AN INVITATION TO WEB GEOMETRY

From Abel’s addition Theorem to the algebraization of codimension one webs

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June 2, 2009
Para Dayse e Jorge,
os meus Pastores.
J.V.P.

Pour Min’
L.P.
Preface

The first purpose of this book is to serve as supporting material for a mini-course on web geometry to be delivered at the 27th Brazilian Mathematical Colloquium which will take place at IMPA in last week of July 2009. But, in its almost 250 pages there is much more than one can possibly cover in five lectures of one hour each. The abundance of material is due to the second purpose of this text: convey some of the beauty of web geometry and to provide an account, as self-contained as possible, of some of the exciting advancements the field has witnessed in the last few years.

We have tried to write a book which is not very demanding in terms of pre-requisites. It is true that at some points familiarity with the basic language of algebraic/complex geometry is welcome but, except at very few passages, not much more is needed. An effort has been made to explain, even if sometimes superficially, every single unusual concept appearing in the text.

At an early stage of this project we decided to use the third instead of the first person. In retrospect, it is hard to understand why two authors, none of them particularly comfortable with the English language, took this decision. Today, the only explanation that comes to mind is a subconscious attempt to expire the sins of two bad writers. We apologize for the awkwardness of the prose and hope that those more familiar with English than us will find some amusement with the clumsiness of it.

This text would take much longer to come to light without the invitation of Márcio Gomes Soares to submit a mini-course proposal to the 27th Brazilian Mathematical Colloquium. Besides Soares,
we would like to thank Hernan Falla Luza and Paulo Sad, whom
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translating to English a draft of the introduction originally written in
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Jorge Vitório Pereira and Luc Pirio
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Introduction

It seems impossible to grasp the ins and outs of a mathematical field without setting it back in its historical context. An attempt, certainly incomplete and biased, is made in the next few pages. At the end of the Introduction, one finds a description of the contents of this book, and suggestions on how to use it.

Historical Notes

If the birth of web geometry can be ascribed to the middle of the 1930s in Hambourg (see below), some precedents can be found as early as the middle of the XIXth century. The concepts and problems of web geometry springs from two different fields of the XIXth century mathematics: projective differential geometry and nomography.

It is mainly from the first that web geometry comes from. At that time, projective differential geometry mainly consisted of the study of projective properties of curves and surfaces in $\mathbb{R}^3$, that is of their differential properties that are invariant up to homographies.

Gaussian geometry, which had appeared before, studied the properties of (curves and) surfaces in ordinary euclidian space that are invariant up to isometric transformations. Gauss and other mathematicians pointed out how useful the first and second fundamental forms are for the study of surfaces. They also brought to light the relevance of derived concepts, such as the principal, asymptotic and conjugated directions. When considering the integral curves of these tangent direction fields, the mathematicians of the time were con-
sidering what they called 2-nets of “lines” on surfaces, that is the data of 2 families of curves, or in more modern terms, 2-webs. It is when they endeavored to generalize these constructions to the projective differential geometry that some 3-nets projectively attached to surfaces in $\mathbb{R}^3$ quite naturally made their appearance (for instance, Darboux introduced a 3-web called after him in [41]; see also Section 1.4.4 in this book).

These webs were useful back then because they encoded properties of the surfaces under study. Thomsen’s paper [106] is a good illustration of this fact. In this article, Thomsen shows that a surface area in $\mathbb{R}^3$ is isothermally asymptotic\(^1\) if and only if its Darboux 3-web is hexagonal\(^2\). At that time, the study of 3-webs on surfaces from the point of view of projective differential geometry was on the agenda.

Thomsen’s result has this particular feature of characterizing the geometric-differential property of being isothermally asymptotic by a closedness property of more topological nature that is (or not) verified by a configuration traced on the surface itself. It is this feature which struck some mathematicians and led to the study of webs at the beginning of the 1930s.

\(\S\)

The second source of web geometry is nomography. This discipline, nowadays practically extinct, belonged to the field of applied mathematics in the 1900s. It was established as an autonomous mathematical discipline by M. d’Ocagne. It consisted in a method of “graphical calculus” which allowed engineers to calculate rather fast. To explain its principle (which to-day appears rather naive), let $F(a_1, a_2, a_3) = 0$ be a law linking three physical variables. Is there a quick and accurate way to determine one variable say $a_i$ from the

\(^1\) Geometers of the XIXth century had established a very rich “bestiary” of surfaces in $\mathbb{R}^3$. The isothermally asymptotic surfaces (or “$F$-surfaces”) formed one of the classes in their classification (see [50] for a modern definition.)

\(^2\) Thomsen’s result applies to real surfaces in $\mathbb{R}^3$ thus his statement is different as one takes place at a neighborhood of an elliptic point or a hyperbolic point of the considered surface.
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other two: $a_j$ and $a_k$? To solve this problem, people used nomograms. A nomogram is a graphic which represents curves according to values of the variables $a_1$, $a_2$ and $a_3$. For instance, to find the value of $a_1$ in function of values $\alpha_2$ and $\alpha_3$ of the variables $a_2$ and $a_3$ (respectively), one has to find the intersection point of the curves $a_2 = \alpha_2$ and $a_3 = \alpha_3$. Through (or near) this point goes a curve $a_1 = \alpha_1$, and $\alpha_1$ is the sought value.

Figure 1: A nomogram from a book by M. D'Ocagne.

What nowadays seems to be far from actual mathematics was once an important part of the mathematical culture. It was probably after considering some results of nomography that Hilbert formulated the thirteenth of the famous 23 problems that he stated at the International Congress of Mathematics of 1900.

The main disadvantage of nomography was the problem of its readability. Of course, the nomograms where the curves coincided with (pieces of) lines were easier to use. Hence the problem to know whether it is possible to linearize the curves of a given nomogram. Or equivalently, whether it is possible to linearize a 3-web of curves on the plane. For more precisions on the links between nomography
and web geometry, the interested reader can consult [2].

**Birth of web geometry: Spring of 1927 in Naples**

Thomsen’s paper [106] is considered as the birth of web geometry. According to Blaschke (see the beginning of the foreword in [18]) this paper is the result of their Spring walks on Posillipo hill, at the vicinity of Naples, in 1927. Even if it concerns the study of some surfaces in $\mathbb{R}^3$, it shows clearly that a plane configuration made of three families of curves (i.e. a 3-web) admits local analytic invariants. It seems that the equivalence between the vanishing of the curvature of a 3-web (which is a condition of analytic nature) and the hexagonality condition (which is a property seemingly of topological nature)$^3$ struck these two mathematicians and led them (with others) to study the matter.

**Early developments: Hamburg school (1927-1938)**

A short time after Thomsen’s paper [106] was published, a group led by Blaschke was set up in Hamburg to do research on webs. Blaschke and his coworkers$^4$ found many results which established web geometry as a discipline. It is a remarkable fact that a series of more than 60 papers were published in a variety of journals between 1927 and 1938, mainly by members of the Hamburg school of web geometry, under the common label of “Topologie Fragen der Differentialgeometrie”$^5$.

Their work focused on three main directions:

- The study of webs from the differential geometry viewpoint, through the analytical invariants which can be associated to them;

- The study of the relations between webs and abstract geometric configurations linked to the algebraic theory of (quasi-)groups;

$^3$See Theorem 1.2.4 farther in this book.

$^4$Bol, Chern, Mayrhofer, Podehl, Walberer were active members of this group. Kähler, Zariski, Reidemeister and others also worked on this subject but in a occasional way.

$^5$In English, “Topological questions of differential geometry”.
Figure 2: A 3-web with vanishing curvature.

- The interpretation of web geometry as a relative of projective algebraic geometry, notably via the notion of abelian relation.

This book focuses on the latter direction of study, and will not expand on the two former ones, due to lack of space and of competence as well. It deals with the links between webs and algebraic geometry, which have their origins in results obtained by Blaschke, Bol and Howe. At the beginning, these results were mainly about planar webs. Firstly, Blaschke came up with an interpretation of a theorem by Graf and Sauer [56] in the framework of web geometry. This theorem says that a linear 3-web carrying an abelian relation is constituted by the tangents to a plane algebraic curve of class 3. Later, as soon as 1932, Blaschke and Howe [17] generalized this theorem to the case of linear $k$-webs carrying at least one complete abelian
relation, thus bringing to light the usefulness of the notion of abelian relation. Bol’s result giving the explicit bound $\frac{1}{2}(k - 1)(k - 2)$ on the dimension of the space of abelian relations of a planar $k$-web appeared shortly afterwards in [19] and allowed to define the rank of a web. Using this formalism, Howe noticed that Lie’s result about the surfaces of double translation can be understood in the framework of web geometry as the striking fact that a planar 4-web of rank 3 is algebraizable. The relationship between the planar webs of maximal rank and Abel’s Theorem was reported the following year by Blaschke in [14], which brought up the final definition of the notion of algebraic web. In the same paper, Blaschke expounded the generalization of Lie’s Theorem to the 5-webs of maximal rank, a result which was later proved by Bol to be incorrect. Surprisingly, he also exhibited Bol’s 5-web $B_5$ as an example of non algebraizable 5-web of rank 5, while it is of maximal rank 6 (see below).

In 1933, Blaschke set about studying webs in dimension three. In [13], he established a bound $\pi(3, k)$ on the rank of a $k$-web of hypersurfaces in $\mathbb{C}^3$. One year later, Bol gave in [20] one of the most important results obtained at the time: for $k \geq 6$, a $k$-web of hypersurfaces on $\mathbb{C}^3$ of maximal rank $\pi(3, k)$ is algebraizable. This success certainly played a role in Blaschke’s attempt in [14, 15] to obtain algebraization results for planar webs of maximal rank. Only in 1936 it was made clear that the result he was looking for were unattainable. In [21], Bol realized that $B_5$ carries one more abelian relation, related to Abel’s five terms equation for the dilogarithm; hence it is an instance of a 5-web of rank 6, which is not algebraizable.

In the year of 1936, Chern defended his PhD dissertation on webs, written under Blaschke’s direction. He then published two papers. The 60th issue of the “Topologische Fragen der Differentialgeometrie” series [29] is of special interest here. Generalizing Blaschke’s result, he obtains a bound on the rank of a web of codimension one in arbitrary dimension$^6$ which now bears his name.

Thus, in 1936, most of the notions studied in this book had been brought to light. A general survey of the state the art then can be

\footnote{See Theorem 2.2.8 in this book.}
found in the third part of the book [18], to which the reader is asked to refer.

Finally, this very year is when Blaschke shifted his interest from web geometry to integral geometry. Few members of the Hamburg school worked again on webs, with the notable exceptions of Blaschke and Chern, but this time in a different way (see below).

