Cells in Coxeter Groups

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Let $W$ be a Coxeter group with generating set $S$ and defining relations of the form $(st)^{m_{st}} = 1$ for pairs of generators $s, t \in S$. In 1979 paper Kazhdan and Lusztig have defined a partition of $W$ into various classes of subsets called \textit{cells}.
Let $W$ be a Coxeter group with generating set $S$ and defining relations of the form $(st)^{m_{st}} = 1$ for pairs of generators $s, t \in S$. In 1979 paper Kazhdan and Lusztig have defined a partition of $W$ into various classes of subsets called *cells*.

Cells can be visualized via the action of $W$ on its Tits cone:

The cells of $\tilde{A}_2$, $\tilde{C}_2$, and $\tilde{A}_3$
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- singularities of Schubert varieties (Kazhdan-Lusztig, 79);
- representations of p-adic groups (Lusztig, 83);
- characters of finite groups of Lie type (Lusztig, 84);
- the geometry of nilpotent orbits in simple complex Lie algebras (Lusztig, 89; Bezrukavnikov-Ostrik, 04).
For the class of *crystallographic* Coxeter groups, which includes *Weyl groups* and *affine Weyl groups*, cells have connections to many areas of mathematics, e.g.:
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- the geometry of nilpotent orbits in simple complex Lie algebras (Lusztig, 89; Bezrukavnikov-Ostrik, 04).

To illustrate the last we mention an important *result of Lusztig*:

If $W$ is the affine Weyl group attached to the simple complex algebraic group $G$ with Lie algebra $\mathfrak{g}$, then the two-sided cells are in bijection with the set $O(L\mathfrak{g})$ of nilpotent orbits of the group dual to $G$. 
Definitions

$(W, S)$ is a \textit{Coxeter system}.

$\mathcal{H}$ is the \textit{Hecke algebra} of $W$ over $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$.
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H is the **Hecke algebra** of W over \( \mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}] \)

\((T_w)_{w \in W}\) is the **standard basis** of H:

- \( T_x T_y = T_{xy}, \) if \( l(xy) = l(x) + l(y) \);
- \( T_s^2 = q + (q - 1)T_s, \) if \( s \in S \)
Definitions

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$(C_w)_{w \in W}$ is the **Kazhdan-Lusztig basis**:

$$C_w = \sum_{y \leq w} (-1)^{l(w) - l(y)} q^{l(w)/2 - l(y)} P_{y, w}(q^{-1}) T_y,$$

where

$$P_{y, w} = \mu(y, w) q^{\frac{1}{2}(l(w) - l(y) - 1)} + \text{lower degree terms}$$

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\(P_{y, w}\) define preorders \(\leq_L, \leq_R, \leq_{LR}\) and the associated equivalence relations \(\sim_L, \sim_R, \sim_{LR}\) on \(W\). The equivalence classes are called left cells (resp. right cells, resp. two-sided cells).
Definitions

Multiplication:

\[ C_x C_y = \sum_z h_{x,y,z} C_z, \quad h_{x,y,z} \in A \]

\( a(z) \) is the smallest integer such that \( q^{-a(z)/2} h_{x,y,z} \in A^- = \mathbb{Z}[q^{-1/2}] \) for all \( x, y \in W \).
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If the function \( a \) is bounded on \( W \), then for every \( x, y, z \in W \)

\[ h_{x,y,z} = \gamma_{x,y,z} q^{a(z)/2} + \delta_{x,y,z} q^{a(z)-1/2} + \text{lower degree terms}. \]
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If the function \( a \) is bounded on \( W \), then for every \( x, y, z \in W \)

\[ h_{x,y,z} = \gamma_{x,y,z} q^{\frac{a(z)}{2}} + \delta_{x,y,z} q^{\frac{a(z)-1}{2}} + \text{lower degree terms.} \]

\( D_i := \{ z \in W \mid l(z) - a(z) - 2\delta(z) = i \} \), where \( \delta(z) = \deg(P_{e,z}). \)

The set \( D = D_0 \) consists of \textit{distinguished involutions} of \( W \).

