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Contents

Part II Applications

Part I

Continuous-time Markov chains

1

Continuous-time Markov Chains

c01

We introduce in this chapter continuous-time Markov chains and present their main properties. We assume that the reader is familiar with the theory of discrete-time Markov chains, which can be found, for example, in Chung [1967], Freedman [1971], or in Norris [1998].

1 Markov Chains

bbsec01

Fix a countable set E endowed with the discrete topology. The elements of E, called configurations or points, are denoted by the Greek letters η, ξ and ζ. $p_t(\eta,\xi)$

bs09

Definition 1.1. *A set of functions $p_t : E \times E \to \mathbb{R}_+$, $t \geq 0$, is a transition probability if for all $\eta, \xi \in E$, $s, t \geq 0$,*

(a) *$p_0(\eta,\xi) = \delta_{\eta,\xi}$, where $\delta_{\eta,\xi}$ represents the delta of Kroenecker;*

(b) *$p_t(\eta,\xi) \geq 0$, $\sum_{\zeta \in E} p_t(\eta,\zeta) = 1$;*

(c) *$p_{t+s}(\eta,\xi) = \sum_{\zeta \in E} p_t(\eta,\zeta)\, p_s(\zeta,\xi)$;*

(d) *For each $(\eta,\xi) \in E \times E$, the function $t \mapsto p_t(\eta,\xi)$ is right-continuous at $t = 0$.*

Property (c) is known as the Chapman-Kolmogorov equations. It follows from the previous conditions that, for each $(\eta,\xi) \in E \times E$, the function $t \mapsto p_t(\eta,\xi)$ is right-continuous at every $t > 0$. Indeed, by properties (b) and (c), for $t \geq 0$, $s > 0$,

$$p_{t+s}(\eta,\xi) - p_t(\eta,\xi) = \sum_{\zeta \neq \eta} p_s(\eta,\zeta)\left\{ p_t(\zeta,\xi) - p_t(\eta,\xi) \right\}.$$

By property (b), the absolute value of this expression is bounded by

$$2\left\{ 1 - p_s(\eta,\eta) \right\},$$

which, by (d), vanishes as $s \downarrow 0$. In particular, the function $t \mapsto p_t(\eta, \xi)$ is measurable.

A set of functions $p_t : E \times E \to \mathbb{R}_+$, $t \geq 0$, is said to be a *substochastic* transition probability if it fulfills all the previous conditions with (b) replaced by

$$p_t(\eta, \xi) \;\geq\; 0 \;, \quad \sum_{\zeta \in E} p_t(\eta, \zeta) \;\leq\; 1 \;.$$

Let Ω be a nonempty set. Elements of Ω are represented by ω. Recall that a *filtration* on Ω is an increasing sequence of σ-algebras of subsets of Ω, and that a probability space is a triple $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbf{P}$ is a probability measure defined on $\mathcal{F}$. In analogy with this definition we introduce the notion of *Markov spaces*.

$\mathcal{F}_t, \mathcal{F}_\infty$

$\boxed{\text{bbs01}}$ **Definition 1.2.** *A Markov space is a triple* $(\Omega, (\mathcal{F}_t : t \geq 0), \{\mathbb{P}_\eta : \eta \in E\})$, *where* $(\mathcal{F}_t : t \geq 0)$ *is a filtration on* Ω *and* $\{\mathbb{P}_\eta : \eta \in E\}$ *is a family of probability measures defined on* $\mathcal{F}_\infty$, *the smallest* σ-algebra which contains all sets $\mathcal{F}_t$.

A family $(\eta(t) : t \geq 0)$ of E-valued random variables defined on the probability space $(\Omega, \mathcal{F}_\infty)$ is said to be *adapted* to the filtration $\mathcal{F}_t$ if for each $t \geq 0$ $\eta(t)$ is $\mathcal{F}_t$-measurable.

$\boxed{\text{bs07}}$ **Definition 1.3.** *Let* $p_t(\eta, \xi)$ *be a transition probability. A sequence of E-valued random variables* $(\eta(t) : t \geq 0)$ *defined on a Markov space* $(\Omega, (\mathcal{F}_t : t \geq 0), \{\mathbb{P}_\eta : \eta \in E\})$ *is a continuous-time Markov chain with transition probability* $p_t(\eta, \xi)$ *if*

(a) *The family $(\eta(t) : t \geq 0)$ is adapted to the filtration $\mathcal{F}_t$;*

(b) *For all $\eta \in E$, $\mathbb{P}_\eta[\eta(0) = \eta] = 1$;*

(c) *For all $\eta \in E$, the trajectories $\eta(t)$ are right-continuous, $\mathbb{P}_\eta$-almost surely;*

(d) *For all $\eta, \xi \in E$, s, $t \geq 0$,*

$$\mathbb{P}_\eta[\eta(t+s) = \xi \,|\, \mathcal{F}_t] \;=\; p_s(\eta(t), \xi) \;, \qquad (1.1) \quad \boxed{\text{b14}}$$

$\mathbb{P}_\eta$-*almost surely.*

We shall abreviated in the sequel continuous-time Markov chain by Markov chain. Property (c) states that for each $\eta \in E$, there exists a set $\mathcal{A} \in \mathcal{F}_\infty$ such that $\mathbb{P}_\eta[\mathcal{A}] = 1$ and for each $\omega \in \mathcal{A}$, the function $\eta(\,\cdot\,, \omega) : \mathbb{R}_+ \to E$ is right-continuous.

$\mathcal{F}_t^\eta$

Denote by $\mathcal{F}_t^\eta$, $t \geq 0$, the σ-algebra spanned by $\eta(s)$, $0 \leq s \leq t$. The filtration $(\mathcal{F}_t^\eta : t \geq 0)$ is called the *natural* filtration. A collection of E-valued random variables $(\eta(t) : t \geq 0)$ which fulfills (1.1) for some filtration $\mathcal{F}_t$ which contains the natural filtration, $\mathcal{F}_t \supset \mathcal{F}_t^\eta$, and some substochastic

transition probability p_t is said to satisfy the Markov property with respect to the filtration $\mathcal{F}_t$ and the substochastic transition probability p_t.

Markov chains do exist! We construct below a Markov chain from two parameters: a positive function $\lambda : E \to (0, \infty)$ and a *transition matrix* p, which is a function $p : E \times E \to \mathbb{R}_+$ such that

$$p(\eta, \xi) \; \geq \; 0 \; , \quad \sum_{\zeta \in E} p(\eta, \zeta) = 1 \; , \quad \eta, \xi \in E \; . \tag{1.2}$$ $\boxed{\texttt{b01}}$

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a discrete-time Markov chain $Y = (Y_n : n \geq 0)$ with transition matrix $p(\eta, \xi)$ given by (1.2). Assume that

(a) Y_n is a *recurrent* chain;

(b) $\mathbb{P}[Y_0 = \eta] > 0$ for all $\eta \in E$;

(c) A sequence $(\mathfrak{e}_n : n \geq 0)$ of i.i.d. mean-one, exponential random variables independent of Y is also defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

$\mathfrak{e}_n$

Define the probability measure $\mathbb{P}_\eta$, $\eta \in E$, on $(\Omega, \mathcal{F})$ by

$$\mathbb{P}_\eta[\,\cdot\,] \;=\; \mathbb{P}[\,\cdot \,|\, Y_0 = \eta] \; .$$

Expectation with respect to $\mathbb{P}$, $\mathbb{P}_\eta$ is denoted by $\mathbb{E}$, $\mathbb{E}_\eta$, respectively. $\mathbb{P}_\eta$

Fix $\zeta \in E$. We associate to every sample path of Y the sequence of random S_j
times $(T_n : n \geq 0)$ given by

$$T_n \;=\; \frac{\mathfrak{e}_n}{\lambda(Y_n)} \; . \quad \text{Let} \quad S_0 \;=\; 0 \; , \quad S_j \;=\; \sum_{k=0}^{j-1} T_k \; . \tag{1.3}$$ $\boxed{\texttt{b20}}$

Since Y is recurrent, $\lim_{n \to \infty} S_n = \sum_{i \geq 0} T_i = \infty$, $\mathbb{P}_\zeta$-almost surely. In particular, the time-change

$$N(t) \;=\; \min\{n \geq 0 : \sum_{i=0}^{n} T_i > t\} \tag{1.4}$$ $\boxed{\texttt{b05}}$

is $\mathbb{P}_\zeta$-almost surely finite for every $t \geq 0$ and

$$\eta(t) \;=\; Y_{N(t)}$$

is a right-continuous trajectory with left limits well defined for all $t \geq 0$. We may also express $\eta(t)$ in terms of the random times S_j and the Markov chain Y_j as

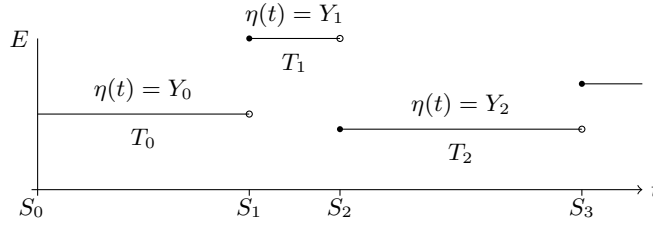$$\eta(t) \;=\; \sum_{j \geq 0} Y_j \, \mathbf{1}\{S_j \leq t < S_{j+1}\} \; , \tag{1.5}$$ $\boxed{\texttt{b45}}$

where $\mathbf{1}\{B\}$ stands for the indicator function of the set B. The right-hand side is a function of t and of the sequences $(Y_k : k \geq 0)$, $(T_k : k \geq 0)$, represented by $\mathbf{F} : \mathbb{R}_+ \times E^{\mathbb{N}} \times (0, +\infty)^{\mathbb{N}} \to E$:

$$\mathbf{F}(t \, ; \, (\eta_k : k \geq 0) \, ; \, (t_k ; k \geq 0) \,) \; := \; \sum_{j \geq 0} \eta_j \, \mathbf{1}\{s_j \leq t < s_{j+1}\} \, , \qquad (1.6) \quad \boxed{\texttt{bb51}}$$

where $s_0 = 0$, $s_j = \sum_{0 \leq i < j} t_i$. Hence, for every $\xi \in E$, $t \geq 0$,

$$\mathbb{P}_\zeta[\eta(t) = \xi] \; = \; \mathbb{P}_\zeta\big[\mathbf{F}(t; (Y_k : k \geq 0); (T_k : k \geq 0) \,) = \xi\big] \, . \qquad (1.7) \quad \boxed{\texttt{b47}}$$

Note that $\eta(0) = \zeta$, $\mathbb{P}_\zeta$-almost surely. Figure 1.1 illustrates a typical trajectory.

Fig. 1.1. A trajectory of the continuous-time process $\eta(t)$. $\boxed{\texttt{figb1}}$

$\mathbf{1}\{B\}$

$\mathbf{F}$

$\boxed{\texttt{bs10}}$ **Proposition 1.4.** *For η, $\xi \in E$, $t \geq 0$, let*

$$p_t(\eta, \xi) \; = \; \mathbb{P}_\eta[\eta(t) = \xi] \, .$$

Then, p_t is a transition probability and for all s, $t \geq 0$, η, $\xi \in E$,

$$\mathbb{P}_\eta[\eta(t + s) = \xi \, | \, \mathcal{F}_t^\eta] \; = \; p_s(\eta(t), \xi) \, . \qquad (1.8) \quad \boxed{\texttt{b02}}$$

Moreover, the sequence of random variables $(\eta(t) : t \geq 0)$ defined on $(\Omega, (\mathcal{F}_t^\eta : t \geq 0), \{\mathbb{P}_\eta : \eta \in E\})$ is an E-valued continuous-time Markov chain with transition probability given by $\{p_t : t \geq 0\}$.

Proof. We first prove (1.8). Fix $\eta \in E$. Since, by definition, $\eta(t)$ is $\mathcal{F}_t^\eta$-measurable, to prove (1.8) we only have to show that for every event A in $\mathcal{F}_t^\eta$,

$$\mathbb{P}_\eta[\eta(t + s) = \xi \, , \, A] \; = \; \mathbb{E}_\eta\big[\mathbb{P}_{\eta(t)}[\eta(s) = \xi] \, \mathbf{1}\{A\}\big] \, .$$

Since the σ-algebra $\mathcal{F}_t^\eta$ is generated by the variables $\eta(s)$, $0 \leq s \leq t$, by Dynkin's $\pi - \lambda$ theorem, it is enough to prove this identity for sets of the form $A = \{\eta(s_1) = \zeta_1, \ldots, \eta(s_\ell) = \zeta_\ell, \eta(t) = \zeta\}$ for $\ell \geq 0$, $0 \leq s_1 < \cdots < s_\ell < t$, $\zeta_1, \ldots, \zeta_\ell, \zeta \in E$. Assuming that A takes this form, we rewrite the left-hand side of the previous formula as

$$\sum_{j\geq 0} \mathbb{P}_\eta[\eta(t+s) = \xi\,,\, A\,,\, S_j \leq t < S_{j+1}]\,. \tag{1.9}$$

b44

On the set $S_j \leq t < S_{j+1}$, $\eta(t) = Y_j$. More generally, on this event, the set A can be expressed in terms of the variables $(Y_0, \ldots, Y_j, T_0, \ldots, T_{j-1})$. For example, the event $\{\eta(s_i) = \zeta_i\}$ can be represented as

$$\bigcup_{k=0}^{j-1} \Big\{ \{S_k \leq s_i < S_{k+1}\} \cap \{Y_k = \zeta_i\} \Big\}\ \cup\ \Big\{ \{S_j \leq s_i\} \cap \{Y_j = \zeta_i\} \Big\}\,.$$

Therefore, the event A can be replaced by a set A' expressed only in terms of the variables $(Y_0, \ldots, Y_j, T_0, \ldots, T_{j-1})$. Moreover, since $\eta(t) = \zeta$ on the set A and $\eta(t) = Y_j$ on the set $S_j \leq t < S_{j+1}$, $Y_j = \zeta$ on A'.

Let $\mathcal{F}_j^{Y,\mathfrak{e}}$, $j \geq 0$, the σ-algebra spanned by $\{Y_0, \ldots, Y_j, \mathfrak{e}_0, \ldots, \mathfrak{e}_{j-1}\}$. Note that $\mathcal{F}_j^{Y,\mathfrak{e}}$ coincides with the σ-algebra spanned by $\{Y_0, \ldots, Y_j, T_0, \ldots, T_{j-1}\}$. In view of the previous paragraph, taking a conditional expectation with respect to $\mathcal{F}_j^{Y,\mathfrak{e}}$, we may write the probability appearing in (1.9) as

$$\mathbb{E}_\eta\Big[\mathbf{1}\{A'\,,\, S_j \leq t\}\, \mathbb{P}_\eta[\eta(t+s) = \xi\,,\, t < S_{j+1}\,|\,\mathcal{F}_j^{Y,\mathfrak{e}}]\Big]\,.$$

Consider the previous conditional probability. Since $S_{j+1} = S_j + T_j$, we may rewrite $S_{j+1} > t$ as $T_j > t - S_j$ and $\eta(t+s)$ as $\mathbf{F}(s; \{Y_{j+k} : k \geq 0\}; \{T_j - (t - S_j), T_{j+k} : k \geq 1\})$, where the function $\mathbf{F}$ has been introduced in (1.6). The variable S_j is measurable with respect to $\mathcal{F}_j^{Y,\mathfrak{e}}$ and has to be treated below as a constant. With this notation, the previous conditional probability becomes

$$\mathbb{P}_\eta[\,\mathbf{F}(s) = \xi\,,\, T_j > t - S_j\,|\,\mathcal{F}_j^{Y,\mathfrak{e}}]\,, \tag{1.10}$$

bc01

where we wrote $\mathbf{F}(s)$ for $\mathbf{F}(s; \{Y_{j+k} : k \geq 0\}; \{T_j - (t - S_j), T_{j+k} : k \geq 1\})$. Given $\mathcal{F}_j^{Y,\mathfrak{e}}$, $Y_k' = Y_{j+k}$, $k \geq 0$, is a discrete-time Markov chain which starts from Y_j and whose transition matrix $p(\eta, \xi)$ is given by (1.2), and $\mathfrak{e}_k' = \mathfrak{e}_{j+k}$ are i.i.d. mean one exponential random variables, independent from the sequence Y_k'. The previous conditional probability is therefore equal to

$$\mathbb{P}_{Y_j}\Big[\mathbf{F}\big(s; \{Y_k' : k \geq 0\}; \{T_0' - (t - S_j), T_k' : k \geq 1\}\big) = \xi\,,\, T_0' > t - S_j\Big]\,, \tag{1.11}$$

bc02

where $T_k' = \mathfrak{e}_k'/\lambda(Y_k')$ and S_j is treated as a constant and is not integrated. Note that the probability measure $\mathbb{P}_\eta$ has been replaced by $\mathbb{P}_{Y_j}$, as the discrete-time Markov chain Y_k' starts from Y_j. We present at the end of this section, in (1.13), a detailed derivation of this identity.

