4

Topology

Consider the birth and death chain $\eta(t)$ on $E_N = \{0, \dots, N\}$ defined in Chapter **??** associated to the functions $H(x) = x^\alpha(1-x)^\alpha$, $\alpha > 1$, and $\Phi(x) = 1$, $x \in [0,1]$. To define the jump rates of this chain, let π_N be the probability measure given by

$$\pi_N(k) \;=\; \frac{1}{Z_N}\,\frac{1}{H_N(k)}\;,$$

where $H_N(k) = H(k/N)$, $1 \le k \le N-1$, $H_N(0) = H_N(N) = N^\alpha$, and where Z_N is the partition function $Z_N = \sum_{0\le k\le N} H_N(k)^{-1}$. The jump rates $R_N(x,y)$ of $\eta(t)$ are defined for $\eta(t)$ to be reversible with respect to π_N:

$$R_N(k,k+1) \;=\; 1\;, \quad R_N(k+1,k) \;=\; \frac{\pi_N(k)}{\pi_N(k+1)}\;, \quad 0 \le k < N\;,$$

$R(k,j) = 0$ otherwise.

Fix a sequence $1 \ll \ell_N \ll N$ and let $\mathcal{E}^1_N = \{0, \dots, \ell_N\}$, $\mathcal{E}^2_N = \{N - \ell_N, \dots, N\}$, $\mathcal{E}_N = \mathcal{E}^1_N \cup \mathcal{E}^2_N$, $\Delta_N = E_N \setminus \mathcal{E}_N$. Let $\Psi_N : E_N \to \{1, 2, \mathfrak{d}\}$ be the order parameter:

$$\Psi_N(\eta) \;=\; \sum_{j=1}^{2} j\,\mathbf{1}\{\eta \in \mathcal{E}^j_N\} \;+\; \mathfrak{d}\,\mathbf{1}\{\eta \in \Delta_N\}\;,$$

and let $\mathbb{X}^N(t) = \Psi_N(\eta(tN^{\alpha+1}))$ be the value of the order parameter at time t, where time has been speeded-up by $N^{\alpha+1}$. We prove in Chapter **??** that on the time-scale $N^{\alpha+1}$, $\eta(t)$ evolves as a symmetric Markov chain on $\{1,2\}$.

Far from the boundary, in the interval $\{\epsilon N, \dots, (1-\epsilon)N\}$, $\epsilon > 0$, the dynamics of the birth and death chain $\eta(t)$ corresponds to the one of a weakly asymmetric random walk. Hence, in the diffusive time scale N^2 in the interval $\{\epsilon N, \dots, (1-\epsilon)N\}$ the birth and death chain $\eta(t)$ evolves as a Brownian motion with a drift. In constrast, close to the boundaries the chain has a drift of order one in the direction of the boundary, which increases as it approaches the boundary. This drift encloses the process in a microscopic neighborhood of the boundary, which is only surmounted in a time scale of order $N^{\alpha+1}$.

This model presents, therefore, two macroscopic time scales: the diffusive one, N^2, which corresponds to the time needed to reach the boundary from the bulk, and the longer time scale $N^{\alpha+1}$, which is the time needed to escape from a microscopic neighborhood of a boundary site. The graph of a typical realization of $\mathbb{X}^N(t)$ is depicted in Figure **??**. As $N \uparrow \infty$, the length of the excursions to $\mathfrak{d}$ decreases to vanish in the limit. These evanescent excursions to $\mathfrak{d}$ precludes the convergence of the process $\mathbb{X}^N(t)$ in the Skorohod topology. We present in this chapter a weaker topology, tailor maded to handle such cases which are typical in the metastable context.

The chapter is organized as follows. For a metric space $\mathbb{M}$, denote by $D([0,T],\mathbb{M})$, $T > 0$, the space of right-continuous functions $x : [0,T] \to \mathbb{M}$ with left-limits. We introduce in 1.8 a metric $\mathbf{d}$ in a subspace of $D([0,T], S_{\mathfrak{d}})$, where $S_{\mathfrak{d}}$ is the one-point compactification of $\mathbb{N}$. The completion of this subspace with respect to the metric $\mathbf{d}$ consists of trajectories $x : [0,T] \to S_{\mathfrak{d}}$ which at each point $t \in (0,T)$ may have at most two left-limits and two right-limits, on in $\mathbb{N}$ and the other one equal to $\mathfrak{d}$, the point added to $\mathbb{N}$ to turn it into a compact metric space. The space of such trajectories is denoted by $E([0,T], S_{\mathfrak{d}})$. We introduce this space in Section 1 below and examine the properties

$D([0,T],\mathbb{M})$

1 The space $E([0,T], S_{\mathfrak{d}})$

$d(k,j)$

Assume now that the order parameter takes a countable number of values, $S = \mathbb{N} = \{1, 2, \dots\}$. Let $S_{\mathfrak{d}}$ be the one-point compactification of S: $S_{\mathfrak{d}} = S \cup \{\mathfrak{d}\}$, $\mathfrak{d} = \infty$, where the metric in $S_{\mathfrak{d}}$ is given by $d(k,j) = |k^{-1} - j^{-1}|$. Generic elements of the set $S_{\mathfrak{d}}$ are denoted by the symbols $\mathfrak{n}$, $\mathfrak{m}$.

We adopt the following nomenclature. A sequence of real numbers $\{t_j : j \geq 1\}$ is said to the be *increasing* if $t_j < t_{j+1}$ for all j, with a similar convention for decreasing sequences. This sequence is said to the be *non-decreasing* if $t_j \leq t_{j+1}$ for all j. We write $t_j \uparrow t$ to say that the increasing sequence t_j converges to t and $t_j \downarrow t$ to say that the decreasing sequence t_j converges to t. Similarly, a function $f : [a,b] \to \mathbb{R}$ is said to be increasing, decreasing if $f(s) < f(t)$, $f(s) > f(t)$, respectively, for $s < t$.

Definition 4.1. *A measurable function $x : [0,T] \to S_{\mathfrak{d}}$ is said to have a* soft *left-limit at $t \in (0,T]$ if one of the following two alternatives holds*

(a) *The trajectory x has a left-limit at t, denoted by $x(t-)$;*
(b) *The set of cluster points of $x(s)$, $s \uparrow t$, is a pair formed by $\mathfrak{d}$ and a point in S, denoted by $x(t\ominus)$.*

A soft *right-limit at $t \in [0,T)$ is defined analogously. In this case, the right-limit, when it exists, is denoted by $x(t+)$, and the cluster point of the sequence*

$x(s)$, $s \downarrow t$, which belongs to S when the second alternative is in force is denoted by $x(t \oplus)$.

$x(t\oplus)$
$x(t\ominus)$

More concisely, a trajectory x has a soft left-limit at t if and only if there exists $n \in S$ such that for all $m \geq 1$, there exists $\delta > 0$ for which $x(s) \in \{n\} \cup S_m^c$ for all $t - \delta < s < t$.

The second alternative in the previous definition asserts that there exist $n \in S$ and two increasing sequences t_j, $t'_j \uparrow t$ such that $\lim_j x(t_j) = n$, $\lim_j x(t'_j) = \mathfrak{d}$. Moreover, if $x(t''_j)$ converges for some sequence $t''_j \uparrow t$, $\lim_j x(t''_j) \in \{n, \mathfrak{d}\}$.

We call $x(t \ominus)$ the *finite* soft left-limit of x at t. Whenever we refer to $x(t-)$ it means that x has a left-limit at t. Similarly, when we refer to $x(t \ominus)$, it is understood that x has not a left-limit at t, but that the alternative (b) of the previous definition is in force. An analogous convention is adopted for $x(t+)$ and $x(t \oplus)$.

Remark 4.2. Since $S_{\mathfrak{d}}$ is a compact set, to prove that x has a soft right-limit at t we only have to show uniqueness of limit points in S. In other words, we have to prove that if t_j and t'_j are sequences decreasing to t and if $x(t_j)$, $x(t'_j)$ converge to $m \in S$, $n \in S$, respectively, then $m = n$.

Definition 4.3. *A trajectory $x : [0,T] \to S_{\mathfrak{d}}$ which has a soft right-limit at t is said to be soft right-continuous at t if one of the following three alternatives holds*

(a) *$x(t+)$ exists and is equal to $\mathfrak{d}$;*
(b) *$x(t+)$ exists, belongs to S, and $x(t+) = x(t)$;*
(c) *$x(t\oplus)$ exists and $x(t\oplus) = x(t)$.*

A trajectory $x : [0,T] \to S_{\mathfrak{d}}$ which is soft right-continuous at every point $t \in [0,T]$ is said to be soft right-continuous.

A trajectory x is soft right-continuous at t if and only if there exists $n \in S$ such that for all $m \geq 1$, there exists $\delta > 0$ for which $x(s) \in \{n\} \cup S_m^c$ for all $t \leq s < t + \delta$.

Note that if x is soft right-continuous at t and if $x(t+) = \mathfrak{d}$, then $x(t)$ may be different from $x(t+)$. In contrast, if $x(t) = \mathfrak{d}$, then $x(t+)$ exists and $x(t+) = \mathfrak{d} = x(t)$.

Clearly, if x is soft right-continuous at t, for every $m \geq 1$, there exists $\epsilon > 0$ such that for all $t \leq s < t + \epsilon$,

$$x(s) = x(t) \text{ or } x(s) \geq m \; . \tag{1.1}$$

Similarly, if x has a soft left-limit at t, there exists $n \in S$ with the following property. For every $m \geq 1$, there exists $\epsilon > 0$ such that for all $t - \epsilon < s < t$,

$$x(s) = n \text{ or } x(s) \geq m \; . \tag{1.2}$$

Let $\mathbb{E}([0,T],S_{\mathfrak{d}})$ be the space of soft right-continuous trajectories $x : [0,T] \to S_{\mathfrak{d}}$ with soft left-limits. $\mathbb{E}([0,T],S_{\mathfrak{d}})$

Fix a trajectory x in $\mathbb{E}([0,T],S_{\mathfrak{d}})$ such that $x(t) = \mathfrak{d}$ for some $t \in [0,T]$. Since it is soft right-continuous, by Definition 4.3,

$$x(t+) \text{ exists and } x(t+) = \mathfrak{d}. \tag{1.3}$$

S_m

Let $S_m = \{1, \dots, m\}$, $m \geq 1$. For a trajectory x in $\mathbb{E}([0,T],S_{\mathfrak{d}})$, $t \in [0,T]$, $m \geq 1$, let

$$\sigma^x_m(t) \ := \ \sup\{s \leq t : x(s) \in S_m\} \ . \tag{1.4}$$

If the set $\{s \leq t : x(s) \in S_m\}$ is empty, we set $\sigma^x_m(t) = 0$, but this convention does not play any role below and we could have defined $\sigma^x_m(t)$ in another way. When there is no ambiguity and it is clear to which trajectory we refer, we denote $\sigma^x_m(t)$ by $\sigma_m(t)$.

σ_m

Fix $t \in (0,T]$ and $m \geq 1$. Suppose that $\sigma_m(t) > 0$ and that $x(\sigma_m(t)) \notin S_m$, so that $x(s) \notin S_m$ for $\sigma_m(t) \leq s \leq t$. By (1.2), there exist $n \in S$ and $\epsilon > 0$ such that for each $s \in (\sigma_m(t) - \epsilon, \sigma_m(t))$ either $x(s) = n$ or $x(s) > m$. By definition of $\sigma_m(t)$ we must have $n \in S_m$. Moreover, $x(\sigma_m(t)-) = n$ if x has a left-limit at $\sigma_m(t)$, and $x(\sigma_m(t)\ominus) = n$ if not.

$\mathfrak{R}_m$

Let $\mathfrak{R}_m x$ be the trajectory which records the last site visited in S_m: $(\mathfrak{R}_m x)(t) = 1$ if $x(s) \notin S_m$ for $0 \leq s \leq t$, and

$$(\mathfrak{R}_m x)(t) \ = \ \begin{cases} x(\sigma_m(t)) & \text{if } x(\sigma_m(t)) \in S_m, \\ x(\sigma_m(t)-) & \text{if } x(\sigma_m(t)) \notin S_m \text{ and } x(\sigma_m(t)-) \text{ exists}, \\ x(\sigma_m(t)\ominus) & \text{otherwise}, \end{cases} \tag{1.5}$$

if there exists $0 \leq s \leq t$ such that $x(s) \in S_m$.