Every left cell of \( W \) contains a unique \( d \in D \) (Lusztig, 87).
The cells of $\tilde{G}_2$ (Lusztig, 85)
The cells of the modular group \((2, 3, \infty)\) (Gunnells)
The cells of the group \((2, 2, 2, 3)\) (Gunnells)
The cells of the Hurwitz group (2, 3, 7) (Gunnells)
Lego game

A left cell (green) of the Hurwitz group
Definitions

Let $w \in W$. $Z(w)$ denotes the set of all $v \in W$ such that $w = x \cdot v \cdot y$ for some $x, y \in W$ and $v \in W_I$ for some $I \subset S$ with $W_I$ finite.
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Let \( w \in W \). \( Z(w) \) denotes the set of all \( v \in W \) such that \( w = x.v.y \) for some \( x, y \in W \) and \( v \in W_I \) for some \( I \subset S \) with \( W_I \) finite.

\( v \in Z(w) \) is **maximal in \( w \)** if it is not a proper subword of any other \( v' \in Z(w) \) for any reduced of \( w \) of the form \( x.v.y \).
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\( Z = Z(W) := \bigcup_{w \in W} Z(w), \ D_f := D \cap Z, \ D_f^* = D_f \setminus S. \)
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We will call \( w = x.v.y \) **rigid at** \( v \) if \( v \in D_f \), \( v \) is maximal in \( w \), and for every reduced expression \( w = x'.v'.y' \) with \( v' \in D_f \) and \( a(v') \geq a(v) \), we have \( l(x) = l(x') \) and \( l(y) = l(y') \):
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\[
\text{not rigid}
\]
Conjectures

Our goal is to detect an *inductive structure* inside $\mathcal{D}$ and to describe an *explicit relationship* between the elements of $\mathcal{D}$ and equivalence relations on $W$ which define its partition into cells. To this end we state two conjectures:

Conjecture 1. ("distinguished involutions")

Let $v = xv_1x^{-1} \in D$ with $v_1 \in D \cdot f$ and $a(v) = a(v_1)$, and let $v' = sv$. with $s \in S$. Then if $sv_1$ is rigid at $v_1$, we have $v' \in D$.

Conjecture 2. ("basic equivalences")

Let $w = xv_0$ with $v_0 \in D \cdot \text{max}$ in $w$. Let $u = xv_1x^{-1} \in D$ satisfy $a(u) \leq a(v_0)$, and let $w' = wu. v_01$ where $v_01$ is a product of $a(u) - 1$ simple reflections from $R(v_0)$. Assume $a(w') = a(w)$ and $v_{0''}xv_1$ is rigid at $v_1$ for every $v_{0''}$ such that $v_0 = v_0v_{0''}$, $l(v_{0''}) = l(v_01)$. Then $\mu(w, w') \neq 0$ and $w \sim_{R} w'$. 
Conjectures

Our goal is to detect an *inductive structure* inside $\mathcal{D}$ and to describe an *explicit relationship* between the elements of $\mathcal{D}$ and equivalence relations on $\mathcal{W}$ which define its partition into cells. To this end we state two conjectures:

**Conjecture 1.** ("distinguished involutions") Let $v = x \cdot v_1 \cdot x^{-1} \in \mathcal{D}$ with $v_1 \in \mathcal{D}_f^\bullet$ and $a(v) = a(v_1)$, and let $v' = s \cdot v \cdot s$ with $s \in S$. Then if $sxv_1$ is rigid at $v_1$, we have $v' \in \mathcal{D}$. 
Conjectures

Our goal is to detect an *inductive structure* inside $D$ and to describe an *explicit relationship* between the elements of $D$ and equivalence relations on $W$ which define its partition into cells. To this end we state two conjectures:

**Conjecture 1.** ("distinguished involutions") Let $v = x.v_1.x^{-1} \in D$ with $v_1 \in D_f^\bullet$ and $a(v) = a(v_1)$, and let $v' = s.v.s$ with $s \in S$. Then if $s xv_1$ is rigid at $v_1$, we have $v' \in D$.

**Conjecture 2.** ("basic equivalences") Let $w = x.v_0$ with $v_0 \in D_f$ maximal in $w$. Let $u = x.v_1.x^{-1} \in D$ satisfy $a(u) \leq a(v_0)$, and let $w' = w.u.v_{01}$ where $v_{01}$ is a product of $a(u) - 1$ simple reflections from $R(v_0)$. Assume $a(w') = a(w)$ and $v''_0 xv_1$ is rigid at $v_1$ for every $v''_0$ such that $v_0 = v'_0.v''_0$, $l(v''_0) = l(v_{01})$. Then $\mu(w, w') \neq 0$ and $w \sim_R w'$. 
Example

Let $W$ be the affine group of type $A_4$:

\[ v_1 = s_4 s_0 s_4 s_2 \]

is the longest element of $W$ with

\[ I = \{ s_0, s_2, s_4 \} \]

so $v_1 \in D_f$.