By the loss of memory of exponential distributions, for every bounded function f, mean λ^{-1} exponential random variable $\mathfrak{e}$, and $r > 0$,

$$E[f(\mathfrak{e} - r)\,\mathbf{1}\{\mathfrak{e} > r\}]\ =\ e^{-\lambda r}\, E[f(\mathfrak{e})]\,.$$

Hence, the probability appearing in the penultimate displayed formula is equal to

$$e^{-\lambda(Y_j)(t-S_j)} \, \mathbb{P}_{Y_j}\big[\mathbf{F}\big(s; \{Y'_k : k \geq 0\}; \{T'_k : k \geq 0\}\big) = \xi\big]$$
$$= \; e^{-\lambda(Y_j)(t-S_j)} \, \mathbb{P}_{Y_j}[\eta(s) = \xi] \, .$$

Up to this point, we have shown that (1.9) is equal to

$$\sum_{j \geq 0} \mathbb{E}_\eta\Big[\mathbf{1}\{A', \, S_j \leq t\} \, e^{-\lambda(Y_j)(t-S_j)} \, \mathbb{P}_{Y_j}[\eta(s) = \xi]\Big] \, .$$

Since $Y_j = \zeta$ on the set A', we may replace in the previous expression $\mathbb{P}_{Y_j}[\eta(s) = \xi]$ by $\mathbb{P}_\zeta[\eta(s) = \xi]$ and drop out this probability from the expectation. On the other hand, repeating the previous argument based on the loss of memory of exponential distributions, we obtain that

$$\mathbb{E}_\eta\big[\mathbf{1}\{A', \, S_j \leq t\} \, e^{-\lambda(Y_j)(t-S_j)}\big] \; = \; \mathbb{P}_\eta\big[A', \, S_j \leq t < S_{j+1}\big] \, .$$

Hence, the sum (1.9) is equal to

$$\mathbb{P}_\zeta[\eta(s) = \xi] \sum_{j \geq 0} \mathbb{P}_\eta\big[A', \, S_j \leq t < S_{j+1}\big] \, .$$

At this point, we may replace back A' by A and sum over j to conclude the proof of (1.8).

We claim that $p_t(\eta, \xi)$ is a transition probability. Condition (a) of Definition 1.1 follows from the definition of $\eta(0)$ and the one of $\mathbb{P}_\eta$, while condition (b) follows from the fact the sequence S_j diverges: for every $\eta \in E$, $t \geq 0$,

$$\sum_{\xi \in E} p_t(\eta, \xi) \; = \; \sum_{\xi \in E} \mathbb{P}_\eta[\eta(t) = \xi] \; = \; \sum_{\xi \in E} \sum_{j \geq 0} \mathbb{P}_\eta[\eta(t) = \xi, \, S_j \leq t < S_{j+1}] \, .$$

On the set $\{S_j \leq t < S_{j+1}\}$, $\eta(t) = Y_j$ and $\sum_{\xi \in E} \mathbb{P}_\eta[Y_j = \xi] = 1$. The previous sum is thus equal to $\sum_{j \geq 0} \mathbb{P}_\eta[S_j \leq t < S_{j+1}] = 1$. Condition (c) of Definition 1.1 corresponds to the Markov property (1.8):

$$p_{t+s}(\eta, \xi) \; = \; \mathbb{P}_\eta[\eta(t+s) = \xi] \; = \; \sum_{\zeta \in E} \mathbb{P}_\eta[\eta(t) = \zeta, \, \eta(t+s) = \xi] \, .$$

Taking the conditional expectation with respect to $\mathcal{F}_t^\eta$, by (1.8), the previous sum is equal to

$$\sum_{\zeta \in E} \mathbb{P}_\eta[\eta(t) = \zeta] \, \mathbb{P}_\zeta[\eta(s) = \xi] \; = \; \sum_{\zeta \in E} p_t(\eta, \zeta) \, p_s(\zeta, \xi) \, .$$

Finally, condition (d) of Definition 1.1 can be proven from the non-degeneracy of the function λ. For $\eta, \xi \in E$,

$$p_t(\eta, \xi) \;=\; \mathbb{P}_\eta[\eta(t) = \xi] \;=\; \mathbb{P}_\eta[\eta(t) = \xi,\, T_0 > t] \;+\; \mathbb{P}_\eta[\eta(t) = \xi,\, T_0 \le t]\,.$$

The first term on the right-hand side is equal to $\delta_{\eta,\xi}(1 - \mathbb{P}_\eta[T_0 \le t])$, while the second term is bounded by $\mathbb{P}_\eta[T_0 \le t]$. This shows that the absolute value of $p_t(\eta, \xi) - \delta_{\eta,\xi}$ is bounded by $\mathbb{P}_\eta[T_0 \le t] = 1 - e^{-\lambda(\eta)t}$, which vanishes as $t \downarrow 0$. This proves that $p_t(\eta, \xi)$ is a transition probability.

It remains to prove that $(\eta(t) : t \ge 0)$ is an E-valued continuous-time Markov chain on $(\Omega, (\mathcal{F}_t^\eta : t \ge 0), \{\mathbb{P}_\eta : \eta \in E\})$ with transition probability $(p_t : t \ge 0)$. Conditions (a), (b) and (c) of Definition 1.3 hold by construction, while conditions (d) has been derived in the first part of the proof. $\square$

Fix a configuration $\eta \in E$. Denote by τ_1 the time of the first jump of $\eta(t)$:

$$\tau_1 \;:=\; \inf\big\{\, t > 0 : \eta(t) \ne \eta(0) \,\big\}\,. \tag{1.12}$$

By construction, $\tau_1 = S_1 = T_0 = \mathfrak{e}_0/\lambda(Y_0)$. Since, under $\mathbb{P}_\eta$, $Y_0 = \eta$ and $\mathfrak{e}_0$ is a mean-one exponential random variable, τ_1 is an exponential random variable with mean $\lambda(\eta)^{-1}$. Hence, the process remains at η a random time exponentially distributed and $\lambda(\eta)$, introduced in (1.2), represents the parameters of the exponential distribution. The parameters $\lambda(\eta)$, $\eta \in E$, are called the *holding rates* of the Markov chain $\eta(t)$.

The variable $\eta(\tau_1)$ stands for the site visited after the first jump. By construction, $\eta(T_1) = Y_1$. Since Y_n is a discrete-time Markov chain, $\mathbb{P}_\eta[\eta(\tau_1) = \xi] = \mathbb{P}_\eta[Y_1 = \xi] = p(\eta, \xi)$. Thus, $p(\eta, \xi)$ represents the probability that the process $\eta(t)$ jumps from η to ξ. The matrices $p(\eta, \xi)$, $R(\eta, \xi) = \lambda(\eta)\,p(\eta, \xi)$ are called, respectively, the *jump probabilities* and the *jump rates* of the chain.

brm1 *Remark 1.5.* We only used the assumption that the discrete-time Markov chain associated to the transition matrix $p(\eta, \xi)$ is recurrent to ensure that the sequence of random times S_j defined above (1.4), increases to $+\infty$ $\mathbb{P}_\eta$-almost surely. This assumption could be replaced by the assumption that the holding rates $\lambda(\eta)$ are bounded, $\sup_{\eta \in E} \lambda(\eta) < \infty$.

brm1 *Remark 1.6.* In the above construction, we may assume, without loss of generality, that the transition matrix $p(\eta, \xi)$, introduced in (1.2), vanishes on the diagonal, $p(\eta, \eta) = 0$. Indeed, assume that this is not the case, $p(\eta, \eta) > 0$ for some $\eta \in E$. Assume that the discrete-time Markov chain Y_n starts from η, $Y_0 = \eta$. Let $\mathfrak{N}$ be the time the chain Y_n jumps to $E \setminus \{\eta\}$:

$$\mathfrak{N} \;=\; \min\{n \ge 1 : Y_n \ne \eta\}\,,$$

and let Y_1' be the first site visited, $Y_1' = Y_{\mathfrak{N}}$. The random variable $T_0' = \sum_{0 \le j < \mathfrak{N}} T_j$ represents the jump time of $\eta(t)$:

$$T_0' \;:=\; \sum_{j=0}^{\mathfrak{N}-1} T_j \;=\; \inf\{t \ge 0 : \eta(t) \ne \eta\}\,,$$

and Y_1' the value of the chain at time T_0': $Y_1' = \eta(T_0')$.

It is clear that $\mathbb{P}_\eta[Y_1' = \xi] = p(\eta, \xi)/[1 - p(\eta, \eta)]$. We claim that T_0' has an exponential distribution of parameter $\lambda(\eta)[1-p(\eta, \eta)]$. To verify this assertion we compute the Laplace transform of T_0'. Fix $\theta > 0$ and observe that

$$\mathbb{E}_\eta\big[e^{-\theta T_0'}\big] \;=\; \mathbb{E}_\eta\big[\exp\{-\theta \sum_{j=0}^{\mathfrak{N}-1} T_j\}\big] \;=\; \sum_{n \geq 1} \mathbb{E}_\eta\big[\mathbf{1}\{\mathfrak{N} = n\} \exp\{-\theta \sum_{j=0}^{n-1} T_j\}\big] \;.$$

Denote by $\mathcal{G}$ the σ-algebra spanned by the Markov chain Y_n. We may rewrite the previous sum as

$$\sum_{n \geq 1} \mathbb{E}_\eta\Big[\mathbf{1}\{\mathfrak{N} = n\} \,\mathbb{E}_\eta\Big[\exp\{-\theta \sum_{j=0}^{n-1} T_j\} \,|\, \mathcal{G}\Big]\Big]$$

because $\mathfrak{N}$ is $\mathcal{G}$-measurable. Given $\mathcal{G}$, on the set $\mathfrak{N} = n$, the random variables T_j, $0 \leq j < n$ are i.i.d. mean $\lambda(\eta)^{-1}$ exponential random variables. The previous sum is thus equal to

$$\sum_{n \geq 1} \mathbb{E}_\eta\Big[\mathbf{1}\{\mathfrak{N} = n\} \Big(\frac{\lambda(\eta)}{\theta + \lambda(\eta)}\Big)^n\Big] \;=\; \frac{[1 - p(\eta, \eta)]\, u}{1 - u\, p(\eta, \eta)} \;,$$

as $\mathfrak{N}$ is a mean $[1 - p(\eta, \eta)]^{-1}$ geometric random variable. Here $u = \lambda(\eta)/[\theta + \lambda(\eta)]$. The previous ratio is equal to

$$\frac{\lambda(\eta)[1 - p(\eta, \eta)]}{\theta + \lambda(\eta)[1 - p(\eta, \eta)]} \;,$$

which is the Laplace transform of an exponential random variable of parameter $\lambda(\eta)[1 - p(\eta, \eta)]$. Since the Laplace transform characterizes the distribution, the claim is proved.

This computation indicates that the continuous-time Markov chain $\{\eta'(t) : t \geq 0\}$ constructed from the discrete-time Markov chain Y_n', whose transition matrix $p'(\eta, \xi)$ is given by

$$p'(\eta, \xi) \;=\; \frac{p(\eta, \xi)}{1 - p(\eta, \eta)} \;, \quad \eta \neq \xi \;, \quad p'(\eta, \eta) \;=\; 0 \;,$$

and holding rates $\lambda' : E \to \mathbb{R}_+$ by

$$\lambda'(\eta) \;=\; [1 - p(\eta, \eta)]\, \lambda(\eta) \;,$$

has the same distribution as $\eta(t)$. This claim can be checked, We leave the details to the reader. This proves the assertion made at beginning of the remark because $p'(\eta, \eta) = 0$.

We turn to the identity between (1.10) and (1.11). Recall the notation introduced in the proof of Proposition 1.4. We claim that for every $j \geq 0$ and $r \geq 0$,

$$\mathbb{P}_\eta\Big[\mathbf{F}(s; \{Y_{j+k} : k \geq 0\}; \{T_j - r, T_{j+k} : k \geq 1\}) = \xi \,, \, T_j > r \,\big|\, \mathcal{F}_j^{Y,\mathfrak{e}} \Big]$$
$$= \mathbb{P}_{Y_j}\Big[\mathbf{F}\big(s; \{Y_k : k \geq 0\}; \{T_0 - r, T_k : k \geq 1\}\big) = \xi \,, \, T_0 > r \Big] \,. \qquad (1.13) \quad \boxed{\text{bb52}}$$

Since the right-hand side is measurable with respect to $\mathcal{F}_j^{Y,\mathfrak{e}}$, we have just to show that for every set A in $\mathcal{F}_j^{Y,\mathfrak{e}}$,

$$\mathbb{P}_\eta\Big[\mathbf{F}(s; \{Y_{j+k} : k \geq 0\}; \{T_j - r, T_{j+k} : k \geq 1\}) = \xi \,, \, T_j > r \,, \, A \Big]$$
$$= \mathbb{E}_\eta\Big[\mathbf{1}\{A\} \, \mathbb{P}_{Y_j}\Big[\mathbf{F}\big(s; \{Y_k : k \geq 0\}; \{T_0 - r, T_k : k \geq 1\}\big) = \xi \,, \, T_0 > r \Big] \Big] \,.$$

By Dynkin's π-λ theorem we may assume that the set A is of the form $A_1 \cap A_2$, where A_1, A_2 depend only on $\{Y_i : 0 \leq i \leq j\}$, $\{\mathfrak{e}_i : 0 \leq i \leq j{-}1\}$, respectively. On the other hand, the set $\{\mathbf{F}(s; \{Y_{j+k} : k \geq 0\}; \{T_j - r, T_{j+k} : k \geq 1\}) = \xi\} \cap \{T_j > r\}$ is a function of the variables $\{(Y_{j+k}, \mathfrak{e}_{j+k}) : k \geq 0\}$, denoted by f.

With this notation, the left hand side of the previous displayed equation becomes

$$\mathbb{E}_\eta\big[f \, \mathbf{1}\{A_1\} \, \mathbf{1}\{A_2\} \big] \,. \qquad (1.14) \quad \boxed{\text{bb01}}$$

Take the conditional expectation with respect to $\mathcal{F}_j^Y$, the σ-algebra spanned by $Y_0, \ldots, Y_j$. Since the sequence $\mathfrak{e}_k$ is independent of the chain Y_k and since A_1 is measurable with respect to $\mathcal{F}_j^Y$, the previous expectation is equal to

$$\mathbb{E}_\eta\Big[\mathbf{1}\{A_1\} \, \mathbb{E}_\eta\big[f \,|\, \mathcal{F}_j^Y \big] \Big] \, \mathbb{P}_\eta[A_2] \; = \; \mathbb{E}_\eta\Big[\mathbf{1}\{A_1\} \, \mathbf{1}\{A_2\} \, \mathbb{E}_\eta\big[f \,|\, \mathcal{F}_j^Y \big] \Big] \,.$$

As the sequence $\mathfrak{e}_k$ is independent of the chain Y_k, by the Markov property for the chain Y_k,

$$\mathbb{E}_\eta\Big[f(\{Y_{j+k} : k \geq 0\} \,, \, \{\mathfrak{e}_{j+k} : k \geq 0\}) \,|\, \mathcal{F}_j^Y \Big]$$
$$= \mathbb{E}_{Y_j}\Big[f(\{Y_k : k \geq 0\} \,, \, \{\mathfrak{e}_{j+k} : k \geq 0\}) \Big] \,.$$

Since the sequence $\mathfrak{e}_k$ is independent of the chain Y_n and is identically distributed, we may replace $\{\mathfrak{e}_{j+k} : k \geq 0\}$ by $\{\mathfrak{e}_k : k \geq 0\}$ in this formula. We have thus proved that (1.14) is equal to

$$\mathbb{E}_\eta\Big[\mathbb{E}_{Y_j}\Big[f(\{Y_k : k \geq 0\} \,, \, \{\mathfrak{e}_k : k \geq 0\}) \Big] \mathbf{1}\{A_1\} \, \mathbf{1}\{A_2\} \Big] \,,$$

as claimed.