**Web geometry in mid XXth century (1938-1960)**

Blaschke strongly supported exchanges between mathematicians. From 1927 to 1960 he travelled a lot and had the opportunity to give lectures about web geometry in numerous countries (for instance in Romania, Greece, Spain, Italy, the United States, India, Japan), thus inspiring people with a variety of nationalities and backgrounds to do research on web geometry.\(^7\)

It seems that Blaschke went to Italy many times during this period. As a by-product, an Italian school of web geometry developed at that time. Bompiani, Terracini and Buzano were its most prominent contributors. Their work was chiefly about the links between geometry of planar webs and the projective differential geometry of surfaces. In the 1950s and 1960s a second Italian school of web geometry appeared, probably thanks to Bompiani’s influence. He, Vaona and Villa (among others) published papers on the projective deformation of planar 3-webs, but with no major outcome.

It also must be mentioned the work of the Romanian mathematicians Pantazi and Mihăileanu. During the 1930s and 1940s, they obtained interesting results on how to determine the rank of planar webs. These results were published as short notes in Romanian journals (see [84, 78]) and were then forgotten.

The war and later Blaschke’s political stance during the war (see [101, p. 423]) put an end to his influence for some time. When things went back to normal, he gave lectures again, on webs among other

\(^7\)For instance, it is in attending to some conferences given by Blaschke at Pekin in 1933 that Chern became interested in web geometry and decided to go to study at Hamburg.
things. Although he didn’t obtain new results, these lectures induced new researches once more, for instance Dou in Barcelona [42, 43, 44] and Ozkân in Turkey [83].

**Russian school (from 1965 onwards)**

More than at the Hamburg school, it is at the Moscow school of differential geometry that the Russian school of web geometry, led first by Akivis, and then by Akivis and Goldberg, seems to have its origin. Under the influence of the work of Élie Cartan, a Russian school of differential geometry developed in USSR at the instigation of Finikov from the 1940s onwards. Projective differential geometry was studied in full generality and involved the study of some nets (which could be called webs but only in a weak sense) projectively attached to (analytic) projective subvarieties. It is probably this fact which led to the study of webs for their own sake in arbitrary dimension and/or codimension, from the 1960s onwards. Akivis was joined by Goldberg quite early. They explored several directions in web geometry, published many papers and had many students.

The work of this school led chiefly dealt with the differential geometry of webs and with the interactions between webs and the theory of quasigroups. The links with algebraic geometry were not their major concern. Their results had little influence in the West for two main reasons: (1) their papers were in Russian, hence they were not distributed in the West; (2) the method they used was the Cartan-Laptev method\(^8\), which non specialists do not understand easily.

The reader who wishes to get an outline of the methods and results of this school may consult the books [4] and [55].

**Chern’s and Griffith’s work (1977-1980)**

Throughout his professional life Chern kept being interested in webs, particularly in the notion of web of maximal rank, as shown in [33],

\(^8\)Cartan-Laptev method is a reinterpretation/generalization of the methods of the mobile frame and of equivalence of Élie Cartan, by the Russian geometer G. Laptev.
INTRODUCTION

[34] and [36]. This point can be illustrated by quoting the last lines of [36]:

Due to my background I like algebraic manipulation, as Griffiths once observed. Local differential geometry calls for such works. But good local theorems are difficult to come by. The problem on maximal rank webs discussed above\(^9\) is clearly an important problem, and will receive my attention.

My mathematical education goes on.

In 1978, he resumed working on webs of maximal rank jointly with Griffiths. In the long paper [30], they set about demonstrating that a \(k\)-web of codimension one and of maximal rank \(\pi(n,k)\) is algebraizable when \(n > 2\) and \(k \geq 2n\). Their proof is not complete (cf. [31]) and it is necessary to make an extra non-natural assumption to ensure the validity of the result. They also got a sharp bound for the rank of webs of codimension two in [32]. Griffiths’s interest on the subject probably came from the links between web geometry and algebraization results like the converse of Abel’s Theorem discussed in Chapter 4. Although he published no other paper on the subject, he kept being interested in webs since he discussed them in the opening lecture he gave for the bicentennial of Abel’s birth in Oslo (transcribed in [74]).

Although it contains a non-trivial mistake, the paper [30] has been quite influential in web geometry. It has popularized the subject and led the Russian school to pay attention to the notions of abelian relations and rank. It is probably from [30] that Trépreau has taken up Bol’s method to obtain a proof of the result originally aimed at by Chern and Griffiths. The present book would not exist if [30] had not been written. The readers should read it, as it contains a masterfully written introductory part putting things in perspective, and offers different proofs of many of the results included here.

\(^9\)He is referring to the classification of webs of maximal rank.
Recent developments (since 1980)

A number of new results in web geometry have been obtained in the last twenty years. Here only, and certainly not all, results related to rank, abelian relations and maximal rank webs will be mentioned.

The abelian relations of Bol’s web all come (after analytic prolongation) from its dilogarithmic abelian relation, which thus appears more fundamental than the other relations. In 1982, in [52], Gelfand and MacPherson found a geometric interpretation of this relation. In it Bol’s web appears as defined on the space of projective configurations of 5 points of $\mathbb{RP}^2$. In [40], Damiano considers, for $n \geq 2$ a curvilinear $(n + 3)$-web $D_n$ naturally defined on the space of projective configurations of $n + 3$ points in $\mathbb{RP}^n$. He shows that this web is of maximal rank and gives a geometric interpretation of the “main abelian relation” of $D_n$, thus obtaining a family of exceptional webs which generalizes Bol’s web.

From 1980 to 2000, Goldberg studied the webs of codimension strictly bigger than one from the point of view of their rank. He obtained many results, most of which are expounded in [55]. More recently, he started studying planar webs under the same viewpoint in collaboration with Lychagin.

At the beginning of the 1990s, Hénaut started studying webs in the complex analytical realm. He published about 15 papers on the matter. His research is mainly about rank and abelian relations, and is concerned with webs of arbitrary codimension as well as planar webs.\(^\text{10}\) The papers [65, 66, 68] have to be mentioned, as related with the topic of this book. At the time when he started working, the field attracted little attention. Without any doubt his tenacity played a major role to popularize web geometry in France, and in other countries as well. With Nakai he co-organized the conference \textit{Géométrie des tissus et équations différentielles}, held at the CIRM in 2003, which was attended by researchers from all over the world and was for some mathematicians (particularly for the first author of this book) an opportunity to have their first contact with web geometry.

In 2001, the second author [90, 92] and Robert [100] independently showed that the Spence-Kummer 9-web associate to the trilogarithm

\(^{10}\)For an outline of the results he obtained before 2000, see [67].
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is an example of exceptional web. Within a short time were published several papers [91, 93, 77, 88] bringing to light a myriad of exceptional webs.

In 2005, Trêpreau provided a proof of the result which Chern and Griffiths aimed at in [30], i.e. the algebraization of maximal rank $k$-webs on $(\mathbb{C}^n, 0)$, when $k \geq 2n$ and $n > 2$.

It seems to us that nowadays the study of webs is undergoing a revival, as testifies the Bourbaki seminar [87] devoted to the results hitherto mentioned. Mathematicians with the most diverse backgrounds now publish papers on the matter. A few recent articles, but not all of them, are mentioned in this book. The readers are invited to consult the literature in order to get a better acquaintance with the advancement of the researches.

Contents of the chapters

The table of contents tells rather precisely what the book is about. The following descriptions give additional information.

Chapter 1 is introductory, and describes the basic notions of web geometry. The content of this chapter is for the main part quite well known, except for the notion of duality for global webs on projective spaces $\mathbb{P}^n$, which appears to be new when $n > 2$. A short survey of this notion is presented in Section 1.4.3. The first two sections, more specifically Section 1.1 and Section 1.2, are of rather elementary nature and might be read by an undergraduate student.

Chapter 2 is about the notions of abelian relation and rank. It offers an outline of Abel’s method to determine the abelian relations of a given planar web. It also gives a description of the abelian relations of planar webs admitting an infinitesimal symmetry. The most important results in this chapter are Chern’s bound on the rank (Theorem 2.2.8) and the normal form for the conormals of a web of maximal rank (Proposition 2.3.10). This last result is demonstrated through a geometric approach based on classical concepts and results from projective algebraic geometry which are described in detail.
Chapter 3 is devoted to Abel’s notorious addition Theorem. It first deals with the case of smooth projective curves, then tackles the general case after introducing the notion of abelian differentials. Section 3.3 gives a rather precise description of the Castelnuovo curves, hence of some algebraic webs of maximal rank. Section 3.4 expounds new results: an (easy) variant of Abel’s Theorem (Proposition 3.4.1), which is combined with Chern’s bound on the rank so as to obtain bounds on the genus of curves included in abelian varieties (cf. Theorem 3.4.5).

Chapter 4 is where the converse to Abel’s Theorem is demonstrated. Its proof is given through a reduction to the plane case which is then treated using a classical argument that can be traced back to Darboux. Then a presentation of some algebraization results follows. Important concepts as Poincaré’s and canonical maps for webs are discussed in this chapter. Our only contribution is of formal nature and is situated in Section 4.3, where we endeavor to work as intrinsically as possible.

Chapter 5 is entirely devoted to Trépreau’s algebraization result. The proof that is presented is essentially the same as the original one [107]. The only “novelty” in this chapter is Section 5.1.2, where a geometric interpretation of the proof is given. As in the preceding chapter, an effort was made to formulate some of the results and their proofs as intrinsically as possible.

Chapter 6 takes up the case of planar webs of maximal rank, more specifically of exceptional planar webs. Classical criteria which characterize linearizable webs on the one hand, and maximal rank webs on the other hand are explained. Then the existence of exceptional planar $k$-webs, for arbitrary $k \geq 5$, is established through the study of webs admitting infinitesimal automorphisms. The classification of the so called CDQL webs on compact complex surfaces obtained recently by the authors is also reviewed. The chapter ends with a brief discussion about all the examples of planar exceptional webs we are aware of.
How to use this book

There are numerous ways to use this book. Without any effort a few examples come to mind: feed up a fire, prop up a rickety table, impress friends, or even study its mathematical content. For those most interest in the latter, a few lines are dropped below.

The logical organization of this book is rather simple: the readers with enough time to spare can read it from cover to cover. Those mostly interest in Bol-Trépreau’s algebraization Theorem, may find useful the graph below which suggests a minimal route towards it.
Those anxious to learn more about exceptional webs might prefer to use instead the following graph as a reading guide.

The authors plan to maintain at http://www.impa.br/~jvp a web page containing a list of corrections and additional material related to the content of this book. Contributions of mistakes as well as of complementary material are welcome.
Conventions

All the definitions, including this one, are presented in bold case and have a corresponding entry at the remissive index.

