geometries will be specified by imposing an appropriate probability distribution D, derived from E, on R^n.

Given a probability distribution D on R^n, we say that a vector basis U is an *optimal basis* for D, if for any other basis W, and for every $1 \le k \le n$:

$$E((X - X_{(U,k)})^2) \le E((X - X_{(W,k)})^2) \tag{3}$$

where the expectation is taken over D. This is called optimality in the Mean Square Error (MSE) sense.

3. PRINCIPAL COMPONENT ANALYSIS

Approximation of random signals has been studied extensively in signal processing. A well-known optimality result, which we will rely on heavily, is related to the so-called Principal Component Analysis (PCA) procedure.

Assume a random vector $X \in R^n$ drawn from D having zero mean vector and covariance matrix C. Define $\{\Phi_i | i=1,..,n\}$ to be the eigenvectors of C, with corresponding eigenvalues $\{\lambda_i \mid i=1,...,n\}$, ordered such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The matrix Φ is called the *Principal Component Analysis* of X, sometimes also known as the *Karhunen-Loeve* or *Hotelling transform* of X.

It is well known [Jolliffe 1986] that the PCA is optimal in the MSE sense, as defined in (3). Note that Φ does not depend on k – the number of vectors used for the reconstruction, so the optimal basis for reconstruction from $k+1$ basis vectors contains the optimal basis for reconstruction from k vectors.

When the class of meshes is finite and given, containing T meshes, for example an animation sequence of a mesh with a fixed connectivity, this implies a uniform distribution on this finite set. In this case the PCA of that class is equivalent to the numerical Singular Value Decomposition (SVD) procedure [Press *et al.* 1987] on a matrix of size $3n \times T$, consisting of T columns, where the i'th column contains the coordinate values of the i'th mesh in the sequence. This was proposed by [Alexa & Muller 2000] for compressing animated meshes.

To compute the PCA of the infinite continuum of meshes with a given connectivity, where none of them is explicitly given, we must first make some assumptions about the distribution D of this family, and then compute the covariance matrix C of this distribution. We will do this and then show that C is essentially identical to the inverse of the mesh Laplacian matrix, up to a constant factor. Due to this special relationship, both matrices have identical eigenvectors (in opposite order), from which our optimality theorem will follow.

4. 1D MESHES

Before we proceed, a note about the connection between the proof for one dimension, and the proof for two dimensions is in order. Basically, for both cases we prove the same theorem – that $L = \alpha C^{-1}$, for some constant α. While for the one dimensional case the theorem can be proved relatively easily, in two dimensions we resort to asymptotic approximations.

4.1 The Model

In one dimension, a mesh is simply an ordered set of points on the real line. There is only one "universal" (regular) connectivity – each vertex is connected to exactly two neighbours, except the two boundary vertices. For example, for $n=5$, the connectivity graph is as shown in Fig 3.

Figure 3: Example of the connectivity graph for n=5 in one dimension

We say that a vector $X = \{X_1, X_2, ..., X_n\}$ is a *valid* geometry in 1D iff $b_0 \leq X_1 \leq X_2 \leq \ldots \leq X_n \leq b_1$, where b_0 and b_1 are fixed boundary points. This is equivalent to saying that the "chain" connectivity does not "fold" on itself.

4.2 The Geometry Distribution

Let $U_1, U_2, \dots, U_n$ be independent random variables, distributed uniformly on (a,b), and let $a \le U_{(1)} \le U_{(2)} \le \dots \le U_{(n)} \le b$ be these values after sorting them in increasing order. $U_{(1)}, U_{(2)}, \dots, U_{(n)}$ are called *uniform order statistics* [David 1981]. We assume the valid geometries are distributed like the uniform order statistics on (b_0, b_1). Such a distribution will be uniform over all the valid geometries, because every valid geometry, i.e. every order statistics vector, can be generated by $n!$ combinations of the original variables, which are uniformly distributed. The mean geometry for this distribution is the grid geometry, where the distances between the vertices are equal.

4.3 The Optimal Basis

We now proceed to the main theorem in its one-dimensional version.

Theorem 1: The optimal basis for the decomposition of one dimensional meshes are the eigenvectors of the symmetric Laplacian $\{\Psi_i | i=1..n\}$, ordered such that the corresponding eigenvalues are $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$.