2 Strong Markov Property

secb02

Let E be a countable set. Unless otherwise stated, we consider in this section an E-valued, continuous-time, right-continuous process $(\eta(t) : t \geq 0)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and adapted to a filtration $(\mathcal{F}_t : t \geq 0)$, where $\mathcal{F}_t \subset \mathcal{F}$ for all t.

A function $T : \Omega \to [0, \infty]$ is said to be a *stopping time* with respect to the filtration $\mathcal{F}_t$ if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. When it is clear from the context to which filtration we refer, we simply say that T is a stopping time.

The sets $\{T < t\}$, $\{T = t\}$ belong to $\mathcal{F}_t$ if T is a stopping time since

$$\{T < t\} \;=\; \bigcup_{n \geq 1} \{T \leq t - (1/n)\} \;\in\; \mathcal{F}_t \,, \tag{2.1}$$

b48

and $\{T = t\} = \{T \leq t\} \setminus \{T < t\}$.

For a subset A of E, denote by H_A the hitting time of the set A:

$$H_A \;=\; \inf\{t > 0 : \eta(t) \in A\} \,,$$

and by H_A^+ the time of the first return to A:

$$H_A^+ \;=\; \inf\{t > \tau_1 : \eta(t) \in A\} \,,$$

where τ_1 is the time of the first jump, introduced in (1.12).

bs06   **Lemma 1.7.** *For every subset A of E, H_A and H_A^+ are stopping times with respect to the natural filtration $\mathcal{F}_t^\eta$.*

Proof. Since η is right-continuous and E countable, for every $t \geq 0$,

$$\{H_A \leq t\} \;=\; \bigcup_{\substack{0 \leq s < t \\ s \in \mathbb{Q}}} \{\eta(s) \in A\} \;\cup\; \{\eta(t) \in A\} \;\in\; \mathcal{F}_t^\eta \,.$$

By the same reasons,

$$\{H_A^+ \leq t\} \;=\; \{\eta(t) \in A\,,\, \eta(t) \neq \eta(0)\} \;\cup\; \bigcup_{\substack{0 < s < t \\ s \in \mathbb{Q}}} \Big[\{\eta(s) \neq \eta(0)\} \cap \{\eta(t) \in A\}\Big]$$

$$\cup\; \bigcup_{\substack{0 < s < r < t \\ r, s \in \mathbb{Q}}} \Big[\{\eta(s) \neq \eta(0)\} \cap \{\eta(r) \in A\}\Big] \,.$$

The right-hand side clearly belongs to $\mathcal{F}_t^\eta$. $\square$

Observe that the denumerability of the set E played an important role in the proofs above as we used the fact that the trajectories $\eta(t)$ are piecewise constant to the right (for all $t \geq 0$ there exists $\epsilon > 0$ such that $\eta(s) = \eta(t)$ for $t \leq s < t + \epsilon$).

Recall from (1.12) that we denote by τ_1 the time of the first jump. Denote by $(\tau_k : k \geq 1)$ the times of the successive jumps:

$$\tau_{k+1} := \inf \left\{ t > \tau_k : \eta(t) \neq \eta(\tau_k) \right\} , \quad k \geq 1 . \qquad (2.2) \quad \boxed{\text{bb12}}$$

We claim that the jump times $(\tau_j : j \geq 1)$ are stopping times with respect to the natural filtration. We prove this assertion for $j = 1$ and leave the general statement to the reader. For every $t \geq 0$,

$$\{\tau_1 \leq t\} = \{\eta(t) \neq \eta(0)\} \cup \bigcup_{\substack{0 < s < t \\ s \in \mathbb{Q}}} \{\eta(s) \neq \eta(0)\} \in \mathcal{F}_t^\eta . \qquad (2.3) \quad \boxed{\text{bb11}}$$

For a stopping time T with respect to a filtration $\mathcal{F}_t$, denote by $\mathcal{F}_T$ the subset of events A in $\mathcal{F}$ such that

$$A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0 .$$

The set $\mathcal{F}_T$ is a σ-algebra which represents the events which occured before T. If S and T are two stopping times with respect to the same filtration and $S \leq T$, it is not difficult to show that

$$\mathcal{F}_S \subset \mathcal{F}_T . \qquad (2.4) \quad \boxed{\text{b50}}$$

When the filtration $\mathcal{F}_t$ is the natural filtration of a Markov chain $\eta(t)$, we represent the σ-algebra $\mathcal{F}_T$ by $\mathcal{F}_T^\eta$.

$\boxed{\text{bs12}}$ **Lemma 1.8.** *Let S, T be two stopping times with respect to the natural filtration $\mathcal{F}_t^\eta$ of a right-continuous process $\eta(t)$. The sets $\{S \leq T\}$, $\{S < T\}$ and the random variable $\eta(T)$ are $\mathcal{F}_T^\eta$ measurable.*

Proof. We prove the measurability of the set $\{S \leq T\}$ and leave the other case to the reader. It is enough to show that $\{T < S\}$ belongs to $\mathcal{F}_T^\eta$. Since $\eta(t)$ is right-continuous, for each $t \geq 0$,

$$\{T < S\} \cap \{T \leq t\} = \left[\{T = t\} \cap \{S > t\} \right] \cup \bigcup_{\substack{0 < s < t \\ s \in \mathbb{Q}}} \{T < s < S\} .$$

This set belongs to $\mathcal{F}_t^\eta$ in view of (2.1).

To prove the last assertion of the lemma, we need to show that $\{\eta(T) = \eta\}$ belongs to $\mathcal{F}_T^\eta$ for every $\eta \in E$. Fix $\eta \in E$ and note that

$$\{T < t\} = \bigcup_{m \geq 1} \bigcap_{n > m} \bigcup_{k=0}^{[2^n t] - 1} \{k/2^n \leq T < (k+1)/2^n\} ,$$

where $[a]$ represents the integer part of a. Denote the set $\{k/2^n \leq T < (k+1)/2^n\}$ by $A_{n,k}(T)$. By the right continuity of η,

$$\{\eta(T) = \eta\} \cap \{T < t\} \;=\; \bigcup_{m \geq 1} \bigcap_{n > m} \bigcup_{k=0}^{[2^n t]-1} \big[A_{n,k}(T) \cap \{\eta((k+1)/2^n) = \eta\}\big] \,,$$

which belongs to $\mathcal{F}_t^\eta$. Therefore,

$$\{\eta(T) = \eta\} \cap \{T \leq t\} \;=\; \big[\{T = t\} \cap \{\eta(t) = \eta\}\big] \;\cup\; \big[\{\eta(T) = \eta\} \cap \{T < t\}\big]$$

belongs to $\mathcal{F}_t^\eta$, which concludes the proof of the lemma in view of (2.1). $\square$

We return to the framework of Markov chains.

bs13 | **Definition 1.9.** *A Markov chain $(\eta(t) : t \geq 0)$ defined on a space $(\Omega, (\mathcal{F}_t : t \geq 0), \{\mathbb{P}_\eta : \eta \in E\})$ with transition probability $p_t(\eta, \xi)$ is said to be a* strong Markov process *if for every $\eta, \xi \in E$, $s \geq 0$ and stopping time T (with respect to the filtration $\mathcal{F}_t$),*

$$\mathbb{P}_\eta[\eta(T + s) = \xi \,|\, \mathcal{F}_T] \;=\; p_s(\eta(T), \xi) \,,$$

on $\{T < \infty\}$, $\mathbb{P}_\eta$ almost surely.

bs11 | **Lemma 1.10.** *Let $(\eta(t) : t \geq 0)$ be a continuous-time Markov chain defined on a space $(\Omega, (\mathcal{F}_t : t \geq 0), \{\mathbb{P}_\eta : \eta \in E\})$ with transition probability $p_t(\eta, \xi)$, as introduced in Definition 1.3. Then, $\eta(t)$ is a strong Markov process.*

Proof. Fix $\eta, \xi \in E$ and $s \geq 0$. By Lemma 1.8, $p_s(\eta(T), \xi)$ is $\mathcal{F}_T$ measurable. It remains to show that for every set A in $\mathcal{F}_T$,

$$\mathbb{P}_\eta\big[\eta(T + s) = \xi \,, A \,, T < \infty\big] \;=\; \mathbb{E}_\eta\big[\mathbf{1}\{A \,, T < \infty\} \, p_s(\eta(T), \xi)\big] \,. \qquad (2.5) \qquad \boxed{\text{b49}}$$

Assume first that the stopping time T takes values on a countable set $\mathbb{T}$. In this case, the left hand side of (2.5) is equal to

$$\sum_{t \in \mathbb{T}} \mathbb{P}_\eta\big[\eta(t + s) = \xi \,, A \,, T = t\big] \,.$$

Since T is a stopping time, $A \cap \{T = t\}$ belongs to $\mathcal{F}_t$. By the Markov property (1.1), the previous expression is equal to

$$\sum_{t \in \mathbb{T}} \mathbb{E}_\eta\big[\mathbf{1}\{A \,, T = t\} \, p_s(\eta(t), \xi)\big] \;=\; \mathbb{E}_\eta\big[\mathbf{1}\{A \,, T < \infty\} \, p_s(\eta(T), \xi)\big] \,,$$

which proves (2.5).

Fix a stopping time T and define the function $T_n : \Omega \to [0, \infty]$ by $T_n = 0$ on the set $\{T = 0\}$, $T_n = \infty$ on $\{T = \infty\}$ and

$$T_n \;=\; \sum_{k \geq 0} \frac{k+1}{2^n} \mathbf{1}\{k/2^n < T \leq (k+1)/2^n\}$$

on the set $\{0 < T < \infty\}$. The reader will check that T_n is a stopping time which assumes countably many values, that $T \le T_n$ and that T_n decreases to T on the set $\{T < \infty\}$.

Since $T \le T_n$, by (2.4), $A \in \mathcal{F}_T \subset \mathcal{F}_{T_n}$. Therefore, by the first part of the proof,

$$\mathbb{P}_\eta\big[\eta(T_n + s) = \xi\,,\, A\,,\, T_n < \infty\big] \;=\; \mathbb{E}_\eta\big[\mathbf{1}\{A\,,\, T_n < \infty\}\, p_s(\eta(T_n), \xi)\big]\,.$$

Since $\{T_n < \infty\} = \{T < \infty\}$, and since T_n decreases to T as $n \uparrow \infty$, by the right continuity of $\eta(\cdot)$ the left hand side of the previous identity converges to the left hand side of (2.5) as $n \uparrow \infty$. By similar reasons and by the dominated convergence theorem, the right-hand side of the previous identity converges to the right-hand side of (2.5) as $n \uparrow \infty$. $\square$

3 Some Examples and Minimal Chains

secb04

We present in this section some examples of Markov chains which exhibit pathologies which one would not suspect may exist. In Example 1.13 and 1.14 the chain explodes in finite time, while in Example 1.17 it remains in each configuration a mean-0 exponential time.

While discrete-time Markov chains are described by their transition matrices, only elementary continuous-time Markov chains have transition probabilities which can be computed explicitly. This means that continuous-time Markov chains have to be characterized differently.

Proposition 1.12 states that transition probabilities $p_t(\eta, \xi)$ are differentiable at $t = 0$. Denote by $Q(\eta, \xi)$ its derivative and let $q(\eta) = -Q(\eta, \eta)$. It is clear that $q(\eta)$ and $Q(\eta, \xi)$, $\eta \ne \xi$, are non-negative. According to Proposition 1.12, $Q(\eta, \xi)$ is finite out of the diagonal, $q(\eta)$ belongs to $[0, +\infty]$, and $\sum_{\xi \in E} Q(\eta, \xi) \le 0$ if $q(\eta) < \infty$.

Examples 1.13 and 1.14 show that the matrix $Q(\eta, \xi)$ does not characterize the transition probability $p_t(\eta, \xi)$, but if we impose further conditions it does. This is the content of Proposition 1.15. To complete the picture, Proposition 1.16 asserts that given a transition probability p_t such that $q(\eta) < \infty$ for all configurations η, there always exists a Markov chain $\eta(t)$ whose transition probability is p_t. To show that some Markov chains do not fulfill the condition $q(\eta) < \infty$, we present in Example 1.17 a chain in which $q(\eta) = \infty$ for all configurations $\eta \in E$.

We start proving that the transition probability of the Markov chain defined by (1.5) is differentiable at $t = 0$.

bbs05
Lemma 1.11. *Fix a transition matrix $p(\eta, \xi)$ and a positive function $\lambda : E \to (0, \infty)$. Let $\eta(t)$ be the Markov chain introduced in Proposition 1.4. Denote its transition probability by $p_t(\eta, \xi)$. Then, $p_t(\eta, \xi)$ is differentiable at $t = 0$ and its derivative, denoted by $p_0'(\eta, \xi)$, is given by*

$$p_0'(\eta, \eta) \;=\; -\lambda(\eta)\left[1 - p(\eta, \eta)\right]\,, \quad p_0'(\eta, \xi) \;=\; \lambda(\eta)\, p(\eta, \xi)\,, \quad \eta \ne \xi \,\in\, E\,.$$

Proof. Fix two configurations η, $\xi \neq \eta \in E$. If $p(\eta, \eta) = 1$, by construction, under $\mathbb{P}_\eta$, $\eta(t) = \eta$ for all $t \geq 0$. In particular, $p_t(\eta, \eta) = 1$ and $p_t(\eta, \xi) = 0$ for all $t \geq 0$. The assertion of the lemma follows.

