Adaptive Polygonization of Implicit Surfaces

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à Introduction

Polygonization of surfaces has many practical applications in computer graphics and geometric modeling. It basically consist in computing a piecewise linear approximation for a smooth surface *f* described either by implicit, *f(x,y,z)=0*, or parametric functions. This procedure involves basically two basic operations: *Sampling* and *Structuring*. Sampling generates a set of points on the surface and structuring links those points to construct a mesh. In general, algorithms can be classified according to how they implement these operations.

à Motivation

Polygonization (or tesselation) of a surface *f* is implemented in Mathematica by the functions *ContourPlot3D* and *ListContourPlot3D.* ContourPlot3D is based on a uniform decomposition of the three-dimensional space into cubes of the same size called *cells*. The function that describes the surface geometry is evaluated at node points of a regular grid. If there is a change on the function sign at node points of a cell's edge, the intersection point **P** of the edge-surface is computed as a follows for the segment $\overset{\bullet\bullet}{\mathbf{ab}}$:

$$\mathrm{a} = \frac{f@aD}{f@aD-f@bD}$$

$$P = P_a + \mathrm{a}\ \mathrm{H}P_b - P_a\mathrm{L}$$

Uniform decomposition of the ambient space are straightforward to implement, but produce polygonal meshes that are not adapted to the implicit surface. Such solution is only acceptable for shapes with regular features, where the surface curvature is almost constant [LV2]. The fixed sampling rate causes oversampling of areas with low curvature and undersampling of areas with high curvature.

In this report we present an algorithm based in an adaptive sampling rate that varies spatially according to local surface complexity. Adaptive polygonization algorithms are more complex than uniform algorithms because they must deal with two relatted problems:

i) Ensure optimal sampling. Guarantees a faithful geometric approximation, depends on the adaptation criteria
ii) Enforce correct topology. Guarantees the consistency of the mesh and depends on the structuring mechanism

local changes to the sampling rate may affect the global mesh topology. For example, different levels of subdivision in two adjacent cells could create a crack along the boundary between them.

Following are some examples comparing the current implementation of ContourPlot3D and an adaptive polygonization.

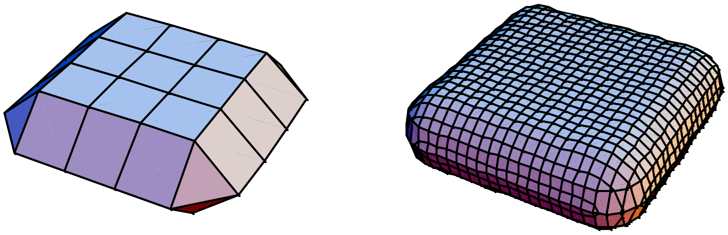
```
<<Graphics`ContourPlot3D`
```

```
<< Polygonize`Polygonize3Dtrm`
```

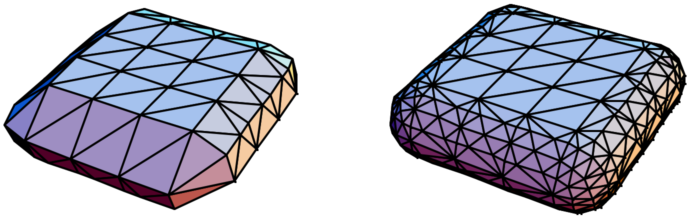
```
<< "D:\Wolfram\Polygonize\Polygonize3Dtrm.m"
```

```
square@x_, y_, z_D :=
 8If@Abs@xD £ 1., x, x • Abs@xDD, If@Abs@yD £ 1., y, y • Abs@yDD, 0<
MyOffset@r_, x_, y_, z_D :=
 Sqrt@Apply@Plus, Map@#1^2 &, 8x, y, z< - square@x, y, zDDDD - r
MySphere@r_, x_, y_, z_D := x^2 + y^2 + z^2 - r^2
```

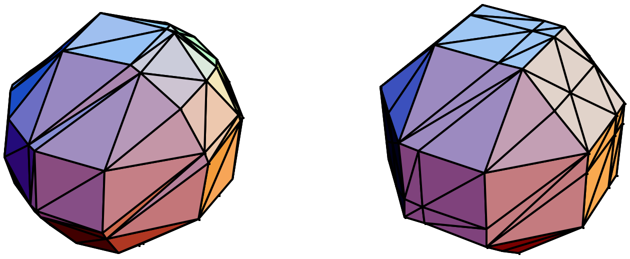
```
Show@
  GraphicsArray@
   ContourPlot3D@MyOffset@0.5, x, y, zD, 8x, -1.6, 1.6<, 8y, -1.6, 1.6<,
      8z, -.5, .5<, Boxed ® False, PlotPoints ® 86, 6, 3<, MaxRecursion ® #,
      DisplayFunction ® IdentityD & •ž 80, 1<DD;
```



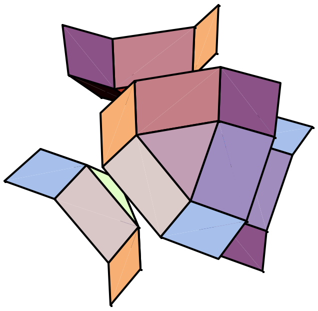
```
Show@GraphicsArray@
   Polygonize3D@MyOffset@0.5, x, y, zD, 8x, -1.6, 1.6<, 8y, -1.6, 1.6<,
      8z, -.5, .5<, PlotPoints ® 86, 6, 3<, MaxAdaptiveRecursion ® #,
      ToleranceSize ® 1 • 5, FlatnessAngle ® 30, PartitionStyle ® 2,
      Boxed ® False, DisplayFunction ® IdentityD &  •ž 80, 10<DD;
```



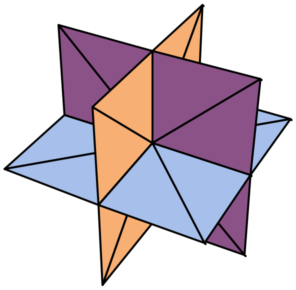
```
Show@GraphicsArray@Polygonize3D@MySphere@1.0, x, y, zD, 8x, -1, 1<,
      8y, -1, 1<, 8z, -1, 1<, IntersectionStyle ® #, PlotPoints ® 4,
      Boxed ® False, DisplayFunction ® IdentityD & •ž 80, 1<DD;
```



```
ContourPlot3D@x y z, 8x, -1.0, 1.0<, 8y, -1.0, 1.0<, 8z, -1.0, 1.0<,
  Contours ® 80.<, Boxed ® False, MaxRecursion ® 1, PlotPoints ® 3D;
```



```
Polygonize3D@x y z, 8x, -1.0, 1.0<, 8y, -1.0, 1.0<,
  8z, -1.0, 1.0<, CFDecomposition ® 81, 2, 3, 4<, Contours ® 80.<,
  Boxed ® False, PlotPoints ® 3, MaxAdaptiveRecursion ® 1D;
```



à Adaptive Polygonization

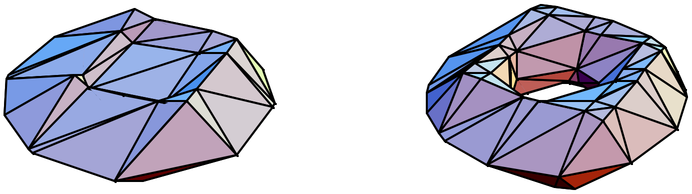
The present work is based in the paper "A Unified Approach for Hierarchical Adaptive Tesselation of Surfaces", [LV1].