Unless stated otherwise all the geometric entities like curves, surfaces, varieties and manifolds considered in this text are reduced and complex holomorphic. Curves, surfaces and varieties may be singular, and may have several irreducible components. The manifolds are smooth connected varieties.

Web geometry lies on the interface of local differential geometry and projective algebraic geometry. Throughout the text, the reader will be confronted with both local non-algebraic subvarieties of the projective space as well as with global, and hence algebraic and compact, projective subvarieties. A projective curve, surface, variety, or manifold will mean a compact curve, surface, variety, or manifold contained in some projective space. Beware that some authors use the term projective to qualify any subvariety, compact or not, algebraic or not, of a given projective space.

Throughout there will be references to points $x \in (\mathbb{C}^n, 0)$ and properties of germs at the point $x$. The point $x$ has to be understood as a point at a sufficiently small neighborhood of the origin and the property as a property of some representative of the germ defined in this very same sufficiently small neighborhood.

If $n$ is a positive integer, $\mathbb{N}$ will stand for the set $\{1, \ldots, n\}$. For any positive integer $q$, $\mathbb{C}_q[x_1, \ldots, x_n]$ will stand for the vector space of degree $q$ homogeneous polynomials in $x_1, \ldots, x_n$. The span of a subset $S$ of projective space or of a vector space will be denoted by $\langle S \rangle$. 
Chapter 1

Local and global webs

In its classical form web geometry studies local configurations of finitely many smooth foliations in general position. In Section 1.1 the basic definitions of our subject are laid down and the algebraic webs are introduced. These are among the most important examples of the whole theory.

Germs of webs defined by few foliations in general position are far from being interesting. Basic results from differential calculus imply that the theory is locally trivial. As soon as the number of foliations surpasses the dimension of the ambient manifold this is no longer true. The discovery in the last years of the 1920 decade of the curvature for 3-webs on surfaces is considered as the birth of web geometry. In Section 1.2 this curvature form is discussed and an early emblematic result of theory that characterizes its vanishing is presented.

Although the emphasis of the theory is local the most emblematic examples are indeed globally defined on projective manifolds. In Section 1.3 the basic definitions are extended to encompass both germs of singular as well as global webs. Certainly more demanding than the previous sections, Section 1.3 should be read in parallel with Section 1.4 where the algebraic webs are revisited from a global viewpoint and is discussed how one can associated webs to linear systems on surfaces.
CHAPTER 1: LOCAL AND GLOBAL WEBS

1.1 Basic definitions

1.1.1 Germs of smooth webs

A germ of smooth codimension one $k$-web

$$W = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$$

on $(\mathbb{C}^n, 0)$ is a collection of $k$ germs of smooth codimension one holomorphic foliations such that their tangent spaces at the origin are in general position, that is, for any number $m$ of these foliations, $m \leq n$, the corresponding tangent spaces at the origin have intersection of codimension $m$.

Usually the foliations $\mathcal{F}_i$ are presented by germs of holomorphic 1-forms $\omega_i \in \Omega^1(\mathbb{C}^n, 0)$, non-zero at $0 \in \mathbb{C}^n$ and satisfying Frobenius integrability condition $\omega_i \wedge d\omega_i = 0$. To present a germ of smooth web and keep track of its defining 1-forms two alternative notations will be used: $W = W(\omega_1, \omega_2, \cdots, \omega_k)$ or $W = W(\omega_1, \cdots, \omega_{k})$. While the latter is self-explanatory the former presents $W$ as an object defined by an element of $\text{Sym}^k \Omega^1(\mathbb{C}^n, 0)$. Notice that the general position assumption translates into

$$\omega_{i_1} \wedge \cdots \wedge \omega_{i_m}(0) \neq 0$$

where $\{i_1, \ldots, i_m\}$ is any subset of $\mathbb{k}$ of cardinality $m \leq \min\{k, n\}$.

Since the foliations $\mathcal{F}_i$ are smooth they can be defined by level sets of submersions $u_i : (\mathbb{C}^n, 0) \to \mathbb{C}$. When profitable to present the web in terms of its defining submersions $W = W(u_1, \ldots, u_k)$ will be used.

The germs of quasi-smooth webs on $(\mathbb{C}^n, 0)$ are defined by replacing the general position hypothesis on the tangent spaces at zero by the weaker condition of pairwise transversality. Explicitly, a germ of quasi-smooth $k$-web $W = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ on $(\mathbb{C}^n, 0)$ is a collection of smooth foliations such that $T_0 \mathcal{F}_i \neq T_0 \mathcal{F}_j$ whenever $i$ and $j$ are distinct elements of $\mathbb{k}$.

There are similar definitions for webs of arbitrary (and even mixed) codimensions. Although extremely rich, the theory of webs of arbitrary codimension will not be discussed in this book.
It is also interesting to study webs in different categories. For instance one can paraphrase the definitions above to obtain differentiable, formal, algebraic, ... webs. This text, unless stated otherwise, will stick to the holomorphic category.

1.1.2 Equivalence and first examples

Local web geometry is ultimately interested in the classification of germs of smooth webs up to the natural action of $\text{Diff}(\mathbb{C}^n,0)$ – the group of germs of biholomorphisms of $(\mathbb{C}^n,0)$. If $\varphi \in \text{Diff}(\mathbb{C}^n,0)$ is a germ of biholomorphism then the natural action just referred to is given by

$$\varphi^* \mathcal{W}(\omega_1 \cdots \omega_k) = \mathcal{W}(\varphi^*(\omega_1 \cdots \omega_k)).$$

The germs of $k$-webs $\mathcal{W}(\omega_1 \cdots \omega_k)$ and $\mathcal{W}'(\omega'_1 \cdots \omega'_k)$ will be considered biholomorphically equivalent if

$$\varphi^*(\omega_1 \cdots \omega_k) = u \cdot (\omega'_1 \cdots \omega'_k)$$

for some germ of biholomorphism $\varphi$ and some germ of invertible function $u \in \mathcal{O}^*_n$. In other words, there exists a permutation $\sigma \in S_k$ – the symmetric group on $k$ elements – such that the germs of 2-forms $\varphi^*(\omega_i \wedge \omega'_\sigma(i))$ are identically zero for every $i \in k$.

Figure 1.1: There is only one smooth 2-web.

Clearly the biholomorphic equivalence defines an equivalence relation on the set of smooth $k$-webs on $(\mathbb{C}^n,0)$. When the dimension of the space is greater than or equal to the number of defining foliations, that is when $n \geq k$, there is just one equivalence class. Indeed, if one
considers a smooth $k$-web defined by $k$ submersions $u_i : (\mathbb{C}^n, 0) \to \mathbb{C}$ then the map $U : (\mathbb{C}^n, 0) \to C^k$, $U = (u_1, \ldots, u_k)$ is a submersion thanks to the general position hypothesis. The constant rank Theorem ensures the existence of a biholomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ taking the function $u_i$ to the coordinate function $x_i$ for every $i \in \mathbb{C}$. Symbolically, $\varphi^* u_i = x_i$.

When the number of defining foliations exceeds the dimension of the space by at least two ($k \geq n + 2$) then one can see the existence of a multitude of equivalence class by the following considerations.

For a $k$-web $W = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$, the tangent spaces of the foliations $\mathcal{F}_i$ at the origin determine a collection of $k$ unordered points in $PT_0^* (\mathbb{C}^n, 0) = \mathbb{P}^{n-1}$. The set of isomorphism classes of $k$ unordered points in general position in a projective space $\mathbb{P}^{n-1}$ is the quotient of the open subset $U$ of $(\mathbb{P}^{n-1})^k$ parametrizing $k$ distinct points in general position by the action

$$((\sigma, g), (x_1, \ldots, x_k)) \mapsto (g(x_{\sigma(1)}), \ldots, g(x_{\sigma(k)}))$$

of the group $G = \mathfrak{S}_k \times \text{PGL}(n, \mathbb{C})$.

When $k \leq n + 1$ the action of $G$ on $U$ is transitive and there is exactly one isomorphism class. When $k \geq n + 2$ the action is locally free (the stabilizer of any point in $U$ is finite) and in particular the set of isomorphism classes of $k$ unordered points in $\mathbb{P}^{n-1}$ has dimension $(k - n - 1)(n - 1)$.

If $W$ and $W' = \varphi^* W$ are two biholomorphically equivalent $k$-webs on $(\mathbb{C}^n, 0)$ then their tangent spaces at the origin determine two sets of $k$ points on $\mathbb{P}^{n-1}$ which are isomorphic through $[d\varphi(0)]$, the projective automorphism determined by the projectivization of the linear map $d\varphi(0)$. It is then clear that for $k \geq n + 2$ there are many non equivalent germs of smooth $k$-webs on $(\mathbb{C}^n, 0)$.

§

It is tempting to infer from the discussion above that there is only one equivalence class of smooth $(n + 1)$-webs on $(\mathbb{C}^n, 0)$ using the following fallacious argument: (a) to a $(n + 1)$-webs on $(\mathbb{C}^n, 0)$ one can associate $n + 1$ sections of $PT^* (\mathbb{C}^n, 0)$; (b) since there is only one isomorphism class of unordered $(n + 1)$ points in general position
in \( \mathbb{P}^{n-1} \) these sections can be sent, through an biholomorphism of \( \mathbb{P}T^*(\mathbb{C}^n,0) \), to the constant sections \([dx_1],\ldots,[dx_n],[dx_1+\cdots+dx_n]\); (c) therefore (a) and (b) implies that every smooth \((n+1)\)-web is equivalent to the web \(W(dx_1,\ldots,dx_n,dx_1+\cdots+dx_n)\).

While (a) and (b) are sound, the conclusion (c) is completely unjustified. The point is that the automorphism used in (b) is not necessarily induced by a biholomorphism \( \varphi \in \text{Diff} (\mathbb{C}^n,0) \). To wit, every biholomorphism \( \Phi : \mathbb{P}T^*(\mathbb{C}^n,0) \rightarrow \mathbb{P}T^*(\mathbb{C}^n,0) \) can be written in the form

\[
\Phi(x,v) = (\varphi(x),[A(x) \cdot v])
\]

where \( \varphi \in \text{Diff} (\mathbb{C}^n,0) \) and \( A \in \text{GL}(n,\mathfrak{o}(\mathbb{C}^n,0)) \). But for very few of them \([A(x) \cdot v] = [d\varphi(x) \cdot v] \).

It will be shown in Section 1.2 that not every 3-web on \((\mathbb{C}^2,0)\) is equivalent to the parallel 3-web \(W(dx,dy,dx+dy)\).