Assume that $p(\eta, \eta) < 1$, and recall from (1.12) that we represent by τ_1 the time of the first jump of $\eta(t)$. By (2.3), τ_1 is a stopping time. Hence, by the strong Markov property at time τ_1, $p_t(\eta, \xi) = \mathbb{P}_\eta[\eta(t) = \xi]$ is equal to

$$\mathbb{P}_\eta\big[\tau_1 > t \,,\, \eta(t) = \xi\big] \;+\; \mathbb{P}_\eta\big[\tau_1 \leq t \,,\, \eta(t) = \xi\big]$$
$$= \delta_{\eta,\xi}\,\mathbb{P}_\eta[\tau_1 > t] \;+\; \mathbb{E}_\eta\Big[\mathbf{1}\{\tau_1 \leq t\}\,p_{t-\tau_1}(\eta(\tau_1), \xi)\Big] \,.$$

By Exercise 1.30, $\mathbb{P}_\eta[\eta(\tau_1) = \eta] = 0$ and $\mathbb{P}_\eta[\eta(\tau_1) = \zeta] = p_\star(\eta, \zeta) := p(\eta, \zeta)/[1 - p(\eta, \eta)]$ for $\zeta \neq \eta$. Moreover, under $\mathbb{P}_\eta$, τ_1 is independent of $\eta(\tau_1)$ and is distributed as an exponential random variable of parameter $\lambda_\star(\eta) := \lambda(\eta)\,[1 - p(\eta, \eta)]$. Thus,

$$p_t(\eta, \xi) \;=\; \delta_{\eta,\xi}\,e^{-\lambda_\star(\eta)t} \;+\; \int_0^t \lambda_\star(\eta)\,e^{-\lambda_\star(\eta)s} \sum_{\zeta \in E} p_\star(\eta, \zeta)\,p_{t-s}(\zeta, \xi)\,ds \,. \quad (3.1) \quad \boxed{\text{bb04}}$$

It follows from this equation, from the fact that $p_r(\zeta, \xi) \leq 1$, $\sum_\zeta p_\star(\eta, \zeta) = 1$ and from the inequality $1 - e^{-x} \leq x$, $x \geq 0$, that

$$\frac{1}{t}\big\{1 - p_t(\eta, \eta)\big\} \;\leq\; \lambda_\star(\eta) \,, \quad \frac{1}{t}\,p_t(\eta, \xi) \;\leq\; \lambda_\star(\eta) \,, \quad \eta \neq \xi \in E \,. \quad (3.2) \quad \boxed{\text{bb05}}$$

To prove that $t^{-1}p_t(\eta, \xi)$, $\eta \neq \xi \in E$, converges to $\lambda(\eta)p(\eta, \xi) = \lambda_\star(\eta)p_\star(\eta, \xi)$ as $t \downarrow 0$, fix $\epsilon > 0$ and consider a finite subset A of E such that $\sum_{\zeta \in A^c} p_\star(\eta, \zeta) \leq \epsilon$. Assume that A contains ξ. Since $p_{t-s}(\zeta, \xi) \leq 1$, $1 - e^{-x} \leq x$, $x \geq 0$,

$$\frac{1}{t}\int_0^t \lambda_\star(\eta)\,e^{-\lambda_\star(\eta)s} \sum_{\zeta \in A^c} p_\star(\eta, \zeta)\,p_{t-s}(\zeta, \xi)\,ds \;\leq\; \epsilon\,\lambda_\star(\eta) \,.$$

On the other hand, by (3.2), for each fixed pair (ζ, ξ), $p_t(\zeta, \xi)$ is continuous at $t = 0$. There exists therefore t_0, such that

$$\max_{\zeta \in A \setminus \{\xi\}} \sup_{0 \leq t \leq t_0} p_t(\zeta, \xi) \;\leq\; \epsilon \,, \quad \sup_{0 \leq t \leq t_0} [1 - p_t(\xi, \xi)] \;\leq\; \epsilon \,.$$

Hence, for $t \leq t_0$,

$$\frac{1}{t}\int_0^t \lambda_\star(\eta)\,e^{-\lambda_\star(\eta)s} \sum_{\zeta \in A} p_\star(\eta, \zeta)\,p_{t-s}(\zeta, \xi)\,ds \;=\; p_\star(\eta, \xi)\,\frac{1 - e^{-\lambda_\star(\eta)t}}{t} + O(\epsilon) \,,$$

where $O(\epsilon)$ is a remainder whose absolute value is bounded by $2\lambda_\star(\eta)\epsilon$. This proves that $t^{-1}p_t(\eta, \xi)$, $\eta \neq \xi \in E$, converges to $\lambda_\star(\eta)p_\star(\eta, \xi)$ as $t \downarrow 0$. The proof that $t^{-1}[1 - p_t(\eta, \eta)]$, converges to $\lambda_\star(\eta)$ as $t \downarrow 0$ is similar. $\square$

Actually, all transition probabilities are differentiable at $t = 0$ and not only the one resulting from the construction presented in Proposition 1.4.

Proposition 1.12. *Let $p_t(\eta, \xi)$ be a transition probability. Then,*

$$Q(\eta, \xi) := \frac{d}{dt}p_t(\eta, \xi)\Big|_{t=0} = p_0'(\eta, \xi)$$

exists for all η, $\xi \in E$. Moreover, $0 \leq Q(\eta, \xi) < \infty$ for $\eta \neq \xi$ and $q(\eta) = -Q(\eta, \eta) \in [0, \infty]$. Finally, $\sum_{\xi \in E} Q(\eta, \xi) \leq 0$ if $q(\eta) < \infty$.

The proof of this proposition can be found in Section II.2 of Chung [1967] or in Section 5.2 of Freedman [1971]. The matrix Q is called the *Q-matrix* associated to the transition probability $p_t(\eta, \xi)$. By Exercise 1.31, the parameter $q(\eta)$ represents the holding rate at η.

A configuration η is called *stable* if $q(\eta) < \infty$ and *instantaneous* if $q(\eta) = \infty$. If $q(\eta) = 0$, the configuration η is said to be *absorbing*. By equation (12) of Section 5.2 in Freedman [1971], a configuration η is absorbing if and only if $p_t(\eta, \eta) = 1$ for all $t \geq 0$.

The next two examples show that the matrix Q does not determine the transition probability $p_t(\eta, \xi)$ and that we may have $\sum_{\xi \in E} Q(\eta, \xi) < 0$ for a stable configuration η. Example 1.17 illustrates the fact that all configurations of a Markov chain may be instantaneous.

Example 1.13 A birth process. Let $E = \mathbb{N} \cup \{0\}$ and consider a function $r : E \to (0, \infty)$ such that

$$\sum_{k \in E} \frac{1}{r(k)} < \infty . \tag{3.3}$$

Let Ω be an abstract set and $\mathcal{F}$ a σ-algebra of subsets of Ω. Let $\mathfrak{e}(k) : \Omega \to \mathbb{R}$, $k \in E$, be measurable functions. Assume that there are probability measures $\mathbb{P}_\ell$, $\ell \in E$, on $(\Omega, \mathcal{F})$ which turn the random variables $\mathfrak{e}(k)$, $k \in E$, independent and under which $\mathfrak{e}(k) = 0$, $k < \ell$, $\mathbb{P}_\ell$-almost surely, and $\mathfrak{e}(k)$, $k \geq \ell$, have an exponential distribution of parameter $r(k)$. It is not difficult to construct a product space $(\Omega, \mathcal{F})$ which can carry these probability measures. Let $S_0 = 0$,

$$S_j = \sum_{i=0}^{j-1} \mathfrak{e}(i) , \quad j \geq 1 .$$

In view of (3.3), for all $k \in E$, $\mathfrak{X} := \lim_{\ell \to \infty} S_\ell < \infty$ $\mathbb{P}_k$-almost surely, since $\mathbb{E}_k[\mathfrak{X}] < \infty$, where $\mathbb{E}_k$ stands for the expectation with respect to $\mathbb{P}_k$. $\mathfrak{X}$ is called the *explosion* time of the chain. According to Exercise 1.33, it is a stopping time with respect to the natural filtration.

Let $E_{\mathfrak{d}} = E \cup \{\mathfrak{d}\}$ be the one-point compactification of E with respect to the discrete topology. Define the $E_{\mathfrak{d}}$-valued random variables $\{\eta(t) : t \geq 0\}$ by

$$\eta(t) \;=\; \begin{cases} j & \text{if } S_j \leq t < S_{j+1} \text{ for some } j \geq 0 \ , \\ \mathfrak{d} & \text{if } t \geq \mathfrak{X} \ . \end{cases} \qquad (3.4) \quad \boxed{\texttt{b23}}$$

Fix $\ell \in E$. Under $\mathbb{P}_\ell$, $S_j = 0$ for $0 \leq j \leq \ell$ and $S_{\ell+1} = \mathfrak{e}(\ell)$ is an exponential random variable of parameter $r(\ell)$. Hence, $\eta(t) = \ell$ for $0 \leq t < S_{\ell+1}$.

Let $p_t(j,k) = \mathbb{P}_j[\eta(t) = k]$, $j, k \in E$, $t \geq 0$. By Exercise 1.34, $\mathbb{P}_j[\mathfrak{X} \leq t] > 0$ for all $j \in E$, $t > 0$. Hence, $\sum_{k \geq 0} p_t(j,k) = \mathbb{P}_j[\mathfrak{X} > t] < 1$ and p_t is not a transition probability, but a substochastic transition probability. It is also easy to show that $p_t(j,k)$ is differentiable at $t = 0$ and that its derivative Q satisfies $-Q(k,k) = r(k)$, $Q(k, k+1) = r(k)$, $k \geq 0$, $Q(k,j) = 0$ otherwise.

The reader can check, repeating the arguments presented in the proof of Proposition 1.4, that $\eta(t)$ satisfies the Markov property with respect to the natural filtration and the substochastic transition probability p_t if we add on both sides of (1.1) the condition that the process did not explode by time t: On the set $\{\mathfrak{X} > t\}$, for every $s \geq 0$ and $\xi \in E$,

$$\mathbb{P}_\eta[\,\eta(t+s) = \xi \,|\, \mathcal{F}_t^\eta\,] \;=\; p_s(\eta(t), \xi) \ . \qquad (3.5) \quad \boxed{\texttt{bb14}}$$

Since p_t is not a transition probability, $\eta(t)$ is not an E-valued Markov chain, but it can be turned into a Markov chain by adding the point $\mathfrak{d}$ to the configuration space.

Instead of defining $\eta(t)$ as $\mathfrak{d}$ after the explosion time $\mathfrak{X}$, we could have restarted the process afresh from a fixed point j_0, or from a point j chosen according to some probability measure μ on E. Repeating this procedure each time the process explodes, we construct an E-valued Markov chain defined on the entire line $\mathbb{R}_+$. The derivative at time 0 of the transition probability of this Markov chain is not affected by the rule which dictates the behavior after the explosion. In particular, the matrix Q does not depend on the rule and does not determine the transition probability since each rule gives rise to a different Markov chain and a different transition probability. $\triangle$

Minimal chain. Let E be a denumerable set and let $Q_0(\eta, \xi)$ be a real-valued matrix on E such that

$$Q_0(\eta, \xi) \;\geq\; 0 \text{ for } \eta \neq \xi \ , \quad q_0(\eta) = -Q_0(\eta, \eta) \geq 0 \ ,$$
$$\sum_{\xi \in E} Q_0(\eta, \xi) \;=\; 0 \text{ for all } \eta \in E \ . \qquad (3.6) \quad \boxed{\texttt{b13}}$$

Note that $q_0(\eta) < \infty$ since we are assuming the matrix $Q_0(\eta, \xi)$ to be real-valued. Let $\lambda(\eta)$, $p(\eta, \xi)$ be the holding rates, transition matrix given by

$$\lambda(\eta) \;=\; q_0(\eta) \ , \quad p(\eta, \xi) \;=\; \begin{cases} Q_0(\eta, \xi)/q_0(\eta) & \text{if } q_0(\eta) > 0 \text{ and } \xi \neq \eta, \\ 0 & \text{otherwise} \ . \end{cases} \qquad (3.7) \quad \boxed{\texttt{bb02}}$$

If the set E is infinite, let $E_\mathfrak{d} = E \cup \{\mathfrak{d}\}$ be the one-point compactification of E with respect to the discrete topology. The construction presented after (1.2)

togheter with the ones of the previous example permit to define a transition probability $\bar{p}_t(\eta, \xi)$ on $E_{\mathfrak{d}}$ and a $E_{\mathfrak{d}}$-valued Markov chain $(\eta(t) : t \geq 0)$, defined by (3.4), on a Markov space $(\Omega, (\mathcal{F}_t^\eta : t \geq 0), \{\mathbb{P}_\eta : \eta \in E_{\mathfrak{d}}\})$.

This Markov chain is called the *minimal* Markov chain. The stopping time $\mathfrak{X}$ is called the *explosion* time of the mininal chain. By restricting the transition probability $\bar{p}_t$ to E we obtain a substochastic transition probability p_t:

$$p_t(\eta, \xi) \ = \ \bar{p}_t(\eta, \xi) \,, \quad \eta \,, \xi \in E \,,$$

called the *minimal* substochastic transition probability. This minimal substochastic transition probability is a transition probability if and only if $\mathfrak{X} = \infty$ $\mathbb{P}_\eta$-almost surely for all $\eta \in E$. Moreover, the derivative at time 0 of the minimal substochastic transition probability p_t is equal to Q_0.

If E is a finite set, the explosion time is infinite $\mathbb{P}_\eta$-almost surely for all $\eta \in E$, and the construction presented after (1.2) provides a transition probability p_t and an E-valued Markov chain defined for all $t > 0$. In this case also the Q-matrix of p_t is equal to Q_0.

Here is another example to illustrate the fact that the matrix Q does not characterize the transition probabilities $p_t(\eta, \xi)$.

$\boxed{\text{bs22}}$ **Example 1.14** Let E be the set of nonnegative rationals, $E = \mathbb{Q}_+$ endowed with the discrete topology and the usual order. Denote by $E_{\mathfrak{d}} = E \cup \{\mathfrak{d}\}$ the one-point compactification of E. Recall that a function $f : [0, \infty) \to E_{\mathfrak{d}}$ equal to $\mathfrak{d}$ at t, $f(t) = \mathfrak{d}$, is right-continuous at t if for every finite subset A of E, there exists $\delta > 0$ such that $f(s) \notin A$ for $t < s < t + \delta$. This observation extends to left-limits.

Consider a function $r : E \to (0, \infty)$ such that for all $\xi \in E$,

$$\sum_{\eta < \xi} \frac{1}{r(\eta)} \ < \ \infty \,, \quad \sum_{\eta \in E} \frac{1}{r(\eta)} \ = \ \infty \,, \qquad (3.8) \quad \boxed{\text{b08}}$$

where the first sum is carried over all points $\eta \in E$ which are smaller than ξ. Let $\mathfrak{e}(\eta)$, $\eta \in E$, be real-valued measurable functions defined on some space $(\Omega, \mathcal{F})$. Assume that there are probability measures $\mathbb{P}_\eta$, $\eta \in E$, on $(\Omega, \mathcal{F})$ which turn the random variables $\mathfrak{e}(\xi)$, $\xi \in E$, independent and under which $\mathfrak{e}(\xi)$, $\xi < \eta$, are equal to 0 $\mathbb{P}_\eta$-almost surely, and $\mathfrak{e}(\xi)$, $\xi \geq \eta$, have an exponential distribution of parameter $r(\xi)$. It is not difficult to construct a product space $(\Omega, \mathcal{F})$ which can carry these probability measures. Let

$$\mathfrak{E}(\eta) \ = \ \sum_{\zeta < \eta} \mathfrak{e}(\zeta) \,.$$

By (3.8), for all $\xi \in E$, $\mathbb{P}_\xi$-almost surely, $\mathfrak{E}(\eta) < \infty$ for all $\eta \in E$, and $\lim_{\eta \to \infty} \mathfrak{E}(\eta) = \infty$.

Define the random variables $(\eta(t) : t \geq 0)$ by

$$\eta(t) \;=\; \begin{cases} \eta & \text{if } \mathfrak{E}(\eta) \leq t < \mathfrak{E}(\eta) + \mathfrak{e}(\eta) \text{ for some } \eta \in E, \\ \mathfrak{d} & \text{if } t \notin \bigcup_{\eta \in E}[\mathfrak{E}(\eta), \mathfrak{E}(\eta) + \mathfrak{e}(\eta)) \,. \end{cases}$$

Lemma 6.19 in Freedman [1971] asserts that for every $t \geq 0$, $\xi \in E$, $\mathbb{P}_\xi[\eta(t) \in E] = 1$. The reader will verify that $\mathbb{P}_\xi$-almost surely the trajectories $\eta(t)$ are right-continuous, have left-limits, and start from ξ, $\eta(0) = \xi$.

For $\eta, \xi \in E$, let

$$p_t(\xi, \zeta) \;=\; \mathbb{P}_\xi[\eta(t) = \zeta] \,.$$

The proof of Proposition 1.4 shows that p_t is a transition probability and that $(\eta(t) : t \geq 0)$ is an E-valued Markov chain on the Markov space $(\Omega, (\mathcal{F}_t^\eta : t \geq 0), \{\mathbb{P}_\eta : \eta \in E\})$ with transition probability p_t, where $\mathcal{F}_t^\eta$ is the natural filtration.