This algorithm combines hierachical curve sampling with simplicial subdivision. It is composed of three independent operations:

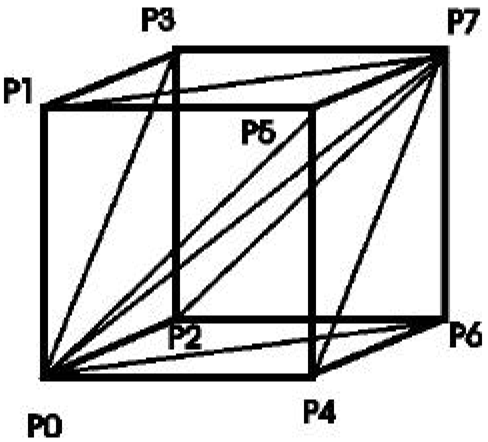
Ÿ Base mesh generation:

A very coarse sampling grid suffices for most implicit shapes of interest. Knowledge about the surface should be used to determine the appropiate uniform sampling rate. For a compact surface of genus 0, the shape should be smaller then the diameter of the largest sphere inscribed in the shape

```
MyTorus@r_, x_, y_, z_D := Hx^2 + y^2 + z^2 - r^2 - 1L^2 - 4 r^2 H1 - z^2L;

Show@GraphicsArray@
  Polygonize3D@MyTorus@2.1, x, y, zD, 8x, -3.1, 3.1<, 8y, -3.1, 3.1<,
     8z, -3.1, 3.1<, PlotPoints ® #, Boxed ® False,
     DisplayFunction ® IdentityD & •ž 884, 4, 3<, 85, 5, 3<<DD;
```



The base mesh generation decomposes the bounding box of the implicit shape using a simplicial cell complex. It then identifies the set of cells that are intersected by the implicit surface and for each of these cells it generates an element of the polygonal mesh approximating the surface. As in [LV2], we use a simplicial decomposition know as Coxeter-Freudenthal subdivision, that is defined as follows: For a cube in R^3 with vertices p_0, $p_{1,...,}$ p_7, it takes the diagonal $p_0\, p_7$ and projects it onto each face of the cube. This gives a triangulation of the faces of the cube. The 3D simplicial cells are constructed by adding to each triangle in a face, the vertex of the diagonal $p_0\, p_7$ that does not belong to it. We obtain:



```
CFdecomp = 881, 5, 6, 8<,
            81, 5, 7, 8<,
            81, 2, 6, 8<,
            81, 2, 4, 8<,
            81, 3, 7, 8<,
            81, 3, 4, 8<<;
```

When a potential hit occurs, the Coxeter-Freudenthal decomposition of the cube is generated and each of its simplices are processed.

Note that the choice of the diagonal $p_0 p_7$ is arbitrary. We can of course chose any of the other three diagonal as well. During the initial subdivision we can choose to test any of these decompositions and select the one that generates the "best" triangulation. The option **CFDecomposition->L**, where L is a sublist of {1,2,3,4}, has precisely this effect.

```
cubepermutations = 8
   81, 2, 3, 4, 5, 6, 7, 8<,
   85, 6, 7, 8, 1, 2, 3, 4<,
   83, 4, 1, 2, 7, 8, 5, 6<,
   82, 1, 4, 3, 6, 5, 8, 7<<;
```

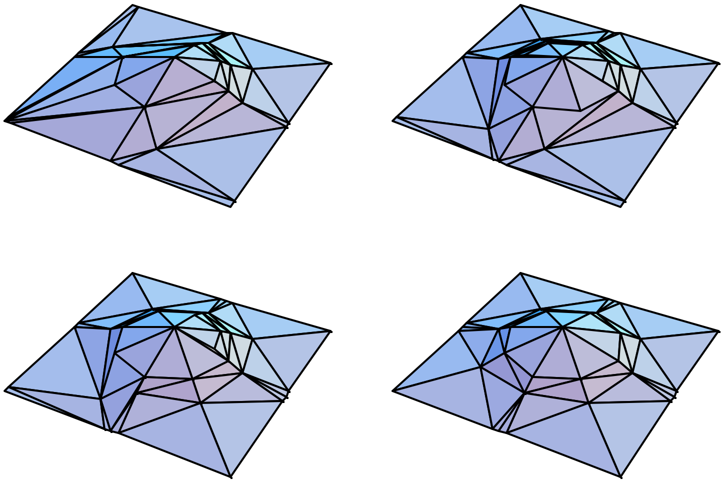
As an example of the effect of this option, consider the following surface:

```
sigmoid@d_ •; d £ 1D := 1 - H4 d^6 - 17 d^4 + 22 d^2L • 9
sigmoid@d_ •; d > 1D := 0.
Saucer@x_, y_, z_D := z - sigmoid@Sqrt@x^2 + y^2DD • 3
```

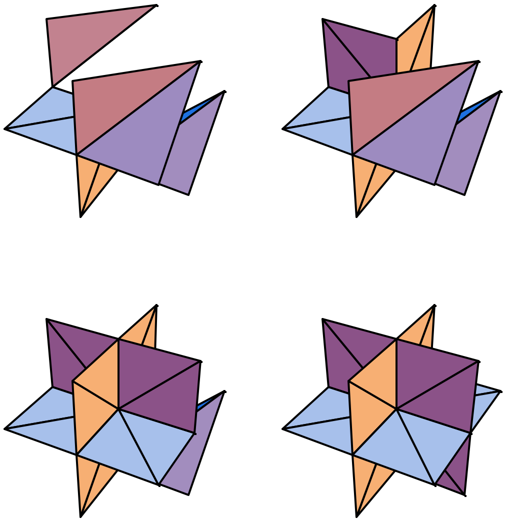
```
8$CFDecomposition<
```

```
881<<
```

```
Show@GraphicsArray@Partition@
   Polygonize3D@Saucer@x, y, zD, 8x, -1, 1<,
      8y, -1, 1<, 8z, -0.01, 0.35<, MaxAdaptativeRecursion ® 10,
      PartitionStyle ® 1, FlatnessAngle ® 20, Boxed ® False,
      PlotPoints ® 3, CFDecomposition ® #, DisplayFunction ® IdentityD & •ž
    881<, 81, 2<, 81, 2, 3<, 81, 2, 3, 4<<, 2DDD;
```

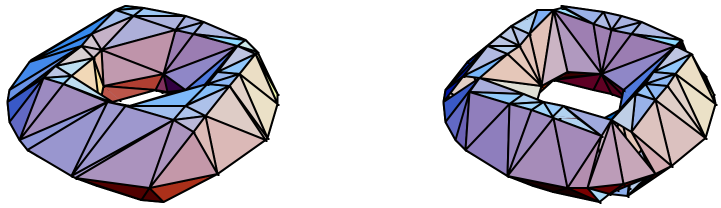


```
Show@GraphicsArray@Partition@
   Polygonize3D@x y z, 8x, -1, 1<, 8y, -1, 1<,
      8z, -1, 1<, PartitionStyle ® 1, Boxed ® False, PlotPoints ® 3,
      CFDecomposition ® #, DisplayFunction ® IdentityD & •ž
    881<, 81, 2<, 81, 2, 3<, 81, 2, 3, 4<<, 2DDD;
```

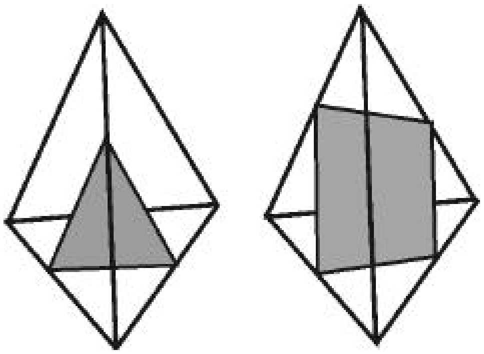


Altough it can considerably improve the quality of the mesh, in some cases it may have the effect of producing "cracks", as the next examples shows:

```
Show@GraphicsArray@
  Polygonize3D@MyTorus@2.0, x, y, zD, 8x, -3.1, 3.1<, 8y, -3.1, 3.1<,
     8z, -3.1, 3.1<, PlotPoints ® 85, 5, 3<, Boxed ® False,
     DisplayFunction ® Identity, CFDecomposition ® #D & •ž 881<, 81, 2, 3, 4<<DD;
```



Based on the sign of *f*, possible intersections of the simplex with the surface are tested. Note that, when the parent cube is intersected by the surfaces only some cells of the simplicial decomposition will be crossed by it. The implicit surface may intersect the edges of a simplex in either three or four points. Orientation of the triangles is asserted by requiring that the normal points in the oposite direction to a negative vertex.



