6

Condensation in Zero-range dynamics

Fix a finite state space S of cardinality $L \geq 2$ and consider a continuous-time Markov chain X_t on S. Denote by $\lambda(x)$, $x \in S$, the holding rates, by $p(x, y)$, $x, y \in S$, the jump probabilities, and by $R(x, y)$ the jump rates:

$$p(x, y) \; \geq \; 0 \;, \quad p(x, x) \; = \; 0 \;, \quad \sum_{y \in S} p(x, y) \; = \; 1 \;, \quad R(x, y) = \lambda(x)\, p(x, y) \;.$$

Assume that the Markov chain is *irreducible* and that the uniform measure on S, denoted by m, $m(x) = 1/L$, $x \in S$, is stationary.

Fix a function $g : \mathbb{N} \to \mathbb{R}$ such that

$$g(0) \; = \; 0 \;, \quad g(k) \; > \; 0 \;, \quad k \geq 1 \;.$$

The zero range process associated to the jump rates R and the rate function g is the Markov dynamics which describes the evolution of interacting particles on the the set S in which a particle jumps from a site $x \in S$ occupied by k particles to a site y at rate $g(k)R(x, y)$.

To define the evolution and its generator, denote by the Greek letters η, ξ and ζ the elements of the configuration space $E = \mathbb{N}^S$. Hereafter, η_x represents the number of particles at site $x \in S$ for the configuration η, and $\sigma^{x,y}\xi$, $x \neq y \in S$, the configuration obtained from a configuration ξ with at least one particle at x, by moving a particle from x to y:

$$(\sigma^{x,y}\xi)_z \; = \; \begin{cases} \xi_x - 1 \text{ for } z = x \\ \xi_y + 1 \text{ for } z = y \\ \xi_z \qquad \text{otherwise} \;. \end{cases}$$

Consider the Markov generator L on E given by

$$(Lf)(\eta) \; = \; \sum_{y \in S} g(\eta_x)\, R(x, y)\, \{f(\sigma^{x,y}\eta) - f(\eta)\} \qquad (0.1)$$

for every function $f : E \to \mathbb{R}$. Hence, in the notation introduced in Chapter 1,

$$\lambda(\eta)$$

$$\lambda(\eta) \;=\; \sum_{x \in S} g(\eta_x)\,\lambda(x) \qquad p(\eta,\xi) \;=\; \begin{cases} \dfrac{g(\eta_x)R(x,y)}{\sum_{z \in S} g(\eta_z)\,\lambda(z)} & \text{if } \xi = \sigma^{x,y}\eta \,, \\ 0 & \text{otherwise.} \end{cases}$$

The zero-range process on E is clearly not irreducible because the total number of particles is conserved by the dynamics. Denote by $E_{S,N}$ the configurations with N particles:

$$E_{S,N} \;=\; \{\eta \in E : \sum_{x \in S} \eta_x = N\} \,.$$

Since the underlying Markov chain X_t is irreducible, if η and ξ are configurations in $E_{S,N}$, there exists $m \geq 1$ and a path $\eta = \zeta_0, \zeta_1, \ldots, \zeta_m = \xi$ such that $p(\zeta_i, \zeta_{i+1}) > 0$, $0 \leq i < m$. The sets $E_{S,N}$ represent therefore the ergodic components of the dynamics. Since each set $E_{S,N}$ is finite, restricted to $E_{S,N}$, the zero-range process is irreducible and positive reccurent. There exists, in particular, a unique stationary probability measure, denoted by μ_N. Since the total number of particles is fixed, in statistical mechanics the measure μ_N is called the *canonical stationary state*.

The probability measure μ_N is given by

$$\mu_N(\eta) \;=\; \frac{1}{Z_{S,N}} \prod_{x \in S} \frac{1}{g!(\eta_x)} \,, \tag{0.2}$$

where $g!(0) = 1$ and $g!(k) = g(1) \cdots g(k)$, $k \geq 1$, and $Z_{S,N}$ is the normalizing factor

$$Z_{S,N} \;=\; \sum_{\eta \in E_{S,N}} \prod_{x \in S} \frac{1}{g!(\eta_x)} \,. \tag{0.3}$$

Since the set S is fixed, most of the times we omit the index S and write $Z_{S,N}$ as Z_N.

To prove this assertion let $R^*(x,y)$ the adjoint jump rates of $R(x,y)$. Since we assumed the uniform measure to be invariant, by xxx, $R^*(x,y) = R(y,x)$. Denote by L_{R^*} the generator defined in (0.1) with R^* replacing R. An elementary computation using the fact that $\sum_{x \in S} R(x,y) = \lambda(y)$, which holds because the uniform measure is stationary for the jump rates R, shows that for every function f, $h : E_{S,N} \to \mathbb{R}$,

$$\langle Lf \,, h \rangle_{\mu_N} \;=\; \langle f \,, L_{R^*}h \rangle_{\mu_N} \,.$$

This proves the assertion, actually, the stronger statement that the adjoint of L in $L^2(\mu_N)$ is L_{R^*}.

If we allow the total number of particles to fluctuate, a simple one-parameter family of stationary states emerges, the *grand canonical stationary states*. Let $Z(\varphi)$ denote the *partition function*:

$$Z(\varphi) \;=\; \sum_{k \geq 0} \frac{\varphi^k}{g!(k)} \,,$$

$g!(k)$
μ_N

φ^*

The parameter $\varphi \geq 0$ is called the *fugacity*. Let φ^* be the radius of convergence of this series. For $\varphi < \varphi^*$, denote by $\hat{\nu}_\varphi$ the probability measure on E defined by

$$\hat{\nu}_\varphi(\eta) \;=\; \frac{1}{Z(\varphi)^L} \prod_{x \in S} \frac{\varphi^{\eta_x}}{g!(\eta_x)} \;.$$

A computation, similar to the one performed for the canonical measure, shows that $\hat{\nu}_\varphi$ is also stationary.

The canonical measure is obtained by conditioning the grand canonical measure on the total number of particles:

$$\mu_N(\eta) \;=\; \hat{\nu}_\varphi\Big(\eta \,\big|\, \sum_{x \in S} \eta_x = N\Big) \;.$$

Note that the right hand side does not depend on the fugacity φ.

Denote by $R(\varphi)$ the mean number of particles under the stationary state $\hat{\nu}_\varphi$:

$$R(\varphi) \;=\; E_{\hat{\nu}_\varphi}[\eta_0] \;=\; \frac{1}{Z(\varphi)} \sum_{k \geq 1} k\, \frac{\varphi^k}{g!(k)} \;=\; \frac{\varphi\, Z'(\varphi)}{Z(\varphi)} \;=\; \varphi\, (\partial_\varphi \log Z)(\varphi) \;.$$

Clearly, $R(0) = 0$ and the derivative $\varphi R'(\varphi)$ is equal to the variance of η_0:

$$\varphi\, R'(\varphi) \;=\; E_{\hat{\nu}_\varphi}[\eta_0^2] \;-\; E_{\hat{\nu}_\varphi}[\eta_0]^2 \;.$$

In particular, R is strictly increasing, and if we denote by ρ^* the limit

$$\rho^* \;=\; \lim_{\varphi \to \varphi^*} R(\varphi) \;,$$

$R : [0, \varphi^*) \to [0, \rho^*)$ is a bijection. Denote by $\Phi : [0, \rho^*) \to [0, \varphi^*)$ its inverse and define ν_ρ, $0 \leq \rho < \rho^*$ by

ν_ρ

$$\nu_\rho \;=\; \hat{\nu}_{\Phi(\rho)} \;.$$

The stationary measures for the zero-range dynamics, ν_ρ, are now index by the conserved quantity, the density of particles,

$$E_{\nu_\rho}[\eta_0] \;=\; E_{\hat{\nu}_{\Phi(\rho)}}[\eta_0] \;=\; R(\Phi(\rho)) \;=\; \rho \;.$$

Phase transition

Fix a real number $\alpha > 0$ and let $g : \mathbb{N} \to \mathbb{R}$ be given by

$$g(0) = 0 \;, \quad g(1) = 1 \;, \quad \text{and} \quad g(n) \;=\; \Big(\frac{n}{n-1}\Big)^\alpha \;, \; n \geq 2 \;,$$

so that $g!(n) = n^\alpha$, $n \geq 1$. In particular,

$$Z(\varphi) \ = \ \sum_{k \geq 0} \frac{\varphi^k}{k^\alpha} \ ,$$

and the radius of convergence $\varphi^* = 1$. Figure 0.1 below presents the graph of $\log Z(\varphi)$ in the cases $0 < \alpha \leq 1$, $1 < \alpha \leq 2$, $\alpha > 2$. In the first case $\log Z(\varphi)$ and its derivative diverges as $\varphi \to \varphi^* = 1$ so that $\rho^* = \infty$. If $1 < \alpha \leq 2$, $\log Z(\varphi)$ converges to a finite limit as $\varphi \to \varphi^* = 1$, but its derivative diverges. Hence, also in this case $\rho^* = \infty$. Finally, if $\alpha > 2$, both $\log Z(\varphi)$ and its derivative converge to a finite limit as $\varphi \to \varphi^* = 1$ so that $\rho^* < \infty$.
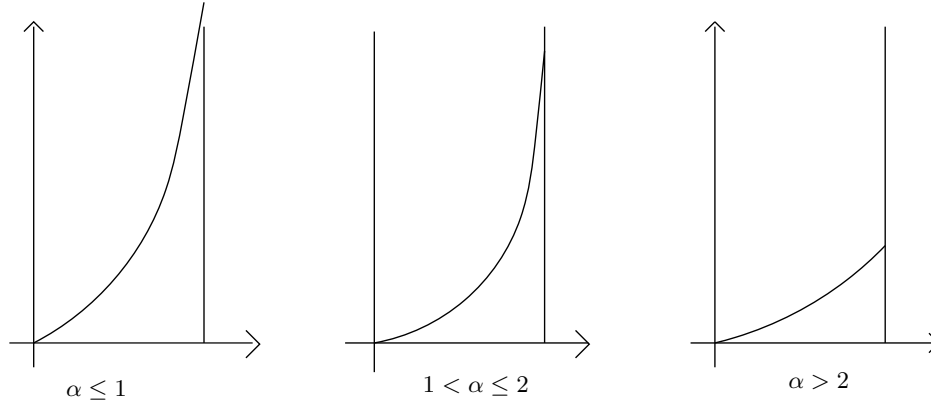

$\alpha \leq 1$	$1 < \alpha \leq 2$	$\alpha > 2$

Fig. 0.1. The graph of $\log Z(\varphi)$ in the cases $0 < \alpha \leq 1$, $1 < \alpha \leq 2$ and $\alpha > 2$.