As stated in Section 3, the optimal basis is the PCA, which is the matrix of the eigenvectors of the covariance matrix of the mesh geometry distribution. Let $X = \{X_1, X_2, \dots, X_n\}$ be a random valid geometry with boundaries b_0 and b_1, and C its covariance matrix. Assume $j \le i$, since $C_{ij} = C_{ji}$. Now by definition

$$C_{ij} = E(X_i X_j) - E(X_i)E(X_j)$$

The first step in finding the optimal basis, is finding the covariance matrix C. In order to do that, we will need some more definitions:

Define $X_0 = b_0$, $X_{n+1} = b_1$, and $Y_i = X_i - X_{i-1}$ for $i=1..n+1$. Y_i are called *uniform spacings*. It is well known [Pyke 1965] that Y_i are interchangeable random variables, namely, that they are identically distributed, and all pairs of variable have the same joint distribution:

$$\begin{aligned} Var(Y_i) &= Var(Y_1) = v(n) && \forall i \in \{1,\dots,n+1\} \\ Cov(Y_i, Y_j) &= Cov(Y_1, Y_2) = c(n) && \forall i, j \in \{1,\dots,n+1\}, i \ne j \end{aligned} \tag{4}$$

for some functions $v(n)$ and $c(n)$.

Lemma 1.1:

If X is a random valid 1D geometry, C its covariance matrix, and $j \le i$, then:

$$C_{ij} = jv(n) + (ij - j)c(n)$$

Proof:

From the definition of Y_i we have that:

$$X_i = b_0 + \sum_{k=1}^{i} Y_k \tag{5}$$

Then, for $j \le i$, by substituting (5) in the covariance definition, and since the Y_i are interchangeable as defined in (4), we have:

$$
\begin{aligned}
C_{ij} &= E(X_i X_j) - E(X_i)E(X_j) = \sum_{k=1}^{i}\sum_{r=1}^{j}\left[E(Y_k Y_r) - E(Y_k)E(Y_r)\right] = \\
&= \sum_{r=1}^{j}\left[E(Y_r^2) - E(Y_r)^2\right] + \sum_{k=1}^{i}\sum_{\substack{r=1 \\ r\neq k}}^{j}\left[E(Y_k Y_r) - E(Y_k)E(Y_r)\right] = \\
&= \sum_{r=1}^{j} var(Y_1) + \sum_{k=1}^{i}\sum_{\substack{r=1 \\ r\neq k}}^{j} cov(Y_1, Y_2) = \\
&= jv(n) + (ij - j)c(n)
\end{aligned} \tag{6}
$$

■

We will now relate the variance and covariance of the uniform spacings, which will allow us to simplify the expression in (6).

Lemma 1.2:

If $v(n)$ and $c(n)$ are the variance and covariance respectively of uniform spacings as defined in (4), then:

$$c(n) = -\frac{1}{n}v(n)$$

Proof:

Consider the following expression:

$$E[(\sum_{i=1}^{n+1} Y_i)^2] - [E(\sum_{i=1}^{n+1} Y_i)]^2$$

On the one hand, developing the expression by expanding the sums and using the linearity of the expectancy yields:

$$
\begin{aligned}
E[(\sum_{i=1}^{n+1} Y_i)^2] - [E(\sum_{i=1}^{n+1} Y_i)]^2 &= \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} E(Y_i Y_j) - \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} E(Y_i)E(Y_j) = \\
&= \sum_{i=1}^{n+1}\sum_{j=1}^{n+1}[E(Y_i Y_j) - E(Y_i)E(Y_j)] = \\
&= \sum_{j=1}^{n+1}\left[E(Y_j^2) - E(Y_j)^2\right] + \sum_{i=1}^{n+1}\sum_{\substack{j=1 \\ i\neq j}}^{n+1}\left[E(Y_i Y_j) - E(Y_i)E(Y_j)\right] = \\
&= \sum_{j=1}^{n+1} var(Y_1) + \sum_{i=1}^{n+1}\sum_{\substack{j=1 \\ i\neq j}}^{n+1} cov(Y_1, Y_2) \\
&= (n+1)v(n) + n(n+1)c(n)
\end{aligned} \tag{7}
$$

On the other hand, since all Y_i have a constant sum ($b_1 - b_0$), we have:

$$E[(\sum_{i=1}^{n+1} Y_i)^2] - [E(\sum_{i=1}^{n+1} Y_i)]^2 = (b_1 - b_0)^2 - (b_1 - b_0)^2 = 0 \cdot \tag{8}$$

Comparing (7) and (8), we have:

$$
\begin{aligned}
&(n+1)v(n) + n(n+1)c(n) = 0 \\
&c(n) = -\frac{1}{n}v(n)
\end{aligned} \tag{9}
$$

■

Note that the covariance of two spacings is always negative. Intuitively, this is because one spacing may grow only at the expense of the other spacings, because of their constant sum.

By using the two previous lemmas, we can now simplify the expression of the covariance, by substituting (9) in (6), to get:

$$C_{ij} = j(n-i+1)\frac{v(n)}{n} \quad \text{for every } 1 \le j \le i \le n$$

So, the covariance matrix C is

(10)
$$C_{ij} = \begin{cases} j(n-i+1)\dfrac{v(n)}{n} & 1 \le j \le i \le n \\ C_{ji} & 1 \le i < j \le n \end{cases}$$

Now let us examine the matrix product $L \cdot C$. The valence d_i for all $2 \le i \le n$-1 is 2, since every vertex has exactly two neighbours, except the vertices near the boundaries. Hence the Laplacian is the (n-2)x(n-2) matrix, since n-2 is the number of interior vertices (excluding the boundary vertices):

$$L = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Note the first and last rows do not sum to zero, since they belong to vertices that are neighbours of the boundaries. This property makes the Laplacian invertible.