Note that $(\mathfrak{R}_m x)(0) = x(0)$ if $x(0) \in S_m$ and $(\mathfrak{R}_m x)(0) = 1$ if $x(0) \notin S_m$. The convention that $(\mathfrak{R}_m x)(t) = 1$ if $x(s) \notin S_m$ for $0 \leq s \leq t$ corresponds to assume that the trajectory x is defined for $t < 0$ and that $x(t) = 1$ in this time interval.

Consider a trajectory x in $D([0,T],S_{\mathfrak{d}})$, $m \geq 1$ and $t \in (0,T]$. Assume that $x(t) \notin S_m$ and that there exists $0 \leq s \leq t$ such that $x(s) \in S_m$. Since x is right-continuous, $\sigma_m(t) > 0$ and $x(\sigma_m(t)) = x(\sigma_m(t)+) \notin S_m$. Hence, since x has left-limits, under the above conditions,

$$(\mathfrak{R}_m x)(t) \ = \ x(\sigma_m(t)-) \ . \tag{1.6}$$

Note that we may have $\sigma_m(t) = t$ in this example.

Assertion A *Fix a trajectory x in $\mathbb{E}([0,T],S_{\mathfrak{d}})$. For each $m \geq 1$, $\mathfrak{R}_m x$ is a trajectory in $D([0,T],S_m)$.*

Proof. Fix $m \geq 1$. We first prove the right continuity of $\mathfrak{R}_m x$. Fix $t \in [0,T)$. By (1.1), there exists $\delta > 0$ such that for all $t \leq s \leq t+\delta$, either $x(s) = x(t)$ or $x(s) > m$. Suppose that $x(t)$ belongs to S_m. In this case, $(\mathfrak{R}_m x)(s) = x(t) =$

$(\mathfrak{R}_m x)(t)$ for $t \le s < t+\delta$. On the other hand, if $x(t) \notin S_m$, $x(s) \notin S_m$ for $t \le s < t+\delta$ so that $\sigma_m(s) = \sigma_m(t)$ in this interval. Therefore, in view of (1.5), $(\mathfrak{R}_m x)(s) = (\mathfrak{R}_m x)(t)$ for $t \le s \le t+\delta$. This proves that $\mathfrak{R}_m x$ is right-continuous.

We turn to the proof of the existence of a left limit at $t \in (0,T]$. If $x(t-)$ exists and belongs to S_m, $(\mathfrak{R}_m x)(s) = x(t-)$ for all $s < t$ close enough of t. If $x(t-)$ exists and does not belong to S_m, $\sigma_m(s)$ is constant in an open interval $(t-\delta, t)$, which implies that $(\mathfrak{R}_m x)(s)$ is constant in the same interval. Finally, suppose that $x(t\ominus)$ exists. In view of (1.2), there exists $\delta > 0$ such that for all $t-\delta < s < t$, either $x(s) > m$ or $x(s) = x(t\ominus)$. If $x(t\ominus) \le m$, $(\mathfrak{R}_m x)(s) = x(t\ominus)$ in some interval $(t-\delta', t)$, $\delta' > 0$. If $x(t\ominus) > m$, then $\sigma_m(s)$ is constant in the interval $(t-\delta, t)$, so that $\mathfrak{R}_m x$ is constant in the same interval. This concludes the proof of the assertion. □

The next example shows that the trajectories $\mathfrak{R}_m x$, $m \ge 1$, do not characterize the trajectory x.

Example 4.4. Fix $0 < s < t < T$ and a sequence $\{t_j : j \ge 1\}$ such that $t_1 < T$, $t_j \downarrow t$. Consider the trajectories x, $y \in \mathbb{E}(([0,T], S_{\mathfrak{d}})$ given by

$$x \;=\; \mathbf{1}\{[0,s)\} \;+\; \mathfrak{d}\,\mathbf{1}\{[s,t]\} \;+\; \sum_{j\ge 2} j\,\mathbf{1}\{[t_j, t_{j-1})\} \;+\; \mathbf{1}\{(t_1, T]\}\;,$$

$$y \;=\; \mathbf{1}\{[0,t]\} \;+\; \sum_{j\ge 2} j\,\mathbf{1}\{[t_j, t_{j-1})\} \;+\; \mathbf{1}\{(t_1, T]\}\;.$$

It is clear that $\mathfrak{R}_m x = \mathfrak{R}_m y$ for all $m \ge 1$.

For a trajectory $x \in \mathbb{E}([0,T], S_{\mathfrak{d}})$, let $\sigma^x_\infty(t)$ be the time of the last visit to S: σ_∞

$$\sigma^x_\infty(t) \;:=\; \sup\{s \le t : x(s) \in S\}\;,$$

with the convention that $\sigma^x_\infty(t) = 0$ if $x(s) = \mathfrak{d}$ for $0 \le s \le t$. As before, when there is no ambiguity and it is clear to which trajectory we refer, we denote $\sigma^x_\infty(t)$ by $\sigma_\infty(t)$. $\mathfrak{R}_\infty$

Let $\mathfrak{R}_\infty x$ be the trajectory which records the last site visited in S: $(\mathfrak{R}_\infty x)(t) = 1$ if $x(s) = \mathfrak{d}$ for all $0 \le s \le t$, and

$$(\mathfrak{R}_\infty x)(t) \;=\; \begin{cases} x(\sigma_\infty(t)) & \text{if } x(\sigma_\infty(t)) \in S, \\ x(\sigma_\infty(t)-) & \text{if } x(\sigma_\infty(t)) \notin S \text{ and if } x(\sigma_\infty(t)-) \text{ exists}, \\ x(\sigma_\infty(t)\ominus) & \text{otherwise}, \end{cases}$$

if there exists $0 \le s \le t$ such that $x(s) \in S$. As for the operator $\mathfrak{R}_m$, the convention that $(\mathfrak{R}_\infty x)(0) = 1$ if $x(0) = \mathfrak{d}$ corresponds in assuming that the trajectory is defined for $t < 0$ and that $x(t) = 1$ for $t < 0$. Note that $(\mathfrak{R}_\infty x)(0) \in S$ and that $(\mathfrak{R}_\infty x)(0) = x(0)$ if and only if $x(0) \in S$.

Consider a trajectory x in $D([0,T],S_{\mathfrak{d}})$ and $t \in (0,T]$. Assume that $x(t) \notin S$ and that there exists $0 \le s \le t$ such that $x(s) \in S$. Since x is right-continuous, $\sigma_\infty(t) > 0$ and $x(\sigma_\infty(t)) = x(\sigma_\infty(t)+) \notin S$. Hence, since x has left-limits, under the above conditions,

$$(\mathfrak{R}_\infty x)(t) \;=\; x(\sigma_\infty(t)-)\,. \tag{1.7}$$

$E([0,T],S_{\mathfrak{d}})$

Denote by $E([0,T],S_{\mathfrak{d}})$ the set of trajectories in $\mathbb{E}([0,T],S_{\mathfrak{d}})$ such that $x(0) \in S$ and which fulfill the following condition. If $x(t) = \mathfrak{d}$ for some $t \in (0,T]$, then $\sigma(t) > 0$ and $x(\sigma_\infty(t)) = x(\sigma_\infty(t)-) = \mathfrak{d}$.

Lemma 4.5. *The trajectory $\mathfrak{R}_\infty x$ belongs to $E([0,T],S_{\mathfrak{d}})$.*

Proof. Fix a trajectory x in $\mathbb{E}([0,T],S_{\mathfrak{d}})$. By definition $(\mathfrak{R}_\infty x)(0) \in S$. We first show that $\mathfrak{R}_\infty x$ belongs to $\mathbb{E}([0,T],S_{\mathfrak{d}})$.

We claim that $\mathfrak{R}_\infty x$ has a left-limit at $t \in (0,T]$ if x has one. Suppose first that $x(t-) = \mathfrak{d}$. If there exists $\delta > 0$ such that $x(s) = \mathfrak{d}$ for $s \in (t-\delta, t)$, then σ_∞ is constant in this interval. By definition, $\mathfrak{R}_\infty x$ is constant in the same interval and has therefore a left-limit at t. On the other hand, if there exists a sequence $t_j \uparrow t$ such that $x(t_j) \in S$, $\sigma_\infty(s) \ge t_1$ for $t_1 \le s < t$. As $x(t-) = \mathfrak{d}$, for every $m \ge 1$, there exists $\delta > 0$ such that $x(s) \ge m$ for $t - \delta \le s < t$. Therefore $(\mathfrak{R}_\infty x)(s) \ge m$ for $t^*_\delta \le s < t$, where t^*_δ is the smallest element of the sequence t_j which is greater than $t-\delta$. This proves that $(\mathfrak{R}_\infty x)(t-)$ exists and is equal to $\mathfrak{d}$. Suppose now that $x(t-) \in S$. In this case $x(s) = x(t-) \in S$ for s in some interval $(t-\delta, t)$. In particular, $(\mathfrak{R}_\infty x)(s) = x(s) = x(t-)$ in the same interval, which proves the claim. The trajectory x of Example 4.4 shows that the left-limits of x and $\mathfrak{R}_\infty x$ at some point t may be different.

Suppose now that $x(t\ominus)$ exists and is equal to $n \in S$. By definition there exists a sequence $t_j \uparrow t$ such that $x(t_j) \to n$, which means that $x(t_j) = n$ for j sufficiently large. By definition, $(\mathfrak{R}_\infty x)(t_j) = n$ for the same indices. Fix $m > n$. By (1.2), there exists $\delta > 0$ such that $x(s) = n$ or $x(s) \ge m$ for all $t - \delta < s < t$. Hence, if we denote again by t^*_δ the smallest element of the sequence t_j which is greater than $t - \delta$, for $t^*_\delta < s < t$, $(\mathfrak{R}_\infty x)(s) = n$ or $(\mathfrak{R}_\infty x)(s) \ge m$. This proves that $\mathfrak{R}_\infty x$ has a soft left-limit at t.

The trajectory $\mathfrak{R}_\infty x$ is soft right-continuous. Fix $t \in [0,T)$ and assume that $x(t) = \mathfrak{d}$. If $x(s) = \mathfrak{d}$ in some interval $(t, t+\epsilon)$, σ_∞ and $\mathfrak{R}_\infty x$ are constant on the interval $[t, t+\epsilon)$; while if there exists a sequence $t_j \downarrow t$ such that $x(t_j) \in S$ for all j, $(\mathfrak{R}_\infty x)(t+) = \mathfrak{d}$. In both cases, $\mathfrak{R}_\infty x$ is soft right-continuous at t.

Suppose now that $x(t)$ belongs to S so that $(\mathfrak{R}_\infty x)(t) = x(t) \in S$. By soft right-continuity of x at t, for a fixed $m \ge 1$, there exists $\delta > 0$ such that $x(s) \in \{x(t)\} \cup S^c_m$ for all $t \le s < t + \delta$. By definition of $\mathfrak{R}_\infty x$ the same property holds for $\mathfrak{R}_\infty x$, which proves its soft right-continuity.

We conclude the proof of the lemma showing that $\mathfrak{R}_\infty x$ belongs to $E([0,T],S_{\mathfrak{d}})$. Fix $t \in (0,T]$ such that $(\mathfrak{R}_\infty x)(t) = \mathfrak{d}$. Denote by $\sigma_\infty(t)$, $\hat\sigma_\infty(t)$

the time of the last visit to S before time t of the trajectory x, $\mathfrak{R}_\infty x$, respectively. Clearly $x(t) = \mathfrak{d}$, otherwise $(\mathfrak{R}_\infty x)(t) = x(t) \in S$. We also have that $\sigma(t) > 0$ because if $\sigma(t) = 0$, $(\mathfrak{R}_\infty x)(t)$ would belong to S: $(\mathfrak{R}_\infty x)(t) = 1$ by definition if $x(s) = \mathfrak{d}$ for $0 \le s \le t$, and $(\mathfrak{R}_\infty x)(t) = x(0) \in S$ if $x(s) = \mathfrak{d}$ for $0 < s \le t$. It follows from the definition of $\mathfrak{R}_\infty x$ and from the identity $(\mathfrak{R}_\infty x)(t) = \mathfrak{d}$ that $x(\sigma_\infty(t)) = x(\sigma_\infty(t)-) = \mathfrak{d}$.