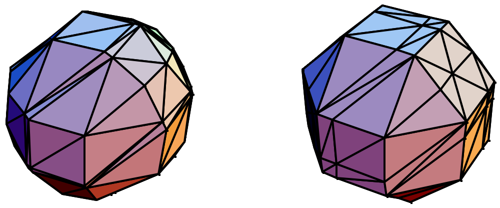
```
tetraedgeslist = 881, 3<, 81, 4<, 81, 2<, 83, 4<, 83, 2<, 82, 4<<;

tetraentries = 88<, 884, 6, 2<<, 885, 4, 1<<, 885, 6, 1<, 81, 6, 2<<,
   886, 5, 3<<, 885, 3, 2<, 82, 4, 5<<, 886, 4, 3<, 83, 4, 1<<,
   882, 1, 3<<, 883, 1, 2<<, 881, 4, 3<, 83, 4, 6<<, 882, 3, 5<, 85, 4, 2<<,
   883, 5, 6<<, 881, 2, 6<, 81, 6, 5<<, 881, 4, 5<<, 882, 6, 4<<, 8<<;
```

Intersection of an edge with the surface can be computed in two different ways. It can be estimated as a proportional values as in ContourPlot3D or by bisection. The number of iteration in the last case is controled with the option **MaxBisectionIter**. The way the intersection is estimated is controled with the option IntersectionStyle->0 for the bisection case and IntersectionStyle->1 for the proportional case.

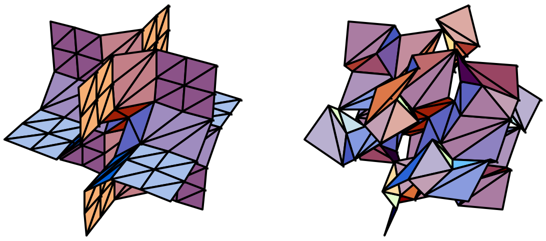
```
8$IntersectionStyle<
```

```
80<
```

```
Show@GraphicsArray@
  Polygonize3D@MySphere@1.0, x, y, zD, 8x, -1, 1<, 8y, -1, 1<, 8z, -1, 1<,
     IntersectionStyle ® #, PlotPoints ® 4, Boxed ® False,
     DisplayFunction ® IdentityD & •ž 80, 1<DD;
```



```
Show@GraphicsArray@
  Polygonize3D@x y z, 8x, -1, 1<, 8y, -1, 1<, 8z, -1, 1<, IntersectionStyle ® #,
     PlotPoints ® 4, CFDecomposition ® 81, 2, 3, 4<,
     Boxed ® False, DisplayFunction ® IdentityD & •ž 80, 1<DD;
```



Ÿ Indexing

```
X = Dim
L = `Log $n_x$p, M = `Log $n_y$p, N = `Log $n_z$p
Index = -8L, M, N, X, X, X<
Corner Vertex Hl, m, nL
    8l, m, n, 0, 0, 0<
Middle Point Hl, m, nL, Hp, qL
    r = p^q, s = p&q
    p' = p^s, q' = q^s
    8l + $s_2$, m + $s_1$, n + $s_0$, r, 0, 0<
Edges Hl, m, nL, Hp, qL, p < q
    8l + $s_2$, m + $s_1$, n + $s_0$, 0, p', q'<
```

Ÿ Edge sampling

The edge representation is constructed using a hierarchical sampling. This scheme generates a multiresolution, adapted, piecewise-linear approximation of the corresponding curve on the surface. This procedure subdivides the edge PQ at its midpoint M, and finds a new sample, the split point T on the surface. It also computes the difference between these two points, the vector D=T-M. Recursion terminates when the maximum sampling density, set by **MaxAdaptiveRecursion** is reached or an edge is found to be *Flat*.

```
8$MaxAdaptiveRecursion<
```

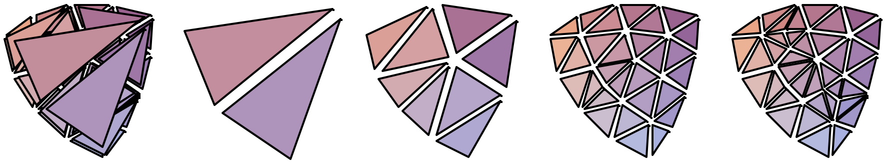
```
81<
```

```
Polygonize3D@MySphere@1.0, x, y, zD, 8x, -1.1, 0.1<,
 8y, -0.1, 1.1<, 8z, -1.1, 0.1<, Statistics ® 1, Boxed ® False,
 FlatnessAngle ® 30, DisplayFunction ® Identity,
 MultiGrid ® 1, PlotPoints ® 2, MaxAdaptiveRecursion ® 5D;
```

```
1.563 Second, 36 Polygons, 762 Function calls, 0.125 WebSizeHKbL, 43.7031
  MemoryHKbL, 4 MaxAdaptive Level, 0 Error, 16.8943 Minimal Edge FoundH%L.
```

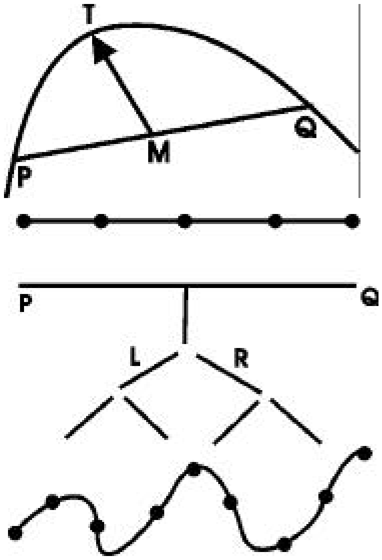
```
test2 = Table@Plot3DWeb@n, 1, 0.2, 0, DisplayFunction ® Identity, Boxed ® FalseD,
   8n, 0, 4<D •• Transpose
Show@GraphicsArray@test2@@1DDDD;
```

```
88… Graphics3D …, … Graphics3D …, … Graphics3D …, … Graphics3D …, … Graphics3D …<,
 852, 2, 8, 28, 36<, 80.384375, 0.384375, 0.13125, 0.0375, 0<<
```



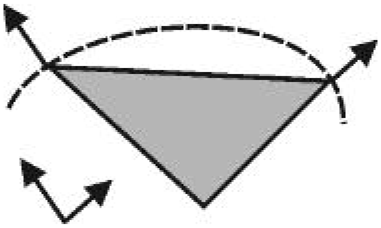
We need to find a point on the implicit surface f^{-1}H0L that is closest to the edge midpoint M. The problem amounts to computing the point T such that f(T)=0 and, at the same time minimizes the distance þT-Mþ. This can be solved using a numerical technique such as gradient descent. However, we employ a physically-based method [LV1]. In its simplest version, the method is:

```
d=SteepestDescentStep
while Èf(x)È>Tolerance
        y ¬ f(x)
        x¬ x-d sign f(y) Ñf(x)
        if sign(y)!= sign(f(x)) then d¬d/2
```



The criteria to classify an edge as **FLAT** or **NOTFLAT** is controlled by the option **AdaptiveStyle**. It can take the values:

0: The surface curvature along an edge is measured by the angle a between the surface normals at the edge endpoints.If the angle is greater than **FlatnessAngle**, the edge is classified as *not flat*.



```
8$FlatnessAngle, $AdaptiveStyle<
```

```
860, 0<
```