### 1.1.3 Algebraic webs

Given a projective curve \( C \subset \mathbb{P}^n \) of degree \( d \) and a hyperplane \( H_0 \in \mathbb{P}^n \) intersecting \( C \) transversely, there is a natural germ of quasi-smooth \( d \)-web \( W_C(H_0) \) on \((\mathbb{P}^n,H_0)\) defined by the submersions \( p_1,\ldots,p_d : (\mathbb{P}^n,H_0) \rightarrow C \) which describe the intersections of \( H \in (\mathbb{P}^n,H_0) \) with \( C \). Explicitly, if one writes the restriction of \( C \) to a sufficiently small neighborhood of \( H_0 \subset \mathbb{P}^n \) as \( C_1 \cup \cdots \cup C_d \), where the curves \( C_i \) are pairwise disjoint curves, then the functions \( p_i \) are defined as \( p_i(H) = H \cap C_i \). The corresponding \( d \)-web is \( W_C(H_0) = W(p_1,\ldots,p_d) \). The \( d \)-webs of the form \( W_C(H_0) \) for some reduced projective curve \( C \) and some transverse hyperplane \( H_0 \) are classically called algebraic \( d \)-webs.

From the definition of \( p_i \) it is clear that the inclusion

\[
p_i^{-1}(p_i(H)) \subset \{ H' \in \mathbb{P}^n \mid p_i(H') \in H \}
\]

holds true for every \( H \in (\mathbb{P}^n,H_0) \) and every \( i \in \mathcal{I} \). In other words the fiber of \( p_i \) through a point \( H \in (\mathbb{P}^n,H_0) \) is contained in the set of hyperplanes that contain the point \( p_i(H) \in C_i \subset C \subset \mathbb{P}^n \). Consequently the fibers of the submersions \( p_i \) are (pieces of) hyperplanes.
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Figure 1.2: On the left $\mathcal{W}_C$ is pictured for a planar reduced cubic curve $C$ formed by a line and a conic. On the right $\mathcal{W}_C$ is drawn for a planar rational quartic $C$.

It is clear from the definition of $\mathcal{W}_C(H_0)$ that when $C$ is a reducible curve with irreducible components $C_1, \ldots, C_m$ then

$$\mathcal{W}_C(H_0) = \mathcal{W}_{C_1}(H_0) \boxtimes \cdots \boxtimes \mathcal{W}_{C_m}(H_0).$$

Moreover, it has not been really used that $C$ is a projective curve. Indeed, if it is agreed to define linear webs as the ones for which all leaves are (pieces of) hyperplanes then the construction just presented establishes an equivalence between linear quasi-smooth $k$-webs on $(\mathbb{P}^n, H_0)$, and $k$ germs of curves in $\mathbb{P}^n$ intersecting $H_0$ transversely in $k$ distinct points.

Back to the case where $C$ is projective, if no irreducible component of $C$ is a line then there is the following alternative description of $\mathcal{W}_C(H_0)$. Let $\check{C}$ be the dual hypersurface of $C$, that is, $\check{C} \subset \mathbb{P}^n$ is the closure of the union of hyperplanes $H \in \mathbb{P}^n$ containing a tangent line of $C$ at some smooth point $p \in C_{sm}$. Symbolically,

$$\check{C} = \bigcup_{p \in C_{sm}} \bigcup_{H \in \mathbb{P}^n, T_pC \subset H} H.$$

The leaves of $\mathcal{W}_C(H_0)$ through $H_0$ are the hyperplanes passing through it and tangent to $\check{C}$ at some point $p \in \check{C}$. 

A similar interpretation holds true when \( C \) does contain lines among its irreducible components. The differences are: the dual of a line is no longer a hypersurface but a \( \mathbb{P}^{n-2} \) linearly embedded in \( \mathbb{P}^n \); and the leaf of a 1-web dual to a line, through a point \( H_0 \in \mathbb{P}^n \) is the hyperplane in \( \mathbb{P}^n \) containing both \( H_0 \) and the dual \( \mathbb{P}^{n-2} \).

### 1.2 Planar 3-webs

This section presents one of the founding stone of web geometry: the characterization of hexagonal planar 3-webs through their holonomy and curvature. The exposition here follows closely [80]. The hexagonality of algebraic 3-webs is also worked out in detail, see Section 1.2.4. For a more leisure account the reader can consult [45, Lecture 18].

#### 1.2.1 Holonomy and hexagonal webs

Let \( \mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3 \) be a germ of smooth 3-web on \((\mathbb{C}^2, 0)\). Denote by \( L_1, L_2, L_3 \) the leaves through 0 of \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \) respectively.

If \( x = x_1 \in L_1 \) is a point sufficiently close to the origin then, thanks to the persistence of transversal intersections under small deformations, the leaf of \( \mathcal{F}_3 \) through it intersects \( L_2 \) in a unique point \( x_2 \) close to the origin. Moreover the map that associates to
$x = x_1 \in L_1$ the point $x_2 \in L_2$ is a germ of biholomorphic map $h_{12} : (L_1, 0) \to (L_2, 0)$.

Analogously there exists a biholomorphism $h_{23} : (L_2, 0) \to (L_3, 0)$ that associates to $x_2 \in L_2$ the point $x_3 \in L_3$ defined by the intersection of $L_3$ with the leaf of $\mathcal{F}_1$ through $x_2 \in L_2$.

Proceeding in this way one can construct a sequence of points $x_1 \in L_1, x_2 \in L_2, x_3 \in L_3, x_4 \in L_1, x_5 \in L_2, x_6 \in L_3, x_7 \in L_1$. The function that associates to the initial point $x = x_1$ the end point $x_7$ is the germ of biholomorphism $h : (L_1, 0) \to (L_1, 0)$ given by the composition

$$h_{31} \circ h_{23} \circ h_{12} \circ h_{31} \circ h_{23} \circ h_{12}.$$

The reader is invited to verify the following properties of the biholomorphism $h$.

(a) If one does the same construction but with the roles of the foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ replaced by the foliations $\mathcal{F}_{\sigma(1)}, \mathcal{F}_{\sigma(2)}, \mathcal{F}_{\sigma(3)}$ – $\sigma$ being a permutation of $\{1, 2, 3\}$ – then the resulting biholomorphism is conjugated to $h^{\text{sign}(\sigma)}$;

(b) If $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a biholomorphism and $W = \varphi^* W$ then the corresponding biholomorphism $\bar{h} : (L_1, 0) \to (L_1, 0)$ for the leaf $L_1 = \varphi^{-1}(L_1)$ of $\mathcal{F}_1 = \varphi^* \mathcal{F}_1$ is equal to $\varphi^{-1} \circ h \circ \varphi$.

It follows from the two properties above that the conjugacy class in $\text{Diff}(\mathbb{C}, 0)$ of the group generated by $h$ is intrinsically attached to $W$. This class is by definition the holonomy of $W$ at $0$. It will be convenient to say that $h$ is the holonomy of $W$ at $0$ instead of repeatedly referring to the conjugacy class of the group generated by it. Hopefully no confusion will arise from this abuse of terminology.

To get a better grasp of the definition of the holonomy of $W$ and to prepare the ground for what is to come, a family of examples parametrized by $\mathbb{C}$ is presented below in the form of a lemma.

**Lemma 1.2.1.** If $k \in \mathbb{C}$ is a complex number and $W_k = W(x, y, x + y + xy(x - y)(k + h.o.t.))$ then the holonomy of $W_k$ is generated by a germ of biholomorphism $h_k : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ which has as first coefficients in its series expansion

$$h_k(x) = x + 4kx^3 + h.o.t..$$
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Proof. Let \( x_1 = (x, 0) \in (L_1, 0) \). To compute \( x_2 \) notice that \( f_k(x, y) = x + y + xy(x - y)(k + h.o.t.) \) is equal to \( x \) when evaluated on both \( x_1 = (x, 0) \) and \( (0, x) \). In other words the leaf of \( \mathcal{F}_3 \), the foliation determined by \( f_k \), through \( x_1 \) cuts the leaf of \( \mathcal{F}_2 \), the foliation determined by \( y \), in \( x_2 = (0, x) \).

From the definition of \( x_3 \) it is clear that its second coordinate is equal to \( x \). To determine its first coordinate one has to solve the implicit equation \( f_k(t, x) = 0 \). A straightforward computation yields \( t = -x - 2kx^3 + h.o.t. \) and consequently \( x_3 = (-x - 2kx^3, x) \) up to higher order terms.

Proceeding in this way one finds

\[
\begin{align*}
    x_4 &= (-x - 2kx^3, 0) \\
    x_5 &= (0, -x - 2kx^3) \\
    x_6 &= (x + 4kx^3, -x - 2kx^3) \\
    x_7 &= (x + 4kx^3, 0)
\end{align*}
\]

up to higher order terms. The details are left to the reader.

In what concerns their holonomy the simplest smooth 3-webs on \((\mathbb{C}^2, 0)\) are the **hexagonal webs**. By definition these are the ones which can be represented in a neighborhood \( U \) of the origin by three pairwise transversal smooth foliations whose germification at any point \( x \in U \) is a germ of 3-web with trivial holonomy. Using the convention about germs spelled out at the end of Section 1.1 the
hexagonal webs on \((\mathbb{C}^2, 0)\) are the ones with trivial holonomy at every point \(x \in (\mathbb{C}^2, 0)\). The guiding example is \(W(x, y, x+y)\), see Figure 1.5 for a proof of its hexagonality.

![Hexagonal Web](image)

**Figure 1.5:** \(W(x, y, x+y)\) is hexagonal.

Beware that hexagonality is much stronger than asking the holonomy to be trivial only at the origin. It is an instructive exercise to produce an example of 3-web having trivial holonomy at zero but non-trivial holonomy at a generic \(x \in (\mathbb{C}^2, 0)\).

### 1.2.2 Curvature for planar 3-webs

Suppose now that a 3-web \(W\) on \((\mathbb{C}^2, 0)\) is presented by its defining 1-forms, that is, \(W = W(\omega_1, \omega_2, \omega_3)\).

There exist invertible functions \(u_1, u_2, u_3 \in \mathcal{O}^*(\mathbb{C}^2, 0)\) for which

\[
    u_1 \omega_1 + u_2 \omega_2 + u_3 \omega_3 = 0. \tag{1.1}
\]

For instance, if \(\delta_{ij}\) are the holomorphic functions defined by the relation \(\delta_{ij} dx \wedge dy = \omega_i \wedge \omega_j\) for \(i, j = 1, 2, 3\) then

\[
    \delta_{23} \omega_1 + \delta_{31} \omega_2 + \delta_{12} \omega_3 = 0.
\]

Although the triple \((\delta_{23}, \delta_{31}, \delta_{12})\) is not the unique solution to equation (1.1), any other will differ from it by the multiplication by an invertible function. In other words, the most general solution of (1.1) is \((u_1, u_2, u_3) = u \cdot (\delta_{23}, \delta_{31}, \delta_{12})\) where \(u \in \mathcal{O}^*(\mathbb{C}^2, 0)\) is arbitrary.
Lemma 1.2.2. Let $\alpha_1, \alpha_2, \alpha_3 \in \Omega^1(\mathbb{C}^2, 0)$ be three 1-forms with pairwise wedge product nowhere zero. If $\alpha_1 + \alpha_2 + \alpha_3 = 0$ then there exists a unique 1-form $\eta$ such that

$$d\alpha_i = \eta \wedge \alpha_i \quad \text{for every } i \in \{1, 2, 3\}.$$ 

Proof. Because the ambient space has dimension two, and the 1-forms $\alpha_i$ are nowhere zero, there exist 1-forms $\gamma_i$ satisfying

$$d\alpha_i = \gamma_i \wedge \alpha_i.$$ 

Notice that the 1-forms $\gamma_i$ can be replaced by $\gamma_i + a_i \alpha_i$, with $a_i \in \mathcal{O}(\mathbb{C}^2, 0)$ arbitrary, without changing the identity above.