Since $x(s) = \mathfrak{d}$ for $\sigma_\infty(t) < s \le t$, and since $x(\sigma_\infty(t)) = x(\sigma_\infty(t)-) = \mathfrak{d}$, we have that $(\mathfrak{R}_\infty x)(s) = \mathfrak{d}$, $\sigma_\infty(t) \le s \le t$, $\hat\sigma_\infty(t) = \sigma_\infty(t)$, $(\mathfrak{R}_\infty x)(\sigma_\infty(t)-) = \mathfrak{d}$. □

Assertion B *Let x be a trajectory in $E([0,T], S_{\mathfrak{d}})$. Then, $\mathfrak{R}_\infty x = x$.*

The proof of this assertion is elementary. It follows from this claim and from Lemma 4.5 that $\mathfrak{R}_\infty : \mathbb{E}([0,T], S_{\mathfrak{d}}) \to E([0,T], S_{\mathfrak{d}})$ is a projection. The next assertion shows that $\mathfrak{R}_m x$ converges pointwisely to x if x belongs to $E([0,T], S_{\mathfrak{d}})$.

Assertion C *Fix a trajectory x in $\mathbb{E}([0,T], S_{\mathfrak{d}})$. Then, $\mathfrak{R}_m x$ converges pointwisely and $\lim_m \mathfrak{R}_m x = \mathfrak{R}_\infty x$.*

Proof. It is clear from the definition of $\mathfrak{R}_m x$ that $\mathfrak{R}_m x \le \mathfrak{R}_{m+1} x$. In particular, the pointwise limit always exists. Fix $0 \le t \le T$ and suppose initially that $x(t) \in S$. In this case, for $m > x(t)$, $(\mathfrak{R}_m x)(t) = (\mathfrak{R}_\infty x)(t)$.

Suppose from now on that $x(t) = \mathfrak{d}$. If $x(s) = \mathfrak{d}$ for $0 \le s \le t$, $(\mathfrak{R}_m x)(t) = 1 = (\mathfrak{R}_\infty x)(t)$ for all $m \ge 1$, while if $x(0) \in S$ and $x(s) = \mathfrak{d}$ for $0 < s \le t$, $(\mathfrak{R}_m x)(t) = x(0) = (\mathfrak{R}_\infty x)(t)$ for all $m \ge x(0)$. We may therefore assume that there exists $0 < s < t$ such that $x(s) \in S$ so that $\sigma_\infty(t) = \sigma^x_\infty(t) > 0$.

If $x(\sigma_\infty(t)) \in S$, for $m > x(\sigma_\infty(t))$ we have that $(\mathfrak{R}_m x)(t) = (\mathfrak{R}_\infty x)(t)$, while if $x(\sigma_\infty(t)) = \mathfrak{d}$ and if $x(\sigma_\infty(t)-)$ exists, $(\mathfrak{R}_\infty x)(t) = x(\sigma_\infty(t)-) = \lim_m (\mathfrak{R}_m x)(t)$. Finally, suppose that $x(\sigma_\infty(t)) = \mathfrak{d}$ and that $x(\sigma_\infty(t)-)$ does not exist. Then, by definition, $(\mathfrak{R}_\infty x)(t) = x(\sigma_\infty(t)\ominus)$ and for $m > x(\sigma_\infty(t)\ominus)$ $(\mathfrak{R}_m x)(t) = x(\sigma_\infty(t)\ominus)$. This proves the assertion. □

The next statement follows from Assertions B and C.

Assertion D *Fix two trajectories x, y in $E([0,T], S_{\mathfrak{d}})$. If $\mathfrak{R}_m x = \mathfrak{R}_m y$ for all m large enough, then $x = y$.*

$\mathbf{d}(x,y)$

For two trajectories x, $y \in \mathbb{E}([0,T], S_{\mathfrak{d}})$, let $d_m(x,y)$

$$\mathbf{d}(x,y) \;=\; \sum_{m\ge 1} \frac{1}{2^m}\, d_m(x,y)\;, \text{ where } d_m(x,y) \;=\; d_S(\mathfrak{R}_m x, \mathfrak{R}_m y)\;. \tag{1.8}$$

Example 4.4 shows that $\mathbf{d}$ is not a metric in $\mathbb{E}([0,T], S_{\mathfrak{d}})$, but the next assertion states that it is a metric in $E([0,T], S_{\mathfrak{d}})$.

Assertion E *The map $\mathbf{d}$ is a metric in $E([0,T], S_{\mathfrak{d}})$.*

Proof. It is clear that $\mathbf{d}$ is finite, non-negative and symmetric, and that $\mathbf{d}$ satisfies the triangular inequality. Suppose that $\mathbf{d}(x,y) = 0$. Then, $\mathfrak{R}_m x = \mathfrak{R}_m y$ for all $m \geq 1$. Hence, by Assertion D, $x = y$. □

Example 4.6. Fix $t_0 < T$ and let $x_n \in D([0,T], S_{\mathfrak{d}})$ be the sequence given by

$$x_n \;=\; \mathbf{1}\{[0,t_0)\} \;+\; n\,\mathbf{1}\{[t_0, t_0 + n^{-1})\} \;+\; \mathbf{1}\{[t_0 + n^{-1}, T]\}\;.$$

While this sequence does not converges in the Skorohod topology, it converges to the constant trajectory equal to 1 in the metric $\mathbf{d}$. This sequence is a caricature of a typical trajectory of the processes examined in this book. In a certain time scale these processes spend a shorter and shorter amount of time on a set which has a vanishing asymptotic probability, but which has to be crossed when moving from one metastable set to another. The unique reason to introduce the metric $\mathbf{d}$ is to define a topology in which such a sequence converges.

In constrast, and as we want, for $\ell \in \mathbb{N}$, $\ell \neq 1$, the sequence

$$y_n \;=\; \mathbf{1}\{[0,t_0)\} \;+\; \ell\,\mathbf{1}\{[t_0, t_0 + n^{-1})\} \;+\; \mathbf{1}\{[t_0 + n^{-1}, T]\}$$

does not converge. This is needed because the points in S represent the metastable sets and we require in the definition of metastability the convergence in the Skorohod topology of the trace of the process on finite subsets of S.

The undesirable aspect of the metric $\mathbf{d}$ is that the sequence

$$z_n \;=\; \mathbf{1}\{[0,t_0)\} \;+\; n\,\mathbf{1}\{[t_0, T]\}$$

also converges to the constant trajectory equal to 1. To exclude such cases, we shall introduce in the next section a subset of trajectories in $E([0,T], S_{\mathfrak{d}})$ which spend only a negligible amount of time in $\mathfrak{d}$ and we shall introduce compactness conditions which ensure that the limit points of a sequence of trajectories belongs to this set. These compactness conditions will exclude sequences as z_n which spend a non-negligible amount of time in a set S_m^c for some m.

$\mathit{\Lambda}$

We conclude this section proving in Proposition 4.8 below that the space $E([0,T], S_{\mathfrak{d}})$ endowed with the metric $\mathbf{d}$ is complete and separable. Denote by $\mathit{\Lambda}$ the set of increasing and continuous functions $\lambda : [0,T] \to [0,T]$ such that $\lambda(0) = 0$, $\lambda(T) = T$. For $\lambda \in \mathit{\Lambda}$, let

$$\|\lambda\|^o \;=\; \sup_{0 \leq s < t \leq T} \Big|\, \log \frac{\lambda(t) - \lambda(s)}{t - s} \,\Big|\;.$$

$\|\lambda\|^o$

Assertion F *Let x be a trajectory in $D([0,T], S_{m+1})$ and fix $\lambda \in \mathit{\Lambda}$. Then, $\mathfrak{R}_m(x \circ \lambda) = (\mathfrak{R}_m x) \circ \lambda$. The same identity holds for a trajectory x in $D([0,T], S_{\mathfrak{d}})$.*

Proof. Since $x \in D([0,T], S_{m+1})$, there exist $k \geq 1$, $0 = t_0 < t_1 < \cdots < t_k = T$, and $\ell_0, \dots, \ell_k \in S_{m+1}$ such that $\ell_i \neq \ell_{i+1}$, $0 \leq i \leq k-2$, and

$$x(t) \;=\; \sum_{i=0}^{k-1} \ell_i \, \mathbf{1}\{[t_i, t_{i+1})\}(t) \;+\; \ell_k \mathbf{1}\{t = t_k\} \,. \tag{1.9}$$

Note that ℓ_{k-1} may be equal to ℓ_k in which case x is left continuous at T. It is easy to obtain from this formula explicit expressions for $\mathfrak{R}_m(x \circ \lambda)$ and for $(\mathfrak{R}_m x) \circ \lambda$ and to check that they are equal.

Consider now a trajectory x in $D([0,T], S_{\mathfrak{d}})$. Fix $\lambda \in \Lambda$ and $m \in S$. Recall that we denote by $\sigma^y_m(t)$ the last visit to S_m before time t for the trajectory y. It is easy to verify that $\sigma^{x\lambda}_m(t) = \lambda^{-1}(\sigma^x_m(\lambda t))$, where $x\lambda = x \circ \lambda$, $\lambda t = \lambda(t)$.

Fix $t \in [0,T]$ and suppose that $x(s) \notin S_m$ for $0 \leq s \leq \lambda(t)$. In this case, $x\lambda(s) \notin S_m$ for $0 \leq s \leq t$ and $(\mathfrak{R}_m(x\lambda))(t) = 1 = (\mathfrak{R}_m x)(\lambda t)$.

If $x(\lambda(t)) \in S_m$, $(\mathfrak{R}_m(x\lambda))(t) = (x\lambda)(t) = (\mathfrak{R}_m x)(\lambda t)$. It remains to consider the case in which $x(s) \in S_m$ for some $0 \leq s \leq \lambda(t)$ and $x(\lambda(t)) \notin S_m$. By (1.6),

$$\begin{aligned}(\mathfrak{R}_m(x\lambda))(t) \;&=\; (x\lambda)(\sigma^{x\lambda}_m(t)-) \;=\; (x\lambda)(\lambda^{-1}(\sigma^x_m(\lambda t))-) \\ &=\; x(\sigma^x_m(\lambda t)-) \;=\; (\mathfrak{R}_m x)(\lambda(t)) \,,\end{aligned}$$

which proves the claim. □

Assertion G *The map $\mathfrak{R}_m : D([0,T], S_{m+1}) \to D([0,T], S_m)$ is continuous.*

Proof. Let x_n be a sequence of trajectories in $D([0,T], S_{m+1})$ which converges in the Skorohod topology to x. We will prove that the sequence of trajectories $\mathfrak{R}_m x_n$ in $D([0,T], S_m)$ converges in the Skorohod topology to $\mathfrak{R}_m x$.

Fix $\epsilon < [m(m+1)]^{-1}$. Since x_n converges to x, there exists n_0 such that for all $n \geq n_0$

$$\max\Big\{ \|x_n - x\lambda\|_\infty \,,\, \|\lambda\|^o \Big\} \;<\; \epsilon \,,$$

for some $\lambda \in \Lambda$, where $\|x_n - x\lambda\|_\infty = \sup_{0 \leq t \leq T} d\big(x_n(t), x\lambda(t)\big)$. Since we chose $\epsilon < [m(m+1)]^{-1}$, we must have that $x_n = x\lambda$ so that $\mathfrak{R}_m x_n = \mathfrak{R}_m(x\lambda)$. Since by Assertion F, $\mathfrak{R}_m(x\lambda) = (\mathfrak{R}_m x) \circ \lambda$, we conclude that

$$d_S(\mathfrak{R}_m x, \mathfrak{R}_m x_n) \;\leq\; \max\Big\{ \|(\mathfrak{R}_m x_n)(t) - (\mathfrak{R}_m x)(\lambda t)\|_\infty \,,\, \|\lambda\|^o \Big\} \;<\; \epsilon \,,$$

which proves the assertion. □

Assertion H *Let y be a trajectory in $D([0,T], S_m)$, $m \geq 2$, and let $x = \mathfrak{R}_{m-1} y$. Suppose that x is discontinuous at $t \in (0,T]$. Then, $y(t) = x(t)$ and y is discontinuous at t.*

Proof. We first show that $y(t) = x(t)$ if x is discontinuous at $t \in (0,T]$. We proceed by contradiction. Fix $t \in (0,T]$ and suppose that $y(t) \neq x(t)$, so that

$y(t) = m$. We want to show that x is continuous at t. Since y belongs to $D([0,T], S_m)$, y can be represented as in (1.9). By definition of $\mathfrak{R}_{m-1}$, the only points where x can be discontinuous are the points t_i, $1 \le i \le k$. If $t = t_i$ and $x(t_i) \neq y(t_i)$, then $y(t_i) = m$, $y(t_{i-1}) \in S_{m-1}$ (because $y(t_{i-1}) \in S_m$ and $y(t_{i-1}) \neq y(t_i) = m$) so that $x(t_i) = y(t_{i-1}) = x(t_{i-1}) = x(t_i-)$ and x is left-continuous at t_i.