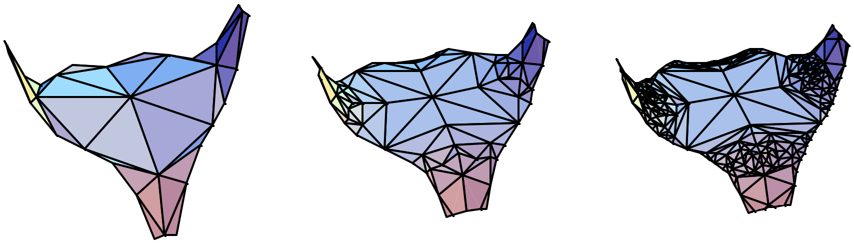
```
Saddle@x_, y_, z_D := z - Hx y L^3
```

```
Show@GraphicsArray@
  Polygonize3D@Saddle@x, y, zD, 8x, -1.1, 1.1<,
     8y, -1.1, 1.1<, 8z, -1, 1<, MaxAdaptiveRecursion ® 10,
     Boxed ® False, CFDecomposition ® 81, 2, 3, 4<, Statistics ® 1,
     DisplayFunction ® Identity, FlatnessAngle ® #D & •ž 860, 30, 15<DD;
```

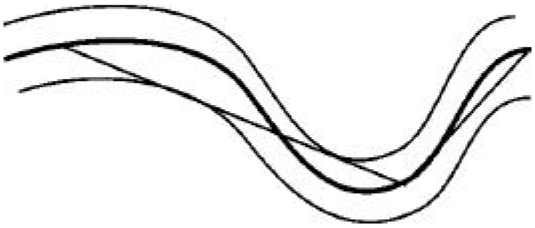
```
1.822 Second, 40 Polygons, 523 Function calls, 0.125 WebSizeHKbL, 5.42188
  MemoryHKbL, 2 MaxAdaptive Level, 0 Error, 6.71193 Minimal Edge FoundH%L.
```

```
4.556 Second, 120 Polygons, 1567 Function calls, 0.125 WebSizeHKbL, 13.5469
  MemoryHKbL, 4 MaxAdaptive Level, 0 Error, 4.02737 Minimal Edge FoundH%L.
```

```
15.312 Second, 460 Polygons, 3883 Function calls, 0.125 WebSizeHKbL, 48.0156
  MemoryHKbL, 6 MaxAdaptive Level, 0 Error, 2.01368 Minimal Edge FoundH%L.
```



1: If the distance from M to T is greater than **TubularPrecision**, the edge is classified as *not flat*.

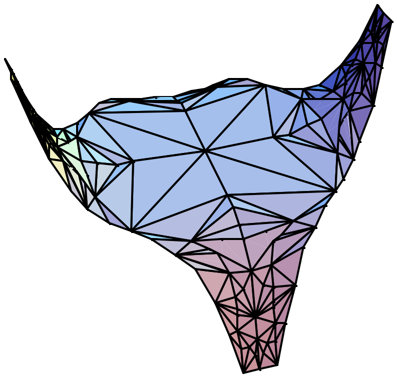


```
8$AdaptiveStyle, $TubularPrecision<
```

```
        1
90, €€€€€€€ =
       100
```

```
Polygonize3D@Saddle@x, y, zD, 8x, -1.1, 1.1<, 8y, -1.1, 1.1<,
 8z, -1, 1<, MaxAdaptiveRecursion ® 10, Boxed ® False,
 CFDecomposition ® 81, 2, 3, 4<, Statistics ® 1, AdaptiveStyle ® 1D;
```

```
11.247 Second, 304 Polygons, 3811 Function calls, 0.125 WebSizeHKbL, 32.1563
  MemoryHKbL, 5 MaxAdaptive Level, 0 Error, 2.5845 Minimal Edge FoundH%L.
```



Also, in order to avoid small edges, we can force an edge to be considered *Flat* if the length of the segment is smaller than a fraction of the coarser cell and is being set by the option **ToleranceSize**. In order to use this feature, the value of the option **PartitionStyle** must be 2 or 3.

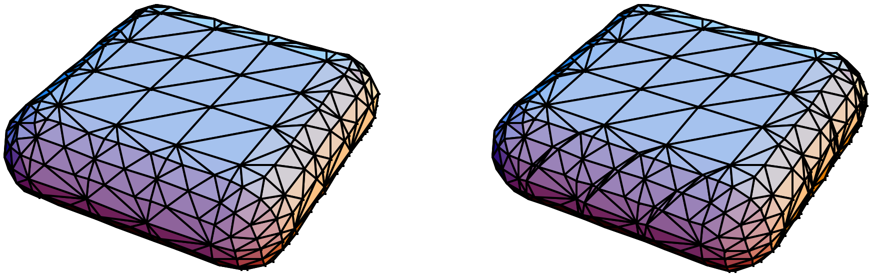
```
8$PartitionStyle, $ToleranceSize<
```

```
     1
90, €€€€=
     6
```

```
Show@GraphicsArray@
  Polygonize3D@MyOffset@0.5, x, y, zD, 8x, -1.6, 1.6<, 8y, -1.6, 1.6<,
    8z, -.5, .5<, PlotPoints ® 86, 6, 3<, MaxAdaptiveRecursion ® 10,
    ToleranceSize ® 1 • #, FlatnessAngle ® 30, PartitionStyle ® 2,
    Boxed ® False, DisplayFunction ® Identity, Statistics ® 1D & •ž 85, 15<DD;
```

```
26.418 Second, 700 Polygons, 11056 Function calls, 0.125 WebSizeHKbL, 68.7188
  MemoryHKbL, 4 MaxAdaptive Level, 0 Error, 9.81683 Minimal Edge FoundH%L.
```

```
31.886 Second, 848 Polygons, 13808 Function calls, 0.125 WebSizeHKbL, 83.1719
  MemoryHKbL, 4 MaxAdaptive Level, 0 Error, 2.5 Minimal Edge FoundH%L.
```



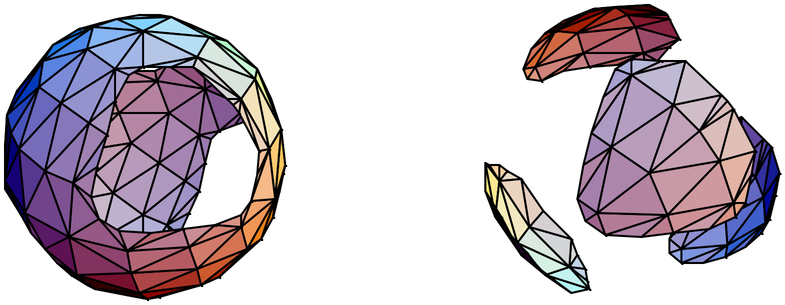
It is also possible to specify a trimming function. Once an edge is classified as Flat, if the option **Trimming** defines a function, the adaptive edge generation subrutine will generate a vertex by finding the intersection of the segment with the specified function.

```
8$Trimming<
```

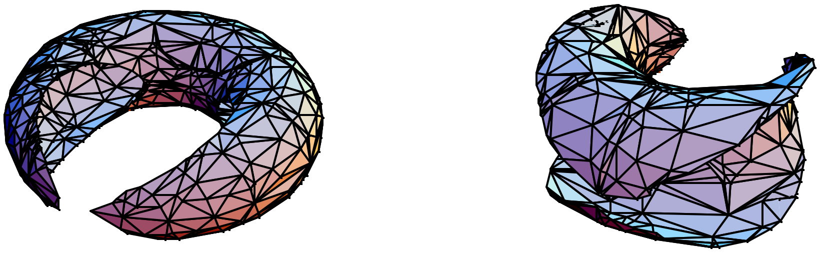
```
80<
```

```
MyPlanes@x_, y_, z_D := x y z
```

```
Show@GraphicsArray@
  Polygonize3D@MySphere@1.0, x, y, zD, 8x, -1, 1<, 8y, -1, 1<, 8z, -1, 1<, Boxed ®
      False, Trimming ® # HMyPlanes@x, y, zD + 0.05L, MaxAdaptiveRecursion ® 10,
     FlatnessAngle ® 30, DisplayFunction ® IdentityD & •ž 8-1, 1<DD;
```