The difference $\gamma_1 - \gamma_2$ is again a 1-form. As such, it can be written as $a_1 \alpha_1 + a_2 \alpha_2$ with $a_1, a_2 \in \mathcal{O}(\mathbb{C}^2, 0)$. Therefore

$$\gamma_1 - a_1 \alpha_1 = \gamma_2 - a_2 \alpha_2.$$ 

If $\eta = \gamma_1 - a_1 \alpha_1 = \gamma_2 - a_2 \alpha_2$ then it clearly satisfies $d\alpha_1 = \eta \wedge \alpha_1$ and $d\alpha_2 = \eta \wedge \alpha_2$. Moreover, since $\alpha_3 = -\alpha_1 - \alpha_2$, it also satisfies $d\alpha_3 = \eta \wedge \alpha_3$. This establishes the existence of $\eta$. For the uniqueness, notice that two distinct solutions $\eta$ and $\eta'$ would verify $(\eta - \eta') \wedge \alpha_i = 0$ for $i = 1, 2, 3$. Any two of these identities are sufficient to ensure that $\eta = \eta'$.

The lemma above applied to $(\delta_{23}\omega_1, \delta_{31}\omega_2, \delta_{12}\omega_3)$ yields the existence of $\eta \in \Omega^1(\mathbb{C}^2, 0)$ for which $d(\delta_{jk}\omega_i) = \eta \wedge (\delta_{jk}\omega_i)$ for any cyclic permutation $(i, j, k)$ of $(1, 2, 3)$. Presenting $\mathcal{W}$ through three others 1-forms – say $\omega'_1 = a_1\omega_1, \omega'_2 = a_2\omega_1$, and $\omega'_3 = a_3\omega_3$ – one sees that the corresponding $\eta'$ relates to $\eta$ through the equation

$$\eta - \eta' = d\log(a_1a_2a_3).$$ 

In particular the 1-form $\eta$ depends on the presentation of $\mathcal{W}$ but in such a way that its differential does not. The 2-form $d\eta$ is, by definition, the curvature of $\mathcal{W}$ and will be denoted by $K(\mathcal{W})$.

It seems appropriate to borrow terminology from the XIX century theory of invariants and say that the 2-form $K(\mathcal{W})$ is a covariant of the web $\mathcal{W}$ since

$$K(\varphi^*\mathcal{W}) = \varphi^*K(\mathcal{W})$$ 

for every germ of biholomorphism $\varphi \in \text{Diff}(\mathbb{C}^2, 0)$. 

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Lemma 1.2.3. If $W = W(x, y, f)$ where $f \in \mathcal{O}(\mathbb{C}^2, 0)$ then

$$K(W) = \frac{\partial^2}{\partial x \partial y} \left( \log \left( \frac{f_x}{f_y} \right) \right) dx \wedge dy.$$

In particular, if $W_k = W(x, y, x + y + xy(x - y)(k + h.o.t.))$ then

$$K(W_k) = 4k dx \wedge dy$$

up to higher order terms.

Proof. Because $(-f_x dx) + (-f_y dy) + (df) = 0$, the 1-form $\eta$ is

$$\partial_x (\log f_y) dx + \partial_y (\log f_x) dy.$$

Hence $K(W)$ is as claimed.

Specializing to $W_k = W_k(x, y, x + y + kxy(x - y))$ it follows that

$$K(W_k) = \frac{\partial^2}{\partial x \partial y} \log \left( \frac{1 + (2xy - y^2)(k + h.o.t.)}{1 - (2xy - x^2)(k + h.o.t.)} \right) dx \wedge dy.$$

The second claim follows from the evaluation of the above expression at zero.

Structure of planar hexagonal 3-webs

The next result can be considered as the founding stone of web geometry. It seems fair to say that it awakened the interest of Blaschke and his coworkers on the subject in the early 1930’s.

Theorem 1.2.4. Let $W = F_1 \boxtimes F_2 \boxtimes F_3$ be a smooth 3-web on $(\mathbb{C}^2, 0)$. The following assertions are equivalent:

(a). the web $W$ is hexagonal;

(b). the 2-form $K(W)$ vanishes identically;

(c). there exists closed 1-forms $\eta_i$ defining $F_i$, $i = 1, 2, 3$, such that $\eta_1 + \eta_2 + \eta_3 = 0$;

(d). the web $W$ is equivalent to $W(x, y, x + y)$. 

Besides Lemma 1.2.1 and Lemma 1.2.3, the proof of Theorem 1.2.4 will also make use of the following.

**Lemma 1.2.5.** Every germ of smooth 3-web $W$ on $(\mathbb{C}^2, 0)$ is equivalent to $W(x, y, f)$, where $f \in \mathcal{O}(\mathbb{C}^2, 0)$ is of the form
\[ f(x, y) = x + y + xy(x - y)(k + h.o.t.) \]
for a suitable $k \in \mathbb{C}$.

**Proof.** As already pointed out in Section 1.1.2 every smooth 2-web on $(\mathbb{C}^2, 0)$ is equivalent to $W(x, y)$. Therefore it can be assumed that $W = W(x, y, g)$ with $g \in \mathcal{O}(\mathbb{C}^2, 0)$ such that $g(0) = 0$. The smoothness assumption on $W$ translates into $dx \wedge dg(0) \neq 0$ and $dy \wedge dg(0) \neq 0$ or, equivalently, both $g_x(0)$ and $g_y(0)$ are non-zero complex numbers.

After pulling back $W$ by $\varphi_1(x, y) = (g_x(0)x, g_y(0)y)$ one can assume that $W$ still takes the form $W(x, y, g)$ but now with the function $g$ having $x + y$ as its linear term.

Let $a(t) = g(t, 0)$ and $b(t) = g(0, t)$. Clearly both $a$ and $b$ are germs of biholomorphisms of $(\mathbb{C}, 0)$. Let $\varphi(x, y) = (a^{-1}(x), b^{-1}(y))$ and set $h(x, y) = \varphi^*g(x, y) = g(a^{-1}(x), b^{-1}(y))$. Notice that $\varphi^*W(x, y, g) = W(x, y, h)$ and that $h$ still has linear term equal to $x + y$. Moreover $h(0, t) = h(t, 0) = h(t, t)/2 = t$ up to higher order terms.

Because the germ $\alpha(t) = h(t, t)$ has derivative at zero of modulus distinct from one it follows from Poincaré Linearization Theorem [7, Chapter 3, §25.B] the existence of a germ of biholomorphism $\phi \in \text{Diff}(\mathbb{C}, 0)$ conjugating $\alpha$ to its linear part. More succinctly,
\[ \phi^{-1} \circ \alpha \circ \phi(t) = 2t. \]

After setting $\varphi(x, y) = (\phi(x), \phi(y))$ and $f = \phi^{-1} \circ h \circ \phi$ one promptly verifies the identities
\[ f(t, 0) = f(0, t) = f(t, t)/2 = t. \]

To conclude the proof it suffices to analyze the implications of the above identities to the series expansion $f(x, y) = \sum a_{ij}x^iy^j$. The reader is invited to fill in the details. \qed
Proof of Theorem 1.2.4

To prove that (a) implies (b) start by applying Lemma 1.2.5 to see that $W$, at any point $p \in (\mathbb{C}^2, 0)$, is equivalent to

$$W(x, y, x + y + xy(x - y)(k + h.o.t.)) \text{ with } k \in \mathbb{C}.$$ 

Because $W$ is hexagonal the holonomy at an arbitrary $p \in (\mathbb{C}^2, 0)$ is the identity. Lemma 1.2.1 implies $k = 0$. Lemma 1.2.3, in its turn, allows one to deduce that $K(W)$ is also zero at an arbitrary point of $(\mathbb{C}^2, 0)$, thus proving that (a) implies (b).

Suppose now that (b) holds true, and assume $W = W(\omega_1, \omega_2, \omega_3)$ with the 1-forms $\omega_i$ satisfying $\omega_1 + \omega_2 + \omega_3 = 0$. Let $\eta$ be the unique 1-form given by Lemma 1.2.2. Because $K(W) = 0$, the 1-form $\eta$ is closed. If

$$\eta_i = \exp \left( - \int \eta \right) \omega_i$$

then

$$d\eta_i = -\eta \wedge \exp \left( - \int \eta \right) \omega_i + \exp \left( - \int \eta \right) d\omega_i = 0$$

because $d\omega_i = \eta \wedge \omega_i$. Moreover

$$\eta_1 + \eta_2 + \eta_3 = \exp \left( - \int \eta \right) (\omega_1 + \omega_2 + \omega_3) = 0.$$ 

This proves that (b) implies (c).

Now assuming the validity of (c), one can define

$$f_i(x) = \int_0^x \eta_i \quad \text{for } i \in \mathbb{Z}.$$ 

Notice that $\varphi(x, y) = (f_1(x, y), f_2(x, y))$ is a biholomorphism, and clearly $\varphi^*W(x, y, x + y) = W(f_1, f_2, f_3)$ since $\eta_1 + \eta_2 = -\eta_3$. Thus $W$ is equivalent to $W(x, y, x + y)$.

The missing implication, (d) implies (a), has already been established in Figure 1.5. \qed
1.2.3 Germs of hexagonal webs on the plane

Having Theorem 1.2.4 at hand it is natural to enquire about germs of smooth $k$-webs on $(\mathbb{C}^2, 0)$, $k > 3$, for which every 3-subweb is hexagonal. The $k$-webs having this property will also be called hexagonal.

The simplest examples of hexagonal $k$-webs are the parallel $k$-webs. These webs $W$ are the superposition of $k$ pencils of parallel lines. They all can be written explicitly as

$$W(\lambda_1 x - \mu_1 y, \ldots, \lambda_k x - \mu_k y)$$

where the pairs $\lambda_i, \mu_i \in \mathbb{C}^2 \setminus \{0\}$ represent the slopes $(\mu_i : \lambda_i) \in \mathbb{P}(\mathbb{C}^2) = \mathbb{P}^1$ of the pencils.

