A Primal–to–Primal Discretization of Exterior Calculus on Polygonal Meshes

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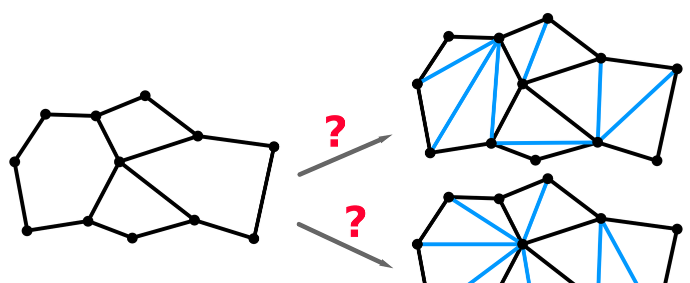
Introduction

Discrete exterior calculus (DEC) offers a coordinate–free discretization of exterior calculus especially suited for computations on curved spaces. As such it is arguably one of the prevalent numerical frameworks to derive discrete differential operators used for geometry processing tasks.

Motivation

The vast majority of work on DEC is restricted to triangle meshes. Our goal is to extend DEC on surface meshes formed by arbitrary polygons. We propose a new discretization for several operators commonly associated to DEC that operate directly on polygons. Our approach offers three main practical benefits:



- By working directly with polygonal meshes, we overcome the ambiguities of subdividing a discrete surface into a triangle mesh.
- Our construction operates solely on primal elements, thus removing any dependency on dual meshes.
- Our method includes the discretization of new differential operators such as the contraction operator.

Results

- We introduce a **polygonal wedge product** compatible with the discrete exterior derivative in the sense that it obeys the Leibniz product rule.
- We define a novel **primal–to–primal Hodge star operator** that is compatible with the polygonal wedge product.
- Using these two operators we derive a **discrete inner product** and a **discrete contraction operator**.