So, by substituting (10) in the product LC:

$$(L \cdot C)_{i,j} = \begin{cases} k(n) & i = j, i \in \{2,...,n-1\} \\ 0 & i \ne j, i \in \{2,...,n-1\} \end{cases}$$

which implies that

$$L \cdot C = k(n) I$$

or

$$L = k(n) C^{-1}$$

From the optimality of PCA, we know that the eigenvectors of C - $\{\Phi_i | i=1..n\}$, ordered such that $\lambda_1 \ge \lambda_2 \ge ... \ge \lambda_n$ are the optimal basis. Since the eigenvectors of C^{-1} are the same as the eigenvectors of C, but the eigenvalues are reciprocated, the optimal basis for the decomposition of 1D meshes are the eigenvectors of the symmetric Laplacian, ordered so the corresponding eigenvalues are $\lambda_1 \le \lambda_2 \le ... \le \lambda_n$.

■

5. 2D MESHES

5.1 The Model

In two dimensions, a geometry (X,Y) is valid for the mesh $M = (V, E, (X,Y))$ if the triangulation of (X,Y) using E does not contain intersecting edges, namely the triangles do not fold over onto each other, or, in other words, all have the same orientation. The coordinates of the boundary vertices of M are

$$(X_{b_1}, X_{b_2}, ..., X_{b_k}) = (x_{b_1}, x_{b_2}, ..., x_{b_k})$$
$$(Y_{b_1}, Y_{b_2}, ..., Y_{b_k}) = (y_{b_1}, y_{b_2}, ..., y_{b_k})$$

and they are fixed for all geometries. A few examples of valid and invalid geometries for the same connectivity and boundaries are shown in Fig. 4.

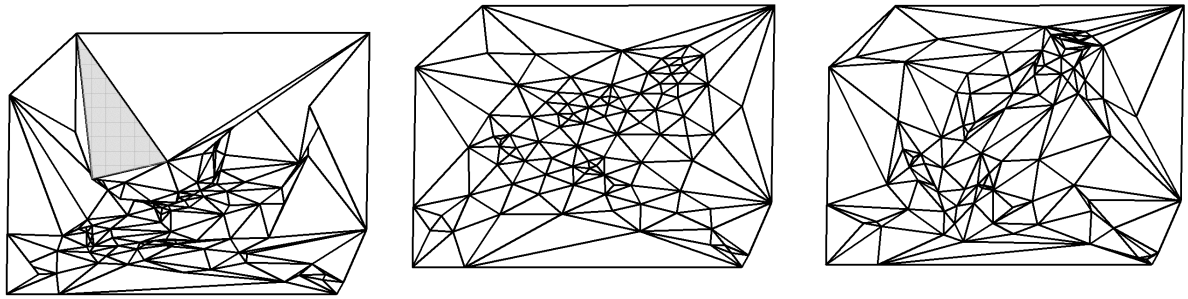


Figure **4:** 2D geometries with common connectivity and boundaries: **(a)** An invalid 2D geometry. The highlighted triangle is folded over. **(b,c)** Valid 2D geometries.

5.2 The Distribution

Following Tutte [1963], Floater [1997] proved that a 2D geometry with a convex boundary is valid if and only if each vertex is a convex combination of its neighbours. This implies that a geometry (X,Y) with boundary (B_x , B_y) is valid, if and only if there exists a matrix W such that

$$\begin{aligned} X &= WX + B_x \\ Y &= WY + B_Y \end{aligned}, \tag{11}$$

where B_x and B_y are

$$B_{xi} = \begin{cases} x_i & i \in \{b_1,...,b_k\} \\ 0 & otherwise \end{cases} \quad B_{yi} = \begin{cases} y_i & i \in \{b_1,...,b_k\} \\ 0 & otherwise \end{cases}$$

and W is

$$W_{ij} = \begin{cases} w_{ij} & (i,j) \in E, i \notin \{b_1,...,b_k\} \\ 0 & otherwise \end{cases}.$$

The weights w_{ij} are positive and normalized such that

$$\sum_{i=1}^{n} W_{ij} = 1 \quad i \notin \{b_1,...,b_k\}.$$

We will call W a *barycentric coordinates matrix.*

From now on, we will only refer to the X dimension, but all the theorems carry over to the Y dimension too.