More generally, if $\mathcal{L}_1, \ldots, \mathcal{L}_k$ are $k$ pairwise distinct pencils of lines on $\mathbb{C}^2$ such that no line joining two base points passes through the origin then $W = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$, seen as a germ at the origin, is also a smooth hexagonal $k$-web.

Figure 1.6: Bol’s 5-web

A less evident family of examples was found by Bol and are the germs of 5-webs defined as follows. Let $\mathcal{L}_1, \ldots, \mathcal{L}_4$ be four pencil of lines satisfying the same conditions as above, plus the extra condition that no three among the four base points of the pencils are collinear. The 5-web obtained from the superposition of $\mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_4$ with the pencil of conics through the four base points is a smooth hexagonal 5-web on $(\mathbb{C}^2, 0)$. According to the relative position of the base points with respect to the origin, one obtains in this a way a two-dimensional family of non-equivalent germs of smooth 5-webs on $(\mathbb{C}^2, 0)$. Any of
these germs will be called Bol’s 5-web $B_5$. The abuse of terminology is justified by the fact they are all germifications of the very same global singular 5-web (a concept to be introduced in Section 1.3.3) defined on $\mathbb{P}^2$.

Anyone taking the endeavor of finding a smooth hexagonal $k$-web on $(\mathbb{C}^2, 0)$ not equivalent to any of the previous examples is doomed to failure. Indeed Bol proved the following

**Theorem 1.2.6.** If $W$ is a smooth hexagonal $k$-web, $k \geq 3$, then $W$ is equivalent to the superposition of $k$ pencil of lines or $k = 5$ and $W$ is equivalent to $B_5$.

A proof will not be presented here. For a recent exposition, with a fairly detailed sketch of proof, see [99].

### 1.2.4 Algebraic planar 3-webs are hexagonal

**Proposition 1.2.7.** If $C \subset \mathbb{P}^2$ is a reduced cubic and $\ell_0 \subset \mathbb{P}^2$ is a line intersecting $C$ transversely then the 3-web $W_C(\ell_0)$ is hexagonal.

The simplest instance of the proposition above is when $C$ is the union of three distinct concurrent lines. In this particular case it can be promptly verified that $W_C(\ell_0)$ is the 3-web $W(x, y, x - y)$ in a suitable affine coordinate system $(x : y : 1) \in \mathbb{P}^2$ without further ado.

In the next simplest instance, $C$ is still the union of three distinct lines but they are no longer concurrent. Then, $W_C(\ell_0)$ is the 3-web $W(x, y, (x - 1)/(y - 1))$ in suitable affine coordinates. The most straightforward way to verify the hexagonality of $W(x, y, (x - 1)/(y - 1))$ consists in observing that the closed differential forms

$\eta_1 = -d \log(x - 1), \eta_2 = d \log(y - 1)$ and $\eta_3 = d \log((x - 1)/(y - 1))$

define the same foliations as the submersions $x, y$ and $(x - 1)/(y - 1)$ and satisfy $\eta_1 + \eta_2 + \eta_3 = 0$. Therefore the 3-web under scrutiny is hexagonal thanks to the equivalence between items (a) and (c) in Theorem 1.2.4.

To deal with the other cubics, one could still try to make explicit three submersions defining the web and work his way to determine a relation between closed 1-forms defining the very same foliations. Once the submersions are determined the second step is rather
straight-forward since the proof of Theorem 1.2.4 gives an algorithmic way to perform it. Besides having many particular cases to treat, the lack of rational parametrizations for smooth cubics would lead one to compute with Weierstrass $\wp$-functions or similar transcendental objects, adding a considerable amount of difficulty to such task. Perhaps the most elementary way to prove the hexagonality of algebraic planar 3-webs relies on the following Theorem of Chasles.

**Theorem 1.2.8.** Let $X_1, X_2 \subset \mathbb{P}^2$ be two plane cubics meeting in exactly nine distinct points. If $X \subset \mathbb{P}^2$ is any cubic containing at least eight of these nine points then it automatically contains all the nine points.

**Proof.** Aiming at a contradiction, suppose that $X$ does not contain $X_1 \cap X_2$. Let $F_1, F_2$ be homogenous cubic polynomials defining $X_1, X_2$ respectively and $G$ be the one defining $X$. Since there are nine points in the intersection of $X_1$ and $X_2$ then, according to Bezout’s Theorem, the curves $X_1$ and $X_2$ must intersect transversely. In particular both curves are smooth at the intersection points. After replacing $X_1$ by the generic member of the pencil \( \lambda F_1 + \mu F_2 = 0 \) one can assume, thanks to Bertini’s Theorem [58, page 137], that $X_1$ is a smooth cubic. Consequently $X_1$ is a smooth elliptic curve.

Consider now the rational function $h : X_1 \to \mathbb{P}^1$ defined as

$$h = \left( \frac{G}{F_2} \right)_{|_{X_1}}.$$

Since $X$ passes through 8 points of $X_2 \cap X_1$ it follows that $h$ has only one zero: the unique point of $X \cap X_1$ that does not belong to $X_1 \cap X_2$. Moreover, the transversality of $X_1$ and $X_2$ ensures that this zero is indeed a simple zero. It follows that $h$ is an isomorphism. Since elliptic curves are not isomorphic to rational curves one arrives at a contradiction that settles the Theorem.

The proof just presented cannot be qualified as elementary since it makes use of Bertini’s and Bezout’s Theorems and some basic facts of differential topology. For an elementary proof and a comprehensive account on Chasles’ Theorem including its distinguished lineage and recent – rather non-elementary – developments the reader is urged to consult [47].
Proof of Proposition 1.2.7

To deduce the hexagonality of $W_C(\ell_0)$ from Chasles’ Theorem start by observing that the leaf of the foliation $F_i$ through $\ell_0 \in \mathbb{P}^2$, denoted by $L_i$, corresponds to lines through the point $p_i = p_i(\ell_0)$. To choose a point $x_1 \in L_1$ is therefore the same as choosing a line through $p_1 \in C_1 \subset \mathbb{P}^2$. If such line is sufficiently close to $\ell_0$ then it cuts $C_3$ in a unique point still denoted by $x_1$. In this way the leaf $L_1$ of $F_1$ can be identified with the curve $C_3$. It will also be useful to identify through the same process $L_2$, the leaf of $F_2$ through $\ell_0 \in \mathbb{P}^2$, with $C_1$ and $L_3$ with $C_2$.

![Figure 1.7: A cubic with two irreducible components.](image)

Now, follow the leaf of $F_3$ through $x_1 \in C_3$ until it meets $L_2$ corresponds to consider the line $x_1p_2$ and intersect it with $C_1$. The intersection point $x_2 = x_1p_2 \cap C_1 \in C_1$ corresponds to a point in $L_2$.

Similarly the sequence of points $x_3, x_4, \ldots, x_7$ appearing in the definition of the holonomy of $W_C(\ell_0)$ can be synthetically obtained as follows:

- $x_3 = x_2p_3 \cap C_2 \in L_3 \simeq C_2$,
- $x_4 = x_3p_1 \cap C_3 \in L_1 \simeq C_3$,
- $x_5 = x_4p_2 \cap C_1 \in L_2 \simeq C_1$,
- $x_6 = x_5p_3 \cap C_2 \in L_3 \simeq C_2$,
- $x_7 = x_6p_1 \cap C_3 \in L_1 \simeq C_3$.

Of course, all the identifications $L_i \simeq C_j$ above, have to be under-
stood as identifications of germs of curves.

\[
\begin{align*}
\text{Figure 1.8: This is not a cubic.}
\end{align*}
\]

Notice that the line \(x_1x_2\) is the same as \(x_1p_2\). Therefore it contains the three points \(x_1, x_2, p_2\). The line \(x_3x_4\) in its turn contains the points \(x_3, x_4, p_1\) and the line \(x_5x_6\) contains the points \(x_5, x_6, p_3\). Thus the reduced cubic \(X_1 = x_0x_1 \cup x_2x_3 \cup x_4x_5\) intersects the cubic \(X_2 = C\) in exactly nine distinct points namely \(p_1, p_2, p_3, x_1, x_2, x_3, x_4, x_5, x_6\). The same reasoning shows that the reduced cubic \(X = x_2x_3 \cup x_4x_5 \cup x_6x_7\) intersects \(X_2 = C\) in the nine points \(p_1, p_2, p_3, x_2, x_3, x_4, x_5, x_6, x_7\). Thus \(X_1 \cap X_2 \cap X\) contains at least eight points. Chasles’ Theorem implies that this eight is indeed a nine and consequently the points \(x_1\) and \(x_7\) must coincide. This is sufficient to prove that the holonomy of \(W_C(\ell_0)\) is the identity.  

\[\square\]

Later on this text the hexagonality of the planar algebraic 3-webs will be established again using Abel’s addition Theorem. Although apparently unrelated both approaches are intimately intertwined. Abel’s addition Theorem can be read as a result about the group structure of the Jacobian of projective curves while Chasles’ Theorem turns out to be equivalent to the existence of an abelian group structure for plane cubics where aligned points sum up to zero.
1.3 Singular and global webs

1.3.1 Germs of singular webs I

It is customary to say that a germ of singular holomorphic foliation is an equivalence class $[\omega]$ of germs of holomorphic 1-forms in $\Omega^1(\mathbb{C}^n,0)$ modulo multiplication by elements of $O^*(\mathbb{C}^n,0)$ such that any representative $\omega$ is integrable ($\omega \wedge d\omega = 0$) and with singular set $\text{sing}(\omega) = \{ p \in (\mathbb{C}^n,0) ; \omega(p) = 0 \}$ of codimension at least two.

An analogous definition can be made for codimension one webs. A germ of singular codimension one $k$-web on $(\mathbb{C}^n,0)$ is an equivalence class $[\omega]$ of germs of $k$-symmetric 1-forms, that is sections of $\text{Sym}^k \Omega^1(\mathbb{C}^n,0)$, modulo multiplication by $O^*(\mathbb{C}^n,0)$ such that a suitable representative $\omega$ defined in a connected neighborhood $U$ of the origin satisfies the following conditions:

(a). the zero set of $\omega$ has codimension at least two;

(b). $\omega$, seen as a homogeneous polynomial of degree $k$ in the ring $O(\mathbb{C}^n,0)[dx_1, \ldots, dx_n]$, is square-free;

(c). (Brill’s condition) for a generic $p \in U$, $\omega(p)$ is a product of $k$ linear forms;

(d). (Frobenius’ condition) for a generic $p \in U$, the germ of $\omega$ at $p$ is the product of $k$ germs of integrable 1-forms.