Can check by direct computation that

\[ s_1 s_4 s_0 s_4 s_2 s_1 s_3 s_1 s_4 s_0 s_4 s_2 s_1 s_3 s_2 s_3 s_1 s_4 s_0 s_4 s_2 s_1 s_3 \in D_f. \]

However,

\[ s_0 s_2 s_3 s_1 s_4 s_0 s_4 s_2 s_1 \not\in D_f \]

This is because

\[ s_0 s_2 s_3 s_1 s_4 s_0 s_4 s_2 s_1 = s_2 s_0 s_3 s_1 s_0 s_4 s_0 s_2 s_1 s_3 s_2 s_0 s_1 s_0 s_4 s_0 s_2 s_1 s_3 \]

where

\[ s_3 s_0 s_1 s_0 \in D_f \]

and

\[ a(s_3 s_0 s_1 s_0) = a(v_1). \]
Example

Let $W$ be the affine group of type $A_4$:

$v_1 = s_4s_0s_4s_2$ is the longest element of $W_i$ with $I = \{s_0, s_2, s_4\}$, so $v_1 \in \mathcal{D}_f$. 
Example

Let $W$ be the affine group of type $A_4$:

$v_1 = s_4s_0s_4s_2$ is the longest element of $W_l$ with $l = \{s_0, s_2, s_4\}$, so $v_1 \in D_f$.

Can check by direct computation that

$s_1 s_4s_0s_4s_2 s_1, s_3s_1 s_4s_0s_4s_2 s_1s_3, s_2s_3s_1 s_4s_0s_4s_2 s_1s_3s_2 \in D$.

However, $s_0s_2s_3s_1 s_4s_0s_4s_2 s_1s_3s_2s_0 \notin D$ (!)
Example

Let $W$ be the affine group of type $A_4$:

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is the longest element of $W_I$ with $I = \{s_0, s_2, s_4\}$, so $v_1 \in \mathcal{D}_f$.

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\[ s_1 s_4 s_0 s_4 s_2 s_1, \ s_3 s_1 s_4 s_0 s_4 s_2 s_1 s_3, \ s_2 s_3 s_1 s_4 s_0 s_4 s_2 s_1 s_3 s_2 \in \mathcal{D}. \]

However, $s_0 s_2 s_3 s_1 s_4 s_0 s_4 s_2 s_1 s_3 s_2 s_0 \notin \mathcal{D}$ (!)

This is because $s_0 s_2 s_3 s_1 s_4 s_0 s_4 s_2$ is not rigid at $v_1$:

\[ s_0 s_2 s_3 s_1 s_4 s_0 s_4 s_2 = s_2 s_0 s_3 s_1 s_0 s_4 s_0 s_2 = s_2 s_3 s_0 s_1 s_0 s_4 s_0 s_2, \]

where $s_3 s_0 s_1 s_0 \in \mathcal{D}_f$ and $a(s_3 s_0 s_1 s_0) = a(v_1)$. 
Results

**Theorem 1.** *(MB-Gunnells)* If Conjectures 1, 2, and standard conjectures* are true then

(1) The set \( D \) of distinguished involutions consists of the union of \( v \in D_f \) and the elements of \( W \) which are obtained from them using Conjecture 1.

(2) Relations described in Conjecture 2 determine the partition of \( W \) into right cells, i.e. \( x \sim_R y \) in \( W \) if and only if there exists a sequence \( x = x_0, x_1, \ldots, x_n = y \) in \( W \) such that \( \{x_{i-1}, x_i\} = \{v, v'\} \) as in the conjecture for every \( i = 1, \ldots, n \).
Theorem 1. (MB-Gunnells) If Conjectures 1, 2, and standard conjectures* are true then