This characterization of valid 2D meshes yields a convenient way to define a probability distribution on them. Instead of specifying exactly the distribution of the valid geometries X (which seems to be quite difficult), we specify the distribution of the barycentric coordinates matrices. We assume that the barycentric coordinates matrices are distributed according to the following stochastic model:

For each interior vertex i, with valence d_i let

$$w_{ij} = D_j^i = U_{(j+1)}^i - U_{(j)}^i \tag{12}$$

where $U_{(j)}^i$ are d_i order statistics over (0,1), with $U_{(0)}^i = 0$, and $U_{(d(i)+1)}^i = 1$. D_j^i are known as *uniform spacings* [Pyke 1965]. This guarantees that the nonzero W_{ij} are indeed positive and all the internal vertices' rows sum to one.

5.3 The Optimal Basis

The 2D version of our optimality theorem is:

Theorem 2: The optimal basis for the decomposition of 2D meshes with connectivity graph $G=(V,E)$ is approximated by the eigenvectors of the symmetric Laplacian of G - $\{\Psi_i | \ i=1..n\}$, ordered such that the corresponding eigenvalues are $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$.

We first address the following question: If the barycentric coordinates matrices W are distributed as in (12), how are the geometries distributed ? Given the matrix W and the boundary B, the geometry X can be calculated by

$$X = (I - W)^{-1} B$$

where I is the identity matrix.

Lemma 2.1:

Let X be the x coordinate of a random valid geometry, distributed as defined in (12). Then X has an approximately multivariate normal distribution.

Proof:

Since the distribution of X is difficult to characterize directly, because of the non-linear dependence of $(I-W)^{-1}$ on the random weights w_{ij}, we characterize the distribution of X via the conditional distributions

$$\left\{(X_i \mid X_j = x_j), j \neq i\right\}.$$

The following Lemma characterizes the multivariate normal distribution via its conditionals:

Lemma 2.2 [Arnold, Castillo & Sarabia 1999]:

Let $X = \{X_1, X_2, \ldots, X_n\}$ be a random vector. If:

1. $X_i \mid X_k = x_j$ is normally distributed for every i, and
2. $E(X_i \mid X_j = x_j)$ is linear in x_j and not constant, for every i,

then X has a multivariate normal distribution.

■

We will show that the conditions of this lemma hold for X.

From the definition of X and W in (11) and (12) respectively, it is easy to see that the conditioned variables $(X_i|X_j = x_j)$ are

(13)
$$X_i = \sum_{j \in N(i)} D_j^i x_j$$

where D_j^i are uniform spacings and $N(i)$ is the set of neighbours of the i-th vertex. Using central limit theorems for functions of uniform spacings [Pyke 1965], it can be shown, that for vertices with large valence:

(14)
$$(X_i \mid X_j = x_j) \sim Normal(\mu_i, \sigma_i^2)$$

where *Normal* is the Gaussian distribution. Hence the first condition of Lemma 2.2 holds for vertices with a large valence.

Since uniform spacings are interchangeable random variables which sum to unity:

$$\sum_{j=1}^{d_i} D_j^i = 1 \qquad \forall i$$

it follows that

(15)
$$E(D_j^i) = E(D_1^i) = \frac{1}{d_i} \qquad \forall j \cdot$$

By substituting the expectation of the spacings (15), in (13), which defines the expectation of X, we get

(16)
$$\begin{aligned} \mu_i = E(X_i \mid X_j = x_j) \quad &= E(\sum_{j \in N(i)} D_j^i x_j) = \\ &= \sum_{j \in N(i)} E(D_j^i) x_j = \\ &= \frac{1}{d_i} \sum_{j \in N(i)} x_j \end{aligned}$$

Note that the x_j are constants, since they are conditioned upon, so $E(D_j^i x_j) = E(D_j^i) x_j$.

This shows that the conditional expectation of X_i given $X_j = x_j$, is linear in x_j. Since the mesh is connected, and there are no isolated vertices, this also shows that the conditional expectation cannot be a constant with respect to the x_j-s. Hence the second condition of Lemma 2.2 also holds.

Applying Lemma 2.2, combined with (14) and (16) implies that, for large valence, X has a multivariate normal distribution, which concludes the proof of Lemma 2.1 ■

The Gaussian distribution of the coordinates can be seen in Fig. 5, where some random values of the coordinates of a vertex are visualized as point clouds. Central limit theorems with asymptotic parameter $d \to \infty$ give very good approximations already for modest values of d. Here the asymptotic parameter is the valence, which is 6 on the average. This seems to be large enough for the normal approximation to be reasonable.

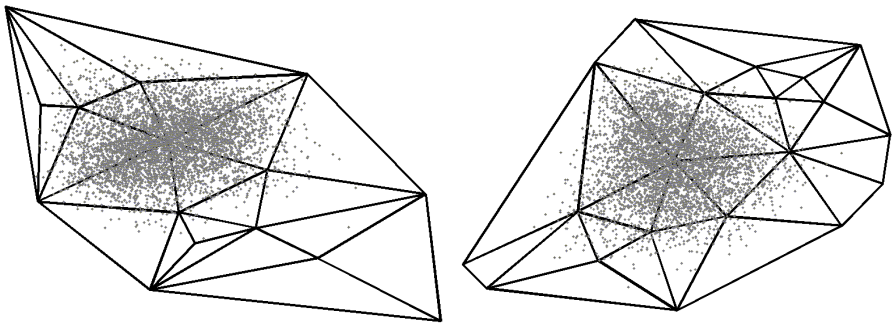


Figure 5: The distribution of a vertex in a random 2D mesh, using the distribution we impose. Each point in the gray cloud represents the location of this vertex in a random valid geometry. The mesh connectivities are drawn in black using the mean valid geometry for the given connectivities.