Equivalence of Ensembles

The equivalence of ensembles asserts that in the thermodynamical limit the canonical stationary state converges to the grand canonical state. More precisely, let $\mathbb{T}_L = \{0, \dots, L-1\}$ be the one-dimensional discrete torus with L sites and let $r : \mathbb{Z} \to \mathbb{R}_+$ be a non-negative function with finite support. The continuous-time random walk in $\mathbb{T}_L$ which jumps from x to y at rate $r(y-x)$ is translation invariant. If it is also irreducible, its unique stationary state is the uniform probability measure m_L.

Consider the zero-range process on $\mathbb{N}^{\mathbb{T}_L}$ in which a particle jumps from a site x to a site y at rate $g(\eta_x)\, r(y-x)$. The unique stationary state on $E_{L,N} = \{\eta \in \mathbb{N}^{\mathbb{T}_L} : \sum_{x \in \mathbb{T}_L} \eta_x = N\}$, denoted by $\mu_{L,N}$, is given by (0.2) with S replaced by $\mathbb{T}_L$.

For $\rho < \rho^*$, denote by ν_ρ the product measure on the infinite space $\mathbb{N}^{\mathbb{Z}}$ whose marginals are given by

$$\nu_\rho\{\eta : \eta_x = k\} \ = \ \frac{1}{Z(\varphi)} \frac{\varphi^k}{g!(k)} \ , \quad \text{where } \varphi = \Phi(\rho).$$

The equivalence of ensembles states that for every $\rho < \rho^*$ and every bounded function $f : \mathbb{N}^{\mathbb{T}_L} \to \mathbb{R}$ which depends on η only through a finite number of variables η_x

$$\lim_{\substack{L \to \infty \\ N/L \to \rho}} E_{\mu_{L,N}}[f] \;=\; E_{\nu_\rho}[f] \,.$$

This means that for very large L, the canonical measure $\mu_{L,N}$ with density $\rho = N/L < \rho^*$ is locally very similar to the grand canonical measure with density ρ.

The proof of the equivalence of ensembles relies on the local central limit theorem and can be found in **?**. A natural question is what happens in the thermodynamical limit when the average density of the canonical measure, N/L, is larger than the critical density ρ^*. To present what has been proved, denote by $\mathbb{K}_L : E_{L,N} \to \cup_{M<N} E_{L-1,M}$ the function which consists in removing the site with the greatest number of particles. If there are two or more sites with this property, we choose at random one of them. **?** and **?** proved that the measure $\mu_{L,N} \mathbb{K}_L^{-1}$ converges to ν_{ρ^*}: for every $\rho > \rho^*$ and every bounded function $f : \mathbb{N}^{\mathbb{T}_L} \to \mathbb{R}$ which depends on η only through a finite number of variables η_x,

$$\lim_{\substack{L \to \infty \\ N/L \to \rho}} E_{\mu_{L,N}}[f] \;=\; E_{\nu_{\rho^*}}[f] \,.$$

Hence, when the density N/L is larger than the critical density ρ^*, if we disregard the site which concentrates the larger number of particles, we observe in the other ones a distribution which is very close to the criticial distribution ν_{ρ^*}. The site with the largest number of particles accomodates a macroscopic number of particles, since $N - (L-1)\rho^* \sim L(\rho - \rho^*)$. This phenomenon is called condensation.

We prove in Proposition 6.2 and in Lemma 6.6 below that even when the total number of sites L is fixed, in the canonical stationary state $\mu_{L,N}$, when $N \uparrow \infty$, all particles concentrate on a single site, while in all the other sites particles are distributed according to the grand canonical measure at critical density.

These results characterize the canonical stationary state above the critical density. Fix L and denote by X the site occupied by the greatest number of particles, called the condensate. By symmetry, X is uniformly distributed. It is natural to examine the time evolution of the condensate. Assume, for instance, that we start from a configuration in which all particles are placed on a site $x \in \mathbb{T}_L$. Denote by H the first time in which all particles occupy the same site $y \neq x$. One would like to derive the distribution of the position of the condensate at time H and to estimate the order of magnitude of H. This is the subject of this chapter.

1 The evolution of the condensate

Assume from now on that the random walk X_t is *reversible* with respect to the uniform measure m on S. For x in S, denote by $\mathbb{P}_x$ the probability measure on the path space $D(\mathbb{R}_+, S)$ induced by the random walk $\{X_t : t \geq 0\}$ starting from x.

$\mathcal{C}(x, y)$

$\mathrm{cap}_S(x, y)$

For two disjoint subsets A, B of S, denote by $\mathrm{cap}_S(A, B)$ the capacity between A and B for the random walk X_t. When $A = \{x\}$ and $B = \{y\}$ are singletons, we represent $\mathrm{cap}_S(\{x\}, \{y\})$ by $\mathrm{cap}_S(x, y)$. Let $\mathcal{C}(x, y)$ be the set of all functions $f : S \to \mathbb{R}$ such that $f(x) = 1$ and $f(y) = 0$, and denote by D_S the Dirichlet form associated to the random walk:

$$D_S(f) \;=\; \frac{1}{2} \sum_{x, y \in S} m(x) \, R(x, y) \, \{f(y) - f(x)\}^2 \qquad (1.1)$$

for $f : S \to \mathbb{R}$. By the Dirichlet principle xxx,

$$\mathrm{cap}_S(x, y) \;=\; \inf_{f \in \mathcal{C}(x, y)} D_S(f) \,. \qquad (1.2)$$

For each $\eta \in E$, let $\mathbf{P}_\eta$ represent the probability on the path space $D(\mathbb{R}_+, E)$ induced by the zero range process $\{\eta(t) : t \geq 0\}$ starting from η. Expectation with respect to $\mathbf{P}_\eta$ is denoted by $\mathbf{E}_\eta$.

Fix a sequence $\{\ell_N : N \geq 1\}$ such that $1 \ll \ell_N \ll N$:

$$\lim_{N \to \infty} \ell_N \;=\; \infty \,, \quad \lim_{N \to \infty} \ell_N / N \;=\; 0 \,. \qquad (1.3)$$

$\mathcal{E}_N^x$

For x in S, let

$$\mathcal{E}_N^x \;:=\; \left\{ \eta \in E_{S,N} : \eta_x \geq N - \ell_N \right\} \,.$$

Obviously, $\mathcal{E}_N^x \neq \varnothing$ for all $x \in S$ and every N large enough.

Condition $\ell_N / N \to 0$ is required to guarantee that on each set $\mathcal{E}_N^x$ the proportion of particles at $x \in S$, η_x / N, is almost one. As a consequence, for N sufficiently large, the subsets $\mathcal{E}_N^x$, $x \in S$, are pairwise disjoint. From now on, we assume that N is large enough so that the partition

Δ_N

$\mathcal{E}_N$

$$E_{S,N} \;=\; \mathcal{E}_N \cup \Delta_N \,, \quad \mathcal{E}_N \;=\; \bigcup_{x \in S} \mathcal{E}_N^x$$

is well defined, where $\Delta_N = \Delta_{S,N}$ is the set of configurations which do not belong to any set $\mathcal{E}_N^x$, $x \in S$, i.e., the subset of configurations with at most $N - \ell$ particles per site:

$$\Delta_N \;=\; \left\{ \eta \in E_{S,N} : \eta_x < N - \ell_N \,, \, \forall x \in S \right\} \,.$$

The assumptions that $\ell_N \uparrow \infty$ is sufficient to prove that $\mu_N(\Delta_N) \to 0$, as we shall see in Proposition 6.2, and to deduce the limit of the capacities stated

$\eta^{\mathcal{E}_N}(t)$

in Theorem 6.7 below. We need, however, further restrictions on the growth of ℓ_N to prove the tunneling behaviour of the zero range processes presented in Theorem 6.1 below.

Denote by $\eta^{\mathcal{E}_N}(t)$ the trace of $\eta(t)$ on $\mathcal{E}_N$. Let $\Psi_N : \mathcal{E}_N \mapsto S$ be the order parameter

$$\Psi_N(\eta) \;=\; \sum_{x \in S} x \, \mathbf{1}\{\eta \in \mathcal{E}_N^x\} \,,$$

and let $\mathbb{X}^N(t) := \Psi_N(\eta^{\mathcal{E}_N}(t))$ be the value of the order parameter at time t.

Theorem 6.1 below asserts that the speeded up non-Markovian process $\mathbb{X}^N(tN^{\alpha+1})$ converges to the random walk $\mathfrak{X}(t)$ on S whose generator $\mathfrak{L}_S$ is given by

$$(\mathfrak{L}_S f)(x) \;=\; \frac{L}{\Gamma(\alpha)\, I_\alpha} \sum_{y \in S} \mathrm{cap}_S(x,y) \left\{ f(y) - f(x) \right\} ,$$

where

$\Gamma(\alpha)$

I_α

$$\Gamma(\alpha) \;:=\; \sum_{j \geq 0} \frac{1}{a(j)} \,, \quad I_\alpha \;:=\; \int_0^1 u^\alpha (1-u)^\alpha \, du \,. \tag{1.4}$$

For x in S, denote by $\mathfrak{P}_x$ the probability measure on the path space $D(\mathbb{R}_+, S)$ induced by the random walk $\{\mathfrak{X}(t) : t \geq 0\}$ starting from x.

Theorem 6.1. *Assume that $L \geq 2$, that (1.3) holds and that*

$$\lim_{N \to \infty} \frac{\ell_N^{1+\alpha(L-1)}}{N^{1+\alpha}} \;=\; 0 \,. \tag{1.5}$$

Then, for each $x \in S$,
*(**M1**) For any sequence $\xi_N \in \mathcal{E}_N^x$, $N \geq 1$, the law of the stochastic process $\{\mathbb{X}_{tN^{\alpha+1}}^N : t \geq 0\}$ under $\mathbf{P}_{\xi_N}$ converges to $\mathfrak{P}_x$ as $N \uparrow \infty$;*
*(**M2**) For every $T > 0$,*

$$\lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_N^x} \mathbf{E}_\eta \left[\int_0^T \mathbf{1}\{\eta^N(sN^{\alpha+1}) \in \Delta_N\} \, ds \right] \;=\; 0 \,.$$

Moreover,

$$\lim_{N \to \infty} \inf_{\eta, \xi \in \mathcal{E}_N^x} \mathbf{P}_\eta \left[H_{\{\xi\}} < H_{\breve{\mathcal{E}}_N^x} \right] \;=\; 1 \,, \tag{1.6}$$

where

$$\breve{\mathcal{E}}_N^x \;:=\; \bigcup_{y \neq x} \mathcal{E}_N^y \,.$$

In order to fulfill conditions (1.3) and (1.5), we can take, for instance,

$$\ell_N \;=\; N^{1/(L-1)}$$

if $L \geq 3$, and $\ell_N = N^{1/[L-(1/2)]}$ if $L = 2$.