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*: • Positivity (Kazhdan-Lusztig’79, Lusztig’85);
• Function $a$ is given by $a(w) = \max_{v \in Z(w) \cap D_f} a(v)$ (cf. Lusztig’03).
Theorem 2. *(MB-Gunnells)* Let \( v = x.v_1.x^{-1} \in \mathcal{D} \) with \( v_1 \in \mathcal{D}_f \), \( a(v) = a(vs) \) and \( \mathcal{L}(vs) \setminus \mathcal{R}(vs) \neq \emptyset \); and let \( v' = s.v.s \). Then if \( v' \) is rigid at \( v_1 \), we have \( v' \in \mathcal{D} \).
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Theorem 3. *(MB-Gunnells)* Let \( w = x.v_0 = t_n \ldots t_1.s_1 \ldots s_1 \) with \( v_0 = s_1 \ldots s_1 \in \mathcal{D}_f \) is maximal in \( w \) and is the longest element of some \( W_I \), and \( a(w) = a(v_0) \); \( u = y.u_0.y^{-1} \in \mathcal{D} \) with \( u_0 \in \mathcal{D}_f \) such that \( a(u) = a(u_0) = l \); and \( w' = w.u.v_01 \) with \( v_01 = s_1 \ldots s_{l-1} \) has \( a(w') = a(w) \) and \( \mathcal{R}(w') \subset \mathcal{R}(w) \). Moreover, assume

1. For any \( v_j = t_j \ldots t_1v_0t_1 \ldots t_j, j = 0, \ldots, n - 1 \) and \( t = t_{j+1} \) or \( t = t_{j-1} \) if \( t_{j-1} \notin \mathcal{R}(v_j) \), we have \( a(v_jt) = a(v_j), \mathcal{L}(v_jt) \setminus \mathcal{R}(v_jt) \neq \emptyset \) and \( tv_jt \) is rigid at \( v_0 \).
2. For any \( u_j = s_{j-1} \ldots s_1us_1 \ldots s_{j-1}, j = 1, \ldots, l - 1 \) with \( u_1 = u \), we have \( a(u_js_j) = a(u_j), \mathcal{L}(u_js_j) \setminus \mathcal{R}(u_js_j) \neq \emptyset \) and \( s_ju_js_j \) is rigid at \( u_0 \).

Then \( \mu(w, w') \neq 0 \) and \( w \sim_R w' \).
Results

Proof of Thm. 2 consists of two steps:

(a) show that $\delta(vs) (= \deg(P_{e,vs})) = \delta(v)$;
(b) show that $\delta(svs) = \delta(vs) + 1$.
Results

Proof of Thm. 2 consists of *two steps*:

(a) show that $\delta(vs)(=\deg(P_{e,vs})) = \delta(v)$;
(b) show that $\delta(svs) = \delta(vs) + 1$.

Proof of (a) is based on (unpublished) correspondence of Springer and Lusztig.

Proof of Thm. 3 uses Thm. 2 and the equality $P_{v_0,v_0}uv_01 = P_{e,v_0}uv_01$ (Kazhdan-Lusztig).
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Proof of Thm. 2 consists of \textit{two steps}:

(a) show that $\delta(vs)(=\deg(P_{e,vs})) = \delta(v)$;
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Proof of (b) uses Kazhdan-Lusztig recursion for $P_{x,w}, x \leq w$:

$$P_{x,w} = q^{1-c} P_{sx,v} + q^c P_{x,v} - \sum_{x \leq z < v \atop sz < z} \mu(z, v) q_z^{-1/2} q_v^{1/2} q^{1/2} P_{x,z},$$

where $w = sv$, $c = 1$ if $sy < y$, $c = 0$ if $sy > y$ and $P_{x,v} = 0$ unless $x \leq v$. It is here, where we need the \textit{extra condition}!
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where \( w = sv, c = 1 \) if \( sy < y \), \( c = 0 \) if \( sy > y \) and \( P_{x,v} = 0 \) unless \( x \leq v \). It is here, where we need the extra condition!

Proof of Thm. 3 uses Thm. 2 and the equality

\[ P_{v_0, v_0uv_01} = P_{e, v_0uv_01} \]

(Kazhdan-Lusztig).
Applications

**Application 1.** Cells in Coxeter groups with equal exponents, e.g. $\tilde{A}_2$, right-angled Coxeter groups (MB’04), etc.
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**Application 2.** If there exist $s, t_1, t_2 \in S$ such that $m(s, t_1) = m(s, t_2) = \infty$ and $m(t_1, t_2)$ is finite, then $W$ has infinitely many one-sided cells.
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Application 2. If there exist $s, t_1, t_2 \in S$ such that $m(s, t_1) = m(s, t_2) = \infty$ and $m(t_1, t_2)$ is finite, then $W$ has infinitely many one-sided cells.

Rem. Application 2 does not cover the Hurwitz group $(2, 3, 7)$. 
Some references


M. Belolipetsky, P. Gunnells, Cells in Coxeter groups, *preprint.*