Fix ξ, ζ in E. By construction, $p_t(\xi, \xi) = \mathbb{P}_\xi[\mathfrak{e}(\xi) > t]$ so that $q(\xi) = -p_0'(\xi, \xi) = r(\xi)$. On the other hand, since $p_t(\xi, \zeta) = \mathbb{P}_\xi[\eta(t) = \zeta]$, $p_t(\xi, \zeta) = 0$ for $\zeta < \xi$ and $p_t(\xi, \zeta) \leq \mathbb{P}_\xi[\mathfrak{e}(\xi) + \mathfrak{e}(\xi') \leq t]$ provided $\xi < \xi' < \zeta$. This proves that $Q(\xi, \zeta) = p_0'(\xi, \zeta) = 0$ for $\xi \neq \zeta$ and provides an example of a Markov chain for which $0 < q(\eta) < \infty$, $Q(\eta, \xi) = 0$ for all points $\eta \neq \xi$. In particular, $\sum_\xi Q(\eta, \xi) < 0$ for all η.

Moreover, if we interchange the position in $\mathbb{R}$ of two points η, $\xi \in E$ and define a new Markov chain accordingly, the derivative at time 0 of the transition probability of this new chain is equal to Q. $\triangle$

We have seen in the previous examples that the Q-matrix does not characterize the transition probability. The next proposition provides sufficient conditions for a substochastic transition probability p_t to be the unique transition probability associated to a given Q-matrix.

For a bounded function $f : E \to \mathbb{R}$ and a matrix Q satisfying (3.6), we denote by $Qf : E \to \mathbb{R}$ the function defined by

$$(Qf)(\eta) \;=\; \sum_{\xi \in E} Q(\eta, \xi)\, f(\xi) \,.$$

Note that the sum is well defined because f is bounded and the absolute value of $Q(\eta, \cdot)$ is summable.

| bs24 | **Proposition 1.15.** *Let Q_0 be a matrix satisfying (3.6). The following statements are equivalent:*

(a) *There exists $\lambda > 0$ for which the equation $(\lambda - Q_0)f = 0$ has only one bounded solution;*
(b) *For all $\lambda > 0$, the equation $(\lambda - Q_0)f = 0$ has only one bounded solution;*
(c) *The minimal substochastic transition probability constructed from Q_0 is a transition probability;*
(d) *The minimal Markov chain constructed from Q_0 is non-explosive;*
(e) *There is at most one substochastic transition probability $p_t(\eta, \xi)$ whose Q-matrix is Q_0.*

Parts of the proof of this Proposition can be found Sections II.18 and II.19 of Chung [1967], in the proof of Theorem 7.51 of Freedman [1971], and in Sections 2.7 and 2.8 of Norris [1998].

The last result of this section states that given a transition probability $p_t(\eta, \xi)$ there always exists a Markov chain $\eta(t)$ whose transition probability is p_t provided all points are stable, $q(\eta) = -p_0'(\eta, \eta) < \infty$ for all $\eta \in E$.

bs23 **Proposition 1.16.** *Let E be a countable state space endowed with the discrete topology and let $p_t(\eta, \xi)$ be a transition probability on E such that $q(\eta) = -p_0'(\eta, \eta) < \infty$ for all $\eta \in E$. There exist a Markov space $(\Omega, (\mathcal{F}_t : t \geq 0), \{\mathbb{P}_\eta : \eta \in E\})$ and a collection $(\eta(t) : t \geq 0)$ of E-valued random variables which is a Markov chain in $(\Omega, (\mathcal{F}_t^\eta : t \geq 0), \{\mathbb{P}_\eta : \eta \in E\})$ with transition probability $p_t(\eta, \xi)$.*

This is Theorem 7.12 in Freedman [1971]. The reader will find in Section II.7 of Chung [1967] and in Chapter 9 of Freedman [1971] a more general version of this statement.

In all previous examples and statements, we assumed all configurations to be stable. We conclude this section with an example where all configurations are instantaneous.

bs25 **Example 1.17** Fix λ, $\mu > 0$, and consider the continuous-time Markov chain on $\{0, 1\}$ such that $\lambda(0) = \lambda$, $\lambda(1) = \mu$, $p(0, 1) = p(1, 0) = 1$. By diagonalizing the matrix

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

the reader can show that the transition probability $p_t(\eta, \xi)$ of this Markov chain is given by

$$p_t(0, 0) = \frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-t(\mu + \lambda)}, \quad p_t(0, 1) = 1 - p_t(0, 0),$$

$$p_t(1, 1) = \frac{\lambda}{\mu + \lambda} + \frac{\mu}{\mu + \lambda} e^{-t(\mu + \lambda)}, \quad p_t(1, 0) = 1 - p_t(1, 1).$$

(3.9) b10

Consider infinitely many independent copies of this chain. Denote by η_i, $i \geq 1$, the i-th coordinate of a sequence $\eta \in \{0, 1\}^{\mathbb{N}}$. Let

$$E = \Big\{ \eta \in \{0, 1\}^{\mathbb{N}} : \sum_{i \geq 1} \eta_i < \infty \Big\}.$$

For $\eta \in E$, denote by $N(\eta)$ the first coordinate which vanishes as well as all the successive ones: $N(\eta) = \min\{k \geq 1 : \eta_j = 0 \text{ for all } j \geq k\}$. For example $N(\underline{0}) = 1$ if $\underline{0}$ represents the configuration with all coordinates equal to 0.

Consider two sequence of non-negative numbers $\{\mu_k : k \geq 1\}$ and $\{\lambda_k : k \geq 1\}$ such that $\mu_k + \lambda_k > 0$ and

$$\prod_{k\geq 1} \frac{\mu_k}{\mu_k + \lambda_k} > 0 \, . \tag{3.10}$$ `b09`

Since $1 - x \leq e^{-x}$,

$$0 < \prod_{k\geq 1} \frac{\mu_k}{\mu_k + \lambda_k} = \prod_{k\geq 1} \Big(1 - \frac{\lambda_k}{\mu_k + \lambda_k}\Big) \leq \exp\Big\{ - \sum_{k\geq 1} \frac{\lambda_k}{\mu_k + \lambda_k} \Big\} \, ,$$

so that

$$\sum_{k\geq 1} \frac{\lambda_k}{\mu_k + \lambda_k} < \infty \, .$$

On a space $(\Omega, \mathcal{F})$, consider independent $\{0,1\}$-valued Markov chains $\{\zeta^k(t) : t \geq 0\}$, $k \geq 1$, which jump from 0 (resp. 1) to 1 (resp. 0) at rate λ_k (resp. μ_k). Define the continuous-time, E-valued process $\eta(t)$ by $\eta(t) = (\zeta^1(t), \zeta^2(t), \dots)$. For a configuration $\eta \in E$, $\eta = (\eta_1, \eta_2, \dots)$, denote by $\mathbb{P}_\eta$ the probability measure on $(\Omega, \mathcal{F})$ under which the process $\zeta^k(t)$ starts from η_k.

Fix $\eta \in E$. We claim that $\mathbb{P}_\eta[\eta(t) \in E] = 1$ for all $t \geq 0$. Indeed, if we denote by $p_t^{(k)}(i,j)$ the transition probability of the process $\zeta^k(t)$, in view of (3.9), for $k \geq N(\eta)$,

$$\mathbb{P}_\eta[\zeta^k(t) = 1] = p_t^{(k)}(0,1) \leq \frac{\lambda_k}{\mu_k + \lambda_k} \, .$$

Since the sequence $\lambda_k/(\mu_k + \lambda_k)$ is summable by Borel-Cantelli lemma,

$$\mathbb{P}_\eta[\eta(t) \notin E] = \mathbb{P}_\eta[\zeta^k(t) = 1 \text{ i.o.}] = 0 \, ,$$

which proves the claim.

For $\eta, \xi \in E$, $t \geq 0$, let

$$\mathbf{p}_t(\eta, \xi) = \prod_{k\geq 1} p_t^{(k)}(\eta_k, \xi_k) \, . \tag{3.11}$$ `b24`

The set of functions $\mathbf{p}_t \colon E \times E \to \mathbb{R}$ is a transition probability. The proof of this assertion is divided in several steps. It is clear that $\mathbf{p}_0(\eta, \xi) = \delta_{\eta, \xi}$ because the transition probabilities $p_t^{(k)}$ satisfy this identity. On the other hand, for all $t \geq 0$, $\mathbf{p}_t(\eta, \xi) \geq 0$, $\eta, \xi \in E$, and

$$\sum_{\zeta \in E} \mathbf{p}_t(\eta, \zeta) = \sum_{\zeta \in E} \prod_{k\geq 1} p_t^{(k)}(\eta_k, \zeta_k) = \prod_{k\geq 1} \sum_{\zeta_k = 0}^{1} p_t^{(k)}(\eta_k, \zeta_k) = 1 \, .$$

It remains to check conditions (c) and (d) of Definition 1.1. We claim that for all $\ell \geq 1$, $\xi^1, \dots, \xi^\ell$ in E, and $0 < t_1 < \cdots < t_\ell$,

$$\mathbb{P}_\eta\big[\eta(t_1) = \xi^1 \, , \, \dots \, , \, \eta(t_\ell) = \xi^\ell\big] = \prod_{j=0}^{\ell-1} \mathbf{p}_{t_{j+1} - t_j}(\xi^j, \xi^{j+1}) \, , \tag{3.12}$$ `b11`

provided $t_0 = 0$, $\xi^0 = \eta$. Condition (c) follows from this identity with $\ell = 1$, 2.

To prove (3.12), fix $\ell \geq 1$, configurations $\xi^1, \ldots, \xi^\ell$ in E, and $0 < t_1 < \cdots < t_\ell$. By definition of $\eta(t)$, for any $\eta \in E$,

$$\mathbb{P}_\eta\big[\eta(t_1) = \xi^1, \ldots, \eta(t_\ell) = \xi^\ell\big] = \prod_{k \geq 1} \mathbb{P}_\eta\big[\zeta^k(t_1) = \xi_k^1, \ldots, \zeta^k(t_\ell) = \xi_k^\ell\big].$$

Expressing the previous probabilities in terms of the transition probability $p_t^{(k)}$ of the Markov chain $\zeta^k(t)$, the right-hand side becomes

$$\prod_{k \geq 1} \prod_{j=0}^{\ell-1} p_{t_{j+1}-t_j}^{(k)}(\xi_k^j, \xi_k^{j+1}) = \prod_{j=0}^{\ell-1} \prod_{k \geq 1} p_{t_{j+1}-t_j}^{(k)}(\xi_k^j, \xi_k^{j+1}),$$

which proves (3.12)

To prove condition (d) of Definition 1.1, we have to show that for every $\eta \in E$,

$$\lim_{t \to 0} \mathbf{p}_t(\eta, \eta) = 1.$$

By definition of $\mathbf{p}_t(\eta, \eta)$,

$$\mathbf{p}_t(\eta, \eta) = \prod_{k \geq 1} \mathbb{P}_\eta[\zeta^k(t) = \eta_k].$$

for every $M \geq N(\eta)$, the right-hand side is equal to

$$\prod_{k=1}^{M-1} \mathbb{P}_\eta[\zeta^k(t) = \eta_k] \prod_{k \geq M} p_t^{(k)}(0,0) \geq \prod_{k=1}^{M-1} \mathbb{P}_\eta[\zeta^k(t) = \eta_k] \prod_{k \geq M} \frac{\mu_k}{\mu_k + \lambda_k},$$

where the inequality follows from (3.9). By (3.10), we may choose M large enough for the second product to be close to 1. Once this has been done, using the fact that $p_t^{(k)}$ is a transition probability for each fixed k, we may choose t small enough for the first product to be close to 1. This proves that $\mathbf{p}_t$ is a transition probability.

Assume that

$$\sum_{k \geq 1} \lambda_k = \infty.$$

Under this further condition all states are instantaneous: $-\mathbf{q}(\eta) := \mathbf{p}_0'(\eta, \eta) = \infty$. Indeed, let $\mathbb{D}(t)$, $t > 0$, the set of all non-negative dyadics less than or equal to t: $\mathbb{D}(t) = \{k/2^N : N \geq 1, k \geq 0, k/2^N \leq t\}$. We first claim that for $t > 0$ and $\eta \in E$,

$$\mathbb{P}_\eta\big[\eta(r) = \eta \text{ for all } r \in \mathbb{D}(t)\big] = 0. \tag{3.13}$$

This is easy. By definition of the process $\eta(t)$,

$$\mathbb{P}_\eta\big[\eta(r) = \eta \text{ for all } r \in \mathbb{D}(t)\big] \leq \mathbb{P}_\eta\big[\zeta^k(r) = 0 \text{ for all } r \in \mathbb{D}(t), k \geq N(\eta)\big].$$

By the independence and by definition of the process $\zeta^k(t)$ this last probability is equal to

$$\prod_{k \geq N(\eta)} e^{-\lambda_k t} \;=\; \exp\big\{-\sum_{k \geq N(\eta)} \lambda_k\, t\big\} \;=\; 0 \,,$$

which proves (3.13).

We further claim that

$$\mathbb{P}_\eta\big[\eta(r) = \eta \text{ for all } r \in \mathbb{D}(t)\,\big] \;=\; e^{-\,\mathbf{q}(\eta)t} \,, \tag{3.14}$$

<div style="text-align:right">`bb13`</div>

where $-\,\mathbf{q}(\eta)$ is the derivative at $t = 0$ of the transition probability $\mathbf{p}_t$ defined by (3.11). By the Markov property, for $N \geq 1$,

$$\mathbb{P}_\eta\big[\eta(k/2^N) = \eta \text{ for } 0 \leq k \leq [t2^N]\,\big] \;=\; \mathbf{p}_{1/2^N}(\eta,\eta)^{[t2^N]} \,.$$

By definition of $\mathbf{q}(\eta)$, the right-hand side converges to $e^{-\mathbf{q}(\eta)t}$ as $N \uparrow \infty$, while the left-hand side converges to the left-hand side of (3.14). By (3.13) and (3.14), all states $\eta \in E$ are instantaneous, as $\mathbf{q}(\eta) = \infty$ for all $\eta \in E$.

We just proved that $\eta(t)$ is not right-continuous at $t = 0$ for the discrete topology. Indeed, the event $\{\eta(t) \text{ is right-continuous at } t = 0\}$ can be represented as

$$\bigcup_{k \geq 1} \big\{\eta(t) = \eta(0) \text{ for all } 0 \leq t < k^{-1}\big\} \,,$$

and we proved that each of these sets has $\mathbb{P}_\eta$-measure zero. In particular, if we want to turn $\eta(t)$ into a Markov chain we need to change the topology. The product topology is the right choice. $\triangle$

Conclusion: We showed in this section that Markov chains may exhibit several different pathologies. To avoid such degeneracies, from now on, we concentrate our attention on minimal chains associated to Q-matrices satisfying the conditions (3.6). More precisely, we shall fix a Q-matrix R satisfying the hypotheses (3.6) and consider the minimal chain associated to R. Proposition 1.15 provides conditions under which the minimal chain is the unique Markov chain whose Q-matrix is R. In particular, in this case we are not losing generality by considering the mininal chain.

4 Canonical Version

`secb05`

Denote by $D([0,\infty),E)$ the set of right-continuous trajectories $x : [0,\infty) \to E$ with left-limits, endowed with the Skorohod topology which turns the space $D([0,\infty),E)$ complete and separable. We refer to Billingsley [1999] for all assertions presented without proofs in this section. Denote by $\mathcal{D}$ the Borel σ-algebra of subsets of $D([0,\infty),E)$.

Let $\mathbf{X}_t : D([0,\infty),E) \to E$, $t \geq 0$, be the evaluation of the trajectory at time t, $\mathbf{X}_t(x) = x(t)$, and denote by $\mathcal{F}_t \subset \mathcal{D}$, $t \geq 0$, the smallest σ-algebra which turns the maps $\mathbf{X}_s$, $0 \leq s \leq t$, measurable.