Property (**M2**) states that in the time scale $N^{\alpha+1}$ at almost every time there is one site occupied by at least $N - \ell_N$ particles. Property (**M1**) describes the evolution in the time scale $N^{\alpha+1}$ of the condensate. It evolves asymptotically as a Markov process on S which jumps from x to y at a rate proportional to the capacity between x and y for the underlying random walk $X(t)$. Property (1.6) guarantees that the process starting in a metastable set $\mathcal{E}_N^x$ thermalizes therein before reaching another metastable set.

In the case where S is the one-dimensional torus with L sites, $S = \mathbb{T}_L$, and $X(t)$ the nearest-neighbor, symmetric random walk, $R(x, x \pm 1) = 1$, $R(x, y) = 0$ if $|y - x| > 1$, the jump probabilities of the condensate are neither nearest-neighbor nor uniform as one would first guess. By Theorem 6.1, the rate at which the condensate jumps from x to y is proportional do the capacity between x and y, which decreases as the inverse of the distance between x and y.

2 Condensation

The main result of this section, Proposition 6.2 below, states that condensation occurs in the stationary state, i.e., that all particles, but a negligible fraction of them, sit on the same site.

Since the stationary measure μ_N is invariant by permutation of the site and since several arguments in this section are inductive in L, we denote in this section $E_{S,N}$ and $Z_{S,N}$ by $E_{L,N}$ and $Z_{L,N}$, respectively.

Proposition 6.2. *For every $\alpha > 1$,*

$$\lim_{N \to \infty} \mu_N(\Delta_N) \;=\; 0 \;.$$

$a(n)$
$a(\eta)$

This result follows from Lemma 6.4 which presents a uniform bound for the stationary measure of the set Δ_N. Let $a(0) = 1$, $a(n) = n^\alpha$ for $n \geq 1$ so that $a(n) = g!(n)$, and let

$$a(\eta) \;:=\; \prod_{x \in S} a(\eta_x) \;.$$

Before providing an estimate on the stationary measure of Δ_N, we show that the sequence $N^\alpha Z_{L,N}$ is bounded below by a strictly positive constant and is bounded above by a finite constant.

Lemma 6.3. *For each $L \geq 2$, there exists a constant $A_L > 0$, which only depends on α and L, such that*

$$1 \;\leq\; N^\alpha \, Z_{L,N} \;\leq\; A_L \;.$$

Proof. Choose x in S and denote by ξ the configuration in $E_{L,N}$ such that $\xi(x) = N$, $\xi(y) = 0$ for $y \neq x$. By definition, $Z_{L,N} \geq 1/a(\xi) = 1/N^\alpha$, which proves the lower bound.

We proceed by induction to prove the upper bound. The estimate clearly holds for $L = 2$. Assume that it is in force for $2 \leq L \leq M$. The identity

$$
N^\alpha Z_{M+1,N} = N^\alpha \Big\{ \frac{1}{N^\alpha} + \sum_{j=0}^{N-1} \frac{(N-j)^\alpha Z_{M,N-j}}{a(j)a(N-j)} \Big\}
$$

permits to extend it to $L = M + 1$. $\square$

Next lemma shows that the measure $\mu_{L,N}$ is concentrated on configurations in which all particles but a finite number accumulate at one site. Let $\Delta_{L,N}(\ell)$, $\ell \geq 1$, be the subset of of all configurations with at most $N - \ell$ particles per site:

$$
\Delta_{L,N}(\ell) = \big\{ \eta \in E_{L,N} : \eta_x \leq N - \ell, \forall x \in S \big\} . \tag{2.1}
$$

Lemma 6.4. *There exists a constant $C_L > 0$, which only depends on α and L, such that for every integer $\ell > 0$,*

$$
\sup_{N > \ell} \mu_{L,N}(\Delta_{L,N}(\ell)) \leq \frac{C_L}{\ell^{\alpha-1}} .
$$

Proof. In view of the previous lemma, it is enough to show that

$$
\sup_{N > \ell} \Big\{ N^\alpha \sum_{\eta \in \Delta_{L,N}(\ell)} \frac{1}{a(\eta)} \Big\} \leq \frac{C_L}{\ell^{\alpha-1}} . \tag{2.2}
$$

We proceed by induction on L. For $L = 2$ the statement is easily checked. Now, suppose the claim holds for $2 \leq L \leq M$. Fix some point x in S_{M+1}. The expression inside braces in the left hand side of (2.2) can be written as

$$
\sum_{\eta \in \Delta_{M+1,N}(\ell)} \frac{N^\alpha}{a(\eta_x)a(N-\eta_x)} \frac{(N-\eta_x)^\alpha}{\prod_{y \neq x} a(\eta_y)} .
$$

This sum is equal to

$$
\Big\{ \sum_{0 \leq i \leq \ell/2} + \sum_{\ell/2 < i \leq N-\ell} \Big\} \frac{N^\alpha}{a(i)a(N-i)} \sum_{\xi \in \Delta_{M,N-i}(\ell-i)} \frac{(N-i)^\alpha}{a(\xi)} , \tag{2.3}
$$

where the second sum is equal to zero if $\{i : \ell/2 < i \leq N - \ell\}$ is empty. Observe that the configurations in $\Delta_{M,N-i}(\ell - i)$ are such that $\eta_x \leq (N-i) - (\ell - i) = N - \ell$, as required. We examine the two terms of this expression separately. By the induction assumption, the first sum is bounded above by

$$\sum_{i=0}^{\ell/2} \frac{N^\alpha}{a(i)a(N-i)} \frac{C_M}{(\ell-i)^{\alpha-1}} \; .$$

By the previous lemma, this sum is less than or equal to

$$\frac{2^{\alpha-1}C_M}{\ell^{\alpha-1}} \sum_{i=0}^{\ell/2} \frac{N^\alpha}{a(i)a(N-i)} \;\leq\; \frac{2^{\alpha-1}C_M N^\alpha Z_{2,N}}{\ell^{\alpha-1}} \;\leq\; \frac{2^{\alpha-1}C_M A_2}{\ell^{\alpha-1}} \; .$$

On the other hand, by Lemma 6.3 and by the induction assumption for $L = 2$, the second term in (2.3) is less than or equal to

$$\sum_{i=(\ell/2)+1}^{N-\ell} \frac{N^\alpha}{a(i)a(N-i)} (N-i)^\alpha Z_{M,N-i} \;\leq\; A_M\, C_2\,(2/\ell)^{\alpha-1} \; .$$

This concludes the proof of the lemma. $\square$

For $N \geq 2$, $0 \leq \ell \leq N$, $x \in S$, denote by $\mathcal{E}_N^x(\ell)$ the set of configurations in $E_{L,N}$ with at least $N - \ell$ particles at site x:

$$\mathcal{E}_N^x(\ell) \;=\; \{\eta \in E_{L,N} : \eta_x \geq N - \ell\} \; .$$

If $\ell < N/2$, the sets $\{\mathcal{E}_N^x(\ell) : x \in S\}$ are pairwise disjoint and

$$E_{L,N} \setminus \bigcup_{x \in S} \mathcal{E}_N^x(\ell) \;=\; \Delta_{L,N}(\ell+1) \; .$$

By symmetry, $\mu_{L,N}(\mathcal{E}_N^x(\ell))$ does not depend on x so that for $N > 2\ell$,

$$1 \;=\; L\,\mu_{L,N}(\mathcal{E}_N^x(\ell)) \;+\; \mu_{L,N}(\Delta_{L,N}(\ell+1)) \; .$$

Hence, by Lemmas 6.4 and 6.3,

$$\sup_{N>2\ell} \left| N^\alpha \sum_{\eta \in \mathcal{E}_N^x(\ell)} \frac{1}{a(\eta)} - \frac{N^\alpha\, Z_{L,N}}{L} \right| \;\leq\; \frac{C_L}{\ell^{\alpha-1}} \;, \tag{2.4}$$

$Z_L(\alpha)$ for some finite constant C_L which depends only on α and L, and which may change from line to line.

Let

$$Z_L(\alpha) \;:=\; L\,\Gamma(\alpha)^{L-1} \; . \tag{2.5}$$

Lemma 6.5. *For every $L \geq 2$,*

$$\lim_{N\to\infty} N^\alpha Z_{L,N} = Z_L(\alpha) \; .$$

Proof. Fix a site x in S. By (2.4),

$$\lim_{N\to\infty} \frac{N^\alpha Z_{L,N}}{L} = \lim_{\ell\to\infty} \lim_{N\to\infty} N^\alpha \sum_{\eta\in\mathcal{E}_N^x(\ell)} \frac{1}{a(\eta)} \; .$$

The previous sum is equal to

$$\sum_{j=0}^{\ell} \frac{N^\alpha}{(N-j)^\alpha} \sum_{\xi\in E_{L-1,j}} \frac{1}{a(\xi)} \; .$$

As $N\uparrow\infty$ and $\ell\uparrow\infty$, this expression converges to

$$\sum_{j\geq 0}\sum_{\xi\in E_{L-1,j}} \frac{1}{a(\xi)} = \sum_{\xi\in\mathbb{N}^{L-1}} \frac{1}{a(\xi)} = \Big(\sum_{j\geq 0}\frac{1}{a(j)}\Big)^{L-1} = \Gamma(\alpha)^{L-1} \; .$$

This concludes the proof of the lemma. $\square$

We conclude this section with an observation which shall not be used in the sequel. We proved in Lemma 6.4 that with a probability close to 1 all particles but a finite number accumulate at one site. Lemma 6.6 below states that given that almost all particles occupy the same site x, in the remaining sites the particles are distributed according to the critical grand canonical measure. Note, by the way, that the critical grand canonical measure $\nu_{\rho^*} = \hat{\nu}_{\varphi^*}$ satisfies

$$\nu_{\rho^*}\{\eta : \eta_y = k\} = \frac{1}{\Gamma(\alpha)}\frac{1}{a(k)} \; .$$

Lemma 6.6. *For every* $x \in S$ *and sequence* ℓ_N *satisfying conditions (1.3),*

$$\lim_{N\to\infty} \sup_{\zeta\in\mathcal{G}_N^x} \Big| \mu_N\big(\eta_z = \zeta_z \, , \, z \neq x \,\big|\, \mathcal{E}_N^x\big) - \prod_{z\neq x}\frac{1}{\Gamma(\alpha)}\frac{1}{a(\zeta_z)} \Big| = 0 \; ,$$

where $\mathcal{G}_N^x := \{\zeta \in \mathbb{N}^{S\setminus\{x\}} : \sum_z \zeta_z \leq \ell_N\}$.