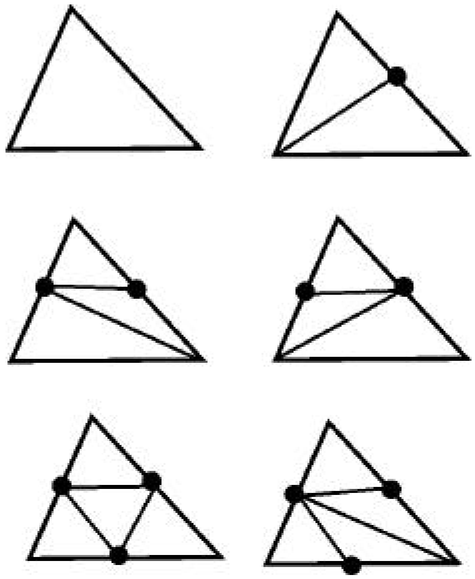
```
Show@GraphicsArray@
  Polygonize3D@MyTorus@2.1, x, y, zD, 8x, -3.1, 3.1<, 8y, -3.1, 3.1<, 8z, -3.1,
      3.1<, PlotPoints ® 85, 5, 3<, Trimming ® # MySphere@2.0, x, y + 1.1, zD,
     MaxAdaptiveRecursion ® 10, Boxed ® False, DisplayFunction ® Identity,
     FlatnessAngle ® 30, PartitionStyle ® 2D & •ž 8-1, 1<DD;
```



Ÿ Cell structuring

The cell structuring operation performs a simplicial decomposition. It recursively subdivides a triangular cell into triangular subcells, using information from its edges. For this purpose, new internal edges are created and adaptively sampled.

The figure below shows the different cases:



note that for the cases where two or three edges are not flat, we have different options in subdividing the cell. The Quality criteria is specified with **QualityForm**. The different options are generated and the best triangulation is selected.

$$Q1 = \frac{l_{min}}{R}$$

$$Q2 = \frac{r}{l_{max}}$$

$$Q3 = \frac{r}{R}$$

$$Q4 = \frac{l_{min}}{l_{max}}$$

$$Q5 = \frac{A}{l^2_{max}}$$

$$Q6 = \mathrm{Min@Cos@}a_i\mathrm{DD}$$

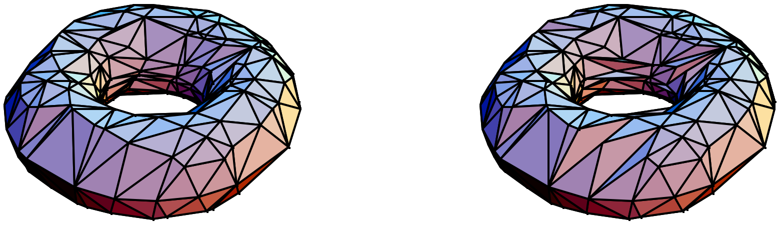
$$Q7 = \mathrm{Max@Cos@}a_i\mathrm{DD}$$

If **PartitionStyle** is equal to 1 or 3, the cells structuring will test all possible subdivisions. This can make the method run much slower than needed. Normally it is only necessary considering the first case of the three nonflat edges case.

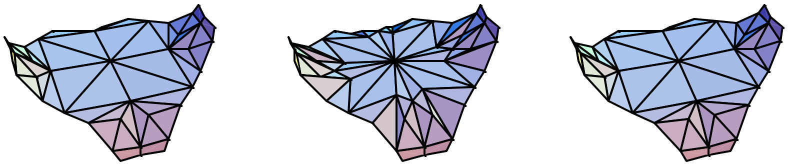
```
8$PartitionStyle, $QualityForm<

80, 4<
```

```
Show@GraphicsArray@
  Polygonize3D@MyTorus@2.1, x, y, zD, 8x, -3.1, 3.1<,
     8y, -3.1, 3.1<, 8z, -3, 3<, MaxAdaptiveRecursion ® 2
     , Boxed ® False, Statistics ® 1, PlotPoints ® 85, 5, 3<,
     DisplayFunction ® Identity, FlatnessAngle -> 30,
     PartitionStyle ® #, QualityForm ® 4D & •ž 82, 3<DD;
```

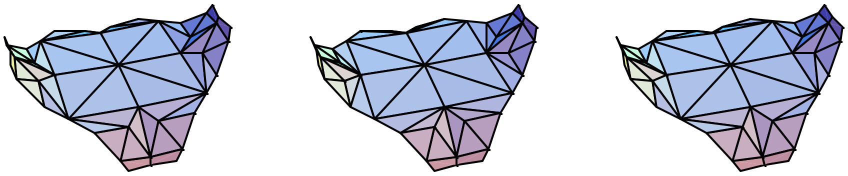
```
20.66 Second, 500 Polygons, 13816 Function calls, 0.125 WebSizeHKbL, 49.0313
  MemoryHKbL, 2 MaxAdaptive Level, 0 Error, 1.8117 Minimal Edge FoundH%L.

20.459 Second, 500 Polygons, 13816 Function calls, 0.125 WebSizeHKbL, 49.0313
  MemoryHKbL, 2 MaxAdaptive Level, 0 Error, 1.8117 Minimal Edge FoundH%L.
```



```
Show@GraphicsArray@
  Polygonize3D@Saddle@x, y, zD, 8x, -1.1, 1.1<, 8y, -1.1, 1.1<, 8z, -1, 1<,
     MaxAdaptiveRecursion ® 3, Boxed ® False, CFDecomposition ® 81, 2, 3, 4<,
     FlatnessAngle ® 30, QualityForm ® #, PartitionStyle ® 2,
     ToleranceSize ® 1 • 5, DisplayFunction ® IdentityD & •ž 81, 2, 3<DD;
```



```
Show@GraphicsArray@
  Polygonize3D@Saddle@x, y, zD, 8x, -1.1, 1.1<, 8y, -1.1, 1.1<, 8z, -1, 1<,
     MaxAdaptiveRecursion ® 3, Boxed ® False, CFDecomposition ® 81, 2, 3, 4<,
     FlatnessAngle ® 30, QualityForm ® #, PartitionStyle ® 2,
     ToleranceSize ® 1 • 5, DisplayFunction ® IdentityD & •ž 84, 5, 6<DD;
```



Ÿ The Algorithm

The basic algoritmis as follows:

```
I) [Initialization]
        Start with a coarse decomposition of the surface
        a) Generate the base mesh
        b) Sample the edges of all cells in the base mesh
II) [Refinement]
        For each cell, test the corresponding surface patch for flatness
        If the patch is not flat, then recursively subdivide the cell
        a) Structure new cells by constructing internal edges
        b) Sample all internal edges
```

The procedures implementing structuring and sampling operations comunicate through a well-defined inyterface: the *edge* and *cell* data structures