We now prove the second claim of the assertion. Fix $t \in (0,T]$ and suppose that x is discontinuous at t. By the first part of the claim, $y(t) = x(t) \in S_{m-1}$. By definition of $\mathfrak{R}_{m-1}$, $y(t-) = x(t-)$ or $y(t-) = m$. In the first case y is discontinuous at t because so is x. In the second case y is also discontinuous at t because $y(t) \in S_{m-1}$. $\square$

Lemma 4.7. *Let $y_m \in D([0,T], S_m)$ a sequence of trajectories such that $\mathfrak{R}_m y_{m+1} = y_m$ for all $m \ge 1$. Then, there exists a trajectory y in $E([0,T], S_{\mathfrak{d}})$ such that $\mathfrak{R}_m y = y_m$ for all $m \ge 1$.*

Proof. Since $\mathfrak{R}_m x \le x$, the sequence y_m is increasing and has therefore a pointwise limit, denoted by y.

Suppose that $y(t) = n \in S$ for some $t \in [0,T]$. In this case $y_m(t) = n$ for all $m \ge n$. Indeed, if $y_{m_0}(t) \neq n$ for some $m_0 > n$, then for all $m \ge m_0$, either $y_m(t) = y_{m_0}(t)$ or $y_m(t) = m > n$, which contradicts the fact that $\lim_m y_m(t) = y(t) = n$.

There exists $1 \le m_0 \le \infty$ such that $y_m(0) = 1$ for $m < m_0$ and $y_m(0) = m_0$ for $m \ge m_0$. For any trajectory x, by our convention in the definition of $\mathfrak{R}_m$,

$$(\mathfrak{R}_m x)(0) \;=\; \begin{cases} x(0) & \text{if } x(0) \le m, \\ 1 & \text{if } x(0) > m. \end{cases}$$

Therefore $y_m(0) = (\mathfrak{R}_m y_{m+1})(0)$ satisfies the relation

$$y_m(0) \;=\; \begin{cases} y_{m+1}(0) & \text{if } y_{m+1}(0) \le m, \\ 1 & \text{if } y_{m+1}(0) = m+1. \end{cases} \tag{1.10}$$

Let $m_0 = \min\{j \ge 1 : y_j(0) \neq 1\}$. Assume that $m_0 < \infty$, otherwise there is nothing to be proven. By (1.10) for $m = m_0 - 1$, $y_{m_0}(0) = m_0$, and by definition of m_0, $y_k(0) = 1$ for $k < m_0$. By (1.10) for $m = m_0$, $y_{m_0+1}(0) = y_{m_0}(0) = m_0$. Repeating this argument, we conclude that $y_k(0) = m_0$ for all $k \ge m_0$, as claimed.

The trajectory y has a soft left-limit at each point $t \in (0,T]$. Fix $t \in (0,T]$ and suppose that there exists an increasing sequence t_j converging to t and such that $y(t_j) \to n \in S$. For j large enough $y(t_j) = n$. We assume, without loss of generality, that this holds for all j: $y(t_j) = n$ for all $j \ge 1$. By the penultimate paragraph, $y_m(t_j) = n$ for all $m \ge n$ and $j \ge 1$. This proves that $y_m(t-) = n$ for all $m \ge n$. In particular, by Remark 4.2, y has a soft left-limit at t.

It is not difficult to construct an example of a sequence y_m for which y has a soft left-limit at $t \in (0,T]$, but not a left-limit, i.e., a sequence y_m for which there exist increasing sequences t_j, t'_j converging to t and such that $y(t_j) \to n \in S$, $y(t'_j) \to \mathfrak{d}$.

The trajectory y is soft right-continuous. Fix $t \in [0,T)$ and suppose that there exists a decreasing sequence t_j converging to t and such that $y(t_j) \to n \in S$. The argument presented above shows that $y_m(t) = n$ for all $m \geq n$, which proves, in view of Remark 4.2, that y has a soft right-limit at t equal to n. Since $y_m(t) = n$ for all $m \geq n$, $y(t) = n$, which proves that y is soft right-continuous at t.

Fix $t \in (0,T]$ and assume that there exists m for which y_m is discontinuous at t. By Assertion H, $y_{m+1}(t) = y_m(t)$ and y_{m+1} is discontinuous at t. Repeating this argument, we conclude that $y_n(t) = y_m(t)$ for all $n \geq m$ so that $y(t) = y_m(t) \in S$.

The trajectory y belongs to $E([0,T],S_{\mathfrak{d}})$. We proved above that $y(0) \in S$. Assume that $y(t) = \mathfrak{d}$ for some $t \in (0,T]$. By the previous paragraph, t is a continuity point of y_m for every m. Denote by $[\ell_m, r_m)$ the largest interval which contains t and in which y_m is constant. ℓ_m is a non-decreasing sequence bounded above by t. Denote by ℓ its limit. It is clear that $\ell = \sigma^y_\infty(t)$, that $y(\ell) = \mathfrak{d}$ and that $y(\ell-) = \mathfrak{d}$, which proves that y belongs to $E([0,T],S_{\mathfrak{d}})$.

It remains to show that $\mathfrak{R}_m y = y_m$ for all $m \geq 1$. Fix $m \geq 1$ and $t \in [0,T]$. If t is a point of discontinuity of y_m, by Assertion H, $y_n(t) = y_m(t)$ for all $n \geq m$ so that $y(t) = y_m(t) \in S_m$ and $(\mathfrak{R}_m y)(t) = y_m(t)$. If t is a continuity point of y_m, as above, let $[\ell_m, r_m)$ the largest constancy interval of y_m which contains t. If $\ell_m > 0$, ℓ_m is a discontinuity point of y_m so that $y(\ell_m) = y_m(\ell_m) \in S_m$. By definition of the sequence y_k, for $k > m$ and $\ell_m \leq s \leq t$, $y_k(s) = y_m(s) = y_m(\ell_m)$ or $y_k(s) > m$. Hence, for $\ell_m \leq s \leq t$, $y(s) = y_m(\ell_m)$ or $y(s) > m$, so that $(\mathfrak{R}_m y)(t) = y_m(\ell_m) = y_m(t)$. If $\ell_m = 0$ and $y_m(0) \neq 1$, the same argument holds since the sequence $y_k(0)$, $k \geq m$, is constant by the assertion above (1.10). If $\ell_m = 0$ and $y_m(0) = 1$, the argument can be adapted even if the sequence $y_k(0)$ may not be constant. By the assertion above (1.10), for $k > m$ and $0 \leq s \leq t$, $y_k(s) = y_m(s) = 1$ or $y_k(s) > m$. Hence, for $0 \leq s \leq t$, $y(s) = 1$ or $y(s) > m$. If $y(s) > m$ for all $0 \leq s \leq t$, by our convention in the definition of $\mathfrak{R}_m$, $(\mathfrak{R}_m y)(t) = 1 = y_m(t)$. If there exists $0 \leq s \leq t$ such that $y(s) = 1$ we also have that $(\mathfrak{R}_m y)(t) = 1 = y_m(t)$. This concludes the proof of the lemma. □

Proposition 4.8. *The space $E([0,T],S_{\mathfrak{d}})$ endowed with the metric $\mathbf{d}(x,y)$ is complete and separable.*

Proof. Consider a Cauchy sequence $\{x_n : n \geq 1\}$ in $E([0,T],S_{\mathfrak{d}})$ for the metric $\mathbf{d}$. By definition of $\mathbf{d}$, for each $m \geq 1$, $\mathfrak{R}_m x_n$ is a Cauchy sequence in $D([0,T],S_m)$ for the metric d_S. Since this space is complete, there exists $y_m \in D([0,T],S_m)$ such that $\mathfrak{R}_m x_n \to y_m$. By Assertion G, $\mathfrak{R}_m y_{m+1} = y_m$. Hence, by Lemma 4.7, there exists $y \in E([0,T],S_{\mathfrak{d}})$ such that $\mathfrak{R}_m y = y_m$ for all $m \geq 1$. Therefore, $\mathfrak{R}_m x_n \to y_m = \mathfrak{R}_m y$, which implies that x_n converges to y in $E([0,T], S_{\mathfrak{d}})$. This proves the completeness.

The separability of $E([0,T], S_{\mathfrak{d}})$ follows from the separability of each set $D([0,T], S_m)$. For each $m \geq 1$, there exists a sequence of trajectories $x_{m,n}$, $n \geq 1$, which is dense in $D([0,T], S_m)$ for the metric d_S. We claim that the countable set of trajectories $x_{m,n}$, $n \geq 1$, $m \geq 1$ is dense.

Fix a trajectory x in $E([0,T], S_{\mathfrak{d}})$ and $\epsilon > 0$. Take $m \geq 1$ such that $2^{-m} < \epsilon$ and $x_{m,n}$ in $D([0,T], S_m)$ such that $d_S(x_{m,n}, \mathfrak{R}_m x) < \min\{\epsilon, [m(m-1)]^{-1}\}$. There exists λ in Λ such that

$$\max\{\|x_{m,n} - (\mathfrak{R}_m x) \circ \lambda\|_\infty\,,\ \|\lambda\|^o\} \ <\ \min\{\epsilon, [m(m-1)]^{-1}\}\ .$$

Since $\|x_{m,n} - (\mathfrak{R}_m x) \circ \lambda\|_\infty < [m(m-1)]^{-1}$, $x_{m,n} = (\mathfrak{R}_m x) \circ \lambda$. Hence, by Assertion F, for $\ell \leq m$, $\mathfrak{R}_\ell x_{m,n} = \mathfrak{R}_\ell[(\mathfrak{R}_m x) \circ \lambda] = (\mathfrak{R}_\ell x) \circ \lambda$. In particular,

$$d_S(\mathfrak{R}_\ell x_{m,n}, \mathfrak{R}_\ell x) \ \leq\ \|\lambda\|^o \ <\ \epsilon\ .$$

Putting together the previous estimates, as $d_S(x,y) \leq 1$ for any pair of trajectories in $D([0,T], S_\ell)$, we obtain that

$$\sum_{\ell \geq 1} \frac{1}{2^\ell}\, d_S(\mathfrak{R}_\ell x_{m,n}, \mathfrak{R}_\ell x) \ \leq\ \sum_{\ell=1}^{m} \frac{1}{2^\ell}\, d_S(\mathfrak{R}_\ell x_{m,n}, \mathfrak{R}_\ell x) \ +\ \epsilon \ \leq\ 2\epsilon\ .$$

This concludes the proof of the proposition. □

2 The space $D^*([0,T], S_{\mathfrak{d}})$

$D^*([0,T], S_{\mathfrak{d}})$

Denote by $D^*([0,T], S_{\mathfrak{d}})$ the subset of all trajectories in $D([0,T], S_{\mathfrak{d}})$ which spend no time at $\mathfrak{d}$ and which are continuous at time T:

$$D^*([0,T], S_{\mathfrak{d}}) \ =\ \Big\{x \in D([0,T], S_{\mathfrak{d}}) : \Lambda_T(x) = 0\,,\ x(T-) = x(T)\Big\}\ ,$$

where

$$\Lambda_T(x) \ =\ \int_0^T \mathbf{1}\{x(s) = \mathfrak{d}\}\, ds\ .$$

$\Lambda_T(x)$

Since a trajectory x in $D^*([0,T], S_{\mathfrak{d}})$ spends no time at $\mathfrak{d}$, $\sigma^x(t) = t$ for all $t \in [0,T]$. In particular, by definition of the map $\mathfrak{R}_\infty$, for x in $D^*([0,T], S_{\mathfrak{d}})$

$$(\mathfrak{R}_\infty x)(t) \ =\ \begin{cases} x(t) & \text{if } x(t) \in S, \\ x(t-) & \text{if } x(t) = \mathfrak{d}. \end{cases} \tag{2.1}$$

Therefore, $(\mathfrak{R}_\infty x)(t) \neq x(t)$ only if $x(t) = \mathfrak{d} \neq x(t-)$ and $(\mathfrak{R}_\infty x)(T) = x(T)$.