Both conditions (c) and (d) are automatic for germs of webs on $(\mathbb{C}^2,0)$ and non-trivial for germs on $(\mathbb{C}^n,0)$ when $n \geq 3$. Notice also that condition (d) implies condition (c). Nevertheless the two conditions are stated independently because condition (c) is of purely algebraic nature (depends only on the value of $\omega$ at $p$) while condition (d) involves the exterior differential and therefore depends not only on the value of $\omega(p)$ but also on the local behavior of $\omega$ near $p$.

A germ of singular web will be called generically smooth if the condition below is also satisfied:

(e). (Generic position) for a generic $p \in U$, any $m \leq n$ linear forms $\alpha_1, \ldots, \alpha_m$ dividing $\omega(p)$ satisfy

$$\alpha_1 \wedge \cdots \wedge \alpha_m \neq 0.$$
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One can rephrase condition (e) by saying that for a generic \( p \in U \) the germ of \( \omega \) at \( p \) defines a smooth web.

Notice that conditions (b) and (c) together imply that every germ of singular web is \textit{generically quasi-smooth}.

§

It is interesting to compare the above definition with the following: a germ of singular codimension \( q \) foliation on \((\mathbb{C}^n, 0)\) is an equivalence class \( \mathcal{F} = [\omega] \) of germs of \( q \)-forms modulo multiplication by elements of \( \mathcal{O}^q_{(\mathbb{C}^n, 0)} \) satisfying ( as above \( \omega \) is a representative of \( \mathcal{F} \) defined on \( U \subset \mathbb{C}^n \) )

(a). the zero set of \( \omega \) has codimension at least two;

(c). (Plücker’s condition) for a generic \( p \in U \), \( \omega(p) \) is a wedge product of \( k \) linear forms \( \alpha_1, \ldots, \alpha_k \);

(d). (Frobenius’ condition) for a generic \( p \in U \), the germ of \( \omega \) at \( p \) is the product of \( k \) germs of 1-forms \( \alpha_1, \ldots, \alpha_k \) and each one of them satisfies \( d\alpha_i \wedge \omega = 0 \).

Notice that the absence of condition (b) is due to the antisymmetric character of \( \Omega^q(\mathbb{C}^n, 0) \). Although apparently similar conditions (c), (d) for codimension \( q \) foliations and \( k \)-webs have rather distinct features.

It is a classical result of Plücker that the \( q \)-form \( \omega(p) \) satisfies condition (c) (foliation version) if only if

\[
(i_v \omega(p)) \wedge \omega(p) = 0 \quad \text{for every} \quad v \in \bigwedge^{q-1} T_p \mathbb{C}^n.
\]

Moreover, varying \( v \in T_p \mathbb{C}^n \), the above formulas are the well known Plücker quadrics and generate the homogeneous ideal defining the locus of completely decomposable \( q \)-forms in \( \bigwedge^q T_p \mathbb{C}^n \), see for instance [58, pages 209–211] or [51, Chapter 3, Theorem 1.5].

Less well known are Brill’s equations describing the locus of completely decomposable \( q \)-symmetric 1-forms, for a modern exposition

\[\text{Here, and throughout, } i_v \text{ denotes interior product.}\]
see [51, chapter 4, section 2]. They differ from Plücker equations in a number of ways: they cannot be so easily described since their definition depends on some concepts of representation theory; they are not quadratic equations, for \( q \)-symmetric 1-forms they are of degree \( q + 1 \); the ideal generated by Brill’s equation is not reduced in general, already for 3-symmetric 1-forms on \( \mathbb{C}^3 \) Brill’s equations do not generate the ideal of the locus of completely decomposable forms, see for instance [39, proposition 2.5].

For condition (d) the situation is even worse. While for alternate \( q \)-forms the integrability condition can be written, see [46, Propositions 1.2.1 and 1.2.2], as

\[
(i_v d\omega) \wedge \omega = 0 \quad \text{for every} \quad v \in q^{-1} T_p \mathbb{C}^n ,
\]

the integrability condition for \( q \)-symmetric 1-forms have not been treated in the literature yet.

### 1.3.2 Germs of singular webs II

There is an alternative definition for germs of singular webs that is in a certain sense more geometric. The idea is to consider the (germ of) web as a meromorphic section of the projectivization of cotangent bundle. This is a classical point of view in the theory of differential equations which has been recently explored in web geometry by Cavalier-Lehmann, see [27]. Beware that the terminology here adopted does not always coincides with the one used in [27].

#### The contact distribution

Let \( \mathbb{P} = \mathbb{P} T^* (\mathbb{C}^n, 0) \) be the projectivization of the cotangent bundle of \( (\mathbb{C}^n, 0) \) and \( \pi : \mathbb{P} \to (\mathbb{C}^n, 0) \) the natural projection. On \( \mathbb{P} \) there is a canonical codimension one distribution, the so called contact distribution \( D \). Its description in terms of a system of coordinates

\footnote{The convention adopted in this text is that over a point \( p \) the fiber \( \pi^{-1}(p) \) parametrizes the one-dimensional subspaces of \( T_p^* (\mathbb{C}^n, 0) \). Beware that some authors consider \( \pi^{-1}(p) \) as a parametrization of the one-dimensional quotients of \( T_p^* (\mathbb{C}^n, 0) \).}
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$x_1, \ldots, x_n$ of $(\mathbb{C}^n, 0)$ goes as follows: if $y_i = \partial_i$ are interpreted as coordinates\(^1\) of the total space of $T^*(\mathbb{C}^n, 0)$ then the lift from $\mathbb{P}$ to $T^*(\mathbb{C}^n, 0)$ of the contact distribution, is the kernel of the 1-form

$$\alpha = \sum_{i=1}^{n} y_i dx_i.$$ \hspace{1cm} (1.2)

The usual way to define $\mathcal{D}$ in more intrinsic terms goes as follows. Recall that the tautological line-bundle $\mathcal{O}_\mathbb{P}(-1)$ is the rank one sub-bundle of $\pi^*T^*(\mathbb{C}^n, 0)$ determined over a point $p = (x, [y]) \in \mathbb{P}$ by the direction parametrized by it. Its dual $\mathcal{O}_\mathbb{P}(1)$ is therefore a quotient of $\pi^*T(\mathbb{C}^n, 0)$. The distribution on $\mathbb{P}$ induced by the kernel of the composition

$$T_\mathbb{P} \xrightarrow{d\pi} \pi^*T(\mathbb{C}^n, 0) \xrightarrow{\cdot} \mathcal{O}_\mathbb{P}(1)$$

is nothing more than the contact distribution $\mathcal{D}$. Notice that the composition is given by the interior product of local sections of $T_\mathbb{P}$ with a twisted 1-form $\alpha \in H^0(\mathbb{P}, \Omega^1_P \otimes \mathcal{O}_\mathbb{P}(1))$, which in local coordinates coincides with the 1-form (1.2). This 1-form is the so called contact form of $\mathbb{P}$.

**Webs as closures of meromorphic multi-sections**

Let now $W \subset \mathbb{P}$ be a subvariety not necessarily irreducible but of pure dimension $d$. Suppose also that $W$ satisfies the following conditions

(a) the image under $\pi$ of every irreducible component of $W$ has dimension $n$;

(b) the generic fiber of $\pi$ intersects $W$ in $k$ distinct smooth points, and at these the differential $d\pi|_W : T_p W \to T_{\pi(p)}(\mathbb{C}^n, 0)$ is surjective; and

(c) the restriction of the contact form $\alpha$ to the smooth part of every irreducible component of $W$ is integrable.

---

\(^1\)In case of confusion, notice that the coordinate functions on a vector space $E$ can be chosen to be elements of $E^*$, that is, linear forms on $E$. 
One can then define a germ of web as the subvarieties $W$ of $\mathbb{P}$ as above. This definition is equivalent to the one laid down in Section 1.3.1. Indeed given a singular $k$-web $[\omega]$ in the sense of §1.3.1 one can consider the closure of its “graph” in $\mathbb{P}$. More precisely, over a generic point $p \in (\mathbb{C}^n, 0)$ the “graph” of $[\omega(p)]$ is formed by the points in $\mathbb{P}T^*_p(\mathbb{C}^n, 0)$ corresponding to the factors of $\omega(p)$. In this way one defines a locally closed subvariety of $\mathbb{P}$ with closure satisfying conditions (a), (b) and (c) above.

Reciprocally the restriction of the contact form $\alpha$ to a subvariety $W \subset \mathbb{P}$ satisfying (a), (b) and (c) above induces a codimension one foliation $\mathcal{F}$ on the smooth part of $W$. Moreover, over regular values of $\pi$ the direct image of $\mathcal{F}$ can be identified with the superposition of $k$ foliations. Since the symmetric product of the $k$ distinct 1-forms defining these foliations is invariant under the monodromy of $\pi$, one ends up with a germ of section of $\text{Sym}^k \Omega^1_{(\mathbb{C}^n, 0)}$ inducing $\pi_*\mathcal{F}$. After cleaning up eventual codimension one components of the zero set one obtains a $k$-symmetric 1-form $\omega$ satisfying the conditions (a), (b), (c) and (d) of Section 1.3.1.

### 1.3.3 Global webs

Although this text is ultimately interested in the classification of germs of smooth webs of maximal rank, a concept to be introduced in Chapter 2, most of the relevant examples are globally defined on projective manifolds. It is therefore natural to lay down the definitions of a global web and related concepts.

A global $k$-web $\mathcal{W}$ on a manifold $X$ is given by an open covering $\mathcal{U} = \{U_i\}$ of $X$ and $k$-symmetric 1-forms $\omega_i \in \text{Sym}^k \Omega^1_X(U_i)$ subject to the conditions:

1. for each non-empty intersection $U_i \cap U_j$ of elements of $\mathcal{U}$ there exists a non-vanishing function $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ such that $\omega_i = g_{ij} \omega_j$;

2. for every $U_i \in \mathcal{U}$ and every $x \in U_i$ the germification of $\omega_i$ at $x$ satisfies the conditions (a), (b), (c) and (d) of Section 1.3.1, in other words, the germ of $\omega_i$ at $x$ is a representative of a germ of a singular web.
The transition functions $g_{ij}$ determine a line-bundle $\mathcal{N}$ over $X$ and the $k$-symmetric 1-forms $\{\omega_i\}$ patch together to form a section of $\text{Sym}^k\Omega^1_X \otimes \mathcal{N}$, that is, $\omega = \{\omega_i\}$ can be interpreted as an element of $H^0(X, \text{Sym}^k\Omega^1_X \otimes \mathcal{N})$. The line-bundle $\mathcal{N}$ will be called the normal bundle of $W$. Two global sections $\omega, \omega' \in H^0(X, \text{Sym}^k\Omega^1_X \otimes \mathcal{N})$ determine the same web if and only if they differ by the multiplication by an element $g \in H^0(X, \mathcal{O}_X^*)$.