Proof. Fix $x \in S$ and $\zeta \in \mathcal{G}_N^x$. Since the canonical measure μ_N is concentrated on configurations with N particles,

$$\mu_N\big(\eta_z = \zeta_z \, , \, z \neq x \,\big|\, \mathcal{E}_N^x\big) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \mu_N\big(\eta_z = \zeta_z \, , \, z \neq x\big) \; .$$

By symmetry, $\mu_N(\mathcal{E}_N^x) = \mu_N(\mathcal{E}_N^y)$, $y \neq x$, so that $L\,\mu_N(\mathcal{E}_N^x) = 1 - \mu_N(\Delta_N)$. On the other hand,

$$\mu_N\big(\eta_z = \zeta_z \, , \, z \neq x\big) = \frac{1}{Z_{L,N}\, a(N-|\zeta|)} \prod_{z\neq x}\frac{1}{a(\zeta_z)} \; ,$$

where $|\zeta| = \sum_{y\neq x}\zeta_y$. Hence, the difference inside the absolute values in the statement of the lemma is equal to

$$\prod_{z \neq x} \frac{1}{a(\zeta_z)} \left\{ \frac{L}{Z_{L,N}\, a(N - |\zeta|)\, [1 - \mu_N(\Delta_N)]} - \frac{1}{\Gamma(\alpha)^{L-1}} \right\}$$

Since $a(\cdot)$ is bounded below by 1, the result follows from Lemmas 6.4 and 6.5.
□

3 Proof of Theorem 6.1

$\mathrm{cap}_N(\mathcal{A}, \mathcal{B})$

The main step in the proof of Theorem 6.1 consists in computing the asymptotic value of the capacities between $\cup_{x \in A} \mathcal{E}_N^x$ and $\cup_{y \in A^c} \mathcal{E}_N^y$ for a non-empty proper subset A of S.

For disjoint subsets $\mathcal{A}, \mathcal{B}$ of $E_{S,N}$ denote by $\mathrm{cap}_N(\mathcal{A}, \mathcal{B})$ the capacity between $\mathcal{A}$ and $\mathcal{B}$. Denote, furthermore, by $\mathcal{C}_N(\mathcal{A}, \mathcal{B})$ the set of functions $F : E_{S,N} \to \mathbb{R}$ which are equal to 1 on $\mathcal{A}$ and which vanish on $\mathcal{B}$:

$\mathcal{C}_N(\mathcal{A}, \mathcal{B})$

$$\mathcal{C}_N(\mathcal{A}, \mathcal{B}) := \{ F : F(\eta) = 1 \,\forall\, \eta \in \mathcal{A} \text{ and } F(\xi) = 0 \,\forall\, \xi \in \mathcal{B} \} .$$

$D_N(F)$

Let D_N be the Dirichlet form associated to the generator L acting on the space of configurations $E_{S,N}$. An elementary computation shows that

$$D_N(F) = \frac{1}{2} \sum_{\substack{x,y \in S \\ x \neq y}} \sum_{\eta \in E_{S,N}} \mu_N(\eta)\, g(\eta_x)\, R(x,y) \left\{ F(\sigma^{xy}\eta) - F(\eta) \right\}^2 ,$$

for every $F : E_{S,N} \to \mathbb{R}$. By the Dirichlet principle,

$$\mathrm{cap}_N(A, B) = \inf \left\{ D_N(F) : F \in \mathcal{C}_N(A, B) \right\} . \tag{3.1}$$

Theorem 6.7. *Assume that $L \geq 2$. Fix a nonempty, proper subset $S^1 \subsetneq S$ and let $S^2 = S \setminus S^1$, $\mathcal{E}^N(S') = \cup_{x \in S'} \mathcal{E}_N^x$, $S' = S^1$, S^2. Then,*

$$\lim_{N \to \infty} N^{1+\alpha} \mathrm{cap}_N\big(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2) \big) = \frac{1}{\Gamma(\alpha)\, I_\alpha} \sum_{x \in S^1, y \in S^2} \mathrm{cap}_S(x, y) .$$

To prove Theorem 6.7, we derive in the enxt two sections a lower and an upper bound for the capacity. In the first part, we need to obtain a lower bound for the Dirichlet form of functions in $\mathcal{C}_N(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2))$. To our advantage, since it is a lower bound, we may neglect some bonds in the Dirichlet form we believe to be irrelevant. On the other hand, and this is the main difficulty, the estimate must be uniform over $\mathcal{C}_N(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2))$. As we shall see below, the proof of a sharp lower bound gives a clear indication of the qualitative behavior of the optimal function for the variational problem (3.1). With this information, we may propose a candidate for the upper bound. Here, in contrast with the first part, we have to estimate the Dirichlet form of a specific function, our elected candidate, but we need to estimate all the Dirichlet form and we can not neglect any bond.

Remark 6.8. By xxx, the equilibrium potential

$$\mathbf{F}_{\mathbf{S^1},\mathbf{S^2}}(\eta) \;=\; \mathbf{P}_\eta\Big[H_{\mathcal{E}_N(S^1)} < H_{\mathcal{E}_N(S^2)}\Big]$$

solves the variational problem (3.1) for the capacity. The candidate proposed in the proof of the upper bound provides, therefore, an approximation, in the Dirichlet sense, of the function equilibrium potential $\mathbf{F}_{\mathbf{S^1},\mathbf{S^2}}$.

We now turn to the proof of Theorem 6.1. In the case where $L = 2$ this result has been proved in Chapter xxx. We may therefore assume that $L \geq 3$. In Chapter xxx, we reduced the proof of the metastability of reversible processes to the verification of three conditions, denoted by **(H0)**, **(H1)** and **(H2)**.

Condition **(H2)** follows immediately from Proposition 6.2 since $\mu_N(\mathcal{E}_N^x) \to 1/L$ for every $x \in S$:

$$\lim_{N\to\infty} \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^x)} \;=\; 0 \;, \quad \forall x \in S \;. \tag{H2}$$

ξ_N^x

We turn to the proof of condition **(H1)**. For each $x \in S$, let $\xi_N^x \in E_{S,N}$ be the configuration with N particles at x. Fix a configuration η in $\mathcal{E}_N^x$. Since $\sum_{y\neq x} \eta_y \leq \ell_N$, by the explicit form of μ_N and Lemma 6.3,

$$\mu_N(\eta) \;\geq\; C_0 \prod_{y\in S\setminus\{x\}} \frac{1}{a(\eta_y)} \;\geq\; \frac{C_0}{\ell_N^{\alpha(L-1)}} \;. \tag{3.2}$$

Here and below, C_0 stands for a constant which does not depend on $N \geq 1$ and whose value may change from line to line. To estimate the capacity, $\mathrm{cap}_N(\eta, \xi_N^x)$, consider a path $\eta^{(j)}$, $0 \leq j \leq p$, from $\eta^{(0)} = \eta$ to $\eta^{(p)} = \xi_N^x$ obtained by moving to x, one by one, each particle. Since there are at most ℓ_N particles to move, we can take a path such that $p \leq L\,\ell_N$. Let F be an arbitrary function in $\mathcal{C}_N(\{\eta\}, \{\xi_N^x\})$. By Cauchy-Schwarz inequality and the explicit expression of the Dirichlet form, since $g(k) \geq 1$, $k \geq 1$,

$$1 \;=\; \Big\{ \sum_{j=0}^{p-1} \big[\, F(\eta^{(j+1)}) - F(\eta^{(j)}) \,\big] \Big\}^2 \;\leq\; C_0\, D_N(F) \sum_{j=0}^{p-1} \frac{1}{\mu_N(\eta^{(j)})} \;.$$

Therefore, by (3.2),

$$\mathrm{cap}_N(\eta, \xi_N^x) \;\geq\; \frac{C_0}{\ell_N^{1+\alpha(L-1)}} \;.$$

The extra factor ℓ_N comes from the length of the path. Condition **(H1)** follows now from this estimate, Theorem 6.7 and condition (1.5):

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}_N^x} \frac{\mathrm{cap}_N(\mathcal{E}_N^x, \breve{\mathcal{E}}_N^x)}{\mathrm{cap}_N(\eta, \xi_N^x)} \;=\; 0 \;, \quad \forall x \in S \;. \tag{H1}$$

Condition **(H0)** follows from Theorem 6.7. Denote by $R_{\mathcal{E}_N}(\cdot,\cdot)$ the jump rates of the trace process $\{\eta^{\mathcal{E}_N}(t) : t \geq 0\}$. For $x \neq y$ in S, let

$$r_N(\mathcal{E}_N^x, \mathcal{E}_N^y) := \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\substack{\eta \in \mathcal{E}_N^x \\ \xi \in \mathcal{E}_N^y}} \mu_N(\eta)\, R_{\mathcal{E}_N}(\eta, \xi)\,.$$

By Lemma xxx,

$$\mu_N(\mathcal{E}_N^x)\, r_N(\mathcal{E}_N^x, \mathcal{E}_N^y) = \frac{1}{2}\Big\{ \mathrm{cap}_N\big(\mathcal{E}_N^x,\, \breve{\mathcal{E}}_N^x\big) + \mathrm{cap}_N\big(\mathcal{E}_N^y,\, \breve{\mathcal{E}}_N^y\big)$$
$$- \mathrm{cap}_N\Big(\mathcal{E}_N(\{x,y\}),\, \mathcal{E}_N(S \setminus \{x,y\})\Big) \Big\}\,.$$

Therefore, by Theorem 6.7, since $\mu_N(\mathcal{E}_N^x)$ converges to L^{-1} for all x in S,

$$\lim_{N \to \infty} N^{1+\alpha} r_N(\mathcal{E}_N^x, \mathcal{E}_N^y) = \frac{L\,\mathrm{cap}_S(x,y)}{\Gamma(\alpha)\, I_\alpha}\,, \quad \forall x,y \in S\,,\ x \neq y\,. \qquad \textbf{(H0)}$$

This concludes the proof of Theorem 6.1 in view of Theorem xxx.