Now that we know the distribution of the geometries, we proceed to compute the covariance matrix, which is the key for proving optimality. The next two lemmas define the structure of K - the inverse of the covariance matrix. Combined, they show that K is essentially identical to the mesh Laplacian.

Lemma 2.3:

Let C be the covariance matrix of the X component of a random valid geometry (X,Y) of a mesh M=(V,E,(X,Y)). Let $K = C^{-1}$. For every (i,j), such that $i \neq j$, and (i,j)$\notin E$, $K_{i,j} = 0$.

Proof:

Let *i,j* be two non-adjacent vertices. Consider the covariance of X_i and X_j conditioned on the rest of the vertices:

$$\begin{aligned} Cov\left(X_i, X_j \mid X_k = x_k, k \neq i, j\right) &= E(X_i X_j \mid X_k = x_k) - E(X_i \mid X_k = x_k)E(X_j \mid X_k = x_k) = \\ &= E(\sum_{r \in N(i)} D_r^i x_r \sum_{m \in N(j)} D_m^j x_m) - E(\sum_{r \in N(i)} D_r^i x_r)E(\sum_{m \in N(j)} D_m^j x_m) = \\ &= \sum_{r \in N(i)} \sum_{m \in N(j)} E(D_r^i x_r D_m^j x_m) - \sum_{r \in N(i)} \sum_{m \in N(j)} E(D_r^i x_r)E(D_m^j x_m) \end{aligned}$$

Since *i* and *j* are not neighbours, D_r^i and D_m^j are differents sets of independent uniform spacings, which implies that:

(17) $$E(D_r^i x_r D_m^j x_m) = E(D_r^i x_r)E(D_m^j x_m)$$

and thus

(18) $$Cov\left(X_i, X_j \mid X_k = x_k, k \neq i, j\right) = 0$$

for every non-adjacent *i* and *j*.

We now need the following lemma which describes a few known properties of the inverse covariance matrix *K* of a multinormal distribution:

Lemma 2.4 [Lauritzen 1996]:

Let *X* be a multivariate normal random variable $X \sim N(\mu,\Sigma)$, and let $K = \Sigma^{-1}$. Then:

1. $K_{ij} = -Cov(X_i, X_j \mid X_{k \neq i,j} = x_k)(K_{ii}K_{jj} - K_{ij}^2)$.
2. $E(X_i \mid X_{j \neq i} = x_j) = E(X_i) + \sum_{j \neq i} \beta_{ij}(x_j - E(X_j))$, $\beta_{ij} = -\frac{K_{ij}}{K_{ii}}$.

Part 1 of Lemma 2.4, and the conditional covariance described in (18) imply that if *i* and *j* are not neighbours, then $K_{ij} = 0$.■

For the entries of *K* corresponding to the edges of the connectivity graph, we need the following Lemma:

Lemma 2.5:

Let *C* be the covariance matrix of the *X* component of a random valid geometry (*X,Y*) of a mesh *M*=(*V,E,*(*X,Y*)). Let $K = C^{-1}$. Then, there exists a constant α, such that:

1. For every $(i,j) \in E$, $K_{ij} = -\alpha$
2. For every $i \in V$, $K_{ii} = \alpha d_i$

Proof:

From part 2 of Lemma 2.4 we know that

$$E(X_i \mid X_{j \neq i} = x_j) = E(X_i) + \sum_{j \neq i} \beta_{ij}(x_j - E(X_j)).$$

On the other hand, as we showed in (16)

$$E(X_i \mid X_{j \neq i} = x_j) = \frac{1}{d_i} \sum_{j \in N(i)} x_j .$$

Since the linear coefficients of x_j must be equal in both expressions, we have

(19) $$\beta_{ij} = -\frac{K_{ij}}{K_{ii}} = \frac{1}{d_i}.$$

C is a covariance, so both *C* and *K* are symmetric, hence

(20) $$\frac{K_{ii}}{d_i} = -K_{ij} = -K_{ji} = \frac{K_{jj}}{d_j}$$

for every $(i,j) \in E$.