Assertion I *The map $\mathfrak{R}_\infty : D^*([0,T], S_{\mathfrak{d}}) \to E([0,T], S_{\mathfrak{d}})$ is one-to-one.*

Proof. Fix two trajectories x, $y \in D^*([0,T],S_{\mathfrak{d}})$ and suppose that $\mathfrak{R}_\infty x = \mathfrak{R}_\infty y$. Let $A = \{t \in [0,T] : x(t) = \mathfrak{d} \text{ or } y(t) = \mathfrak{d}\}$. By (2.1), $x(t) = y(t)$ for $t \notin A$. Hence, since the set A has measure zero and since x and y are right continuous, $x(t) = y(t)$ for $t \in [0,T)$. On the other hand, as we have seen just below (2.1), $x(T) = (\mathfrak{R}_\infty x)(T) = (\mathfrak{R}_\infty y)(T) = y(T)$. □

$E^*([0,T],S_{\mathfrak{d}})$

We denote by $E^*([0,T],S_{\mathfrak{d}})$ the range of the map $\mathfrak{R}_\infty : D^*([0,T],S_{\mathfrak{d}}) \to E([0,T],S_{\mathfrak{d}})$.

Assertion J *A trajectory y in $E([0,T],S_{\mathfrak{d}})$ belongs to $E^*([0,T],S_{\mathfrak{d}})$ if and only if*

(a) *y has left and right-limits at every point;*
(b) *If $y(t+) = \mathfrak{d}$ for some $t \in [0,T)$, then $y(t) = y(t-)$;*
(c) *y is continuous at T;*
(d) *$\Lambda_T(y) = 0$.*

Proof. Fix a trajectory y in $E^*([0,T],S_{\mathfrak{d}})$. Let $x \in D^*([0,T],S_{\mathfrak{d}})$ such that $y = \mathfrak{R}_\infty x$. It follows from (2.1) that $y(t+) = x(t+)$, $y(t-) = x(t-)$, which proves (a). Assume that $y(t+) = \mathfrak{d}$ for some $t \in [0,T)$. As we just have seen, $x(t+) = \mathfrak{d}$. Since x is right continuous, $x(t) = \mathfrak{d}$. Thus, by (2.1), $y(t) = x(t-)$. By the first part of the proof, $x(t-) = y(t-)$, so that $y(t) = y(t-)$, which proves (b). To verify (c), recall from (2.1) that $y(T) = x(T)$ and from the first part of the proof that $y(T-) = x(T-)$. Since x belongs to $D^*([0,T],S_{\mathfrak{d}})$, $x(T) = x(T-)$ so that $y(T) = y(T-)$. Finally, since $y(t) \in S$ whenever $x(t) \in S$, $x(t) = \mathfrak{d}$ if $y(t) = \mathfrak{d}$, and $\Lambda_T(y) \le \Lambda_T(x) = 0$.

Conversely, let y be a trajectory in $E([0,T],S_{\mathfrak{d}})$ which fulfills conditions (a)–(d). Let x be the trajectory defined by $x(t) = y(t+)$, $0 \le t < T$, $x(T) = y(T)$. We claim that $x \in D^*([0,T],S_{\mathfrak{d}})$. By definition, x is right continuous and has left limits, and $x(t+) = y(t+)$, $x(t-) = y(t-)$. Therefore, $x \in D([0,T],S_{\mathfrak{d}})$, and, by assumption (c), $x(T) = x(T-)$.

By definition of x,

$$\Lambda_T(x) \;=\; \int_0^T \mathbf{1}\{y(s+) = \mathfrak{d}\}\, ds\,.$$

Fix $t \in [0,T)$ such that $y(t+) = \mathfrak{d}$. Then, either $y(t) = \mathfrak{d}$ or, by assumption (b), $y(t-) = y(t) \in S$. The first set of points has Lebesgue measure zero because $\Lambda_T(y) = 0$ by assumption (d). The second set is at most countable because y is constant on an interval $[t-\epsilon, t]$ if $y(t-) = y(t) \in S$. This proves that $\Lambda_T(x) = 0$.

It remains to show that $\mathfrak{R}_\infty x = y$. Suppose that $x(t) \in S$. By the definition (2.1) of $\mathfrak{R}_\infty$, $(\mathfrak{R}_\infty x)(t) = x(t) = y(t+)$. Since y is soft right-continuous and since y has a right-limit which belongs to S, $y(t+) = y(t)$, so that $(\mathfrak{R}_\infty x)(t) = y(t)$. Suppose now that $x(t) = \mathfrak{d}$, so that $y(t+) = \mathfrak{d}$. By definition (2.1) of $\mathfrak{R}_\infty$, $(\mathfrak{R}_\infty x)(t) = x(t-) = y(t-)$. Since $y(t+) = \mathfrak{d}$, by assumption (b), $y(t-) = y(t)$ so that $(\mathfrak{R}_\infty x)(t) = y(t)$. □

The set $E^*([0,T],S_{\mathfrak{d}})$ is clearly not closed, but Lemma 4.9 below provides sufficient conditions for the limit x of a converging sequence x_n in $E^*([0,T],S_{\mathfrak{d}})$ to belong to $E^*([0,T],S_{\mathfrak{d}})$.

$D_c([0,T],S_m)$

Denote by $D_c([0,T],S_m)$, $m \geq 1$, the subset of trajectories in $D([0,T],S_m)$ which are continuous at T. Note that $D_c([0,T],S_m)$ is a closed subset of $D([0,T],S_m)$ and that the trajectory $\mathfrak{R}_m x$ belongs to $D_c([0,T],S_m)$ if $x \in E^*([0,T],S_{\mathfrak{d}})$.

For a trajectory $x \in D_c([0,T],S_m)$ and $1 \leq j \leq m$, denote by $\mathfrak{N}_j = \mathfrak{N}_j(x)$ the number of visits to j in the time interval $[0,T]$, and denote by $T_{j,1}, \dots, T_{j,\mathfrak{N}_j}$ the holding times at j. Hence, if the trajectory x is given by

$$x(t) \;=\; \sum_{i=0}^{k-1} \ell_i \, \mathbf{1}\{[t_i, t_{i+1})\}(t) \;+\; \ell_k \mathbf{1}\{[t_k, T]\}\;,$$

where $0 = t_0 < t_1 < \cdots < t_k < T$, and $\ell_i \neq \ell_{i+1}$, $0 \leq i \leq k-1$, and if we denote by I_j the set $\{i \in \{0,\dots,k\} : \ell_i = j\}$, we have that $\mathfrak{N}_j(x) = |I_j|$. Moreover, if $\mathfrak{N}_j \geq 1$ and if $I_j = \{i_1, \dots, i_{\mathfrak{N}_j}\}$, where $i_a < i_{a+1}$ for $1 \leq a < \mathfrak{N}_j$,

$$T_{j,1} \;=\; t(i_1+1) - t(i_1)\;,\; \dots\;,\; T_{j,\mathfrak{N}_j} \;=\; t(i_{\mathfrak{N}_j}+1) - t(i_{\mathfrak{N}_j})\;. \qquad (2.2)$$

In this formula, to avoid small indices we represented t_{i_a} by $t(i_a)$. By convention, $T_{j,\ell} = 0$ for $\ell > \mathfrak{N}_j$.

$\mathfrak{N}_k$

Assertion K *The functionals $\mathfrak{N}_k$, $1 \leq k \leq m$, are continuous with respect to the Skorohod topology in $D_c([0,T],S_m)$, and the sets $\{x : T_{j,\ell} \geq a\}$, $a > 0$, are closed.*

Proof. Fix $1 \leq k \leq m$, and let $\{x_n : n \geq 1\}$ be a sequence in $D_c([0,T],S_m)$ which converges to a trajectory x in the Skorohod topology. Fix $\epsilon < [m(m-1)]^{-1}$. Since x_n converges to x, there exists n sufficiently large and $\lambda \in \Lambda$ such that

$$\|x_n - x\lambda\|_\infty \;<\; \epsilon\;.$$

Since $\epsilon < [m(m-1)]^{-1}$ we have that $x_n = x\lambda$ so that $\mathfrak{N}_k(x\lambda) = \mathfrak{N}_k(x_n)$. Since $\mathfrak{N}_k(x\lambda) = \mathfrak{N}_k(x)$, we conclude that the sequence $\mathfrak{N}_k(x_n)$ is eventually constant and converges to $\mathfrak{N}_k(x)$.

To prove that the sets $\{x : T_{j,\ell} \geq a\}$ are closed, fix $1 \leq j \leq m$, $\ell \geq 1$, $a > 0$, and consider a sequence x_n converging in the Skorohod topology to some trajectory x. Suppose that $T_{j,\ell}(x_n) \geq a$ for all $n \geq 1$ and fix $0 < \epsilon < [m(m-1)]^{-1}$. There exists $\lambda_n \in \Lambda$ such that $\|x_n - x\lambda_n\|_\infty < \epsilon$, $\|\lambda_n\|^o < \epsilon$ for all n large enough. As in the first part of the proof, we deduce from this estimate that $x_n = x\lambda_n$ so that $\mathfrak{N}_j(x_n) = \mathfrak{N}_j(x\lambda_n) = \mathfrak{N}_j(x)$ and $T_{j,\ell}(x_n) = T_{j,\ell}(x\lambda_n)$ for n large enough. Since $T_{j,\ell}(x_n) \geq a$, $\ell \leq \mathfrak{N}_j(x_n) = \mathfrak{N}_j(x)$. Denote by $[s,t)$ the time interval of the ℓ-th visit to j for the trajectory x, so that $T_{j,\ell}(x\lambda_n) = \lambda_n^{-1}(t) - \lambda_n^{-1}(s)$. Since $T_{j,\ell}(x_n) = T_{j,\ell}(x\lambda_n)$ and since $T_{j,\ell}(x_n) \geq a$, $\lambda_n^{-1}(t) - \lambda_n^{-1}(s) \geq a$. However, as $\|\lambda_n\|^o < \epsilon$, $e^{-\epsilon}(t-s) \leq \lambda_n^{-1}(t) - \lambda_n^{-1}(s) \leq e^{\epsilon}(t-s)$. Therefore, $T_{j,\ell}(x) = t - s \geq e^{-\epsilon}[\lambda_n^{-1}(t) - \lambda_n^{-1}(s)] \geq e^{-\epsilon}a$, which proves the assertion. $\square$

Note that in the next lemma all conditions are formulated in terms of the trajectories $\mathfrak{R}_\ell x_n$.

Lemma 4.9. *Let $\{x_n : n \ge 1\}$ be a sequence in $E^*([0,T],S_{\mathfrak{d}})$ which converges to $x \in E([0,T],S_{\mathfrak{d}})$ in the metric* $\mathbf{d}$. *Assume that*

(a)
$$\lim_{m\to\infty} \sup_{\ell\ge 1} \sup_{n\ge 1} \int_0^T \mathbf{1}\{\mathfrak{R}_\ell x_n(s) \ge m\}\, ds \;=\; 0\;;$$

(b) *For each $m \ge 1$, there exists $k_m \in \mathbb{N}$ such that $\mathfrak{N}_m(\mathfrak{R}_\ell x_n) \le k_m$ for all $\ell \ge m$ and $n \ge 1$;*
(c) *For each $m \ge 1$, there exists $\epsilon_m > 0$ such that $T_{m,k}(\mathfrak{R}_\ell x_n) \ge \epsilon_m$ for all $1 \le k \le \mathfrak{N}_m(\mathfrak{R}_\ell x_n)$, $\ell \ge m$ and $n \ge 1$;*
(d) *For all $\ell \ge 1$, $n \ge 1$, $\mathfrak{R}_\ell x_n$ is continuous at T.*

Then, x belongs to $E^*([0,T],S_{\mathfrak{d}})$.

Proof. We need to prove that the trajectory x fulfills conditions (a)–(d) of Assertion J. We first claim that $\Lambda_T(x) = 0$. Fix $\epsilon > 0$. By assumption (a), there exists $m \ge 1$ such that

$$\int_0^T \mathbf{1}\{(\mathfrak{R}_\ell x_n)(s) \ge m\}\, ds \;\le\; \epsilon$$

for all $n \ge 1$, $\ell \ge 1$. Fix $\ell \ge m$. The sequence $\mathfrak{R}_\ell x_n$ converges almost everywhere to $\mathfrak{R}_\ell x$ because it converges in the Skorohod topology. Hence, by Fatou's lemma,

$$\int_0^T \mathbf{1}\{(\mathfrak{R}_\ell x)(s) \ge m\}\, ds \;\le\; \liminf_{n\to\infty} \int_0^T \mathbf{1}\{(\mathfrak{R}_\ell x_n)(s) \ge m\}\, ds \;\le\; \epsilon\,.$$

Since $\mathfrak{R}_\ell x$ converges pointwisely to x, by the dominated convergence theorem,

$$\int_0^T \mathbf{1}\{x(s) \ge m\}\, ds \;\le\; \epsilon\,,$$

so that $\Lambda_T(x) \le \epsilon$.