In Section 1, we constructed a family $(\eta(t) : t \geq 0)$ of E-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If we denote by ω the elements of Ω and by $\eta(t, \omega)$ the value of the random variable $\eta(t)$ at ω, in view of (1.5), for $\mathbb{P}$-almost all ω, the map $t \to \eta(t, \omega)$ is an element of $D([0, \infty), E)$. In particular, the function $\boldsymbol{\eta} : \Omega \to D([0, \infty), E)$, defined by

$$\boldsymbol{\eta}(\omega)(t) \;=\; \eta(t, \omega)\,, \quad t \geq 0\,,$$

is well defined. By [Billingsley, 1999, Theorem 16.6], the map $\boldsymbol{\eta} : (\Omega, \mathcal{F}) \to (D([0, \infty), E), \mathcal{D})$ is measurable. Let $\mathbf{P}_\eta$, $\eta \in E$, be the probability measure on $(D([0, \infty), E), \mathcal{D})$ defined by

$$\mathbf{P}_\eta \;:=\; \mathbb{P}_\eta \circ \boldsymbol{\eta}^{-1}\,.$$

Expectation with respect to $\mathbf{P}_\eta$ is represented by $\mathbf{E}_\eta$. The process $(\mathbf{X}_t : t \geq 0)$ defined on $(D, (\mathcal{F}_t : t \geq 0), \{\mathbf{P}_\eta : \eta \in E\})$ is called the *canonical version* of the Markov chain.

Denote by $(\vartheta(t) : t \geq 0)$ the time shift operators on $D(\mathbb{R}_+, E)$, $\vartheta(t) : D(\mathbb{R}_+, E) \to D(\mathbb{R}_+, E)$, $[\vartheta(t)\, x](s) = x(t + s)$, $x \in D(\mathbb{R}_+, E)$, $s, t \geq 0$. The Markov property can be written as

$$\mathbf{P}_\eta\big[\mathbf{X}_s \circ \vartheta_t = \xi \,\big|\, \mathcal{F}_t\big] \;=\; p_s(\mathbf{X}_t, \xi)\,.$$

For a probability measure μ on E, denote by $\mathbf{P}_\mu$ the measure on $D(\mathbb{R}_+, E)$ defined by

$$\mathbf{P}_\mu \;=\; \sum_{\eta \in E} \mu(\eta)\, \mathbf{P}_\eta\,.$$

Expectation with respect to $\mathbf{P}_\mu$ is represented by $\mathbf{E}_\mu$

5 Recurrent Chains

nsec04

Let R be a Q-matrix satisfying the conditions (3.6), and let $\eta(t)$ be the minimal Markov chain whose Q-matrix is R. All statements of this section refer to this chain even if we do not say it explicitly.

A minimal Markov chain $\eta(t)$ is irreducible if

$$\mathbf{P}_\eta[\eta(t) = \xi] \;>\; 0 \tag{5.1}$$

n08

for all $\eta, \xi \in E$, $t > 0$. An irreducible minimal Markov chain $\eta(t)$ is recurrent if

$$\mathbf{P}_\eta[H_\eta^+ < \infty] \;=\; 1 \tag{5.2}$$

n29

for all $\eta \in E$. We prove below in Lemma 1.18 that if (5.2) holds for one configuration, then it holds for all configurations.

ns10

Lemma 1.18. *Suppose that the chain is irreducible and that* (5.2) *holds for some configuration* $\eta \in E$. *Then,* (5.2) *holds for all configurations* $\xi \in E$.

Proof. Fix a configuration $\eta \in E$, assume that the chain starts from η and denote by H_j, $j \geq 1$, the times of the successive visits to η: $H_0 = 0$, $H_1 = H_\eta^+$ and $H_{j+1} = H_j + H_\eta^+ \circ \vartheta(H_j)$, $j \geq 1$. By (5.2), $\mathbf{P}_\eta[H_\eta^+ < \infty] = 1$, and by (5.1), $\mathbf{P}_\eta[H_\xi < H_\eta^+] > 0$. Since

$$\{H_{j+1} < H_\xi\} \;=\; \{H_j < H_\xi\} \cap \left\{ \{H_\eta^+ < H_\xi\} \circ \vartheta(H_j) \right\}, \qquad (5.3) \quad \boxed{\text{n22}}$$

taking conditional expectation with respect to $\mathcal{F}_{H_j}^\eta$, we obtain that

$$\mathbf{P}_\eta[H_{j+1} < H_\xi] \;=\; \mathbf{P}_\eta[H_j < H_\xi]\,\mathbf{P}_\eta[H_\eta^+ < H_\xi],$$

insomuch that

$$\mathbf{P}_\eta[H_j < H_\xi] \;=\; \mathbf{P}_\eta[H_\eta^+ < H_\xi]^j, \quad j \geq 0. \qquad (5.4) \quad \boxed{\text{n23}}$$

Hence, since $H_j \to \infty$ $\mathbf{P}_\eta$-almost surely and since $\mathbf{P}_\eta[H_\eta^+ < H_\xi] < 1$,

$$\mathbf{P}_\eta[H_\xi = \infty] \;=\; \lim_{j \to \infty} \mathbf{P}_\eta[H_j < H_\xi] \;=\; 0. \qquad (5.5) \quad \boxed{\text{n06}}$$

On the other hand, since $\mathbf{P}_\eta[H_\eta^+ < \infty] = 1$,

$$\mathbf{P}_\eta[H_\xi < H_\eta^+] \;=\; \mathbf{P}_\eta\big[H_\xi < H_\eta^+\,,\, H_\eta^+ < \infty\big].$$

On the set $\{H_\xi < H_\eta^+\}$, $H_\eta^+ = H_\eta \circ \vartheta_{H_\xi}$. Therefore, by the strong Markov property,

$$\mathbf{P}_\eta[H_\xi < H_\eta^+] \;=\; \mathbf{P}_\eta[H_\xi < H_\eta^+]\,\mathbf{P}_\xi[H_\eta < \infty].$$

Since $\mathbf{P}_\eta[H_\xi < H_\eta^+] > 0$, we conclude that $\mathbf{P}_\xi[H_\eta < \infty] = 1$.

Up to this point we proved that $\mathbf{P}_\eta[H_\xi < \infty] = \mathbf{P}_\xi[H_\eta < \infty] = 1$. Under $\mathbf{P}_\xi$

$$H_\xi^+ \;<\; H_\eta \,+\, H_\xi \circ \vartheta_{H_\eta}.$$

Therefore, by the strong Markov property,

$$\begin{aligned}
\mathbf{P}_\xi[H_\xi^+ < \infty] \;&\geq\; \mathbf{P}_\xi\big[H_\eta < \infty\,,\, H_\xi \circ \vartheta_{H_\eta} < \infty\big] \\
&=\; \mathbf{P}_\xi[H_\eta < \infty]\,\mathbf{P}_\eta[H_\xi < \infty] \;=\; 1,
\end{aligned}$$

as claimed. □

If we replace in Example 1.14 the set of non-negative rationals, $\mathbb{Q}_+$, by the set of rationals of the circle, $\mathbb{Q}_{|\mathbb{Z}}$, we obtain an example of irreducible, recurrent Markov chain whose Q-matrix R is such that $R(\eta, \xi) = 0$ for all $\eta \neq \xi$.

Let $\eta(t)$ be an irreducible recurrent chain. Assume that its Q-matrix, denoted by R, satisfies assumption (3.6). Denote by $\xi(t)$ the minimal chain constructed from the Q-matrix R and by $q_t(\eta, \xi)$ the transition probability of $\xi(t)$. The Q-matrix of ξ is R again.

This chain is non-explosive because the explosion time must be greater than the total time spent by the chain on one configuration which is infinite, being the sum of i.i.d. exponential random variables.

Recall from the construction of the Markov chain $\eta(t)$ presented in the first section of Chapter 1 that Y_n represents the discrete-time embedded Markov chain. Denote by $\mathbb{H}_A$, $\mathbb{H}_A^+$, $A \subset E$, the time the embedded chain Y_n hits, returns to the set A, respectively:

$$\mathbb{H}_A \;=\; \min\big\{k \geq 0 : Y_k \in A\big\}\,, \quad \mathbb{H}_A^+ \;=\; \min\big\{k \geq 1 : Y_k \in A\big\}\,. \qquad (5.6)$$ `n31`

`ns11` **Lemma 1.19.** *The continuous-time Markov chain $\eta(t)$ is irreducible if and only if the embedded chain Y_n is irreducible. In this case, the Markov chain $\eta(t)$ is recurrent if and only if the embedded chain Y_n is recurrent.*

Proof. Fix $t_0 > 0$ and two configurations $\eta \neq \xi \in E$. By construction of the chain $\eta(t)$, $\mathbf{P}_\eta[\eta(t_0) = \xi] > 0$ if and only if $\mathbf{P}_\eta[Y_n = \xi] > 0$ for some $n \geq 1$. This proves the first assertion of the lemma.

On the other hand, since under $\mathbf{P}_\eta$,

$$H_\eta^+ \;=\; \sum_{k=0}^{\mathbb{H}_\eta^+ - 1} \frac{\mathfrak{e}_k}{\lambda(Y_k)}\,,$$

$\mathbf{P}_\eta[H_\eta^+ < \infty] = 1$ if and only if $\mathbf{P}_\eta[\mathbb{H}_\eta^+ < \infty] = 1$. □

Denote by Π_η, $\eta \in E$, the measure on E defined by

$$\Pi_\eta(\xi) \;=\; \mathbf{E}_\eta\Big[\int_0^{H_\eta^+} \mathbf{1}\{\eta(s) = \xi\}\, ds \Big]\,, \quad \xi \in E\,. \qquad (5.7)$$

Π_η
$\mathbb{H}_A$
$\mathbb{H}_A^+$ `n30`

By construction, for any non-negative function $f : E \to \mathbb{R}$, any subset A of E, and any configuration $\eta \in A$, $\mathbf{P}_\eta$ almost surely,

$$\int_0^{\mathbb{H}_A^+} f(\eta(s))\, ds \;=\; \sum_{k=0}^{\mathbb{H}_A^+ - 1} \frac{\mathfrak{e}_k}{\lambda(Y_k)}\, f(Y_k)\,.$$

Since $\{\mathfrak{e}_j : j \geq 0\}$ is a sequence of mean-one random variables independent from the discrete-time Markov chain Y_j, by the previous observation, for all $\xi \in E$,

$$\Pi_\eta(\xi) \;=\; \mathbf{E}_\eta\Big[\sum_{k=0}^{\mathbb{H}_\eta^+ - 1} \frac{\mathfrak{e}_k}{\lambda(Y_k)}\, \mathbf{1}\{Y_k = \xi\} \Big] \;=\; \mathbf{E}_\eta\Big[\sum_{k=0}^{\mathbb{H}_\eta^+ - 1} \frac{1}{\lambda(Y_k)}\, \mathbf{1}\{Y_k = \xi\} \Big]\,.$$

Therefore,

$$\lambda(\xi)\, \Pi_\eta(\xi) \;=\; \mathbf{E}_\eta\Big[\sum_{k=0}^{\mathbb{H}_\eta^+ - 1} \mathbf{1}\{Y_k = \xi\} \Big]\,. \qquad (5.8)$$ `n25`

Denote by M_η, $\eta \in E$, the measure on E defined by

$$M_\eta(\eta) \;=\; \lambda(\eta)\, \Pi_\eta(\xi)\,, \quad \xi \in E\,. \tag{5.9}$$ `n32`

By construction,

$$M_\eta(\eta) \;=\; \lambda(\eta)\, \Pi_\eta(\eta) \;=\; 1\,. \tag{5.10}$$ `n27`

`bbs02` **Definition 1.20.** *A measure μ is invariant for the chain $\eta(t)$ if for all $\eta \in E$*

$$\sum_{\xi \in E} \mu(\xi)\, R(\xi, \eta) \;=\; \lambda(\eta)\, \mu(\eta)\,.$$

A measure μ is invariant for the Markov chain $\eta(t)$ if and only if the measure M, defined by $M(\eta) = \mu(\eta)\lambda(\eta)$, is invariant for the embedded chain Y_n:

$$\sum_{\xi \in E} M(\xi)\, p(\xi, \eta) \;=\; \sum_{\xi \in E} \mu(\xi)\, R(\xi, \eta) \;=\; \mu(\eta)\, \lambda(\eta) \;=\; M(\eta)\,. \tag{5.11}$$ `n28`

`ns07` **Lemma 1.21.** *Assume that the chain is recurrent. Each measure Π_η, $\eta \in E$, is invariant for the continuous-time Markov chain $\eta(t)$.*

Proof. Fix $\eta \in E$. In view of (5.9) and (5.11), we have to show that for every $\zeta \in E$,

$$\sum_{\xi \in E} M_\eta(\xi)\, p(\xi, \zeta) \;=\; M_\eta(\zeta)\,. \tag{5.12}$$ `n26`

On the one hand, by (5.8), the left hand side of this identity is equal to

$$\sum_{\xi \in E} \mathbf{E}_\eta \Big[\sum_{k=0}^{\mathbb{H}_\eta^+ - 1} \mathbf{1}\{Y_k = \xi\}\, p(\xi, \zeta) \Big] \;=\; \mathbf{E}_\eta \Big[\sum_{k=0}^{\mathbb{H}_\eta^+ - 1} p(Y_k, \zeta) \Big]\,.$$

On the other hand, as the chain is recurrent, by Lemma 1.19, $\mathbb{H}_\eta^+$ is finite $\mathbf{P}_\eta$-almost surely, so that $Y_0 = Y_{\mathbb{H}_\eta^+} = \eta$. Hence, by (5.8),

$$
\begin{aligned}
M_\eta(\zeta) &= \mathbf{E}_\eta \Big[\sum_{k=0}^{\mathbb{H}_\eta^+ - 1} \mathbf{1}\{Y_k = \zeta\} \Big] \;=\; \mathbf{E}_\eta \Big[\sum_{k=1}^{\mathbb{H}_\eta^+} \mathbf{1}\{Y_k = \zeta\} \Big] \\
&= \sum_{k \geq 1} \mathbf{E}_\eta \Big[\mathbf{1}\{\mathbb{H}_\eta^+ \geq k\}\, \mathbf{1}\{Y_k = \zeta\} \Big]\,.
\end{aligned}
$$

The event $\{\mathbb{H}_\eta^+ \geq k\}$ is measurable with respect to the σ-algebra $\mathcal{F}_{k-1}^Y = \sigma(Y_0, \ldots, Y_{k-1})$. In particular, by the Markov property for the discrete-time chain Y_n, the previous sum is equal to

$$\sum_{k \geq 1} \mathbf{E}_\eta \Big[\mathbf{1}\{\mathbb{H}_\eta^+ \geq k\}\, p(Y_{k-1}, \zeta) \Big]\,.$$

Performing the change of variables $k' = k - 1$, we obtain that the previous sum is equal to

$$\sum_{k\geq 0} \mathbf{E}_\eta\Big[\mathbf{1}\{\mathbb{H}_\eta^+ \geq k+1\}\, p(Y_k, \zeta)\Big] \;=\; \sum_{k\geq 0} \mathbf{E}_\eta\Big[\mathbf{1}\{\mathbb{H}_\eta^+ > k\}\, p(Y_k, \zeta)\Big] \,.$$

This is exactly the expression we obtained for the left hand side of (5.12) and concludes the proof of the lemma. $\square$

Next lemma states that the measure M_η is the minimal one among the invariant measures M for the chain Y_N such that $M(\eta) = 1$.