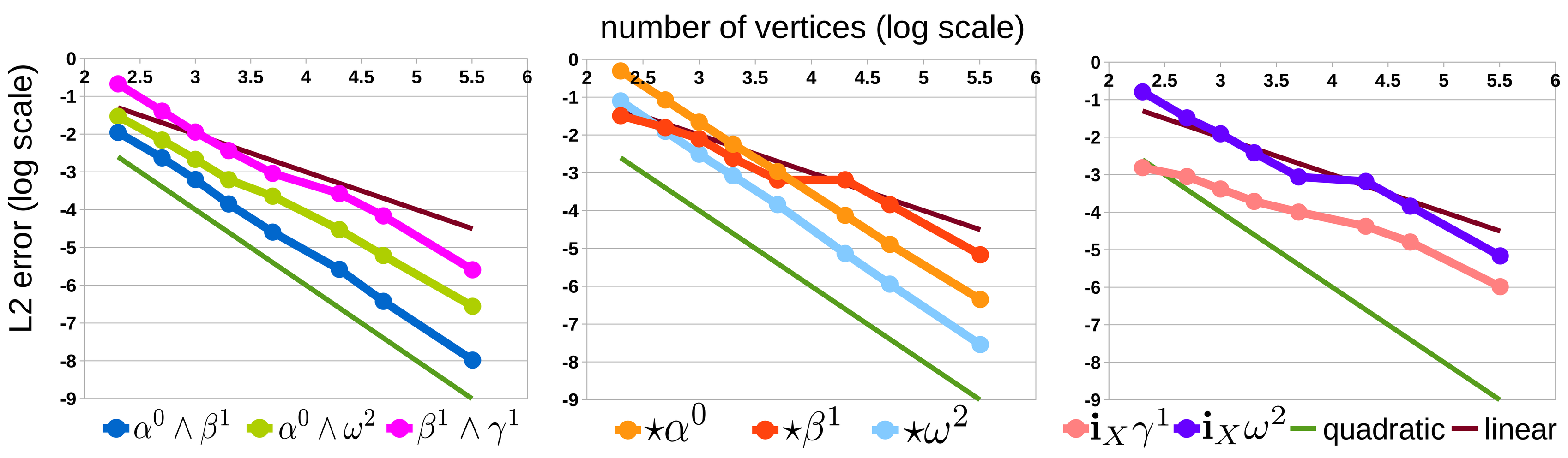


Figure 1: L^2 errors of approximation (axes in $\log_{10}$ scale) of the wedge product (L), Hodge star (C) and contraction operator (R) on a set of irregular polygonal meshes on a torus. Here $\alpha^0 = x^2+y^2$, $\beta^1 = -y dx + x dy$, $\gamma^1 = -2xz dx - 2yz dy + 2(x^2+y^2-\sqrt{x^2+y^2})dz$, $\omega^2 = \frac{\beta\wedge\gamma}{x^2+y^2}$, and $X = (-y, x, 0)$.

Primal–to–Primal Operations and Operators

Discrete Wedge Product

Our wedge product is a product of two forms of degree k and l that returns a form of degree $k+l$ located on primal $(k+l)$–dimensional cells, see Figure 2.

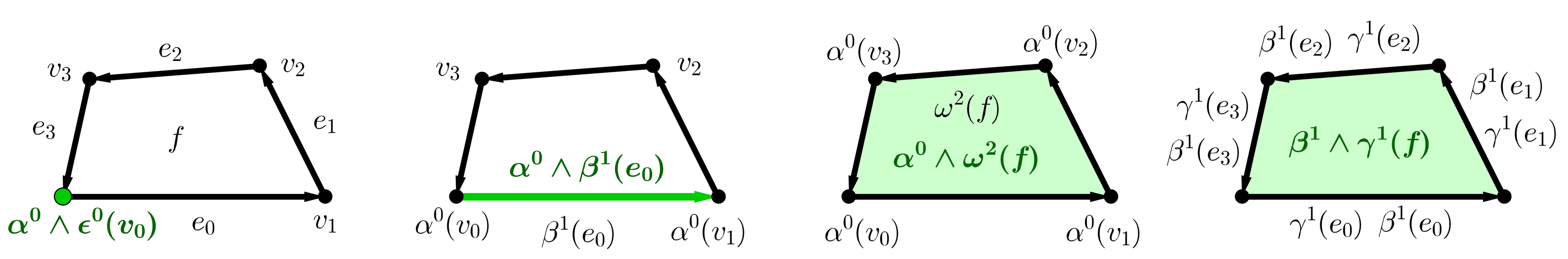


Figure 2: The wedge product on a quadrilateral: The product of two 0–forms is a 0–form located on vertices (far left). The product of a 0–form with a 1–form is a 1–form located on edges (center left). The product of a 0–form with a 2–form is a 2–form located on faces (center right), and the product of two 1–forms is a 2–form located on faces (far right).

In matrix form, the polygonal wedge product reads:

$$\alpha^0\wedge\epsilon^0 = \alpha^0\odot\epsilon^0, \tag{1}$$

$$\alpha^0\wedge\beta^1 = (\mathsf{B}\,\alpha^0)\odot\beta^1, \qquad \mathsf{B}\in\mathbb{R}^{|E|\times|V|}, \mathsf{B}[i,j] = \begin{cases}\frac{1}{2} & \text{if } v_j\prec e_i,\\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

$$\alpha^0\wedge\omega^2 = (\mathsf{fv}\,\alpha^0)\odot\omega^2, \qquad \mathsf{fv}\in\mathbb{R}^{|F|\times|V|}, \mathsf{fv}[i,j] = \begin{cases}\frac{1}{p_i} & \text{if } v_j\prec f_i, f_i \text{ is a } p_i\text{–gon},\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