4 A lower bound for $\mathrm{cap}_N(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2))$

$L_N^{x,y}$

We prove in this section a lower bound for the capacity appearinf in Theorem 6.7. For $\ell \geq 3$ and x, y in S, $x \neq y$, consider the tube $L_N^{x,y}$ defined by

$$L_N^{x,y} = \Big\{ \eta \in E_N : \eta_x + \eta_y \geq N - \ell \Big\}\,.$$

Clearly, $L_N^{x,y} = L_N^{y,x}$ for any $x, y \in S$. We claim that for each $x \in S$ and every N sufficiently large

$$L_N^{x,y} \cap L_N^{x,z} \subset \mathcal{E}_x^N\,, \quad y, z \in S \setminus \{x\}\,.$$

Indeed, let $\eta \in L_N^{x,y} \cap L_N^{x,z}$. Clearly, $\eta_z \leq \ell$ because η belongs to $L_N^{x,y}$. Hence, $\eta_x \geq N - 2\ell$ as η belongs to $L_N^{x,z}$. Since $\ell_N \to \infty$, this shows that $\eta_x \geq N - \ell_N$, for N large enough and we conclude that $\eta \in \mathcal{E}_N^x$. Moreover, it follows from this argument that, for N sufficiently large,

$$L_N^{x,y} \cap L_N^{z,w} \neq \varnothing \quad \text{if and only if} \quad \{x,y\} \cap \{z,w\} \neq \varnothing\,. \qquad (4.1)$$

Proposition 6.9. *Fix a nonempty subset $S^1 \subsetneq S$ and let $S^2 = S \setminus S^1$. Then,*

$$\liminf_{N \to \infty} N^{1+\alpha} \mathrm{cap}_N\big(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2)\big) \geq \frac{1}{\Gamma(\alpha)\, I_\alpha} \sum_{x \in S^1, y \in S^2} \mathrm{cap}_S(x,y)\,.$$

Proof. Fix a function F in $\mathcal{C}_N(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2))$. By definition,

$$D_N(F) = \frac{1}{2} \sum_{z,w \in S} \sum_{\eta \in E_N} \mu_N(\eta) \, r(z,w) \, g(\eta_z) \, \{F(\sigma^{zw}\eta) - F(\eta)\}^2 .$$

We may bound from below the Dirichlet form $D_N(F)$ by

$$\frac{1}{2} \sum_{x \in S^1} \sum_{y \in S^2} \sum_{z,w \in S} \sum_{\eta \in L_N^{x,y}} \mu_N(\eta) \, r(z,w) \, g(\eta_z) \, \{F(\sigma^{zw}\eta) - F(\eta)\}^2 .$$

In this inequality, we are neglecting several terms corresponding to configurations η which do not belong to $\cup_{x \in S^1, y \in S^2} L_N^{x,y}$. On the other hand, some configurations are counted more than once because the sets $\{L_N^{x,y} : x \in S^1, y \in S^2\}$ are not disjoints. However, by (4.1), if $L_N^{x,y}$ and $L_N^{x',y'}$ are different strips and η belongs to $L_N^{x,y} \cap L_N^{x',y'}$ then, without loss of generality, $x = x'$ and $y \neq y'$. In consequence, $\eta_x \geq N - 2\ell$. In particular, for N large enough, η and $\sigma^{zw}\eta$ belong to $\mathcal{E}_N^x$ for all $z, w \in S$, so that $F(\sigma^{zw}\eta) = F(\eta)$ because F is constant on $\mathcal{E}_N^x$.

The proof of the lower bound has two steps. We first use the underlying random walk to estimate the Dirichlet form $D_N(F)$ by the capacity of the random walk multiplied by the Dirichlet form of a zero range process on two sites. Then, we bound by explicit computations the remaining two sites Dirichlet form.

$\mathfrak{d}_x$

Fix $x \in S^1$, $y \in S^2$. Denote by $\mathfrak{d}_x$, $x \in S$, the configuration with a unique particle at x, summation of configurations is performed componentwise. The change of variables $\xi = \eta - \mathfrak{d}_z$ shows that

$$\frac{1}{2} \sum_{z,w \in S} \sum_{\eta \in L_N^{x,y}} \mu_N(\eta) \, r(z,w) \, g(\eta_z) \, \{F(\sigma^{zw}\eta) - F(\eta)\}^2$$

$$= \frac{1}{2 Z_N} \sum_{z,w \in S} \sum_{\substack{\xi \in E_{N-1} \\ \xi + \mathfrak{d}_z \in L_N^{x,y}}} \frac{1}{a(\xi)} \, r(z,w) \, \{F(\xi + \mathfrak{d}_w) - F(\xi + \mathfrak{d}_z)\}^2 .$$

This sum is clearly bounded below by

$$\frac{1}{2 Z_N} \sum_{z,w \in S} \sum_{\substack{\xi \in E_{N-1} \\ \xi_x + \xi_y \geq N - \ell}} \frac{1}{a(\xi)} \, r(z,w) \, \{F(\xi + \mathfrak{d}_w) - F(\xi + \mathfrak{d}_z)\}^2 .$$

Fix a configuration ξ in E_{N-1} such that $F(\xi + \mathfrak{d}_x) \neq F(\xi + \mathfrak{d}_y)$ and consider the function $f : S \to \mathbb{R}$ given by $f(v) = \{F(\xi + \mathfrak{d}_v) - F(\xi + \mathfrak{d}_y)\}/\{F(\xi + \mathfrak{d}_x) - F(\xi + \mathfrak{d}_y)\}$. Note that $f(x) = 1$, $f(y) = 0$. Moreover, if we recall the expression (1.1) of the Dirichlet form of the underlying random walk,

$$\frac{1}{2} \sum_{z,w \in S} r(z,w) \, \{F(\xi + \mathfrak{d}_w) - F(\xi + \mathfrak{d}_z)\}^2$$

$$= L \, D_S(f) \, \{F(\xi + \mathfrak{d}_x) - F(\xi + \mathfrak{d}_y)\}^2 .$$

Since $f(x) = 1$, $f(y) = 0$, the previous expression is bounded below by

$$L \operatorname{cap}_S(x,y) \left\{ F(\xi + \mathfrak{d}_x) - F(\xi + \mathfrak{d}_y) \right\}^2 .$$

Up to this point we proved that the Dirichlet form of F is bounded below by

$$\frac{L}{Z_N} \sum_{x \in S^1, y \in S^2} \operatorname{cap}_S(x,y) \sum_{\substack{\xi \in E_{N-1} \\ \xi_x + \xi_y \geq N - \ell}} \frac{1}{a(\xi)} \left\{ F(\xi + \mathfrak{d}_x) - F(\xi + \mathfrak{d}_y) \right\}^2 .$$

Note that the second sum is very close to the Dirichlet form of a zero-range process on two sites.

S_{xy}

Fix $x \in S^1$, $y \in S^2$ and let $S_{xy} := S \setminus \{x, y\}$. For each $k \geq 0$. Recall that we denote by $E_{S_{xy},k}$ the set of configurations given by

$$E_{S_{xy},k} = \left\{ \zeta \in \mathbb{N}^{S_{xy}} : \sum_{v \in S_{xy}} \zeta_v = k \right\} .$$

For ζ in $E_{S_{xy},k}$, let $G_\zeta : \{0, \ldots, N - k - 1\} \to \mathbb{R}$ be defined as $G_\zeta(i) = F(\xi)$, where $\xi \in E_{N-1}$ is the configuration given by $\xi_v = \zeta_v$, $v \in S_{xy}$, $\xi_x = i$ and $\xi_y = N - k - i$. With this notation, for $x \in S^1$, $y \in S^2$ fixed, we may rewrite the second sum in the previous formula as

$$\frac{L}{Z_N} \sum_{k=0}^{\ell-1} \sum_{\zeta \in E_{S_{xy},k}} \frac{1}{a(\zeta)} \sum_{i=0}^{N-1-k} \frac{1}{a(i)\, a(N - 1 - k - i)} \left\{ G_\zeta(i+1) - G_\zeta(i) \right\}^2 .$$

Note that G_ζ is equal to 0 on the set $\{0, \ldots, \ell_N - k\}$, and equal to 1 on the set $\{N - \ell_N, \ldots, N - k\}$. We may therefore restrict the sum over i to a subset. It is easy to derive a lower bound for

$$\sum_{i = \ell_N - k}^{N - \ell_N - 1} \frac{1}{a(i)\, a(N - 1 - k - i)} \left\{ G_\zeta(i+1) - G_\zeta(i) \right\}^2 .$$

The function G which minimizes this expression is given by $G(N - \ell_N) = 1$,

$$G(i+1) - G(i) = \frac{1}{K_N}\, a(i)\, a(N - 1 - k - i), \quad \ell_N - k \leq i \leq N - \ell_N - 1 ,$$

where K_N is a normalizing constant to ensure the boundary condition $G(\ell_N - k) = 0$. The respective lower bound is

$$\Xi_N(x,y) := \left\{ \sum_{i = \ell_N - k}^{N - \ell_N - 1} a(i)\, a(N - 1 - k - i) \right\}^{-1} .$$

This expression depends on the configuration ζ only through its number of particles. Moreover, for every fixed k, $N^{1+2\alpha} \Xi_N(x,y)$ converges to I_α^{-1} as $N \uparrow \infty$.

In conclusion,

$$N^{\alpha+1} D_N(F) \geq L \sum_{x \in S^1, y \in S^2} \mathrm{cap}_S(x, y) \frac{N^{\alpha+1}}{Z_N} \sum_{k=0}^{\ell-1} \sum_{\zeta \in E_{S_{xy}, k}} \Xi_N(x, y) \frac{1}{a(\zeta)} \cdot$$

By Lemma 6.5 and by the asymptotic behavior of $\Xi_N(x, y)$, as $N \uparrow \infty$, the right hand side converges to

$$\frac{L}{I_\alpha Z_L(\alpha)} \sum_{x \in S^1, y \in S^2} \mathrm{cap}_S(x, y) \sum_{k=0}^{\ell-1} \sum_{\zeta \in E_{S_{xy}, k}} \frac{1}{a(\zeta)} \cdot$$

Recall that ℓ is a free parameter introduced in the definition of the strip $L_N^{x, y}$. Thus, letting $\ell \uparrow \infty$, by the definition (1.4) of $\Gamma(\alpha)$, the second sum in the last expression converges to

$$\sum_{k \geq 0} \sum_{\zeta \in E_{S_{xy}, k}} \frac{1}{a(\zeta)} = \prod_{z \in S_{xy}} \sum_{j \geq 0} \frac{1}{a(j)} = \Gamma(\alpha)^{L-2} \cdot$$

By (2.5), $Z_L(\alpha) = L\Gamma(\alpha)^{L-1}$, which concludes the proof of the proposition. $\square$

5 An upper bound for $\mathrm{cap}_N(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2))$

We prove in this section an upper bound for the capacity between $\mathcal{E}_N(A)$ and $\mathcal{E}_N(A^c)$ for a proper, non-empty subset A of S.