We now show that x has left and right limits and that condition (b) of Assertion J is in force. Since x belongs to $E([0,T],S_{\mathfrak{d}})$ to prove the first claim it is enough to exclude the possibility that x has a finite soft limit at some point $t \in [0,T]$. Fix $m \ge 1$. By assumptions (b) and (c) of this lemma, there exist $k_m \ge 1$ and $\epsilon_m > 0$ such that $\mathfrak{N}_m(\mathfrak{R}_\ell x_n) \le k_m$ and $T_{m,k}(\mathfrak{R}_\ell x_n) \ge \epsilon_m$ for all $1 \le k \le \mathfrak{N}_m(\mathfrak{R}_\ell x_n)$, $\ell \ge m$, $n \ge 1$. Since $\mathfrak{R}_\ell x_n$ converges in the Skorohod topology to $\mathfrak{R}_\ell x$, by Assertion K, $\mathfrak{N}_m(\mathfrak{R}_\ell x) \le k_m$ and $T_{m,k}(\mathfrak{R}_\ell x) \ge \epsilon_m$ for all $1 \le k \le \mathfrak{N}_m(\mathfrak{R}_\ell x)$, $\ell \ge m$. As the sequence $\mathfrak{N}_m(\mathfrak{R}_\ell x)$ increases with ℓ, it is constant for ℓ large enough. Denote by $[s_1^\ell, t_1^\ell), \dots, [s_N^\ell, t_N^\ell)$ the $N = \mathfrak{N}_m(\mathfrak{R}_\ell x)$ time-intervals in which $\mathfrak{R}_\ell x$ visits m. Since $T_{m,k}(\mathfrak{R}_\ell x) \ge \epsilon_m$ for all k, $t_i^\ell \ge$

$s_i^\ell + \epsilon_m$. By Assertion H, $s_i^{\ell+1} = s_i^\ell$, $1 \le i \le N$, and $t_i^{\ell+1} \le t_i^\ell$. Since $\mathfrak{R}_\ell x$ converges pointwisely to $\mathfrak{R}_\infty x = x$, the set $\{s \in [0,T] : x(s) = m\}$ is the union of N disjoint intervals of lenght greater or equal to ϵ_m, which are closed at the left boundary and open or closed at the right boundary. In particular, m can not be the finite soft limit of x at some point $t \in [0,T]$. Since this holds for every m, x does not have a left or a right finite soft limit at some time $t \in [0,T]$. This proves condition (a) of Assertion J.

We turn to condition (b) of Assertion J. Suppose that $x(t+) = \mathfrak{d}$ for some $t \in [0,T)$. If $x(t) = \mathfrak{d}$, since $x \in E([0,T], S_\mathfrak{d})$ and $\Lambda_T(x) = 0$, $\sigma_\infty(t) = t$ and, by definition of the set $E([0,T], S_\mathfrak{d})$, $x(t-) = x(t)$. If $x(t) = m \in S$, since $x(t+) = \mathfrak{d}$, t is the right endpoint of an interval $[s_i, t_i]$ obtained as the limit of the intervals $[s_i^\ell, t_i^\ell)$ introduced in the previous paragraph. Since the interval is not degenerate, $x(t-) = m = x(t)$, which proves condition (b) of Assertion J.

We finally prove condition (c) of Assertion J. Suppose that $x(T) = k \in S$. In this case, since the set $\{s \in [0,T] : x(s) = k\}$ is the union of a finite number of disjoint intervals of positive lenght, x is continuous at T. Suppose now that $x(T) = \mathfrak{d}$. By assumption (d) of this lemma, $(\mathfrak{R}_\ell x_n)(T) = (\mathfrak{R}_\ell x_n)(T-)$ for all $\ell \ge 1$, $n \ge 1$. Since $\mathfrak{R}_\ell x_n$ converges to $\mathfrak{R}_\ell x$ in the Skorohod topology, the continuity at T is inherited by $\mathfrak{R}_\ell x$. Denote by $[a_\ell, T]$ the constancy interval of $\mathfrak{R}_\ell x$ and fix $m \ge 1$. Since $x(T) = \mathfrak{d}$ and since $(\mathfrak{R}_\ell x)(T)$ converges to $x(T)$, there exists $\ell_0 \ge 1$ such that for all $\ell \ge \ell_0$, $(\mathfrak{R}_\ell x)(T) \ge m$. By definition of a_ℓ and since $x \ge \mathfrak{R}_\ell x$, for all $a_\ell \le t \le T$, $x(t) \ge (\mathfrak{R}_\ell x)(t) = (\mathfrak{R}_\ell x)(T) \ge m$. This proves that $x(T-) = \mathfrak{d} = x(T)$. Condition (c) of Assertion J is therefore in force, which concludes the proof of the lemma. □

Corollary 4.10. *Let x be a trajectory in $E([0,T], S_\mathfrak{d})$ which satisfies conditions* (b)–(d) *of the previous lemma and such that $\Lambda_T(x) = 0$. Then, x belongs to $E^*([0,T], S_\mathfrak{d})$.*

Proof. By the proof of Lemma 4.9, x satisfies conditions (a)–(c) of Assertion J. Since condition (d) of this assertion holds by assumption, the corollary is proved. □

3 Weak Convergence of Probability Measures.

We examine in this section the weak convergence of probability measures on $E([0,T], S_\mathfrak{d})$.

Fix $m \ge 1$ and consider a sequence x_n in $D([0,T], S_m)$ converging to x in the Skorohod topology. Then, x_n converges to x in $E([0,T], S_\mathfrak{d})$. Indeed,

$$\begin{aligned}\mathbf{d}(x_n, x) \;&=\; \sum_{\ell \ge 1} \frac{1}{2^\ell}\, d_S(\mathfrak{R}_\ell x_n, \mathfrak{R}_\ell x) \\ &=\; \frac{1}{2^m}\, d_S(x_n, x) \;+\; \sum_{\ell=1}^{m} \frac{1}{2^\ell}\, d_S(\mathfrak{R}_\ell x_n, \mathfrak{R}_\ell x)\;.\end{aligned}$$

By hypothesis and by Assertion G, this sum vanishes as $n \uparrow \infty$.

Let $F : E([0,T], S_{\mathfrak{d}}) \to \mathbb{R}$ be a continuous function for the soft topology. Then, its restriction to $D([0,T], S_m)$, $m \ge 1$, is continuous for the Skorohod topology. Indeed, consider a sequence x_n converging in $D([0,T], S_m)$ to x. By the previous paragraph, x_n converges to x in the soft topology of $E([0,T], S_{\mathfrak{d}})$. Since F is continuous in this topology, $F(x_n)$ converges to $F(x)$.

Lemma 4.11. *A sequence of probability measures P_n on $E([0,T], S_{\mathfrak{d}})$ converges weakly to a measure P if and only if for each $m \ge 1$ the sequence of probability measures $P_n \circ \mathfrak{R}_m^{-1}$ defined on $D([0,T], S_m)$ converges weakly to $P \circ \mathfrak{R}_m^{-1}$ with respect to the Skorohod topology.*

Proof. Suppose that the sequence P_n converges weakly to P and fix $m \ge 1$. Since $\mathfrak{R}_m : E([0,T], S_{\mathfrak{d}}) \to D([0,T], S_m)$ is continuous for the soft topology, $P_n \circ \mathfrak{R}_m^{-1}$ converges weakly to $P \circ \mathfrak{R}_m^{-1}$.

Conversely, suppose that $P_n \circ \mathfrak{R}_m^{-1}$ converges weakly to $P \circ \mathfrak{R}_m^{-1}$ for every $m \ge 1$. Fix a bounded, uniformly continuous function $F : E([0,T], S_{\mathfrak{d}}) \to \mathbb{R}$ and $\epsilon > 0$. Since F is uniformly continuous, there exists $\delta > 0$ such that $|F(y) - F(x)| \le \epsilon$ if $\mathbf{d}(x,y) \le \delta$. Let $m \ge 1$ such that $2^{-(m-1)} < \delta$. Since $\mathbf{d}(x, \mathfrak{R}_m x) \le 2^{-(m-1)} < \delta$, the difference $E_{P_n}[F(x)] - E_{P_n}[F(\mathfrak{R}_m x)]$ is absolutely bounded by ϵ, uniformly in n. A similar estimate holds for P replacing P_n.

We have shown right before the lemma that $F : D([0,T], S_m) \to \mathbb{R}$ is continuous for the Skorohod topology. As $P_n \circ \mathfrak{R}_m^{-1}$ converges weakly to $P \circ \mathfrak{R}_m^{-1}$ in the Skorohod topology, and since F is bounded and continuous, there exists n_0 such that for all $n \ge n_0$, $|E_{P_n}[F(\mathfrak{R}_m x)] - E_P[F(\mathfrak{R}_m x)]| \le \epsilon$. Putting togheter the previous estimates we conclude that for all $n \ge n_0$,

$$\big| \, E_{P_n}[F(x)] - E_P[F(x)] \, \big| \;\le\; 3\,\epsilon \;,$$

which concludes the proof of the lemma. □

Proposition 4.12. *Let $\{P_n : n \ge 1\}$ be a sequence of probability measures on $E^*([0,T], S_{\mathfrak{d}})$ which converges weakly to a measure P in $E([0,T], S_{\mathfrak{d}})$ endowed with the soft topology. Assume that*

(a)

$$\lim_{m\to\infty} \limsup_{\ell\to\infty} \limsup_{n\to\infty} E_{P_n}\Big[\int_0^T \mathbf{1}\{(\mathfrak{R}_\ell x)(s) \ge m\}\, ds \Big] \;=\; 0\;;$$

(b) *For each $m \ge 1$,*

$$\lim_{k\to\infty} \limsup_{\ell\to\infty} \limsup_{n\to\infty} P_n\big[\, \mathfrak{N}_m(\mathfrak{R}_\ell x) \ge k \,\big] \;=\; 0\;;$$

(c) *For each $m \ge 1$,*

$$\lim_{\epsilon\to 0} \limsup_{\ell\to\infty} \limsup_{n\to\infty} P_n\Big[\bigcup_{k=1}^{\mathfrak{N}_m(\mathfrak{R}_\ell x)} \{T_{m,k}(\mathfrak{R}_\ell x) < \epsilon\} \Big] \;=\; 0\;;$$

(d) *For every ℓ, $n \geq 1$,*

$$P_n\big[\, (\mathfrak{R}_\ell x)(T) = (\mathfrak{R}_\ell x)(T-) \,\big] \;=\; 1\;.$$

Then, P is concentrated on $E^*([0,T], S_{\mathfrak{d}})$.