Ÿ Data Structures

```
struct VertexElement={vertexid vid;
    Attributes attributes {FunctionValue,Region,Contact,Normal vector};
    Coordinate p;
    edgesid*edges;};
```

```
?? Polygonize`DataHandlingv3`private`vvH
```

```
Polygonize`DataHandlingv3`private`vvH@-21504D =
 8-21504, 82.63, 1, False, 80.57735, 0.57735, 0.57735<<, 81.1, 1.1, 1.1<, 8<<

Polygonize`DataHandlingv3`private`vvH@-20992D =
 8-20992, 81.42, 1, False, 80.707107, 0.707107, 0.0007064<<, 81.1, 1.1, 0.<, 8<<

Polygonize`DataHandlingv3`private`vvH@-20480D =
 8-20480, 82.63, 1, False, 80.577735, 0.577735, -0.57658<<, 81.1, 1.1, -1.1<, 8<<

Polygonize`DataHandlingv3`private`vvH@-19456D =
 8-19456, 81.42, 1, False, 80.707107, 0.0007064, 0.707107<<, 81.1, 0., 1.1<, 8<<

Polygonize`DataHandlingv3`private`vvH@-18944D =
 8-18944, 80.21, 1, False, 80.999999, 0.000999, 0.000999<<, 81.1, 0., 0.<, 8<<

Polygonize`DataHandlingv3`private`vvH@-18432D =
 8-18432, 81.42, 1, False, 80.707813, 0.000707106, -0.706399<<, 81.1, 0., -1.1<, 8<<
```

```
struct FacetElement={facetid fid;
    Attributes attributes {Adaptive Level,Child facets,error,aspect ratio,{child aspect ratio},Parent facet ID};
    edgesid*edges {Adaptive edges IDs,Index};
    vertexid*vertices;};
```

```
Polygonize`DataHandlingv3`private`ffH
```

```
Polygonize`DataHandlingv3`private`ffH@2D =
 82, 81, 8<, 0., 0.756132, 8<, 0<, 881, 2, 3<, 81, 1, 1<<, 8-8706, -8195, -7<<

Polygonize`DataHandlingv3`private`ffH@3D =
 83, 81, 8<, 0., 0.756132, 8<, 0<, 884, 2, 5<, 81, 1, 1<<, 8-10241, -8195, -7<<

Polygonize`DataHandlingv3`private`ffH@4D =
 84, 81, 8<, 0., 0.756132, 8<, 0<, 886, 7, 3<, 81, 1, 1<<, 8-8706, -518, -7<<

Polygonize`DataHandlingv3`private`ffH@5D =
 85, 81, 8<, 0., 0.756132, 8<, 0<, 888, 7, 9<, 81, 1, 1<<, 8-2564, -518, -7<<

Polygonize`DataHandlingv3`private`ffH@6D =
 86, 81, 8<, 0., 0.756132, 8<, 0<, 8810, 11, 5<, 81, 1, 1<<, 8-10241, -2053, -7<<

Polygonize`DataHandlingv3`private`ffH@7D =
 87, 81, 8<, 0., 0.756132, 8<, 0<, 8812, 11, 9<, 81, 1, 1<<, 8-2564, -2053, -7<<

Polygonize`DataHandlingv3`private`ffH@8D =
 88, 81, 8<, 0., 0.853575, 8<, 0<, 8813, 6, 14<, 81, 1, 1<<, 8-10753, -8706, -518<<

Polygonize`DataHandlingv3`private`ffH@9D =
 89, 81, 8<, 0., 0.853575, 8<, 0<, 8815, 8, 14<, 81, 1, 1<<, 8-10753, -2564, -518<<
```

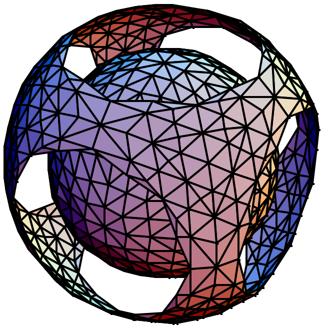
```
struct AdaptiveEdge={adaptiveid aid;
   vid p,q;
   double e;
   TreeNode*T;}
struct TreeNode={vectorid t;
     Vector d;
     double e;
     TreeNode*L, *R;};
```

```
Polygonize`DataHandlingv3`private`aaH


Polygonize`DataHandlingv3`private`aaH@1D = 81, -8706, -8195,
  0.078125, 816, 80.0000934693, -0.0721992, -0.0298459<, 0.078125, 8<, 8<<<

Polygonize`DataHandlingv3`private`aaH@2D = 82, -8195, -7,
  0.046875, 817, 8-0.0141765, -0.0315934, -0.0315934<, 0.046875, 8<, 8<<<

Polygonize`DataHandlingv3`private`aaH@3D =
 83, -7, -8706, 0.125, 818, 8-0.0405627, -0.11106, -0.0405627<, 0.125, 8<, 8<<<

Polygonize`DataHandlingv3`private`aaH@4D = 84, 16, 17, 0., 8<<

Polygonize`DataHandlingv3`private`aaH@5D = 85, 17, 18, 0., 8<<

Polygonize`DataHandlingv3`private`aaH@6D = 86, 18, 16, 0., 8<<

Polygonize`DataHandlingv3`private`aaH@7D = 87, -10241, -8195,
  0.078125, 819, 80.0000934693, -0.0298459, -0.0721992<, 0.078125, 8<, 8<<<

Polygonize`DataHandlingv3`private`aaH@8D =
 88, -7, -10241, 0.125, 820, 8-0.0405627, -0.0405627, -0.11106<, 0.125, 8<, 8<<<
```

Ÿ Examples

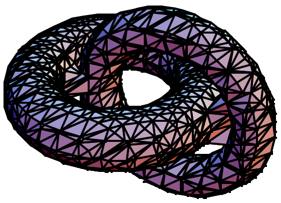
```
Polygonize3D@MySphere@1, x, y, zD, 8x, -1.1, 1.1<, 8y, -1.1, 1.1<,
 8z, -1.1, 1.1<, Contours ® 80.0, 0.9<, Boxed -> False, Statistics ® 1,
 FlatnessAngle ® 15, CFDecomposition ® 81, 2, 3, 4<, MaxAdaptiveRecursion ® 10D;
```

```
61.819 Second, 1488 Polygons, 27258
  Function calls, 0.125 WebSizeHKbL, 149.133 MemoryHKbL, 3
  MaxAdaptive Level, 0 Error, 4.6223 Minimal Edge FoundH%L.
```

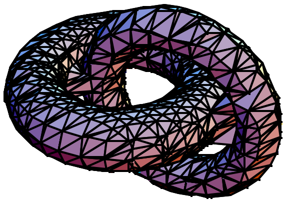


```
MyTorusxy@a_, b_, x_, y_, z_D := Hx^2 + y^2 + z^2 + a^2 - b^2L^2 - 4 a^2 Hx^2 + y^2L
MyTorusxz@a_, b_, x_, y_, z_D := Hx^2 + y^2 + z^2 + a^2 - b^2L^2 - 4 a^2 Hx^2 + z^2L
MyTorusyz@a_, b_, x_, y_, z_D := Hx^2 + y^2 + z^2 + a^2 - b^2L^2 - 4 a^2 Hy^2 + z^2L
```

```
Polygonize3D@Min@MyTorusxy@2.0, 0.7, x, y, zD, MyTorusxz@2, 0.5, x - 1.5, y, zDD,
 8x, -3, 4<, 8y, -3, 3<, 8z, -3, 3<, PlotPoints ® 815, 15, 15<,
 MaxAdaptiveRecursion ® 1, PartitionStyle ® 0, ToleranceSize ® 1 • 5, Boxed ® FalseD;
```



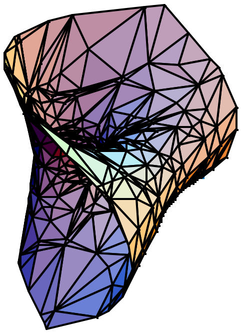
```
Polygonize3D@Min@MyTorusxy@2.0, 0.7, x, y, zD, MyTorusxz@2, 0.5, x - 1.5, y, zDD,
 8x, -3, 4<, 8y, -3, 3<, 8z, -3, 3<, PlotPoints ® 815, 15, 15<,
 PartitionStyle ® 2, ToleranceSize ® 1 • 3, Boxed ® False, MaxAdaptiveRecursion ® 1D;
```



```
Polygonize3D@MyTorusxy@2.0, 0.7, x, y, zD - MyTorusxz@2, 0.5, x - 1.5, y, zD,
 8x, -3, 4<, 8y, -3, 3<, 8z, -3, 3<, PlotPoints ® 85, 5, 5<,
 MaxAdaptiveRecursion ® 10, PartitionStyle ® 2,
 ToleranceSize ® 1 • 40, Boxed ® False, FlatnessAngle ® 30,
 QualityForm ® 2, DisplayFunction ® Identity, MultiGrid ® 1D
```