$\boxed{\text{ns08}}$ **Lemma 1.22.** *Let M be an invariant measure for the embedded discrete-time Markov chain Y_n such that $M(\eta) = 1$. Then, $M_\eta(\xi) \leq M(\xi)$ for all $\xi \in E$.*

Proof. By assumption and since $M(\eta) = 1$, for each $\xi \in E$,

$$M(\xi) \;=\; \sum_{\zeta \in E} M(\zeta)\, p(\zeta, \xi) \;=\; \sum_{\zeta \neq \eta} M(\zeta)\, p(\zeta, \xi) \;+\; p(\eta, \xi) \,.$$

Replacing $M(\zeta)$ by $\sum_{\zeta_2 \in E} M(\zeta_2) p(\zeta_2, \zeta)$ and separating the term $\zeta_2 = \eta$ from the others, we rewrite the previous sum as

$$\sum_{\zeta_1, \zeta_2 \neq \eta} M(\zeta_2)\, p(\zeta_2, \zeta_1)\, p(\zeta_1, \xi) \;+\; \sum_{\zeta_1 \neq \eta} p(\eta, \zeta_1)\, p(\zeta_1, \xi) \;+\; p(\eta, \xi) \,.$$

The sum of the second and third term can be written as

$$\mathbf{E}_\eta\Big[\mathbf{1}\{Y_1 = \xi\} + \mathbf{1}\{Y_2 = \xi\}\,\mathbf{1}\{\mathbb{H}_\eta^+ > 2\}\Big] \;=\; \sum_{k=0}^{1} \mathbf{E}_\eta\Big[\mathbf{1}\{Y_k = \xi\}\,\mathbf{1}\{\mathbb{H}_\eta^+ > k\}\Big] \,.$$

Iterating this procedure, one obtains that

$$M(\xi) \;\geq\; \sum_{k=0}^{n} \mathbf{E}_\eta\Big[\mathbf{1}\{Y_k = \xi\}\,\mathbf{1}\{\mathbb{H}_\eta^+ > k\}\Big]$$

for any $n \geq 1$. The inequality replaced the identity because we removed the sum which carries the measures M. Letting $n \uparrow \infty$ yields that

$$M(\xi) \;\geq\; \sum_{k \geq 0} \mathbf{E}_\eta\Big[\mathbf{1}\{Y_k = \xi\}\,\mathbf{1}\{\mathbb{H}_\eta^+ > k\}\Big] \;=\; \mathbf{E}_\eta\Big[\sum_{k=0}^{\mathbb{H}_\eta^+ - 1} \mathbf{1}\{Y_k = \xi\}\Big] \;=\; M_\eta(\xi)$$

as claimed. $\square$

$\boxed{\text{ns12}}$ **Corollary 1.23.** *Suppose that the Markov chain $\eta(t)$ is irreducible and recurrent. Then, two invariant measures for $\eta(t)$ may only differ by a multiplicative constant.*

Proof. In view of Lemma 1.19 and of (5.11), it is enough to prove this statement for irreducible and recurrent discrete-time Markov chains.

Let M be an invariant measure for the embedded chain Y_n which is not identically equal to 0. Fix a configuration η such that $M(\eta) \neq 0$. Multiplying M by $M(\eta)^{-1}$ we obtain a new invariant measure which differs from M by a scalar multiple and such that $M(\eta) = 1$. We may therefore assume without loss of generality that $M(\eta) = 1$ for some η.

Fix such a measure. By Lemma 1.22, $M_\eta \leq M$. Therefore, $M^\star = M - M_\eta$ is also an invariant measure, and $M^\star(\eta) = 0$. We claim that $M^\star(\xi) = 0$ for all $\xi \in E$. Indeed, fix a configuration ξ. As the chain Y_n is irreducible, there exists $n \geq 1$ such that $\mathbf{P}_\xi[Y_n = \eta] > 0$. Therefore, since $M^\star$ is an invariant measure,

$$M^\star(\eta) \;=\; \sum_{\zeta \in E} M^\star(\zeta)\,\mathbf{P}_\zeta[Y_n = \eta] \;\geq\; M^\star(\xi)\,\mathbf{P}_\xi[Y_n = \eta]\,.$$

This shows that $M^\star(\xi) = 0$ and concludes the proof of the lemma. $\square$

6 Positive-recurrent Chains

We assume in this section that the chain $\eta(t)$ is irreducible. An irreducible Markov chain is positive-recurrent if

$$\mathbf{E}_\eta[H_\eta^+] \;<\; \infty \tag{6.1}$$

for all $\eta \in E$. We prove below in Lemma 1.24 that if (6.1) holds for one configuration, then it holds for all configurations.

Lemma 1.24. *Suppose that the chain is irreducible and that* (6.1) *holds for some configuration* $\eta \in E$. *Then,* (6.1) *holds for all configurations* $\xi \in E$.

Proof. Fix a configuration $\xi \in E$ and recall the definition of the sequence of stopping times $\{H_j : j \geq 0\}$ introduced above. We first show by induction that

$$\mathbf{E}_\eta\big[H_j \mathbf{1}\{H_j < H_\xi\}\big] \;=\; j\,\mathbf{P}_\eta[H_\eta^+ < H_\xi]^{j-1}\,\mathbf{E}_\eta\big[H_1 \mathbf{1}\{H_1 < H_\xi\}\big] \tag{6.2}$$

for all $j \geq 1$. This equation holds trivially for $j = 1$. Assume that it holds for some $j \geq 1$. Since on the set $\{H_{j+1} < H_\xi\}$, $H_{j+1} = H_j + H_1 \circ \vartheta_{H_j}$, by the strong Markov property and by (5.3),

$$\begin{aligned}
\mathbf{E}_\eta\big[H_{j+1} \mathbf{1}\{H_{j+1} < H_\xi\}\big] \;=\;&\; \mathbf{E}_\eta\big[H_j \mathbf{1}\{H_j < H_\xi\}\big]\,\mathbf{P}_\eta[H_\eta^+ < H_\xi] \\
&+\; \mathbf{E}_\eta\big[H_1 \mathbf{1}\{H_1 < H_\xi\}\big]\,\mathbf{P}_\eta[H_j < H_\xi]\,.
\end{aligned}$$

By (5.4), the last term of the second line is equal to $\mathbf{P}_\eta[H_1 < H_\xi]^j$, while by the induction assumption the first expectation on the right-hand side is equal

to the right-hand side of (6.2). Summing the two lines we conclude the proof of (6.2).

We claim that

$$\mathbf{E}_\eta[H_\xi] \;<\; \infty \;. \tag{6.3}$$ $\boxed{\text{n20}}$

Indeed, by (5.5), $H_\xi < \infty$, $\mathbf{P}_\eta$-almost surely. Therefore, since $H_j \to \infty$, $\mathbf{P}_\eta$-almost surely,

$$\begin{aligned}
\mathbf{E}_\eta[H_\xi] \;&=\; \sum_{j\geq 0} \mathbf{E}_\eta\big[H_\xi\,\mathbf{1}\{H_j < H_\xi < H_{j+1}\}\big] \\
&\leq\; \sum_{j\geq 0} \mathbf{E}_\eta\big[H_{j+1}\,\mathbf{1}\{H_j < H_\xi < H_{j+1}\}\big] \;.
\end{aligned}$$

As above, write $H_{j+1} = H_j + H_1 \circ \vartheta_{H_j}$ and apply the strong Markov property to rewrite the previous sum as

$$\begin{aligned}
\sum_{j\geq 0} &\mathbf{E}_\eta\big[H_j\,\mathbf{1}\{H_j < H_\xi\}\big]\,\mathbf{P}_\eta[H_\xi < H_1] \\
&+\; \sum_{j\geq 0} \mathbf{P}_\eta\big[H_j < H_\xi\big]\,\mathbf{E}_\eta\big[H_1\,\mathbf{1}\{H_\xi < H_1\}\big] \;.
\end{aligned}$$

By (6.2), the first sum is bounded by $\mathbf{P}_\eta[H_\xi < H_\eta^+]^{-1}\mathbf{E}_\eta[H_\eta^+]$, and, by (5.4), the second sum is bounded by the same expression. This proves (6.3) in view of (5.5).

We finally claim that

$$\mathbf{E}_\xi[H_\eta] \;<\; \infty \;. \tag{6.4}$$ $\boxed{\text{n24}}$

Indeed, $H_\eta^+ \geq H_\eta^+\,\mathbf{1}\{H_\xi < H_\eta^+\} \geq H_\eta \circ \vartheta_{H_\xi}\,\mathbf{1}\{H_\xi < H_\eta^+\}$. Therefore, by the strong Markov property,

$$\mathbf{E}_\eta[H_\eta^+] \;\geq\; \mathbf{E}_\xi[H_\eta]\,\mathbf{P}_\eta[H_\xi < H_\eta^+] \;.$$

As the process is irreducible, $\mathbf{P}_\eta[H_\xi < H_\eta^+] > 0$, which proves (6.4).

We are now in a position to prove the lemma. Since $H_\xi^+ \leq H_\eta + H_\xi \circ \vartheta_{H_\eta}$, by the strong Markov property,

$$\mathbf{E}_\xi[H_\xi^+] \;\leq\; \mathbf{E}_\xi[H_\eta] \;+\; \mathbf{E}_\eta[H_\xi] \;.$$

This expression is finite in view of (6.3) and (6.4). $\quad\square$

In the positive-recurrent case the measure Π_η introduced in (5.7) is finite:

$$\Pi_\eta(E) \;=\; \mathbf{E}_\eta\big[H_\eta^+\big] \;<\; \infty \;.$$

In particular, normalizing the measure Π_η, we obtain an invariant probability measure, denoted by π_η:

$$\pi_\eta(\xi) \;=\; \frac{\mathbf{E}_\eta\Big[\int_0^{H_\eta^+}\mathbf{1}\{\eta(s) = \xi\}\,ds\Big]}{\mathbf{E}_\eta[H_\eta^+]} \;, \quad \xi \in E \;. \tag{6.5}$$ $\boxed{\text{n03}}$

ns13 **Lemma 1.25.** *Suppose that the chain $\eta(t)$ is irreducible and positive-recurrent. Then, π_η is the unique invariant probability measure.*

Proof. Suppose that μ is an invariant probability measure for the chain $\eta(t)$. By (5.11), $M(\xi) = \lambda(\xi)\mu(\xi)$ and $\tilde{M}_\eta(\xi) = \lambda(\xi)\pi_\eta(\xi)$ are invariant measures for the embedded chain Y_n. By Corollary 1.23 M and $\tilde{M}_\eta$ differ at most by a scalar multiple. This property clearly extends to μ and π_η. Since both are probability measures, they must coindice. □

ns14 **Lemma 1.26.** *Suppose that the chain $\eta(t)$ is non-explosive and that there exists an invariant probability measure. Then, the chain is positive-recurrent.*

Proof. Denote by π the invariant probability measure and recall from Section 1.3 that we denote by $\mathfrak{X}$ the explosion time. Fix a configuration $\eta \in E$. We have that

$$H_\eta^+ \wedge \mathfrak{X} \;=\; \sum_{k=0}^{\mathbb{H}_\eta - 1} \frac{\mathfrak{e}_k}{\lambda(Y_k)} \;.$$

The inequality is false if we replace on the left hand side $H_\eta^+ \wedge \mathfrak{X}$ by H_η^+ because if the process explodes before returning to η, $H_\eta^+ > \mathfrak{X} = \sum_{0 \le k < \mathbb{H}_\eta} \mathfrak{e}_k/\lambda(Y_k)$.
By the previous displayed equation and by Tonelli's theorem,

$$\mathbf{E}_\eta\big[H_\eta^+ \wedge \mathfrak{X}\big] \;=\; \mathbf{E}_\eta\Big[\sum_{k=0}^{\mathbb{H}_\eta - 1} \frac{\mathfrak{e}_k}{\lambda(Y_k)} \Big] \;=\; \sum_{\xi \in E} \frac{1}{\lambda(\xi)} \mathbf{E}_\eta\Big[\sum_{k=0}^{\mathbb{H}_\eta - 1} \mathbf{1}\{Y_k = \xi\} \Big] \;.$$

Let Π be the measure on E defined by $\Pi(\xi) = \pi(\xi)\,\lambda(\xi)$. By (5.11), Π is an invariant measure for the embdedded chain Y_n and so is the measure $\hat{\Pi}$ defined by $\hat{\Pi}(\xi) = \Pi(\xi)/\Pi(\eta)$. This latter measure is such that $\hat{\Pi}(\eta) = 1$. In particular, by (5.8), (5.9) and Lemma 1.22,

$$\mathbf{E}_\eta\Big[\sum_{k=0}^{\mathbb{H}_\eta - 1} \mathbf{1}\{Y_k = \xi\} \Big] \;=\; M_\eta(\xi) \;\le\; \hat{\Pi}(\xi) \;=\; \frac{\Pi(\xi)}{\Pi(\eta)} \;.$$

Putting togheter the previous two estimates, we obtain that

$$\mathbf{E}_\eta\big[H_\eta^+ \wedge \mathfrak{X}\big] \;\le\; \sum_{\xi \in E} \frac{1}{\lambda(\xi)} \frac{\Pi(\xi)}{\Pi(\eta)} \;=\; \frac{1}{\Pi(\eta)} \sum_{\xi \in E} \pi(\xi) \;.$$

As π is a probability measure and the chain is non-explosive,

$$\mathbf{E}_\eta\big[H_\eta^+\big] \;=\; \mathbf{E}_\eta\big[H_\eta^+ \wedge \mathfrak{X}\big] \;\le\; \frac{1}{\lambda(\eta)\,\pi(\eta)} \;,$$

as claimed. □

Example 3.5.4 in Norris [1998] shows that the assumption that the chain is non-explosive is needed in the previous result.

7 Stationary States

nsec05

We prove in this section that a measure is invariant for a minimal, irreducible, recurrent Markov chain if and only if it is stationary. We conclude the section introducing reversible states.

bbs03 **Definition 1.27.** *Let $\eta(t)$ be a minimal Markov chain. A measure μ on E is a stationary state for the chain $\eta(t)$ if for every $\eta \in E$ and $t \geq 0$,*

$$\mu(\eta) \;=\; \sum_{\xi \in E} \mu(\xi)\, p_t(\xi, \eta) \;.$$

bbs04 **Lemma 1.28.** *Consider an E-valued minimal, irreducible, recurrent Markov chain $\eta(t)$. A measure μ on E is a stationary state for $\eta(t)$ if it is an invariant state. Conversely, if μ is a stationary state and the jump rate λ is summable with respect to μ, then the measure μ is an invariant state.*

Let μ be an invariant state. Recall from xxx that λ is summable with respect to μ if and only if the measure $M(\eta) = \mu(\eta)\,\lambda(\eta)$ is a finite measure. We the embedded chain Y_k is positive-recurrent.

Proof (Proof of Lemma 1.28). Fix an invariant state μ. We will prove that μ is stationary. Fix a configuration $\eta \in E$. By Corollary 1.23, μ is a multiple of the measure Π_η introduced in (5.7). It is therefore enough to prove that Π_η is a stationary state.

We first claim that for every $t > 0$,

$$\Pi_\eta(\xi) \;=\; \mathbf{E}_\eta\Big[\int_t^{t+H_\eta^+} \mathbf{1}\{\eta(s) = \xi\}\, ds \Big]\,, \quad \xi \in E \;. \tag{7.1}$$ 　bb03

Indeed, since $[0, t) \cup [t, t + H_\eta^+) = [0, H_\eta^+) \cup [H_\eta^+, t + H_\eta^+)$,

$$\mathbf{E}_\eta\Big[\int_0^t \mathbf{1}\{\eta(s) = \xi\}\, ds \Big] \;+\; \mathbf{E}_\eta\Big[\int_t^{t+H_\eta^+} \mathbf{1}\{\eta(s) = \xi\}\, ds \Big]$$

$$=\; \Pi_\eta(\xi) \;+\; \mathbf{E}_\eta\Big[\int_{H_\eta^+}^{t+H_\eta^+} \mathbf{1}\{\eta(s) = \xi\}\, ds \Big] \;.$$

By the strong Markov property,

$$\mathbf{E}_\eta\Big[\int_{H_\eta^+}^{t+H_\eta^+} \mathbf{1}\{\eta(s) = \xi\}\, ds \Big] \;=\; \mathbf{E}_\eta\Big[\int_0^t \mathbf{1}\{\eta(s) = \xi\}\, ds \Big] \;,$$

which proves (7.1).