Proof. It is not difficult to show that for each $m \leq \ell$ the map $y \to \int_0^T \mathbf{1}\{y(s) \geq m\}\, ds$ is continuous in $D([0,T], S_\ell)$. Therefore, the map $y \to \int_0^T \mathbf{1}\{(\mathfrak{R}_\ell y)(s) \geq m\}\, ds$ is bounded and continuous in $E([0,T], S_{\mathfrak{d}})$. By assumption (a), given $\epsilon > 0$, there exists m_0 such that for all $m \geq m_0$,

$$\limsup_{\ell\to\infty} E_P\Big[\int_0^T \mathbf{1}\{(\mathfrak{R}_\ell x)(s) \geq m\}\, ds\Big] \;\leq\; \epsilon\;.$$

Since $\mathfrak{R}_\ell x$ increases pointwisely to $\mathfrak{R}_\infty x = x$, by the monotone convergence theorem,

$$E_P\big[\Lambda_T(x)\big] \;\leq\; E_P\Big[\int_0^T \mathbf{1}\{x(s) \geq m\}\, ds\Big] \;\leq\; \epsilon\;.$$

Letting $\epsilon \downarrow 0$, we conclude that $E_P[\Lambda_T(x)] = 0$, i.e., that

$$P[\Lambda_T(x) = 0] \;=\; 1\;. \tag{3.1}$$

By Assertion K, the functionals $\mathfrak{N}_m$, $m \geq 1$, are continuous for the Skorohod topology. The sets $\{x \in D([0,T], S_\ell) : \mathfrak{N}_m(x) \geq k\} = \{x \in D([0,T], S_\ell) : \mathfrak{N}_m(x) \leq k-1\}^c$ are therefore open and, by assumption (b), for every $m \geq 1$,

$$\lim_{k\to\infty} \limsup_{\ell\to\infty} P\big[\,\mathfrak{N}_m(\mathfrak{R}_\ell x) \geq k\,\big] \;=\; 0\;. \tag{3.2}$$

As $\mathfrak{N}_m(\mathfrak{R}_\ell x)$ is a non-decreasing sequence in ℓ, the set $\{\mathfrak{N}_m(\mathfrak{R}_\ell x) \geq k\}$ is contained in $\{\mathfrak{N}_m(\mathfrak{R}_{\ell+1} x) \geq k\}$. Thus, for every $m \geq 1$,

$$P\Big[\bigcap_{k\geq 1}\bigcup_{\ell\geq m}\{\mathfrak{N}_m(\mathfrak{R}_\ell x) \geq k\}\Big] \;=\; \lim_{k\to\infty}\lim_{\ell\to\infty} P\Big[\mathfrak{N}_m(\mathfrak{R}_\ell x) \geq k\Big] \;=\; 0\;,$$

where the last equality follows from (3.2). Since this identity holds for every $m \geq 1$,

$$P\Big[\bigcap_{m\geq 1}\bigcup_{k\geq 1}\bigcap_{\ell\geq m}\{\mathfrak{N}_m(\mathfrak{R}_\ell x) \leq k\}\Big] \;=\; 1\;. \tag{3.3}$$

A straightforward modification of the proof of Assertion K shows that for every $\ell \geq m$, the set $\bigcap_{k=1}^{\mathfrak{N}_m(y)}\{T_{m,k}(y) \geq \epsilon\}$ is closed in $D([0,T], S_\ell)$. Therefore, by assumption (c),

$$\lim_{\epsilon\to 0}\limsup_{\ell\to\infty} P\Big[\bigcup_{k=1}^{\mathfrak{N}_m(\mathfrak{R}_\ell x)}\{T_{m,k}(\mathfrak{R}_\ell x) < \epsilon\}\Big] \;=\; 0\;.$$

Since the duration of the visits to a point m may only decrease as ℓ increases, $\bigcup_{k=1}^{\mathfrak{N}_m(\mathfrak{R}_\ell x)}\{T_{m,k}(\mathfrak{R}_\ell x) < \epsilon\} \subset \bigcup_{k=1}^{\mathfrak{N}_m(\mathfrak{R}_{\ell+1} x)}\{T_{m,k}(\mathfrak{R}_{\ell+1} x) < \epsilon\}$. In particular, by the previous displayed equation,

$$P\Big[\bigcap_{j\ge 1}\bigcup_{\ell\ge m}\bigcup_{k=1}^{\mathfrak{N}_m(\mathfrak{R}_\ell x)}\Big\{T_{m,k}(\mathfrak{R}_\ell x) < \frac{1}{j}\Big\}\Big] \;=\; 0\,.$$

Since this equation holds for every $m \ge 1$, we conclude that

$$P\Big[\bigcap_{m\ge 1}\bigcup_{j\ge 1}\bigcap_{\ell\ge m}\bigcap_{k=1}^{\mathfrak{N}_m(\mathfrak{R}_\ell x)}\Big\{T_{m,k}(\mathfrak{R}_\ell x) \ge \frac{1}{j}\Big\}\Big] \;=\; 1\,. \tag{3.4}$$

Finally, as the set $\{x \in D([0,T], S_\ell) : x(T) = x(T-)\}$ is closed, by assumption (d), for every $\ell \ge 1$,

$$P\big[\,(\mathfrak{R}_\ell x)(T) = (\mathfrak{R}_\ell x)(T-)\,\big] \;=\; 1\,,$$

so that

$$P\Big[\bigcap_{\ell\ge 1}\{(\mathfrak{R}_\ell x)(T) = (\mathfrak{R}_\ell x)(T-)\}\Big] \;=\; 1\,. \tag{3.5}$$

Denote by A the intersection of the events with full measure appearing in (3.1), (3.3), (3.4), (3.5). By Corollary 4.10, any trajectory in A belongs to $E^*([0,T], S_{\mathfrak{d}})$. This proves the proposition. $\square$

In view of condition (b), to prove condition (c) of Proposition 4.12, it is enough to show that for each $k, m \ge 1$,

$$\lim_{\epsilon\to 0}\limsup_{\ell\to\infty}\limsup_{n\to\infty} P_n\big[\,T_{m,k}(\mathfrak{R}_\ell x) < \epsilon\,\big] \;=\; 0\,. \tag{3.6}$$

We conclude this section with two remarks needed later. Fix a trajectory x in $E([0,T], S_{\mathfrak{d}})$ and $m \ge 1$. Then,

$$\mathbf{d}(x, \mathfrak{R}_m x) \;\le\; \frac{1}{2^{m-1}}\,\cdot \tag{3.7}$$

This bound follows from the observation that $\mathfrak{R}_k\mathfrak{R}_m x = \mathfrak{R}_k x$ for $k \le m$, and from the fact that $d_S(y,z) \le 1$ if y and z are trajectories in $D([0,T], S_\ell)$ for some $\ell \ge 1$.

Let x, y be two trajectories in $D([0,T], S_{\mathfrak{d}})$ such that $d_S(x,y) < [m(m+1)]^{-1}$ for some $m \ge 1$. Then,

$$\mathbf{d}(x,y) \;\le\; \frac{1}{m(m+1)} \;+\; \frac{1}{2^m}\,\cdot \tag{3.8}$$

Indeed, since $d_S(x,y) < [m(m+1)]^{-1}$, by definition of the Skorohod metric there exists an increasing function $\lambda : [0,T] \to [0,T]$ such that

$$\max\Big\{\|x - y\circ\lambda\|_\infty\,,\,\|\lambda\|^o\Big\} \;<\; \frac{1}{m(m+1)}\;\cdot$$

Since $\|x - y\circ\lambda\|_\infty < [m(m+1)]^{-1}$, if $x(s) \le m$ for some $s \in [0,T]$, then $(y\circ\lambda)(s) = x(s)$ and, conversely, if $(y\circ\lambda)(r) \le m$ for some $r \in [0,T]$, then $x(r) = (y\circ\lambda)(r)$. It follows from these relations that $\mathfrak{R}_k x = \mathfrak{R}_k(y\circ\lambda)$ for all $k \le m$. Hence, in view of Assertion F, $\mathfrak{R}_k x = (\mathfrak{R}_k y)\circ\lambda$, and

$$d_S(\mathfrak{R}_k x, \mathfrak{R}_k y) \;\le\; \max\Big\{\|\mathfrak{R}_k x - (\mathfrak{R}_k y)\circ\lambda\|_\infty\,,\,\|\lambda\|^o\Big\} \;<\; \frac{1}{m(m+1)}\;\cdot$$

To conclude the proof of (3.8) it remains to recall that $d_S(\mathfrak{R}_k x, \mathfrak{R}_k y) \le 1$ for all k.

4 Applications

In view of Lemma 4.11 and of Proposition 4.12, to prove that a sequence of probability measures P_n in $E^*([0,T], S_{\mathfrak{d}})$ converges in the soft topology to a probability measure P in $E^*([0,T], S_{\mathfrak{d}})$, we have first to show that the projections $P_n \circ \mathfrak{R}_m^{-1}$, $m \ge 1$, converge in the Skorohod topology of $D([0,T], S_m)$ to $P\circ\mathfrak{R}_m^{-1}$, and then to prove that the assumptions (a)–(c) of Proposition 4.12 are fulfilled. We show in this section, by inspecting three examples, that the conditions (a)–(c) of Proposition 4.12 follow from the convergence of the order parameter to a Markov process and from the fact that asymptotically the process spends a negligible amount of time on $\varDelta_N$.

1. Random walks among traps. Consider the random walk among traps $\eta(t) = \eta^N(t)$ introduced in Chapter **??**, and recall that we denoted by π_N the stationary state. Fix $T > 0$ and denote by $\mathbb{Q}^N_k$, $k \ge 1$, the probability measure on $D([0,T], S_{\mathfrak{d}})$ induced by the random walk $Z^N(t) = \varPsi_N(\eta(\beta_N t))$ starting from k. Note that time has been speeded-up by $\beta_N = v_{\ell_N}(x^N_1)^{-1}$, where $v_{\ell_N}(x^N_1)$ is the probability to escape from the ball of radius ℓ_N centered at the deepest trap x^N_1:

$$v_{\ell_N}(x^N_1) \;=\; \mathbb{P}_{x^N_1}\big[H_{B(x^N_1,\ell_N)^c} < H^+_{x^N_1}\big]\;.$$

Note also that the measure $\mathbb{Q}^N_k$ is concentrated on the set $D([0,T], S_{|V_N|})$, where V_N represents the set of vertices of the graph in which the evolution takes place.

β_N

It is clear from this last observation that $\varLambda_T(x) = 0$, $\mathbb{Q}^N_k$–almost surely. On the other hand, if we denote by τ_j, $j \ge 1$, the holding times of the trajectory $x(t)$, $x(t)$ is discontinuous at T if and only if $\tau_1 + \cdots + \tau_j = T$ for some j. Since, $\mathbb{Q}^N_k[\tau_1 + \cdots + \tau_j = T] = 0$ for each $j \ge 1$, $\mathbb{Q}^N_k$ is concentrated on the set $D^*([0,T], S_{\mathfrak{d}})$.

Denote by P_N the probability measure on $E([0,T], S_{\mathfrak{d}})$ defined by $P_N = \mathbb{Q}^N_k \circ \mathfrak{R}_\infty^{-1}$. By the last observation, P_N is concentrated on $E^*([0,T], S_{\mathfrak{d}})$. We claim that the sequence P_N fulfills all the assumptions of Proposition 4.12. We start with assumption (a). Since $\mathfrak{R}_\ell x \le x$, it is enough to show that

$$\lim_{m\to\infty} \limsup_{N\to\infty} E_{P_N} \Big[\int_0^T \mathbf{1}\{x(s) \ge m\}\, ds \Big] \;=\; 0\;. \tag{4.1}$$

By definition of P_N,

$$\begin{aligned} E_{P_N} \Big[\int_0^T \mathbf{1}\{x(s) \ge m\}\, ds \Big] \;&=\; E_{\mathbb{Q}_k^N} \Big[\int_0^T \mathbf{1}\{x(s) \ge m\}\, ds \Big] \\ &\le\; \frac{1}{\pi_N(k)} \sum_{j\ge 1} \pi_N(j)\, E_{\mathbb{Q}_j^N} \Big[\int_0^T \mathbf{1}\{x(s) \ge m\}\, ds \Big]\;. \end{aligned}$$

Since π_N is the stationary state, the previous sum is equal to $T\,\pi_N\{S_{m-1}^c\}$, where, we recall, $S_m = \{1, \dots, m\}$. As, for every $k \ge 1$,

$$\lim_{m\to\infty} \limsup_{N\to\infty} \frac{\pi_N\{S_m^c\}}{\pi_N(k)} \;=\; 0\;,$$

condition (4.1) is in force.

We first prove Conditions (b) and (c) of Proposition 4.12 under the assumption that $\beta := \sup_{N\ge 1} \beta_N$ is finite. This is the case of the random walk on a torus $\mathbb{T}_N^d$ in dimension $d \ge 3$.

Since $\mathfrak{N}_m(\mathfrak{R}_\ell x) \le \mathfrak{N}_m(x)$, $\ell \ge 1$, to prove condition (b) of Proposition 4.12, it is enough to show that for each $m \ge 1$,

$$\lim_{j\to\infty} \limsup_{N\to\infty} P_N \big[\mathfrak{N}_m(x) \ge j \big] \;=\; 0\;. \tag{4.2}$$

The above probability is equal to $\mathbb{Q}_k^N[\mathfrak{N}_m(x) \ge j]$. Denote by τ_i^m, $i \ge 1$, the holding times at m. This is a sequence of i.i.d. mean $\beta_N^{-1} W_m$ exponential random variables. Since $\{\mathfrak{N}_m(x) \ge j\} \subset \{\tau_1^m + \cdots + \tau_j^m \le T\}$, the previous probability is bounded by $\mathbb{Q}_k^N\big[\tau_1^m + \cdots + \tau_j^m \le T\big] \;\le\; P\big[T_1 + \cdots + T_j \le T\big]$, where T_i, $i \ge 1$, is a sequence of i.i.d. mean $\beta^{-1} W_m$ exponential random variables. This expression vanishes as $j \uparrow \infty$, which proves (4.2).