Proposition 6.10. *Fix a nonempty subset $S^1 \subsetneq S$ and let $S^2 = S \setminus S^1$. Then,*

$$\limsup_{N \to \infty} N^{1+\alpha} \mathrm{cap}_N(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2)) \leq \frac{1}{\Gamma(\alpha) I_\alpha} \sum_{x \in S^1, y \in S^2} \mathrm{cap}_S(x, y) \cdot$$

In view of the variational formula for the capacity, to obtain an upper bound for $\mathrm{cap}_N(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2))$, we need to choose a suitable function belonging to $\mathcal{C}_N(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2))$ and to compute its Dirichlet form. Recalling the proof of the lower bound, we expect this candidate to depend on the function which solves the variational problem for the capacity of the underlying random walk and on the optimal function for the zero range process on two sites.

To introduce the candidate, fix $x \in S^1$, $y \in S^2$ and recall the definition of the tube $L_N^{x, y}$. In view of the proof of the lower bound, the optimal function $F \in \mathcal{C}_N(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2))$ on the tube $L_N^{x, y}$ should satisfy

$$F(\xi + \mathfrak{d}_w) - F(\xi + \mathfrak{d}_z) = \{\mathbf{f_{xy}}(w) - \mathbf{f_{xy}}(z)\} \{F(\xi + \mathfrak{d}_x) - F(\xi + \mathfrak{d}_y)\}$$
$$= \{\mathbf{f_{xy}}(w) - \mathbf{f_{xy}}(z)\} \{G(\xi_x + 1) - G(\xi_x)\} ,$$

where $\mathbf{f_{x,y}}$ is the function which solves the variational problem (1.2) in $\mathcal{C}(x,y)$ for the capacity of the underlying random walk, and G is the optimal function for the two-sites zero-range process, which already appeared in the proof of the lower bound.

Since, on the tube $L_N^{x,y}$, $\sum_{z\neq x,y} \xi_z \leq \ell_N$ and G is a smooth function, by paying a small cost we may replace ξ_x in the previous formula by $\xi_x + \sum_{z\in A} \xi_z$ for any suitable set $A \subset S \setminus \{x,y\}$. Therefore, on the strip $L_N^{x,y}$ a reasonable candidate to solve the optimal problem for the capacity is

$$\hat{F}_{xy}(\xi) := \sum_{j=1}^{L-1} \{\mathbf{f_{xy}}(z_j) - \mathbf{f_{xy}}(z_{j+1})\} \, G\big(\xi_{z_1} + \cdots + \xi_{z_j}\big) \,,$$

where $x = z_1, z_2, \ldots, z_L = y$ is an enumeration of S such that $\mathbf{f_{xy}}(z_j) \geq \mathbf{f_{x,y}}(z_{j+1})$ for $1 \leq j < L$. A calculation shows that this function has the properties listed in the previous paragraph.

Since the tubes $L_N^{x,y}$, $x \in S^1$, $y \in S^2$, are essentially disjoints, the candidate F should be equal to $\hat{F}_{xy}$ on each tube $L_N^{x,y}$ and equal to some appropriate convex combination of these functions on the complement.

We hope that this informal exposition will help the reader to understand the definition of the candidate we now present. We first define an approximation of the optimal function for the Dirichlet problem in a two sites zero-range process.

$H(t)$ Fix $0 < \epsilon < 1/8$. Let $H : [0,1] \to [0,1]$ be the smooth function given by

$$H(t) := \frac{1}{I_\alpha} \int_0^{\phi(t)} u^\alpha (1-u)^\alpha \, du \,,$$

where I_α is the constant defined in (1.4) and $\phi : [0,1] \to [0,1]$ is a smooth

$\phi(t)$ bijective function such that

$$\phi(t) + \phi(1-t) = 1 \text{ for every } t \in [0,1] \text{ and } \phi(s) = 0 \ \forall s \in [0,4\epsilon].$$

It can be easily checked that

$$H(t) + H(1-t) = 1 \,, \quad \forall t \in [0,1] \,, \tag{5.1}$$

$H|_{[0,4\epsilon]} \equiv 0$ and $H|_{[1-4\epsilon,1]} \equiv 1$. The function H is a smooth approximation of the function G which appeared in the proof of the lower bound.

At the end of the proof of the upper bound, we also need $\phi(t)$ to satisfy the following bounds:

$$\sup \big\{ \phi'(u) : u \in [0,1] \big\} \ \leq \ 1 + \sqrt{\epsilon} \,. \tag{5.2}$$

It follows from this assumption, the fact that $\phi(\epsilon) = 0$ and the mean value theorem that

$$\sup \big\{ \frac{\phi(u)}{u - \epsilon} : u \in [2\epsilon, 1] \big\} \ \leq \ 1 + \sqrt{\epsilon} \,. \tag{5.3}$$

Assumption (5.2) can easily be accomplished since $(1 + \sqrt{\epsilon})$ times the length of the interval $[4\epsilon, 1 - 4\epsilon]$ is strictly greater than 1 for ϵ small enough.

The function H is an approximation of the optimal function for the Dirichlet problem of a zero-range process on two sites. To obain from H a solution of the Dirichlet problem for the zero-range process in many sites, we need to incorporate to H the geometry in S created by the random walk. This is achieved below with the help of the optimal functions $\mathbf{f_{xy}}$ of the Dirichlet problem associated to the random walk. $\mathbf{f_{xy}}$

For each pair $x \neq y \in S$, consider the function $\mathbf{f_{xy}} : S \to [0, 1]$ in $\mathcal{C}(x, y)$ such that

$$D_S(\mathbf{f_{xy}}) \;=\; \mathrm{cap}_S(x, y) \,.$$

By xxx, $\mathbf{f_{xy}}(z)$ is equal to the probability that the random walk with generator $\mathcal{L}_S$ reaches x before y when it starts from z: $\mathbf{f_{xy}}(z) = \mathbb{P}_z[H_x < H_y]$. Fix an enumeration

$$x \;=\; z_1, \;\; z_2, \;\; \dots, \;\; z_L = y \tag{5.4}$$

of S satisfying $\mathbf{f_{xy}}(z_j) \geq \mathbf{f_{xy}}(z_{j+1})$ for $1 \leq j \leq L - 1$. This enumeration depends on the pair (x, y). To stress this dependence we sometimes denote z_j by $z_j^{x,y}$. Since $\mathbf{f_{xy}}(z) = \mathbb{P}_z[H_x < H_y] = 1 - \mathbb{P}_z[H_y < H_x] = 1 - \mathbf{f_{yx}}(z)$, once the enumeration $x = z_1^{x,y}, \dots, z_L^{x,y} = y$ has been fixed, we may set $z_j^{y,x} = z_{L+1-j}^{x,y}$ to obtain an enumeration for the pair (y, x), $y = z_1^{y,x}, \dots, z_L^{y,x} = x$, satisfying $\mathbf{f_{yx}}(z_j) \geq \mathbf{f_{yx}}(z_{j+1})$ for $1 \leq j \leq L - 1$.

Let $\mathcal{D} \subset \mathbb{R}_+^S$ be the compact subset

$$\mathcal{D} \;:=\; \Big\{ u \in \mathbb{R}_+^S : \sum_{x \in S} u_x = 1 \Big\} \,.$$

For $x \neq y \in S$ and $\delta > 0$, let $\mathcal{L}_\delta^{xy}$, $\mathcal{L}_\delta^x$ be the closed subsets of $\mathcal{D}$ defined by $\mathcal{L}_\delta^{xy}$

$$\mathcal{L}_\delta^{xy} \;:=\; \{u \in \mathcal{D} : u_x + u_y \geq 1 - \delta\} \,, \quad \mathcal{L}_\delta^x \;:=\; \bigcup_{y \neq x} \mathcal{L}_\delta^{xy} \,.$$

Note that $\mathcal{L}_\delta^{xy} = \mathcal{L}_\delta^{yx}$. Define $F_{xy} : \mathcal{D} \to [0, 1]$ as the smooth function

$$F_{xy}(u) \;:=\; \sum_{j=1}^{L-1} \{\mathbf{f_{xy}}(z_j) - \mathbf{f_{xy}}(z_{j+1})\} \, H\Big(\sum_{i=1}^{j} u_{z_i} \Big) \,.$$

Actually, we will only be interested in the value of F_{xy} in the band $\mathcal{L}_{2\epsilon}^{xy}$. The identities (5.1), $\mathbf{f_{xy}}(z) = 1 - \mathbf{f_{yx}}(z)$, and $z_j^{y,x} = z_{L+1-j}^{x,y}$ yield that

$$F_{xy}(u) \;+\; F_{yx}(u) \;=\; 1 \quad \text{for all } u \in \mathcal{D}. \tag{5.5}$$

On the other hand, since H vanishes on $[0, 4\epsilon]$ and is identical to 1 on $[1 - 4\epsilon, 1]$, $F_{xy}(u) = 1$ if $u_x \geq (1 - 4\epsilon)$ and $F_{xy}(u) = 0$ if $u_x \leq 2\epsilon$, $u \in \mathcal{L}_{2\epsilon}^{xy}$. F_x

Define $F_x : \mathcal{L}_{2\epsilon}^x \to [0, 1]$ by

$$F_x(u) := F_{xy}(u) \quad \text{if } u \in \mathcal{L}_{2\epsilon}^{xy}. \tag{5.6}$$