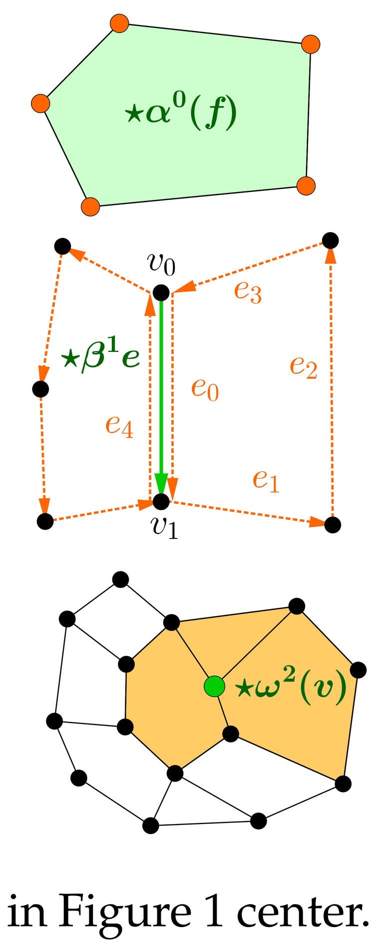
$$(\beta^1\wedge\gamma^1)|_f = (\beta^1|_f)^\top\mathsf{R}(\gamma^1|_f), \qquad \mathsf{R}\in\mathbb{R}^{p\times p}, \mathsf{R} = \sum_{a=1}^{\lfloor\frac{p-1}{2}\rfloor}\left(\frac{1}{2}-\frac{a}{p}\right)\mathsf{R}_a, f \text{ is a } p\text{–gon and} \tag{4}$$

$$\mathsf{R}_a[k,j] = \begin{cases}1 & \text{if } e_j \text{ is } (k+a)\text{–th halfedge of } f,\\ -1 & \text{if } e_j \text{ is } (k-a)\text{–th halfedge of } f,\\ 0 & \text{otherwise.}\end{cases} \tag{5}$$

where $\odot$ is the Hadamard (element–wise) product.

Our wedge product is a skew–commutative bilinear operation, but it is not associative in general. Convergence tests indicate at least linear convergence to analytical solutions, Figure 1 left depicts decreasing L^2 errors of approximation on a torus.

Discrete Hodge Star Operator



The discrete Hodge star operator is defined as

$$\star\alpha^0 = \mathsf{W}_F\,\mathsf{fv}\,\alpha^0, \qquad \mathsf{W}_F\in\mathbb{R}^{|F|\times|F|},\ \mathsf{W}_F[i,i] = |f_i|,$$

$$(\star\beta^1)|_f = \mathsf{W}_E\,\mathsf{R}^\top(\beta|_f), \qquad \mathsf{W}_E\in\mathbb{R}^{p\times p},\ \mathsf{W}_E[i,j] = \frac{\langle e_i, e_j\rangle}{|f|},$$

$$\star\omega^2 = \mathsf{W}_V^{-1}\,\mathsf{fv}^\top\,\omega^2, \qquad \mathsf{W}_V\in\mathbb{R}^{|V|\times|V|},\ W_V[i,i] = \sum_{f_k\succ v_i}\frac{|f_k|}{p_k}.$$

On 1–forms it is first defined per a p–polygonal face f. If e is not a boundary edge, it has two adjacent faces, we compute the values of $\star\beta^1$ on corresponding halfedges, sum their values with appropriate orientation sign and divide by 2. E.g. in the inset, $e=(v_0,v_1)$ has halfedges e_0, e_4, thus $\star\beta(e) = \frac{\star\beta(e_0)-\star\beta(e_4)}{2}$. Numerically, our approximation exhibits at least linear convergence on all tested cases, as illustrated in Figure 1 center.