For the diagonal of K, it is easy to see, that if the mesh is connected, all the values K_{ii}/d_i must equal a constant α that does not depend on i: Define $K_{11}/d_1 = \alpha$. From (20) we have that for all the neighbours $j \in N(1)$, $K_{jj}/d_j = K_{11}/d_1 = \alpha$. The same can be done inductively for the neighbours of j and so on. Finally, since the mesh is connected, every vertex i has a path to the first vertex, so it must hold that:

(21) $$\frac{K_{ii}}{d_i} = \alpha$$

for every $i \in V$. Substituting (21) in (20) implies

(22) $$K_{ij} = -\alpha$$

for every $(i, j) \in E$ ■

Combining Lemmas 2.3 and 2.5, we conclude:

Let C be the covariance matrix of the X component of a random valid geometry (X,Y) of a mesh $M=(V,E,(X,Y))$. Let $K = C^{-1}$. Then K is:

$$K_{ij} = \begin{cases} -\alpha & (i,j) \in E \\ \alpha d_i & i = j \\ 0 & otherwise \end{cases}$$

Returning to the definition of the symmetric Laplacian in (1), we see that:

(23) $$C^{-1} = K = \alpha L$$

where α is a constant that depends on n alone.

As in the one-dimensional case, this concludes the proof of Theorem 2. ■

Since our result is an asymptotic result, the immediate question arises as to how good this approximation is in practice. Figs.67 and 7 show some experimental results, which give evidence to support the quality of the approximation. Both figures compare the results obtained with the symmetric Laplacian, to the results obtained by an empirical covariance matrix. The empirical covariance matrix is computed by generating a random geometries matrix $X = \{X_1,...,X_m\}$ where each X_i is a geometry generated according to our distribution model. Then we compute $COV_X = E(XX^T)-E(X)E(X^T)$, where $E(X)$ is the mean vector of X. Fig. 6 shows a color-coded map of the difference matrix $\left|L^{-1} - \alpha C\right| / \left|\alpha C\right|$, where the division is performed element by element. The darker the color, the closer this difference is to zero. It is evident that the two matrices are practically identical. Fig. 7 shows the normalized energy contained in the first k coefficients for the spectral decomposition of a geometry vector, and the decomposition using the (eigenvectors of the) empirical covariance matrix. The higher this energy is, the fewer coefficients we need to use to express more features of the geometry. It can be seen that the spectral and the PCA bases give very similar results.

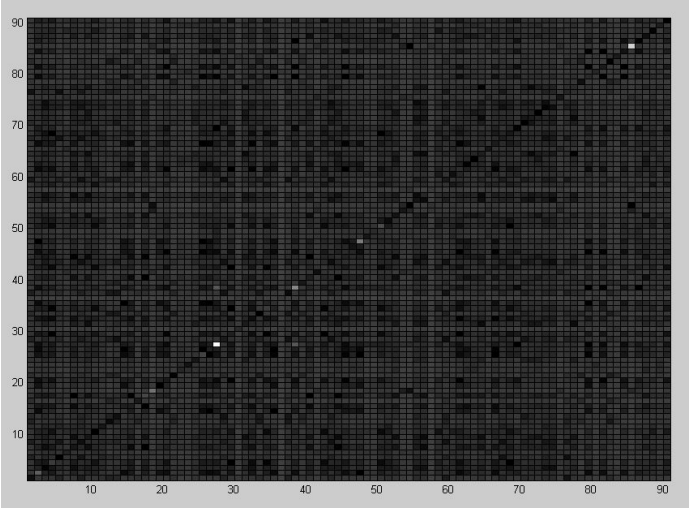


Figure 6**:** The difference matrix $\left|L^{-1}-\alpha C\right|/\left|\alpha C\right|$, color-coded. The darker the color, the closer to zero the value in the difference matrix. The maximal (white) entry is approximately 3.4.

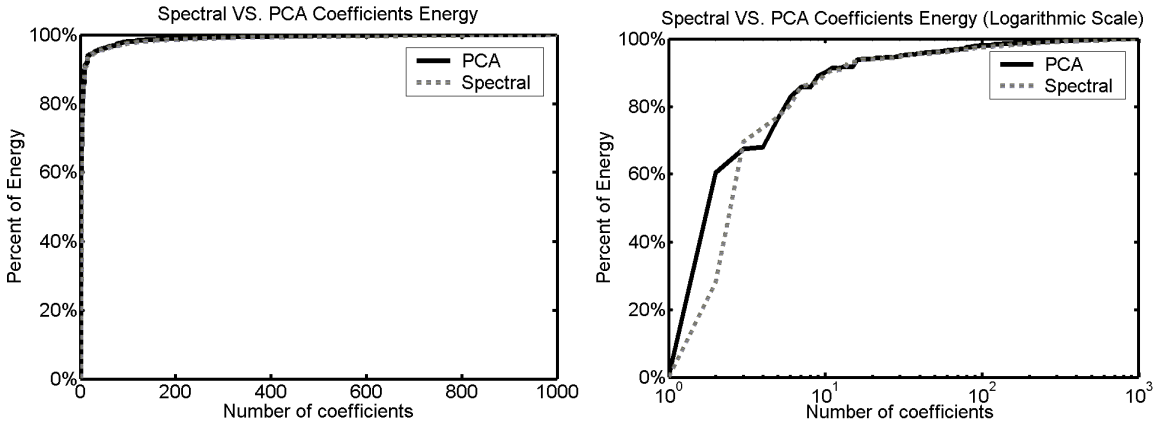


Figure 7: The "energy" contained in the first k coefficients, for a mesh with n=1000 vertices. The graph shows $\Sigma_{i=1\ldots k} c_i^2/\Sigma_{i=1..n} c_i^2$ where c_i are the coefficients of the decomposition by the PCA basis (computed using the empirical covariance matrix), and the spectral basis.