The function F_x is well defined: if u belongs to $\mathcal{L}_{2\epsilon}^{xy} \cap \mathcal{L}_{2\epsilon}^{xz}$, $u_y \leq 2\epsilon$ as $u \in \mathcal{L}_{\epsilon}^{xz}$. Hence, $u_x \geq 1 - 4\epsilon$ so that $F_{xy}(u) = F_{xz}(u) = 1$. Moreover, since $F_x = F_{xy}$ on $\mathcal{L}_{2\epsilon}^{xy}$, and since $F_{xy}(u) = 1$ if $u_x \geq (1 - 4\epsilon)$,

$$F_x(u) = 1 \quad \text{if } u_x \geq 1 - 2\epsilon . \tag{5.7}$$

On the other hand, we have seen that $F_{xy}(u) = 0$ if $u_x \leq 2\epsilon$, $u \in \mathcal{L}_{2\epsilon}^{xy}$. We may therefore extend F_x to the set $\mathcal{B}_x(2\epsilon) = \{u \in \mathcal{D} : u_x \leq 2\epsilon\}$ setting

$$F_x(u) = 0 \quad \text{if } u_x \leq 2\epsilon . \tag{5.8}$$

Up to this point we have defined F_x on the compact space $\mathcal{L}_{2\epsilon}^x \cup \mathcal{B}_x(2\epsilon)$. Since F_x is Lipschitz continuous on $\mathcal{L}_{2\epsilon}^x \cup \mathcal{B}_x(2\epsilon)$, we may extend F_x to $\mathcal{D}$ keeping this property: There exists a finite constant C_ϵ such that for all $u, v \in \mathcal{D}$

$$|F_x(u) - F_x(v)| \leq C_0|u - v| . \tag{5.9}$$

We are finally in a position to define the approximation of the solution of the Dirichlet problem. Fix a nonempty proper subset $S^1 \subsetneq S$ and let $S^2 = S \setminus S^1$. We define the function $F_{S^1} : \mathcal{D} \to \mathbb{R}$ as

$$F_{S^1}(u) := \sum_{x \in S^1} F_x(u) .$$

Herafter, $F_{S^1}(\eta)$, $F_x(\eta)$ and $F_{xy}(\eta)$ stand for $F_{S^1}(\eta/N)$, $F_x(\eta/N)$ and $F_{xy}(\eta/N)$, respectively.

Let

$$D_N^x := \{\eta \in E_N : \eta_x \geq N - 3\ell_N\} , \quad x \in S ,$$

so that $\mathcal{E}_N^x \subset D_N^x$. It follows from (5.7), (5.8) that if $\eta \in D_N^x$ for some $x \in S$ then

$$F_{S^1}(\eta) = \mathbf{1}\{x \in S^1\} = F_{S^1}(\sigma^{zw}\eta) , \tag{5.10}$$

for every $z, w \in S$ and every N large enough. In particular,

$$F_{S^1} \in \mathcal{C}_N\Big(\bigcup_{x \in S^1} D_N^x, \bigcup_{y \in S^2} D_N^y \Big) .$$

Now that we have a candidate, we need to estimate its Dirichlet form. This is done in two steps. We first estimate the pieces outside the tubes $\mathcal{L}_\epsilon^{xy}$, which should not contribute to the total Dirichlet form, and then the pieces in the tubes which are the important ones. For each subset $A \subseteq E_N$ and function $F : E_N \to \mathbb{R}$, let

$$D_N(F; A) := \frac{1}{2}\sum_{\eta \in A}\sum_{z,w \in S} \mu_N(\eta)g(\eta_z)r(z,w)\{F(\sigma^{zw}\eta) - F(\eta)\}^2 .$$

$\mathcal{I}_N^{xy}$

For $x \neq y \in S$, let $\mathcal{I}_N^{xy}$ be a miscroscopic approximation of the band $\mathcal{L}_\epsilon^{xy}$:

$$\mathcal{I}_N^{xy} := \left\{ \eta \in E_N : \eta_x + \eta_y \geq N - \ell_N \right\}.$$

Clearly, $\mathcal{I}_N^{xy} = \mathcal{I}_N^{yx}$, x, $y \in S$ and, for every N large enough, $\eta/N \in \mathcal{L}_\epsilon^{xy}$ if $\eta \in \mathcal{I}_N^{xy}$. Let $\mathcal{I}_N^x := \cup_{y \in S \setminus \{x\}} \mathcal{I}_N^{xy}$.

Lemma 6.11. *There exists a finite constant C_ϵ, independent of N, such that for every $N \geq 1$ large enough,*

$$D_N(F_{S^1}; E_N \setminus \cup_{z \in S^1} \mathcal{I}_N^z) \leq \frac{C_\epsilon}{N^{\alpha+1} \, \ell_N^{\alpha-1}}.$$

Proof. By Cauchy-Schwarz inequality,

$$D_N(F_{S^1}; E_N \setminus \cup_{z \in S^1} \mathcal{I}_N^z) \leq |S^1| \sum_{x \in S^1} D_N(F_x; E_N \setminus \cup_{z \in S^1} \mathcal{I}_N^z)$$

$$\leq |S^1| \sum_{x \in S^1} D_N(F_x; E_N \setminus \mathcal{I}_N^x).$$

It remains to estimate

$$\frac{1}{2} \sum_{\eta \in E_N \setminus \mathcal{I}_N^x} \sum_{z,w \in S} \mu_N(\eta) g(\eta_z) r(z,w) \left\{ F_x(\sigma^{zw}\eta) - F_x(\eta) \right\}^2$$

for each $x \in S^1$.

By properties (5.7) and (5.8), we can restrict the previous sum to configurations $\eta \in E_N \setminus \mathcal{I}_N^x$ satisfying $\epsilon N \leq \eta_x \leq (1-\epsilon)N$. Hence, by (5.9), the last expression is bounded above by

$$\frac{C_\epsilon}{N^2} \sum_{\substack{\eta \in E_N \setminus \mathcal{I}_N^x \\ \epsilon N \leq \eta_x \leq (1-\epsilon)N}} \mu_N(\eta) \leq \frac{C_\epsilon}{Z_N \, N^2} \sum_{i=\epsilon N}^{(1-\epsilon)N} \frac{1}{a(i)} \sum_{\substack{\xi \in E_{S \setminus \{x\}, N-i} \\ \xi_y \leq N-i-\ell_N}} \frac{1}{a(\xi)}.$$

In this formula and below, C_ϵ is a finite constant whose value may change from line to line, but which never depends on N.

By definition (2.1) of $\Delta_{L,N}(\ell)$, this expression can be written as

$$\frac{C_\epsilon}{Z_N \, N^2} \sum_{i=\epsilon N}^{(1-\epsilon)N} \frac{1}{a(i) \, a(N-i)} \left\{ (N-i)^\alpha \sum_{\zeta \in \Delta_{S \setminus \{x\}, N-i}(\ell_N)} \frac{1}{a(\zeta)} \right\}.$$

By Lemmas 6.4 and 6.5, this sum is bounded above by

$$\frac{C_\epsilon}{Z_N \, N^2 \, \ell_N^{\alpha-1}} \sum_{i=\epsilon N}^{(1-\epsilon)N} \frac{1}{a(i) a(N-i)}.$$

To conclude the proof it remains to recall Lemma 6.5. $\square$

Since $\ell_N \uparrow \infty$, by Lemma 6.11,

$$\lim_{N\to\infty} N^{\alpha+1} D_N(F_{S^1}; E_N \setminus \cup_{z\in S^1}\mathfrak{I}_N^z) = 0 . \tag{5.11}$$

It remains to estimate $D_N(F_{S^1}; \cup_{z\in S^1}\mathfrak{I}_N^z)$. The contribution to the total Dirichlet form of the tubes $\mathfrak{I}_N^{x,y}$, from a site $x \in S^1$ to another site $y \in S^1$, vanishes because F_{S^1} is equal to 1 on these strips: for any N large enough,

$$F_{S^1}(\sigma^{zw}\eta) = 1 = F_{S^1}(\eta) \quad \text{for all } \eta \in \bigcup_{x,y\in S^1} \mathfrak{I}_N^{xy} \text{ and } z, w \in S .$$

Indeed, for $x \neq y$ in S^1, by (5.8), (5.6), for $\eta/N \in \mathcal{L}_{2\epsilon}^{xy}$,

$$F_{S^1}(\eta) = F_{xy}(\eta) + F_{yx}(\eta) ,$$

and by (5.5) this last sum is equal to 1. Therefore, by definition of $\mathfrak{I}_N^z$,

$$D_N\Big(F_{S^1}; \bigcup_{z\in S^1} \mathfrak{I}_N^z\Big) = D_N\Big(F_{S^1}; \bigcup_{\substack{x\in S^1 \\ y\in S^2}} \mathfrak{I}_N^{xy}\Big) = \sum_{x\in S^1}\sum_{y\in S^2} D_N\big(F_{S^1}; \mathfrak{I}_N^{xy}\big) .$$

The last identity follows from (5.10) and the relation

$$\mathfrak{I}_N^{x_1 y_1} \cap \mathfrak{I}_N^{x_2 y_2} \subseteq \bigcup_{z\in S} D_N^z \quad \text{for all } x_1, x_2 \in S^1 \text{ and } y_1, y_2 \in S^2 .$$

Therefore, by (5.6) and (5.8) we finally conclude that

$$D_N\Big(F_{S^1}; \bigcup_{z\in S^1} \mathfrak{I}_N^z\Big) = \sum_{x\in S^1}\sum_{y\in S^2} D_N\big(F_{xy}; \mathfrak{I}_N^{xy}\big). \tag{5.12}$$

Lemma 6.12. *For any $x, y \in S$, $x \neq y$,*

$$\limsup_{N\to\infty} N^{\alpha+1} D_N(F_{xy}; \mathfrak{I}_N^{xy}) \leq \frac{(1+\sqrt{\epsilon})^{2\alpha+1}}{I_\alpha \Gamma(\alpha)} \operatorname{cap}_S(x,y) .$$

Proof. Fix x, y in S, $x \neq y$. Let $x = z_1, z_2, \ldots, z_L = y$ be the enumeration established in the definition of F_{xy}, so that $\mathbf{f_{xy}}(z_n) \geq \mathbf{f_{xy}}(z_{n+1})$, $1 \leq n \leq L-1$. Fix two different sites $z_i \neq z_j$ in S with $1 \leq i < j \leq L$. By definition of F_{xy},

$$F_{xy}(\sigma^{z_i z_j}\eta) - F_{xy}(\eta) = \sum_{n=i}^{j-1} \big[\mathbf{f_{xy}}(z_n) - \mathbf{f_{xy}}(z_{n+1})\big] \{H(m_n - N^{-1}) - H(m_n)\} ,$$

where $m_n = N^{-1}\sum_{m=1}^{n} \eta_{z_m}$. The definition of m_n will be affected by the change of variables performed during the proof. These modifications will be pointed out. Thus, by the Cauchy-Schwarz inequality, the sum

$$\sum_{\eta \in \mathcal{J}_N^{xy}} \mu_N(\eta) g(\eta_{z_i}) r(z_i, z_j) \big\{ F_{xy}(\sigma^{z_i z_j} \eta) - F_{xy}(\eta) \big\}^2 \tag{5.13}$$

is bounded above by $\{\mathbf{f_{xy}}(z_i) - \mathbf{f_{xy}}(z_j)\}$ times

$$\sum_{n=i}^{j-1} \big(\mathbf{f_{xy}}(z_n) - \mathbf{f_{xy}}(z_{n+1})\big) \sum_{\eta \in \mathcal{J}_N^{xy}} \mu_N(\eta) g(\eta_{z_i}) r(z_i, z_j) \big\{ H(m_n - N^{-1}) - H(m_n) \big\}^2 .$$