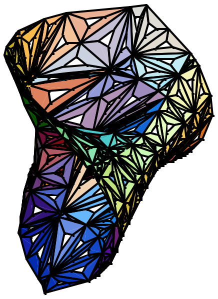
```
… Graphics3D …
```

```
Plot3DWeb@5, 1.0, 0, 0, Boxed ® FalseD
```



```
8… Graphics3D …, 1034, 0.478739<
```

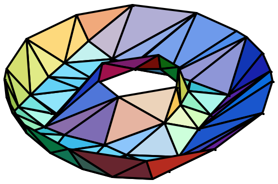
```
Plot3DWeb@1, 1.0, -0.7, 1, Boxed ® FalseD
```



```
8… Graphics3D …, 828, 0.627595<
```

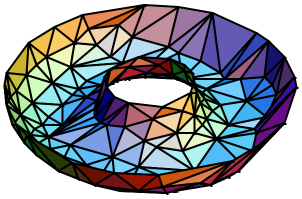
```
Polygonize3D@MyTorus@1.9, x, y, zD, 8x, -3.0, 3.0<,
  8y, -3.0, 3.0<, 8z, -3.0, 0.0<, PlotPoints ® 85, 5, 3<,
  MaxAdaptiveRecursion ® 10, QualityForm ® 4, PartitionStyle ® 2,
  CFDecomposition ® 81<, Boxed ® False, FlatnessAngle ® 30,
  MultiGrid ® 1, ToleranceSize ® 1 • 6, DisplayFunction ® IdentityD;
```

```
Plot3DWeb@1, 1, 0, 1, Boxed ® FalseD
```



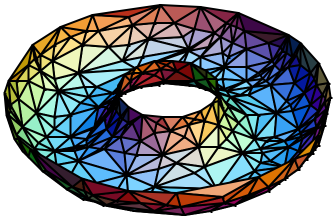
```
8… Graphics3D …, 92, 0.52251<
```

```
Plot3DWeb@2, 1, 0, 1, Boxed ® FalseD •• Timing
```



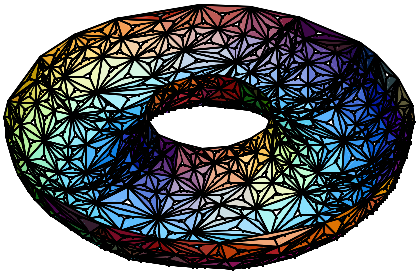
```
81.371 Second, 8… Graphics3D …, 280, 0.217938<<
```

```
Plot3DWeb@4, 1, 0, 1, Boxed ® FalseD •• Timing
```



```
82.684 Second, 8… Graphics3D …, 620, 0<<
```

```
Plot3DWeb@4, 1, -0.8, 1, Boxed ® FalseD •• Timing
```



```
86.309 Second, 8… Graphics3D …, 1860, 0<<
```

Ÿ Performance

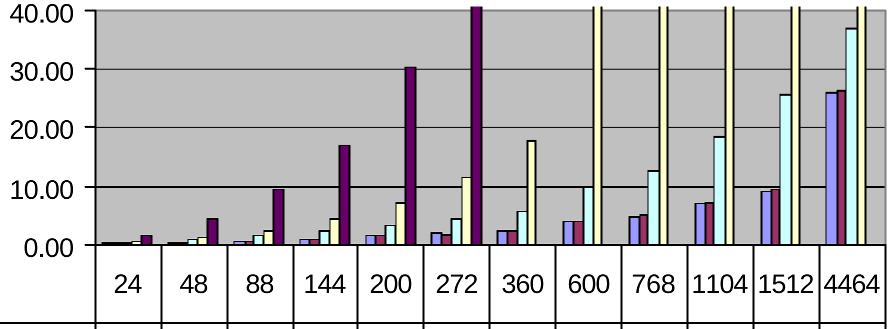
Data structures for vertex, facets and adaptive edges have being implemented in three different ways:

i) Mathematica Lists. Append[] and Position[]
ii) Caching values. See examples above
iii) ExpertBags.

The following are graphs of running times, function calls and memory requirements:

V1: No data structure. Repetive function evaluation.
V1c: No data structure. Cachinf function values f[x_]:=f[x]=...
V2: Data structure implemented using i)
V2c: Data structure implemented using ii)
V3: Data structure implemented using iii)

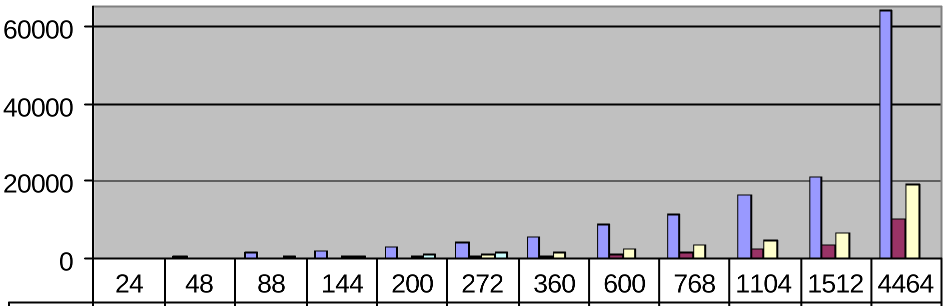
Running Time (Level 1)



	24	48	88	144	200	272	360	600	768	1104	1512	4464
■ V1	0.19	0.37	0.68	0.98	1.38	1.97	2.44	3.93	4.92	7.11	9.02	26.16
■ V1 Cached	0.20	0.39	0.72	1.00	1.39	1.95	2.47	3.88	5.16	7.24	9.28	26.24
□ V2 Cached	0.43	0.82	1.56	2.48	3.38	4.53	5.86	9.73	12.79	18.54	25.75	36.83
□ V2	0.61	1.24	2.35	4.48	7.17	11.42	17.80	42.63	68.56	131.9	241.3	353.6
■ V3	1.63	4.53	9.46	16.98	30.34	48.16						