In view of (3.6) and since $T_{m,j}(\mathfrak{R}_\ell x)$, $j \ge 1$, are identically distributed, to prove condition (c) of Proposition 4.12 we need to show that for each $m \ge 1$,

$$\lim_{\epsilon\to 0} \limsup_{\ell\to\infty} \limsup_{N\to\infty} P_N \big[T_{m,1}(\mathfrak{R}_\ell x) < \epsilon \big] \;=\; 0\;.$$

Since $T_{m,1}(\mathfrak{R}_\ell x) \ge T_{m,1}(x)$, $\ell \ge m \ge 1$, to prove condition (c) of Proposition 4.12 we just have to show that for each $m \ge 1$,

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} P_N \big[T_{m,1}(x) < \epsilon \big] \;=\; 0\;. \tag{4.3}$$

With the notation introduced in the previous paragraph, the probability above is equal to $\mathbb{Q}_k^N[\tau_1^m < \epsilon]$. As τ_1^m is a mean $\beta_N^{-1} W_m$ exponential random variable and as $\beta_N \le \beta$, the previous probability is less than or equal to $P[T < \epsilon]$, where T is a mean $\beta^{-1} W_m$ exponential random variable. This proves condition (c) of Proposition 4.12 in the case where $\sup_N \beta_N < \infty$.

We conclude this section proving conditions (b) and (c) of Proposition 4.12 without the assumption that $\sup_N \beta_N < \infty$. Recall that we denote by A_N the set of the first M_N deepest traps, $A_N = \{x_1^N, \dots, x_{M_N}^N\}$. Let U_1^N be the time of the first visit to A_N, $U_1^N = \inf\{t \ge 0 : \eta(t) \in A_N\}$, and define recursively the sequence of stopping times U_j^N, $j \ge 1$, by

$$U_{j+1}^N \;=\; \inf\Big\{t \ge U_j^N : \eta(t) \in A_N \,,\, \exists U_j^N \le s \le t \text{ s.t. } \eta(s) \notin B_N\Big\}\,,$$

where $B_N = \cup_{i=1}^{M_N} B(x_i^N, \ell_N)$. Hence, the sequence U_j^N represents the succesive visits to the deepest traps after escaping from these traps. We refer to the time interval $[U_j^N, U_{j+1}^N)$ as the j-th excursion.

For $m \ge 1$, let $e_1(m) = \min\{j \ge 1 : \eta(U_j^N) = x_m^N\}$ be the first excursion to the trap x_m^N. Define recursively $e_i(m)$, $i \ge 1$, by

$$e_{i+1}(m) = \min\{j > e_i(m) : \eta(U_j^N) = x_m^N\}\,.$$

Note that we may have $e_{i+1}(m) = e_i(m) + 1$, as the process may escape from the trap x_m^N and then return to it before visiting any other deep trap. We refer to $[U_{e_i(m)}^N, U_{e_{i_1}(m)+1}^N)$ as the i-th excursion to x_m^N.

Let G_i^N, $i \ge 1$, be the number of visits to x_m^N during the i-th excursion to x_m^N, in other words, G_i^N is the number of visits to x_m^N in the time interval $[U_{e_i(m)}^N, U_{e_{i_1}(m)+1}^N)$. The random variables G_i^N, $i \ge 1$, are i.i.d. and have a mean β_N geometric distribution. Let $T_{i,p}^N$, $p \ge 1$, be the p-th holding time at x_m^N after $U_{e_i(m)}^N$. To clarify this definition, observe that the random walk $\eta(t\beta_N)$ remains at x_m^N in the time interval $[U_{e_i(m)}^N, U_{e_i(m)}^N + T_{i,1}^N)$ and that $T_{i+1,p}^N = T_{i,G_i^N+p}^N$. The random variables $T_{i,p}^N$ are i.i.d., have a mean W_m/β_N exponential distribution, and are independent from the sequence G_i^N.

Fix N large enough for $M_N \ge \ell$ so that $\mathfrak{N}_m(\mathfrak{R}_\ell x) \le \mathfrak{N}_m(\mathfrak{R}_{M_N} x)$. In this case,

$$\{\mathfrak{N}_m(\mathfrak{R}_\ell x) \ge j\} \;\subset\; \Big\{\sum_{i=1}^{j}\sum_{p=1}^{G_i^N} T_{i,p}^N \le T\Big\}\,.$$

It follows from the conclusion of the last paragraph that $\sum_{1\le p\le G_i^N} T_{i,p}^N$, $i \ge 1$, forms a sequence of i.i.d. mean W_m exponential random variables. This proves condition (b) of Proposition 4.12.

In view of (3.6) and since $T_{m,j}(\mathfrak{R}_\ell x)$, $j \ge 1$, are identically distributed, to prove condition (c) of Proposition 4.12 we need to show that for each $m \ge 1$,

$$\lim_{\epsilon\to 0}\limsup_{\ell\to\infty}\limsup_{N\to\infty} P_N\big[\,T_{m,1}(\mathfrak{R}_\ell x) < \epsilon\,\big] \;=\; 0\,.$$

Since $T_{m,1}(\mathfrak{R}_\ell x) \ge T_{m,1}(\mathfrak{R}_{M_N} x)$, it is in fact enough to show that for each $m \ge 1$,

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} P_N\big[\, T_{m,1}(\mathfrak{R}_{M_N} x) < \epsilon \,\big] \;=\; 0 \,.$$

This probability is equal to $\mathbb{Q}^N_k[\, T_{m,1}(\mathfrak{R}_{M_N} x) \;<\; \epsilon\,]$ and $T_{m,1}(\mathfrak{R}_{M_N} x) \;\ge\; \sum_{p=1}^{G_1^N} T^N_{1,p}$, a mean W_m exponential random variable. Therefore,

$$P_N\big[\, T_{m,1}(\mathfrak{R}_{M_N} x) < \epsilon \,\big] \;\le\; P[T < \epsilon] \,,$$

where T is a mean W_m exponential random variable, which proves condition (c) of Proposition 4.12.

2. Zero-range processes. Consider the zero-range process $\eta(t) = \eta^N(t)$ introduced in Chapter **??**. We assume that $\eta(t)$ is defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $N > L$, let the projection $\Psi_N : E_{L,N} \to \{1, \dots, L\} \cup \{N\}$ be defined by

$$\Psi_N(\eta) \;=\; \sum_{x=1}^{L} j\, \mathbf{1}\{\eta \in \mathcal{E}^x_N\} \;+\; N \mathbf{1}\{\eta \in \Delta_N\} \,.$$

It could be more natural to define $\Psi_N(\eta)$ as $\mathfrak{d}$ in the set Δ_N. However, with such a definition $\Psi_N(\eta(t))$ would not be a trajectory in $D([0,T], S_{\mathfrak{d}})$ and theory developped in the previous sections could not be applied.

Fix $T > 0$, $1 \le x \le L$, and a configuration η in $\mathcal{E}^x_N$. Denote by $\mathbb{Q}^N_\eta$ the probability measure on $D([0,T], S_{\mathfrak{d}})$ induced by the random walk $\mathbb{X}^N(t) = \Psi_N(\eta(N^{1+\alpha} t))$ starting from η. Note that time has been speeded-up by $N^{1+\alpha}$ and that the measure $\mathbb{Q}^N_\eta$ is concentrated on the set $D([0,T], S_N)$.

It is clear from this last observation that $\Lambda_T(x) = 0$, $\mathbb{Q}^N_\eta-$almost surely. On the other hand, if we denote by τ_j, τ^η_j, $j \ge 1$, the holding times of the processes $\mathbb{X}^N(t)$, $\eta(N^{\alpha+1} t)$, respectively, $\mathbb{X}^N(t)$ is discontinuous at T if and only if $\tau_1 + \cdots + \tau_j = T$ for some j. Since, $\tau_1 + \cdots + \tau_j = \tau^\eta_1 + \cdots + \tau^\eta_k$ for some $k \ge j$ and since $\mathbb{P}[\tau^\eta_1 + \cdots + \tau^\eta_\ell = T] = 0$ for all $\ell \ge 1$, we have that $\mathbb{Q}^N_\eta[\tau_1 + \cdots + \tau_j = T] = 0$ for each $j \ge 1$. Therefore, $\mathbb{Q}^N_\eta$ is concentrated on the set $D^*([0,T], S_{\mathfrak{d}})$.

Denote by P_N the probability measure on $E([0,T], S_{\mathfrak{d}})$ defined by $P_N = \mathbb{Q}^N_\eta \circ \mathfrak{R}^{-1}_\infty$. By the last observation, P_N is concentrated on $E^*([0,T], S_{\mathfrak{d}})$. We claim that the sequence P_N fulfills all the assumptions of Proposition 4.12. We start with assumption (a). As in the previous example, it is enough to show that (4.1) holds. By definition of P_N, for $N \ge m \ge L$,

$$\begin{aligned} E_{P_N}\Big[\int_0^T \mathbf{1}\{x(s) \ge m\}\, ds\Big] \;&=\; E_{\mathbb{Q}^N_\eta}\Big[\int_0^T \mathbf{1}\{x(s) \ge m\}\, ds\Big] \\ &=\; \mathbb{E}_\eta\Big[\int_0^T \mathbf{1}\{\eta(s\, N^{\alpha+1}) \in \Delta_N\}\, ds\Big] \,, \end{aligned}$$

which is the statement of Lemma **??** in **?**.

We turn to condition (b) of Proposition 4.12. As in the example of random walks among traps, it is enough to prove (4.2). Denote by T_j, $j \ge 1$, the holding times between successive visits to the metastable sets: $T_1 = \inf\{t > 0 : \eta(t) \in \mathcal{E}^N\}$,

$$T_{j+1} \;=\; \inf\{t>0 : \eta(\mathbf{T}_j + t) \in \mathcal{E}^N \setminus \mathcal{E}^N_{\eta(\mathbf{T}_j)}\}\;, \quad \mathbf{T}_j \;=\; T_1 + \cdots T_j\;, \quad j \ge 1\;.$$

Denote by $T^{\mathcal{E}}_j$, $j \ge 1$, the same sequence for the trace process $\eta^{\mathcal{E}}(t)$, $T^{\mathcal{E}}_1 = \inf\{t > 0 : \eta^{\mathcal{E}}(t) \in \mathcal{E}^N\}$.

For $1 \le k \le L$, let $e_1(k) = \min\{j \ge 1 : \eta(\mathbf{T}_j) \in \mathcal{E}_k\}$ be the first visit to the metastable set $\mathcal{E}_k$. Define recursively $e_i(k)$, $i \ge 1$, by

$$e_{i+1}(k) = \min\{j > e_i(k) : \eta(\mathbf{T}_j) \in \mathcal{E}_k\}\;.$$

It is clear that $T^{\mathcal{E}}_j \le T_j$, $j \ge 1$, and that $\{\mathfrak{N}_k(\mathbb{X}^N) \ge j\} \subset \{T_{e_1(k)} + \cdots + T_{e_j(k)} \le T\} \subset \{T^{\mathcal{E}}_{e_1(k)} + \cdots + T^{\mathcal{E}}_{e_j(k)} \le T\}$. Since the sequence $T^{\mathcal{E}}_{e_j(k)}$ represents the holding times at k for the process $\mathbb{X}^N(t) = \Psi_N(\eta^{\mathcal{E}}(N^{1+\alpha}t))$, and since the process $\mathbb{X}^N(t)$ converges in the Skorohod topology to a Markov process on $\{1, \dots, L\}$,

$$\limsup_{N\to\infty} \mathbb{P}_\eta\big[T^{\mathcal{E}}_{e_1(k)} + \cdots + T^{\mathcal{E}}_{e_j(k)} \le T\big] \;\le\; P\big[S_1 + \cdots + S_j \le T\big]\;,$$

where S_i, $i \ge 1$, is a sequence of non-degenerate i.i.d. exponential random variables. As $j \uparrow \infty$, this expression vanishes, which proves (4.2).

It remains to prove assertion (c) of Proposition 4.12. As argued in the previous example, it is enough to show that (4.3) holds for every $m \ge 1$. With the notation introduced above, it means that we have to show that

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \mathbb{P}_\eta\big[T_{e_1(m)} < \epsilon\big] \;=\; 0\;.$$

Since $T^{\mathcal{E}}_{e_1(m)} \le T_{e_1(m)}$, it is enough to prove the previous assertion with $T^{\mathcal{E}}_{e_1(m)}$ replacing $T_{e_1(m)}$. This follows from the convergence of $T^{\mathcal{E}}_{e_1(m)}$ to a non-degenerate exponential distribution.