Discrete Inner Product and Contraction Operator

Just like on Riemannian manifolds, we define the **discrete L^2–Hodge inner product** of two l–forms α,β by

$$(\alpha,\beta) := \sum_f(\alpha\wedge\star\beta)(f),$$

thus in matrix form it reads:

$$\mathsf{M}_0 = \mathsf{fv}^\top\,\mathsf{W}_F\,\mathsf{fv}, \qquad \mathsf{M}_1|_f = \mathsf{R}\,\mathsf{W}_E\,\mathsf{R}^\top, \qquad \mathsf{M}_2 = \mathsf{fv}\,\mathsf{W}_V^{-1}\,\mathsf{fv}^\top,$$

where $\mathsf{M}_1|_f$ is the matrix of product of two 1–forms restricted to a face f. Our inner product of 1–forms is identical to the one of [AW11, Lemma 3], that is,

$$\mathsf{R}\,\mathsf{W}_E\,\mathsf{R}^\top = \frac{1}{|f|}B_f B_f^\top$$

where B_f is a matrix with edge midpoint vectors as rows (we take the barycenter of each p–gon f as the center of origin per face).

We define the **discrete contraction operator** using the following property that holds on Riemannian surfaces [Hir03, Lemma 8.2.1]:

$$\mathbf{i}_X\,\alpha^k = (-1)^{k(2-k)}\star(\star\alpha^k\wedge X^\flat), \qquad X^\flat(e) = \int_e\langle X(\mathbf{x}), e'(\mathbf{x})\rangle d(\mathbf{x}).$$

Our $\mathbf{i}_X$ is a linear map sending k–forms to $(k-1)$–forms s.t. $\mathbf{i}_X\,\mathbf{i}_X = 0$, like its continuous analog. Experimental convergence is illustrated in Figure 1 right.

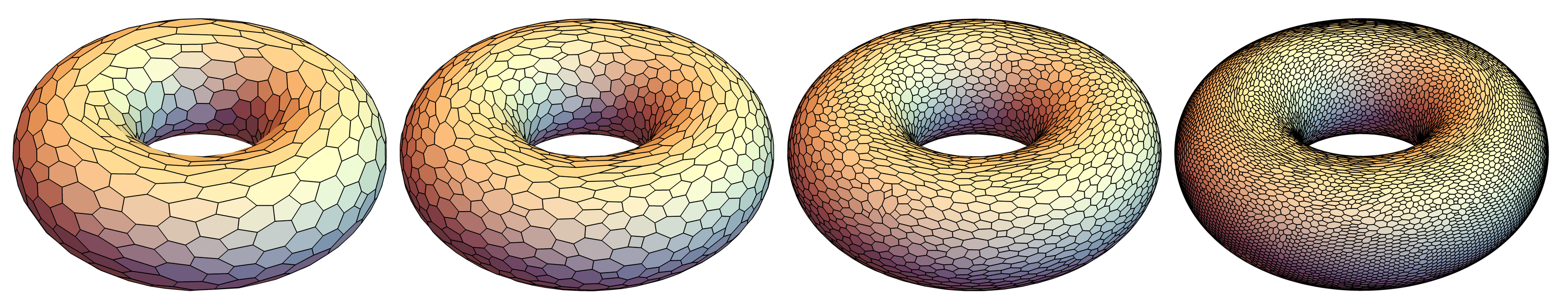


Figure 3: Samples of irregular meshes on a torus: from left to right with 1k, 2k, 5k and 20k vertices.

Ongoing Work

Currently, we are numerically evaluating a **discrete Lie derivative** given by the Cartan's magic formula

$$\pounds_X = \mathbf{i}_X\,d + d\,\mathbf{i}_X\,.$$

We are also starting to study a **discrete Hodge decomposition** using a discrete codifferential

$$\delta(\alpha^k) = (-1)^k\star d\star\alpha^k,$$

where $\star$ is our discrete Hodge star, and using a discrete Laplacian given by

$$\Delta = \delta d + d\delta.$$

Conclusion

- Geometry processing with polygonal meshes is a new developing area, e.g., see discrete Laplacians on general polygonal meshes in [AW11].
- We propose a novel discretization of several operators that act directly on general polygonal meshes, are compatible with each other and easy to implement.
- We believe that the generality of our framework will make it a useful tool in geometry processing tasks and will inspire further research in the area.

References

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