Generating multivariate normal distributions by defining the coefficients of the conditional mean is a well-known model [Besag & Kooperberg 1995], usually referred to as *conditional autoregressive*, or *auto-normal formulation*. Such models are useful in Bayesian networks, where they describe interactions between random variables that have neighborhood relationships.

6. 3D MESHES

6.1 The Model and the Distribution

In three dimensions, matters are more complicated. A *valid* manifold geometry is not well defined, let alone its distribution. Just applying the barycentric method (23) to a (non-planar) 3D boundary results in a smooth, featureless surface interpolating the boundary vertices. This is obviously not rich enough to capture interesting 3D shapes.

A possible natural extension of the 2D model, which allows for 3D features to emerge, is the following linear system: $X = WX + N$, where N is a random displacement vector, independent of X, and the system has fixed

boundary vertices. This means that each vertex is a convex combination of its neighbours up to a displacement. The displacement can be any reasonable random variable, as long as it is independent of X. The variance of the displacement values will influence the smoothnes of the mesh. A smooth mesh can be created by randomizing displacement values on the boundaries, and then computing the displacement values on the internal vertices by computing $N = L^{-1}B_N$ where L is the Laplacian, and N_B are the displacement values on the boundaries. The barycentric coordinates matrix W can be generated by uniform spacings, as in the two dimensional case. See Fig. 8 for examples of 3D meshes with different displacements, and different barycentric coordinates matrices based on this model.

6.2 The Optimal Basis

The optimality proof for the 2D case is based on the following two key properties of the multivariate random variable X: The normal distribution of the conditionals, and the linear conditional expectation. Both those properties carry over to 3D, when using the model we proposed in the previous section, due to the displacements N being independent of X.

1. Normal conditionals – Since N is independent of X, the central limit theorem that was used in the 2D proof is still applicable.
2. Linear conditional expectation – Since N is independent of X, the conditional expectation of any vertex is still linear in the other vertices, and the linear coefficients are the same as in the 2D case.

In conclusion, the proof remains valid for this 3D model.

7 CONCLUSION, DISCUSSION AND FUTURE WORK

We have shown that under a few natural assumptions on the distributions of geometries, given 1D and 2D connectivities, the covariance matrix of the mesh geometries is the inverse of the symmetric Laplacian matrix of that connectivity, hence has the same eigenvectors (in reversed order). This implies that spectral decomposition of meshes, which uses the eigenvectors of the symmetric Laplacian as the basis, is optimal.

Our proof is based heavily on our assumptions about the distribution of meshes in one and two dimensions. The distributions that we impose are obviously not the only ones possible. In one dimension, one may use the same scheme as in two dimensions, and generate the geometries using a random barycentric coordinates matrix. Another possibility is to define X_i to be $\sum_{j=1}^{i} Y_j \Big/ \sum_{j=1}^{n} Y_j$ where Y_i are uniformly distributed random variables. This is guaranteed to generate a monotonically increasing sequence. The optimality proof in the one-dimensional case hinges on the key fact that the random variables $X_i - X_{i-1}$ are interchangeable, hence the proof holds for any distribution that satisfies that condition. Specifically, this is the case for both of the models just described, even though these do not generate uniform distributions on the class of valid geometries. It is encouraging to see that the optimality result is not too sensitive to the geometry distribution.

In two dimensions, there are two main variations on our model. One possibility is not to use barycentric coordinates at all: For example, one can generate random X and Y coordinates, and keep only the (X,Y) vector pairs that form a valid geometry. A geometry will be valid if all the triangles have the same orientation. Obviously,

this is not an efficient method to sample this distribution, since for large n, the probability that a random (X,Y) pair will form a valid geometry is very small. The advantage of this process, however, is that it will generate geometries distributed uniformly over the class of valid geometries, but it is not clear whether our proof extends to this distribution.

Another possibility is to generate random 2D geometries by modifying the distribution of the barycentric coordinates matrix. For example, instead of being uniform spacings, the barycentric coordinates can be $w_{ij} = Y_i/\Sigma_{i=1..d}Y_i$ where Y_i are i.i.d uniform random variables. However, here too it is not clear whether the optimality result holds.

For the same proof to work for other 2D models, the distribution should have the following properties:

- The distribution of the geometry of every vertex conditioned on its neighbours should be (approximately) normal,
- The barycentric coodinates of a given vertex should be identically distributed random variables.

