Introduction

Discrete exterior calculus (DEC) offers a coordinate–free discretization of exterior calculus especially suited for computations on curved spaces. As such it is arguably one of the prevalent numerical frameworks to derive discrete differential operators used for geometry processing tasks.

Motivation

The vast majority of work on DEC is restricted to triangle meshes. Our goal is to extend DEC on surface meshes formed directly on polygons. Our approach offers commonly associated to DEC that operate directly on polygons. Our approach offers three main practical benefits:

- By working directly with polygonal meshes, we overcome the ambiguities of subdividing a discrete surface into a triangle mesh.
- Our construction operates solely on primal elements, thus removing any dependency on dual meshes.
- Our method includes the discretization of new differential operators such as the contraction operator.

Results

- We introduce a polygonal wedge product compatible with the discrete exterior derivative in the sense that it obeys the Leibniz product rule.
- We define a novel primal–to–primal Hodge star operator that is compatible with the polygonal wedge product.
- Using these two operators we derive a discrete inner product and a discrete contraction operator.

Discrete Hodge Star Operator

The discrete Hodge star operator is defined as

\[ \ast \alpha^i = W_f \alpha^i, \quad W_f \in \mathbb{R}^{2\times2}, \quad W_f[i, j] = |f|, \]

\[ \langle \ast \beta^j, \gamma \rangle_f = W_R \langle \beta^j, \gamma \rangle_f, \quad W_R \in \mathbb{R}^{n\times n}, \quad W_R[i, j] = \{(\omega_i, \omega_j)\} \]

\[ \ast \omega^i = W_c^i \omega^i, \quad W_c \in \mathbb{R}^{2\times2}, \quad W_c[i, j] = \sum_{k=0}^{n-1} |f_k|, \]

where \( W_f \) is a matrix with edge midpoint vectors as rows (we take the barycenter of each p-gon \( f \) as the center of origin per face).

\[ M_{i,j} = W_f f_{ij} W_f^{-1} f_{ij}, \]

where \( M_{i,j} \) is the matrix of product of two 1–forms restricted to a face \( f \). Our inner product of 1–forms is identical to the one of [AW11, Lemma 3], that is,

\[ \langle \ast \alpha^i, \beta^j \rangle_f = \frac{1}{|B_f|} \sum_{k=0}^{n-1} \langle \alpha^i, \beta^j \rangle_k \]

where \( B_f \) is a matrix with edge midpoint vectors as rows (we take the barycenter of each p-gon \( f \) as the center of origin per face).

Discrete Inner Product and Contraction Operator

Just like on Riemannian manifolds, we define the discrete L–Hodge inner product of two l–forms \( \alpha, \beta \) by

\[ \langle \alpha, \beta \rangle = \sum_f \langle \alpha, \beta \rangle_f, \]

thus in matrix form it reads:

\[ M_{i,j} = f_{ij} W_f f_{ij} W_f^{-1} f_{ij}, \]

where \( M_{i,j} \) is the matrix of product of two 1–forms restricted to a face \( f \). Our inner product of 1–forms is identical to the one of [AW11, Lemma 3], that is,

\[ \langle \ast \alpha^i, \beta^j \rangle_f = \frac{1}{|B_f|} \sum_{k=0}^{n-1} \langle \alpha^i, \beta^j \rangle_k \]

where \( B_f \) is a matrix with edge midpoint vectors as rows (we take the barycenter of each p-gon \( f \) as the center of origin per face).

Ongoing Work

Currently, we are numerically evaluating a discrete Lie derivative given by the Cartan’s magic formula

\[ L_X = i_X d + d i_X. \]

We are also starting to study a discrete Hodge decomposition using a discrete codifferential

\[ \delta(\alpha^i) = (-1)^{k+1} d \ast d \alpha^i, \]

where \( \ast \) is our discrete Hodge star, and using a discrete Laplacian given by

\[ \Delta = bd + db. \]

Conclusion

- Geometry processing with polygonal meshes is a new developing area, e.g., see discrete Laplacians on general polygonal meshes in [AW11].
- We propose a novel discretization of several operators that act directly on general polygonal meshes, are compatible with each other and easy to implement.
- We believe that the generality of our framework will make it a useful tool in geometry processing tasks and will inspire further research in the area.

References