Fix a configuration $\zeta \in E$ and $t > 0$. By the definition of the measure Π_η introduced in (5.7), we have that

$$\sum_{\xi \in E} \Pi_\eta(\xi) \, p_t(\xi, \zeta) \;=\; \sum_{\xi \in E} \mathbf{E}_\eta \Big[\int_0^{H_\eta^+} \mathbf{1}\{\eta(s) = \xi\} \, ds \Big] \mathbf{P}_\xi[\eta(t) = \zeta] \,.$$

By Tonelli's theorem we may write the previous sum as

$$\int_0^\infty ds \sum_{\xi \in E} \mathbf{P}_\eta \big[H_\eta^+ \geq s \,, \, \eta(s) = \xi \big] \, \mathbf{P}_\xi[\eta(t) = \zeta] \,.$$

Since the event $\{H_\eta^+ \geq s\}$ belongs to the σ-algebra $\mathcal{F}_s^\eta \subset \mathcal{F}_s$, by the Markov property,

$$\mathbf{P}_\eta \big[H_\eta^+ \geq s \,, \, \eta(s) = \xi \,, \, \eta(s+t) = \zeta \big] \;=\; \mathbf{P}_\eta \big[H_\eta^+ \geq s \,, \, \eta(s) = \xi \big] \mathbf{P}_\xi[\eta(t) = \zeta] \,.$$

The penultimate displayed formula is thus equal, after summation over ξ, to

$$\int_0^\infty \mathbf{P}_\eta \big[H_\eta^+ \geq s \,, \, \eta(s+t) = \zeta \big] \, ds \;=\; \mathbf{E}_\eta \Big[\int_t^{t+H_\eta^+} \mathbf{1}\{\eta(s) = \zeta\} \, ds \Big] \,,$$

which, by (7.1), is equal to $\Pi_\eta(\zeta)$.

To prove the converse, assume that μ is a stationary state for the minimal chain $\eta(t)$ and that the jump rate λ is summable with respect to μ. As μ is a stationary state, for all $\eta \in E$, $t > 0$,

$$\sum_{\xi \in E} \mu(\xi) \, \frac{1}{t} \{ \delta_{\eta,\xi} - p_t(\xi, \eta) \} \;=\; 0 \,.$$

By Lemma 1.11, as $t \downarrow 0$, $t^{-1}\{1 - p_t(\eta, \eta)\}$, $t^{-1} p_t(\eta, \xi)$ converge to $\lambda(\eta)$, $R(\eta, \xi)$, respectively. Since λ is summable with respect to μ, by (3.2) and by the dominated convergence theorem, letting $t \downarrow 0$ in the previous displayed sum yields

$$\sum_{\xi \neq \eta} \mu(\xi) \, R(\xi, \eta) \;-\; \lambda(\eta) \, \mu(\eta) \;=\; 0 \,,$$

proving that μ is an invariant state. $\square$

Recall the definition of the measure $\mathbf{P}_\mu$ introduced in xxx. If μ is a stationary state, for every $n \geq 1$, $0 \leq t_1 < \cdots < t_n$, $t \geq 0$, $\xi_1, \ldots, \xi_n \in E$,

$$\mathbf{P}_\mu \big[\eta(t_1 + t) = \xi_1, \ldots, \eta(t_n + t) = \xi_n \big] \;=\; \mathbf{P}_\mu \big[\eta(t_1) = \xi_1, \ldots, \eta(t_n) = \xi_n \big] \,.$$

The distribution of the chain $\eta(t)$ is thus translation invariant under $\mathbf{P}_\mu$: for every $s \geq 0$, $\{\eta(t) : t \geq 0\}$ and $\{\eta(t + s) : t \geq 0\}$ have the same distribution under $\mathbf{P}_\mu$. In consequence, we may assume that the chain is defined in the entire time line $\mathbb{R}$, and for every $t_1 < \cdots < t_n$, $\xi_1, \ldots, \xi_n \in E$,

$$\mathbf{P}_\mu \big[\eta(t_1) = \xi_1, \ldots, \eta(t_n) = \xi_n \big] \;=\; \mathbf{P}_\mu \big[\eta(0) = \xi_1, \ldots, \eta(t_n - t_1) = \xi_n \big] \,.$$

Reversible chains. Fix a Q-matrix R satisfying the conditions (3.6). Assume that the discrete-time Markov chain associated to the transition matrix $p(\eta, \xi)$ is recurrent. Let $\eta(t)$ be the unique Markov chain whose Q-matrix is R.

A measure μ is said to satisfy the detailed balance conditions if

$$\mu(\eta)\, R(\eta, \xi) \;=\; \mu(\xi)\, R(\xi, \eta) \,, \quad \eta \neq \xi \in E \,. \tag{7.2} \quad \boxed{\text{bb06}}$$

Recall that we denote by M_μ the measure defined by $M_\mu(\eta) = \lambda(\eta)\, \mu(\eta)$. A measure μ satisfies the detailed balance conditions if and only if

$$M_\mu(\eta)\, p(\eta, \xi) \;=\; M_\mu(\xi)\, p(\xi, \eta) \,, \quad \eta \neq \xi \in E \,.$$

Since $p^{(n+1)}(\eta, \xi) = \sum_{\zeta \in E} p^{(n)}(\eta, \zeta) p(\zeta, \xi)$, by induction we obtain that for every $n \geq 1$,

$$M_\mu(\eta)\, p^{(n)}(\eta, \xi) \;=\; M_\mu(\xi)\, p^{(n)}(\xi, \eta) \,, \quad \eta \neq \xi \in E \,. \tag{7.3} \quad \boxed{\text{bb08}}$$

It is also clear that if a measure μ satisfies the detailed balance conditions it is then an invariant measure:

$$\sum_{\eta \in E} M_\mu(\eta)\, p(\eta, \xi) \;=\; \sum_{\eta \in E} M_\mu(\xi)\, p(\xi, \eta) \;=\; M_\mu(\xi) \,.$$

$\boxed{\text{bbs06}}$ **Lemma 1.29.** *Let $\eta(t)$ be a non-explosive minimal Markov chain. A measure μ satisfies the detailed balance conditions with respect to the Q-matrix of $\eta(t)$ if and only if for all $t \geq 0$,*

$$\mu(\eta)\, p_t(\eta, \xi) \;=\; \mu(\xi)\, p_t(\xi, \eta) \,, \quad \eta, \xi \in E \,. \tag{7.4} \quad \boxed{\text{bb07}}$$

Proof. Suppose that a measure μ satisfies the detailed balance conditions. Fix two configurations $\eta \neq \xi \in E$ and $t > 0$. Recall that we denote by S_j, $j \geq 0$, the successive jumps of the chain $\eta(t)$. Since the chain is non-explosive, $S_j \to \infty$ $\mathbf{P}_\eta$-almost surely. In particular, $\mu(\eta)\, p_t(\eta, \xi) = \mu(\eta)\, \mathbf{P}_\eta[\eta(t) = \xi]$ is equal to

$$\mu(\eta)\, p_t(\eta, \xi) \;=\; \mu(\eta)\, \mathbf{P}_\eta[\eta(t) = \xi] \;=\; \sum_{j \geq 1} \mu(\eta)\, \mathbf{P}_\eta\big[\eta(t) = \xi \,, S_j \leq t < S_{j+1}\big] \,.$$

On the event $\{S_j \leq t < S_{j+1}\}$, $\eta(t) = Y_j$. Summing over all possible trajectories $Y_1, \ldots, Y_{j-1}$, for a fixed j the previous expression becomes

$$\sum_{\xi_1, \ldots, \xi_{j_1} \in E} \mu(\eta)\, p(\eta, \xi_1) \cdots p(\xi_{j-1}, \xi)\, \mathbf{P}_\eta\Big[\frac{\mathfrak{e}_0}{\lambda(\eta)} + R \leq t < \frac{\mathfrak{e}_0}{\lambda(\eta)} + \frac{\mathfrak{e}_j}{\lambda(\xi)} + R\Big] \,,$$

where $R = \sum_{1 \leq k \leq j-1}[\mathfrak{e}_k / \lambda(\xi_k)]$. In the probability $\mathbf{P}_\eta$, integration is performed with respect to the i.i.d. mean-one exponential random variables $\mathfrak{e}_0 \ldots, \mathfrak{e}_j$. At this point, the index η of $\mathbf{P}_\eta$ has no meaning.

One the one hand, as μ satisfies the detailed balance conditions, in view of (7.3),

$$\mu(\eta)\, p(\eta, \xi_1) \cdots p(\xi_{j-1}, \xi) \;=\; \frac{\lambda(\xi)}{\lambda(\eta)}\, \mu(\xi)\, p(\xi, \xi_{j-1}) \cdots p(\xi_1, \eta) \,.$$

On the other hand, if $\mathfrak{e}, \mathfrak{f}$ are independent, mean-one exponential random variables, for every $u > 0$, $a > 0$, $b > 0$,

$$P\Big[\frac{\mathfrak{e}}{a} \le u < \frac{\mathfrak{e}}{a} + \frac{\mathfrak{f}}{b}\Big] \;=\; \frac{a}{b-a}\big\{e^{-au} - e^{-bu}\big\} \;=\; \frac{a}{b}\, P\Big[\frac{\mathfrak{e}}{b} \le u < \frac{\mathfrak{e}}{b} + \frac{\mathfrak{f}}{a}\Big] \,.$$

Therefore, taking a condition expectation with respect to the σ-algebra spanned by $\{\mathfrak{e}_1, \dots, \mathfrak{e}_{j-1}\}$, we obtain that

$$\mathbf{P}_\eta\Big[\frac{\mathfrak{e}_0}{\lambda(\eta)} + R \le t < \frac{\mathfrak{e}_0}{\lambda(\eta)} + \frac{\mathfrak{e}_j}{\lambda(\xi)} + R\Big]$$
$$= \frac{\lambda(\eta)}{\lambda(\xi)}\, \mathbf{P}_\xi\Big[\frac{\mathfrak{e}_0}{\lambda(\xi)} + R \le t < \frac{\mathfrak{e}_0}{\lambda(\xi)} + \frac{\mathfrak{e}_j}{\lambda(\eta)} + R\Big] \,.$$

For convenience, we replaced the index η of $\mathbf{P}_\eta$ on the left hand side by the index ξ.

It follows from the two identities derived in the previous paragraph that $\mu(\eta)\, p_t(\eta, \xi)$ is equal to

$$\sum_{j \ge 1} \sum_{\xi_1, \dots, \xi_{j_1} \in E} \mu(\xi)\, p(\xi, \xi_{j-1}) \cdots p(\xi_1, \eta)\, \mathbf{P}_\xi\Big[\frac{\mathfrak{e}_0}{\lambda(\xi)} + R \le t < \frac{\mathfrak{e}_0}{\lambda(\xi)} + \frac{\mathfrak{e}_j}{\lambda(\eta)} + R\Big] \,.$$

Since

$$p(\xi, \xi_{j-1}) \cdots p(\xi_1, \eta)\, \mathbf{P}_\xi\Big[\frac{\mathfrak{e}_0}{\lambda(\xi)} + R \le t < \frac{\mathfrak{e}_0}{\lambda(\xi)} + \frac{\mathfrak{e}_j}{\lambda(\eta)} + R\Big]$$
$$= \mathbf{P}_\xi\Big[Y_1 = \xi_{j-1}, \dots, Y_{j-1} = \xi_1, Y_j = \eta, S_{j-1} \le t < S_j\Big] \,,$$

summing over $\xi_1, \dots, \xi_{j_1}$ and over j we conclude that

$$\mu(\eta)\, p_t(\eta, \xi) \;=\; \mu(\xi)\, p_t(\xi, \eta) \,.$$

Conversely, suppose that (7.4) holds for all $t \ge 0$, $\eta, \xi \in E$. Fix $\eta \ne \xi$. Dividing both sides of this identity by t and letting $t \downarrow 0$, in view of Lemma 1.29 we obtain that $\mu(\eta) R(\eta, \xi) = \mu(\xi) R(\xi, \eta)$. $\square$

Let $\eta(t)$ be a non-explosive, minimal Markov chain. Suppose that μ is an invariant probability measure which satisfies the detailed balance conditions with respect to the Q-matrix of $\eta(t)$. It follows from the previous lemma that

$$\mathbf{P}_\mu\big[\eta(t_0) = \xi_0, \dots, \eta(t_n) = \xi_n\big] \;=\; \mathbf{P}_\mu\big[\eta(-t_n) = \xi_n, \dots, \eta(-t_0) = \xi_0\big] \,. \quad (7.5) \quad \boxed{\text{bb09}}$$

Since μ is a stationary state, we may assume that the chain $\eta(t)$ is defined on the time interval $\mathbb{R}$. Let $\eta^*(t)$ be the time-reversed chain: $\eta^*(t) = \eta(-t)$. It follows from (7.5) that if the stationary state μ satisfies the detailed balance conditions, the dynamics of the time-reversed chain coincides with the original dynamics.

8 Exercises

bex1
Exercise 1.30. Let $(\eta(t) : t \geq 0)$ be the Markov chain introduced in Proposition 1.4, and recall that τ_1 represents the time of the first jump. Fix $\eta \in E$ and assume that $p(\eta, \eta) < 1$. Show that τ_1 and $\eta(\tau_1)$ are independent under $\mathbb{P}_\eta$, that τ_1 is distributed according to an exponential random variable of parameter $\lambda(\eta) [1 - p(\eta, \eta)]$ and that $\mathbb{P}_\eta[\eta(\tau_1) = \xi] = p(\eta, \xi)/[1 - p(\eta, \eta)]$ for all $\xi \neq \eta$.

bex6
Exercise 1.31. Let $\eta(t)$ be a Markov chain as in Definition 1.3 and recall that τ_1 represents the time of the first jump. Fix $\eta \in E$. Show that, under the measure $\mathbb{P}_\eta$, τ_1 is exponentially distributed with parameter $q(\eta) = - p_0'(\eta, \eta)$. Show, moreover, that $\eta(\tau_1)$ and τ_1 are independent random variables.

bex5
Exercise 1.32. Recall from (2.2) that $(\tau_k : k \geq 1)$ represents the successive times of jumps of a chain $\eta(t)$. Show that each τ_k is a stopping time with respect to the natural filtration $\mathcal{F}_t^\eta$.

bex2
Exercise 1.33. Show that the explosion time $\mathfrak{X}$, introduced in Example 1.13 is a stopping time with respect to the natural filtration $(\mathcal{F}_t^\eta : t \geq 0)$.

bex3
Exercise 1.34. In example 1.13, show that $\mathbb{P}_j[\mathfrak{X} \leq t] > 0$ for all $j \in E$, $t > 0$. Prove identity (3.5).

bex4
Exercise 1.35. Recall the notation introduced in Definition 1.3. Fix $\eta \in E$. Show that the sequence $(Z_k : k \geq 0)$, defined by $Z_0 = \eta$, $Z_k = \eta(\tau_k)$, $k \geq 1$, forms a discrete-time Markov chain in which the configurations ξ such that $p(\xi, \xi) = 1$ are absorbing points.
with transition probability $p_\star(\xi, \zeta) = p(\xi, \zeta)/[1 - p(\xi, xi)]$

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Notation

$C_b(E)$, 35
$D([0, \infty), E)$, 22
E, 3
G_λ, 40
H_A, H_A^+, 10
I, 36
P_t, 36
S_{xy}, 156
$V_{A,B}$, 67
$[a]$, 12
$\langle f , g \rangle_\pi$, 44
Ω, 3
$\|R\|$, 36
$\|f\|_\infty$, 35
$\| \cdot \|_2$, 44
$\mathbb{H}_A^+$, 25
$\mathbb{H}_A$, 25
$\eta(t)$, 4
η_x, 141
$\ll$, 108
$\mathbf{1}\{B\}$, 5
$\mathbf{E}_\nu$, 51
$\mathbf{P}_\mu$, 23
$\mathbf{P}_\nu$, 51
$\mathcal{D}(L)$, 37
$\mathcal{F}_T^\eta$, 12
$\mathcal{F}_t^\eta$, 4
$\mathcal{F}_T$, 11
$\mathcal{F}_t$, 3
$\mathcal{R}(\lambda - L)$, 37
$\mathfrak{X}$, 17
$\mathfrak{d}_x$, 155
$\nu_{A,B}^*$, 87
ν_{AB}, 86
π, 44
$\sigma^{x,y}\eta$, 141
$\theta(t)$, 53
φ^*, 142
$\vartheta(t)$, 23
$p_t(\eta, \xi)$, 3

Index