Performing the change of variables $\xi = \eta - \mathfrak{d}_{z_i}$, the second sum above is less than

$$r(z_i, z_j) \frac{1}{Z_N} \sum_{\xi \in A_N^{xy}} \frac{1}{a(\xi)} \big\{ H(m_n + N^{-1}) - H(m_n) \big\}^2 ,$$

where $A_N^{xy} := \{\xi \in E_{S,N-1} : \xi_x + \xi_y \geq N - 2\ell_N\}$ and now $m_n = N^{-1} \sum_{m=1}^{n} \xi_{z_m}$. So far, we have shown that (5.13) is bounded above by $r(z_i, z_j) Z_N^{-1} \{\mathbf{f_{xy}}(z_i) - \mathbf{f_{xy}}(z_j)\}$ times

$$\sum_{n=i}^{j-1} \big(\mathbf{f_{xy}}(z_n) - \mathbf{f_{xy}}(z_{n+1})\big) \sum_{\xi \in A_N^{xy}} \frac{1}{a(\xi)} \big\{ H(m_n + N^{-1}) - H(m_n) \big\}^2 .$$

Fix some $i \leq n < j$. The second sum in the above expression may be re-written as

$$\sum_{m=0}^{2\ell_N} \sum_{\zeta \in E_{S_{xy}, m}} \frac{1}{a(\zeta)} \sum_{k=2\epsilon N}^{(1-2\epsilon)N} \frac{1}{a(k) a(N - m - k)} \big\{ H(m_n + N^{-1}) - H(m_n) \big\}^2 , \tag{5.14}$$

where m_n is now $N^{-1}(k + \sum_{\ell=2}^{n} \zeta_{z_\ell})$. The sum in k is carried over the set $\{2\epsilon N, \dots, (1-2\epsilon)N\}$ because H is constant in the intervals $[0, 4\epsilon]$, $[1 - 4\epsilon, 1]$ an there are at most $2\ell_N$ ζ-particles. Let $\phi_k = \phi(N^{-1}(k + \sum_{\ell=2}^{n} \zeta_{z_\ell}))$. With this notation, we may bound the last expression by $N^{-2\alpha} I_\alpha^{-2}$ times

$$\sum_{m=0}^{2\ell_N} \sum_{\zeta \in E_{S_{xy}, m}} \frac{1}{a(\zeta)} \sum_{k=2\epsilon N}^{(1-2\epsilon)N} \int_{\phi_k}^{\phi_{k+1}} u^\alpha (1-u)^\alpha \, du \int_{\phi_k}^{\phi_{k+1}} \frac{u^\alpha (1-u)^\alpha}{(\frac{k}{N})^\alpha (1 - \frac{k+m}{N})^\alpha} \, du .$$

The last integral is bounded above by

$$\{\phi_{k+1} - \phi_k\} \frac{\phi_{k+1}^\alpha (1 - \phi_k)^\alpha}{(\frac{k}{N})^\alpha (1 - \frac{k+m}{N})^\alpha} .$$

Since $\sum_{\ell=2}^{n} \zeta_{z_\ell} = m \leq 2\ell_N$ and $\phi_k = \phi(N^{-1}(k + m))$, in view of (5.3), $\phi_{k+1} \leq (1 + \sqrt{\epsilon}) [N^{-1}(k+1+m) - \epsilon] \leq (1 + \sqrt{\epsilon}) (k/N)$ for N large enough. On the other hand, since $1 - \phi(t) = \phi(1-t)$, by (5.3) $1 - \phi_k = 1 - \phi(N^{-1}(k+m)) = \phi(1 - N^{-1}(k+m)) \leq (1 + \sqrt{\epsilon}) [1 - N^{-1}(k+m) - \epsilon] \leq (1 + \sqrt{\epsilon}) [1 - N^{-1}(k+m)]$. By (5.2), $\phi_{k+1} - \phi_k \leq (1 + \sqrt{\epsilon}) N^{-1}$. In conclusion, the previous displayed

formula is bounded above by $(1 + \sqrt{\epsilon})^{2\alpha+1} N^{-1}$, and (5.14) is less than or equal to.

$$\frac{(1 + \sqrt{\epsilon})^{2\alpha+1}}{I_\alpha N^{2\alpha+1}} \sum_{m=0}^{2\ell_N} \sum_{\zeta \in E_{S_{xy},m}} \frac{1}{a(\zeta)} \leq \frac{(1 + \sqrt{\epsilon})^{2\alpha+1}}{I_\alpha N^{2\alpha+1}} \sum_{m \geq 0} \sum_{\zeta \in E_{S_{xy},m}} \frac{1}{a(\zeta)}$$
$$= \frac{(1 + \sqrt{\epsilon})^{2\alpha+1} \Gamma(\alpha)^{L-2}}{I_\alpha N^{2\alpha+1}} .$$

Up to this point we have shown that (5.13) is bounded above by

$$r(z_i, z_j)\{\mathbf{f_{xy}}(z_i) - \mathbf{f_{xy}}(z_j)\}^2 \Big(\frac{\Gamma(\alpha)^{L-2}\{1 + \sqrt{\epsilon}\}^{2\alpha+1}}{Z_N \, I_\alpha \, N^{2\alpha+1}}\Big) .$$

The same upper bound for (5.13) holds for $j < i$. To conclude the proof of the lemma, it remains to sum over i, j and to recall the statement of Lemma 6.5 and the definition of $\mathbf{f_{xy}}$. □

Proof (of Proposition 6.10). Fix a nonempty subset $S^1 \subsetneq S$ and let $S^2 = S \setminus S^1$. By xxx, the capacity is monotone in each variable. Therefore, since $\mathcal{E}_N(S^i) \subset \cup_{x \in S^i} D_N^x$ and since F_{S^1} belongs to $\mathcal{C}(\cup_{x \in S^1} D_N^x, \cup_{y \in S^2} D_N^y)$,

$$\mathrm{cap}_N\big(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2)\big) \leq \mathrm{cap}_N\Big(\bigcup_{x \in S^1} D_N^x, \bigcup_{y \in S^2} D_N^y\Big) \leq D_N(F_{S^1}) .$$

It follows from (5.11), (5.12) and Lemma 6.12 that

$$\limsup_{N \to \infty} N^{\alpha+1} \mathrm{cap}_N\big(\mathcal{E}_N(S^1), \mathcal{E}_N(S^2)\big) \leq \frac{1}{I_\alpha \Gamma(\alpha)} \sum_{x \in S^1, y \in S^2} \mathrm{cap}_S(x, y) ,$$

as claimed. □

6 Comments and References

?

Condensation has been observed and investigated in shaken granular systems, growing and rewiring networks, traffic flows and wealth condensation in macroeconomics. We refer to the recent review by Evans and Hanney **?**.

Several aspects of the condensation phenomenon for zero range dynamics have been examined. Let the condensate be the site with the maximal occupancy. Precise estimates on the number of particles at the condensated, as well as its fluctuations, have been obtained in **???**. The equivalence of ensembles has been proved by Großkinsky, Schütz and Spohn **?**. Ferrari, Landim and Sisko **?** proved that if the number of sites is kept fixed, as the total number of particles $N \uparrow \infty$, the distribution of particles outside the condensated converges to the grand canonical distribution with critical density. Armendariz

and Loulakis **?** generalized this result showing that if the number of sites L grows with the number of particles N in such a way that the density N/L converges to a value greater than the critical density, the distribution of the particles outside the condensate converges to the grand canonical distribution with critical density.

This article leaves two interesting open questions. The techniques used here rely strongly on the reversibility of the process. It is quite natural to examine the same problem for asymmetric zero range processes where new techniques are required. On the other hand, the number of sites is kept fixed. It is quite tempting to let the number of sites grow with the number of particles. In this case, in the nearest neighbor, symmetric model, for instance, the condensate jumps from one site to another at rate proportional to the inverse of the distance. The rates are therefore not summable and it is not clear if a scaling limit exists.

Simulations for the evolution of the condensated have been performed by Godrèche and Luck **?**. The authors predicted the time scale, obtained here, in which the condensate evolves and claimed that the time scale should be the same for non reversible dynamics.

Remark 6.13. In **?**, it is shown that, in the case the number of sites increases with the number of particles, the highest occupied site contains a nonzero fraction of the particles in the system. This result includes the case $1 < \alpha \leq 2$. In contrast, when the number of sites is kept fixed, it seems to have been unnoticed in the literature that the condensation phenomenon appears also for $1 < \alpha \leq 2$. More precisely, if $1 \ll \ell_N \ll N$, then

$$\lim_{N \to \infty} \mu_N\big(\eta_x \geq N - \ell_N\big) \;=\; 1/L \;, \quad \forall x \in S \;.$$

Moreover, given that particles concentrate on $x \in S$, the distribution of the configuration on $S \setminus \{x\}$ is asymptotically given by the grand-canonical measure determined by m: For any x in S,

$$\lim_{N \to \infty} \sup_{\zeta \in \mathcal{G}_N^x} \left| \mu_N\big(\eta_z = \zeta_z \,,\, z \neq x \,\big|\, \eta_x \geq N - \ell_N\big) - \prod_{z \neq x} \frac{1}{\Gamma_z} \frac{m(z)^{\zeta_z}}{a(\zeta_z)} \right| \;=\; 0 \;,$$

where $\mathcal{G}_N^x := \{\zeta \in \mathbb{N}^{S \setminus \{x\}} : \sum_z \zeta_z \leq \ell_N\}$. There is just a small difference between the cases $1 < \alpha \leq 2$ and $\alpha > 2$. While in the former, the variables $\{\eta_z : z \in S\}$ do not have finite expectation under the critical grand-canonical measure, they do have finite expectation in the latter case.