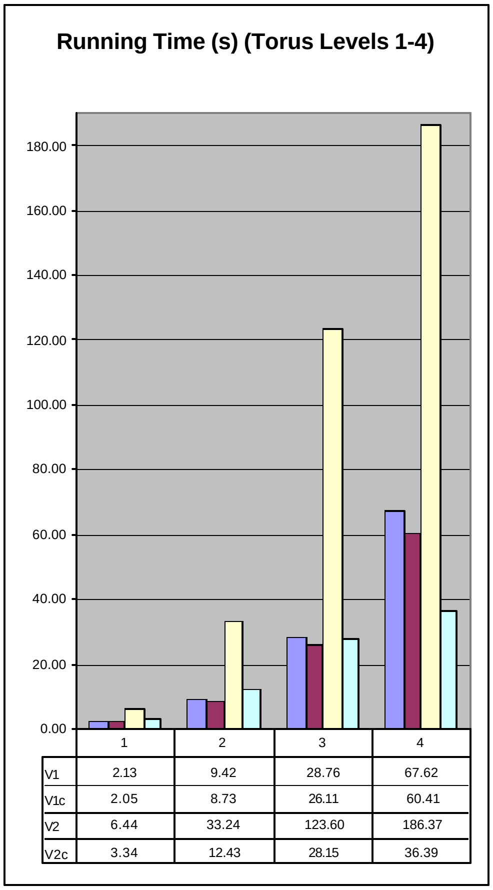
Triangles

Function Calls, Level 1

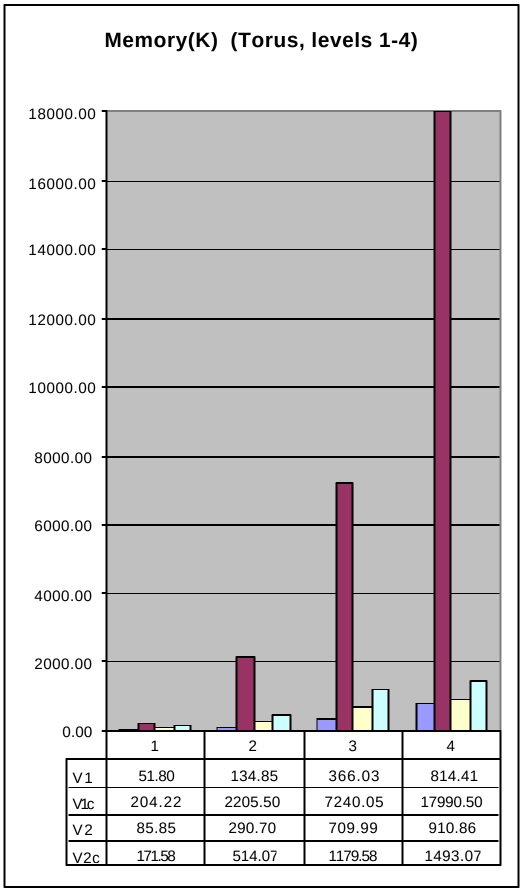


	24	48	88	144	200	272	360	600	768	1104	1512	4464
■ V1	472	928	1692	2316	3324	4700	5864	8848	11772	16640	20908	64096
■ V1c	107	188	320	420	598	834	1025	1544	2017	2868	3583	10487
□ V2	163	292	504	716	1006	1386	1753	2752	3561	5084	6615	19423
□ V3	223	428	686	942	1344	1704						

Triangles


Running Time (s) (Torus Levels 1-4)

	1	2	3	4
V1	2.13	9.42	28.76	67.62
V1c	2.05	8.73	26.11	60.41
V2	6.44	33.24	123.60	186.37
V2c	3.34	12.43	28.15	36.39


Memory(K) (Torus, levels 1-4)

	1	2	3	4
V1	51.80	134.85	366.03	814.41
V1c	204.22	2205.50	7240.05	17990.50
V2	85.85	290.70	709.99	910.86
V2c	171.58	514.07	1179.58	1493.07

Ÿ Current and Future work

Parametric Adaptive polygonization
Adaptive ListContourPlot
Non-Manifold surfaces
4D -Meshes

à Polygonize`Polygonize3D`

Polygonize3D is the tetrahedral analog of CountourPlot3D which also includes adaptive refinement.

Polygonize3D will plot surfaces showing particular values of *f* as a function of *x*, *y* and *z*. Polygonize3D works by dividing the three-dimensional space into cubes and deciding if the surface intersect each cube. If the surface does intersect a cube, Polygonize3D will subdivide this cube further, and so on.

ContourPlot3D@ *f*, 8*x*, *xmin*, *xmax*<, 8*y*, *ymin*, *ymax*<, 8*z*, *zmin*, *zmax*<D	generate a three-dimensional contour plot *f*
Plot3DWeb@*level*, *quality*, *gap*, *normals*D	generates *a* three - dimensionalplot of

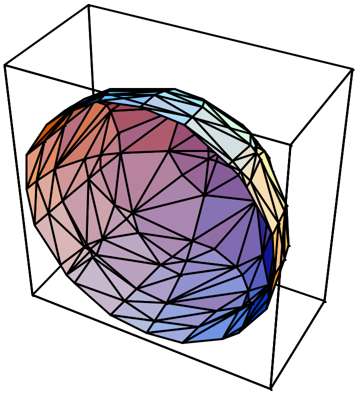
This loads the package

```
<< Polygonize`Polygonize3Dtrm`
```

```
<< "D:\Wolfram\Polygonize\Polygonize3Dtrm.m"
```

This produces a three-dimentional plot of a zero values of the function.

```
Polygonize3D[Cos[Sqrt[ x^2 + y^2 + z^2 ]], {x,-2,2}, {y,0,2}, {z,-2,2},
PlotPoints->5];
```



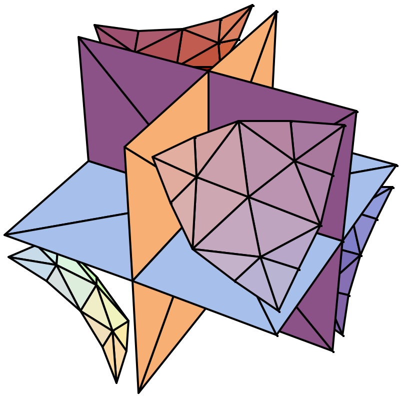
option name	*default value*	
`Contours`	`80.<`	the list of values for the contours to be plotted
`ContourStyle`	`8<`	the list of styles for the contours to be plotted
`MaxRecursion`	`1`	the number of levels of recursion used in each cube
`PlotPoints`	`83,5<`	the number of evaluation points to use in each direction
MaxProjectionIter	$MaxProjectionIter = 10	maximumnumberof iterations forprojectionstep
MaxBisectionIter	$MaxBisectionIter = 10	maximumnumberof iterations forbisectionstep
FlatnessAngle	$FlatnessAngle = 60	Adaptive criteria forAdaptiveStyle -> 0
Tolerance	$Tolerance = 10^{-2}	Tolerance forprojectionand bisection steps
SteepestDescentStep	$SteepestDescentStep = 0.125	Initialstep forthe steepest descent step
GradientStep	$GradientStep = 10^{-3}	delta value for finitedifference approximation
ToleranceAR	$ToleranceAR = - 1.0	Triangle aspect ratio tolerance foradaptive style > 1
ToleranceSize	$ToleranceSize = $\frac{1}{6}$	Triangle size tolerancefor adaptive style > 1
TubularPrecision	$TubularPrecision = 10^{-2}	Adaptive criteria forAdaptiveStyle ® 1
Gap	$Gap = 0	Plotting gap, range@- 1, 1D
CFDecomposition	$CFDecomposition = 81<	Coxeter - Freudenthal decomposition
IntersectionStyle	$IntersectionStyle = 0	Segment - surface intersection, 0 : Bisection subdivision 1 : Proportional scaling
QualityForm	$QualityForm = 4	Factor formcriteria
PartitionStyle	$PartitionStyle = 0	Adaptive Subdivision : 0 :
AdaptiveStyle	$AdaptiveStyle = 0	Adaptive Criteria : 0 : Flatness angle 1 : Tubularprecision
MultiGrid	$MultiGrid = 0	Data stored : 0 : Just higherlevel, initialreset data 1 : All levels , initialreset data 2 : All levels , no initialreset data
Statistics	$Statistics = 0	Verbosereport
Trimming	$Trimming = 0	Trimming function

Options for Polygonize3D

Each value specified in `Contours` generates a different surface. `ContourStyle` colors each surface. To use this option, you must set `Lighting -> False`. `MaxAdaptiveRecursion` sets the number of times you subdivide each initial triangle. However, if the surface does not intersect the cube, the cube is not subdivided into tetrahedrals and no initial triangles are generated. If `MaxAdaptiveRecursion` is greater than 0, adaptive recursion takes place. Polygonize`3D` and `Plot3DWeb` return a `Graphics3D` object. This means the functions will accept any option that can be specified for a `Graphics3D` object.

Here is another plot showing a contour value of -0.2 and 0.0.

```
Polygonize3D@x y z, 8x, -1, 1<, 8y, -1, 1<, 8z, -1, 1<, Contours ® 80., -0.2<,
  CFDecomposition ® 81, 2, 3, 4<, MaxAdaptiveRecursion ® 10,
  FlatnessAngle ® 30, SteepestDescentStep ® 0.125 • 4.0, Boxed ® FalseD;
```



option name	*default value*	
`Contours`	`80.<`	level plotted, 0->All levels
`ContourStyle`	`8<`	Cuttoff criteria foraspect ratio
`MaxRecursion`	`1`	Plotting gap
`PlotPoints`	`83,5<`	Include triangle normals in plotting

Parameters for Plot3DWeb

à Polygonize`DataHandlingv3`

à **Bibliography**

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