If those two properties hold, the geometry has a multinormal distribution, and the proof holds.

Although our 2D optimality result was derived using properties of the normal distribution, which is just an approximation of the true geometry distribution, we believe that a similar result holds for the true distribution of the geometries, as in the 1D case. This, unfortunately, will probably require a completely different proof.

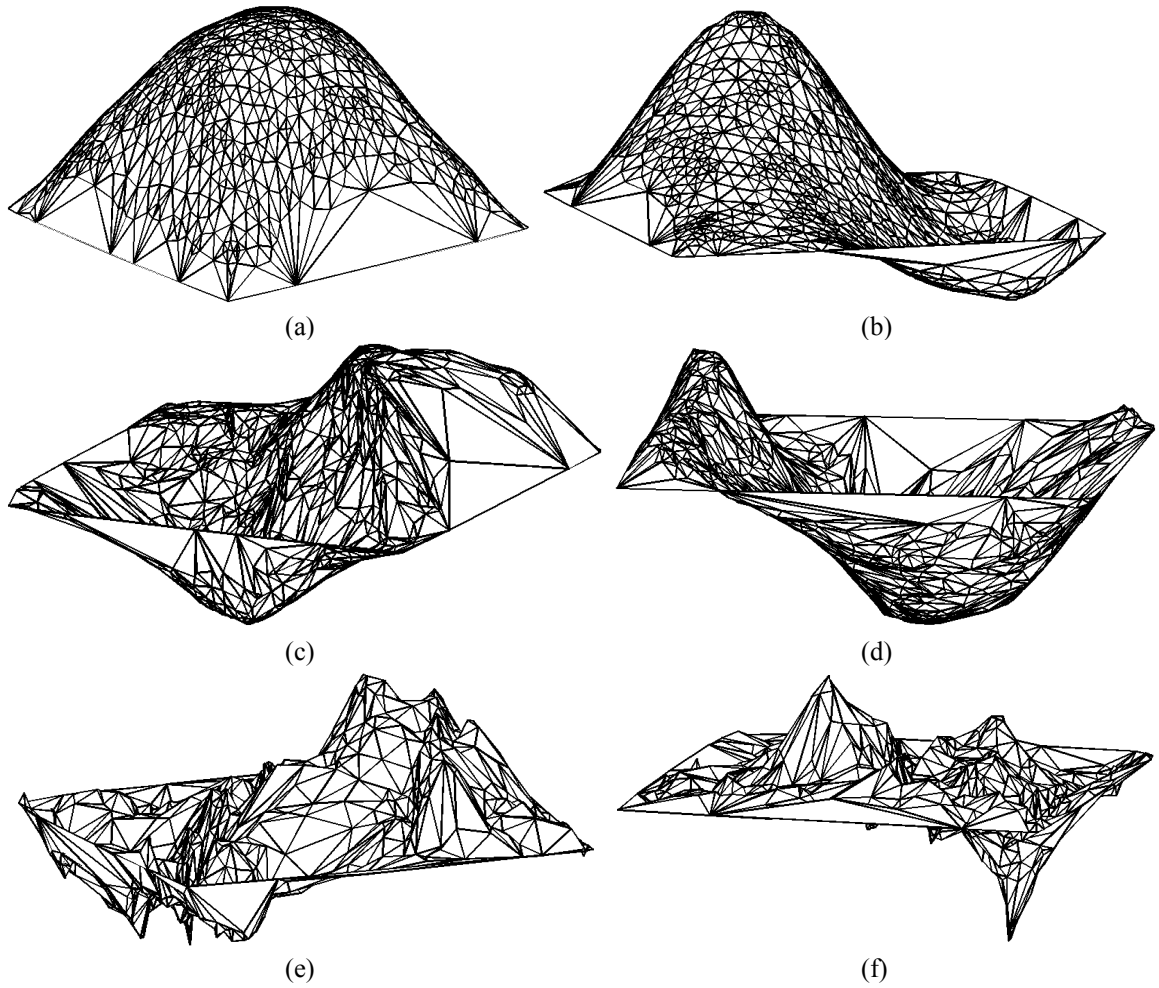


Figure 8: 3D geometries with common connectivity and boundaries. **(a)** The barycentric coordinates matrix is (I-*L*'), where *L*' is the Laplacian *L* normalized to unit row sums, and the displacement has constant variance for all vertices. **(b)** The barycentric coordinates matrix is (I-*L*'), and the displacement is smooth. It was generated by randomizing displacement values on the boundaries, and computing the displacement at each vertex by $\boldsymbol{N} = L^{-1}B_N$. **(c,d)** The displacement is computed as in (b), but random barycentric coordinates matrices are used **(e,f)** Random barycentric coordinates matrix and random Gaussian displacement with variance $1/n^2$ where n is the number of vertices in the mesh.

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