If $X$ is compact, or more generally if the only global sections of $\mathcal{O}_X^*$ are the non-zero constants, then a global $k$-web is nothing more than an element of $\text{Pic}(X)$, with germification of any representative at any point of $X$ satisfying conditions (a), (b), (c) and (d) of Section 1.3.1.

When $X$ is a variety for which every line-bundle has non-zero meromorphic sections one can alternatively define global $k$-webs as equivalence classes $[\omega]$ of meromorphic $k$-symmetric 1-forms modulo multiplication by meromorphic functions such that at a generic point $x \in X$ the germification of any representative $\omega$ satisfies the very same conditions referred to above. The transition to the previous definition is made by observing that a meromorphic $k$-symmetric 1-form $\omega$ can be interpreted as a global holomorphic section of $\text{Sym}^k\Omega^1_X \otimes \mathcal{O}_X((\omega)_0 - (\omega)_{\infty})$ where $(\omega)_0$, respectively $(\omega)_{\infty}$, stands for the zero divisor, respectively polar divisor, of $\omega$.

A $k$-web $W \in \mathbb{P}H^0(X, \text{Sym}^k\Omega^1_X \otimes \mathcal{N})$ is decomposable if there are global webs $W_1, W_2$ on $X$ sharing no common subwebs such that $W$ is the superposition of $W_1$ and $W_2$, that is $W = W_1 \boxtimes W_2$. A $k$-web $W$ will be called completely decomposable if one can write $W = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ for $k$ global foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$ on $X$. Remark that the restriction of a web at a sufficiently small neighborhood of a generic $x \in X$ is completely decomposable.

**Monodromy**

Thanks to condition 1.3.1.(b) the germ of a global $k$-web $W$ at a generic point $x \in X$ is completely decomposable. Moreover the set of points $x \in X$ where $W_x$ is not completely decomposable is a closed analytic subset of $X$. If $U$ is the complement of this subset then for arbitrary $x_0 \in U$ it is possible to write $W_{x_0}$, the germ of $W$ at $x_0$, as $\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. Notice that $W$ does not have to be quasi-smooth at
$x_0 \in U$. It may happen that $T_{x_0}F_i = T_{x_0}F_j$ for some $i \neq j$.

Analytic continuation of this decomposition along paths $\gamma$ contained in $U$ determines an anti-homomorphism from the fundamental group of $U$ to the permutation group on $k$ letters $\mathfrak{S}_k$. Because distinct choices of base points yield conjugated anti-homomorphisms, it is harmless to identify all these anti-homomorphisms and call them the monodromy (anti)-representation of $W$. The image of $\rho_W \subset \mathfrak{S}_k$ is, by definition, the monodromy group of $W$.

The reader is invited to verify the validity of the following proposition.

**Proposition 1.3.1.** The following assertions hold:

(a) If $W$ is not completely decomposable then there exists $\gamma \in \pi_1(U,x_0)$ such that $\rho_W(\gamma)$ is a non-trivial permutation;

(b) Every irreducible component of the complement of $U$ has codimension one.

Proposition 1.3.1 makes clear that $\rho_W$ measures the obstruction to $W$ be completely decomposable.

Alternatively one can also define a global $k$-web on $X$ as a closed subvariety $W \subset \mathbb{P}(TX)$ satisfying the natural global analogues of conditions (a), (b) and (c) of Section 1.3.2. In this alternative take the monodromy is nothing more than the usual monodromy of the projection $\pi|_W : W \to X$.

### 1.3.4 Discriminant

The discriminant locus $\Delta(W)$ of a $k$-web $W$ on a complex manifold $X$ is composed by the set of points where the germ of $W$ is not quasi-smooth. Thinking $W$ as a subvariety $W \subset \mathbb{P}(TX)$, the discriminant

\[\Delta(W) = \{ x \in X | T_xW \text{ is not smooth} \}\]

As usual the fundamental group acts on the right and thus $\rho_W$ is not a homomorphism but instead an anti-homomorphism, that is

\[\rho_W(\gamma_1 \cdot \gamma_2) = \rho_W(\gamma_2) \cdot \rho_W(\gamma_1)\]

for arbitrary $\gamma_1, \gamma_2 \in \pi_1(U,x_0)$.
is precisely the image under the natural \( \pi|_W : W \to X \) of the union of singular points of \( W \) with the critical set of the restriction of \( \pi|_W \) to the smooth part.

From its very definition it is clear that \( \Delta(W) \) is a closed analytic subset with complement contained in the subset \( U \) used in the definition of the monodromy representation. Therefore the monodromy representation can be thought as a anti-homomorphism from \( \pi_1(X \setminus \Delta(W)) \) to \( \mathbb{S}_k \).

For webs \( W \) on surfaces there are simple expressions for their discriminants inherited from the classical invariant theory of binary forms.

The resultant and tangencies between webs on surfaces

Recall that for two homogeneous polynomials in two variables, also known as binary forms,

\[
P = \sum_{i=0}^{m} p_i x^i y^{m-i} \quad \text{and} \quad Q = \sum_{i=0}^{n} q_i x^i y^{n-i}
\]

the resultant \( R[P,Q] \) of \( P \) and \( Q \) is given by the determinant of the Sylvester matrix

\[
\begin{bmatrix}
p_{m} & \cdots & \cdots & p_0 \\
\vdots & & & \vdots \\
p_{m} & \cdots & \cdots & p_0 \\
q_{n} & \cdots & q_0 \\
\vdots & & & \vdots \\
q_{n} & \cdots & q_0
\end{bmatrix}
\]

This is the \((m + n) \times (m + n)\)-matrix formed from the coefficients of \( P \) and \( Q \) as schematically presented above with \( n = \deg(Q) \) rows builded from the coefficients of \( P \) and \( m = \deg(P) \) rows coming from the coefficients of \( Q \).

If \( g(x, y) = (\alpha x + \beta y, \gamma x + \delta y) \) is a linear automorphism of \( \mathbb{C}^2 \) and \( \lambda, \mu \in \mathbb{C}^* \) then the resultant obeys the transformation rules

\[
R[\lambda P, \mu Q] = \lambda^{\deg(Q)} \mu^{\deg(P)} R[P,Q], \\
R[g^* P, g^* Q] = \det(Dg)^{\deg(P) \cdot \deg(Q)} R[P,Q].
\] (1.3)
Moreover $R[P,Q]$ vanishes if and only if $P$ and $Q$ share a common root.

If, for $\ell = 1, 2$, $W_{\ell} = [\omega_{\ell}] \in PH^{0}(S, Sym^{k_{\ell}}\Omega_{S}^{1} \otimes N_{S}^{k_{\ell}})$ is a $k_{\ell}$-web on a surface $S$ then the local defining 1-forms $\omega_{\ell,i} \in Sym^{k_{\ell}}\Omega_{S}^{1}(U_{i})$ can be interpreted as binary forms in the variables $dx, dy$ with coefficients in $O_{S}(U_{i})$. The resultant $R[\omega_{1,i}, \omega_{2,i}]$ is then an element of $O_{S}(U_{i})$ with zero locus coinciding with the tangencies between $W_{1}|_{U_{i}}$ and $W_{2}|_{U_{i}}$.

The transformation rules (1.3) imply that the collection $\{R[\omega_{1,i}, \omega_{2,i}]\}$ patch together to form a global holomorphic section of the line-bundle $K_{S}^{\otimes k_{1}} \cdot k_{2} \cdot N_{1}^{\otimes k_{2}} \otimes N_{2}^{\otimes k_{1}}$. This section is different from the zero section if and only if $W_{1}$ and $W_{2}$ do not share a common subweb since the resultant vanishes only when its parameters share common roots.

If $\text{tang}(W_{1}, W_{2})$ is defined as the divisor locally given by the resultant of the defining $k_{\ell}$-symmetric 1-forms then the discussion just made can be summarized in the following proposition.

**Proposition 1.3.2.** Let $W_{1}$ be a $k_{1}$-web and $W_{2}$ a $k_{2}$-web with respective normal bundles $N_{1}$ and $N_{2}$, both defined on the same surface $S$. If they do not share a common subweb then the identity

$$O_{S}(\text{tang}(W_{1}, W_{2})) = K_{S}^{\otimes k_{1}} \cdot k_{2} \cdot N_{1}^{\otimes k_{2}} \otimes N_{2}^{\otimes k_{1}}$$

holds true in the Picard group of $S$.

**The discriminant of webs on a surface**

By definition, the **discriminant** $\Delta(P)$ of binary form $P$ is

$$\Delta(P) = \frac{R[P, \partial_{x}P]}{n^{2}p_{n}} = \frac{R[P, \partial_{y}P]}{n^{2}p_{0}}.$$ 

Notice that $\Delta(P)$ vanishes if and only if both $P$ and $\partial_{x}P$ share a common root, that is, $P$ has a root with multiplicity greater than one.

The discriminant obeys rules analogous to the ones obeyed by the resultant. Namely

$$\Delta(\lambda P) = \lambda^{2(\deg(P) - 1)} \Delta(P),$$

$$\Delta(g^{*}P) = \det(Dg)^{\deg(P)(\deg(P) - 1)} \Delta(P).$$
If $W$ is a $k$-web, $k \geq 2$, on a surface $S$ then the discriminant divisor of $W$ is, by definition, the divisor locally defined by $\Delta(\omega)$ where as before $\omega \in \text{Sym}^k \Omega^1_S(U_i)$ locally defines $W$. Notice that the support of the discriminant divisor coincides with the discriminant set of $W$ previously defined.

The discussion about the tangency of two webs adapts verbatim to yield the proposition below.

**Proposition 1.3.3.** If $W$ is a $k$-web with normal-bundle $N$ defined on a surface $S$ then

$$\mathcal{O}_S(\Delta(W)) = K_S^\otimes k(k-1) \otimes N^\otimes 2(k-1).$$

### Discriminants of real webs

Due to obvious technical constraints all the pictures of planar webs are drawn over the real plane. In particular the webs portrayed ought to be defined by real analytic $k$-symmetric 1-forms $\omega$ on some open subset $U$ of $\mathbb{R}^2$. Most of the time these 1-forms will be polynomial 1-forms and hence globally defined on $\mathbb{R}^2$.

The sign of the discriminant of $\omega$ at a given point $p \in U$ gives clues about the number of real leaves of $W = [\omega]$ through $p$. For a 2-web $W$ induced by $\omega = adx^2 + bdxdy + cdy^2$ the sign of $\Delta = \Delta(\omega) = b^2 - 4ac$ tells all one may want to know about the number of real leaves: when $\Delta(p) > 0$ there are two real leaves through $p$, and when $\Delta(p) < 0$ there are no real leaves through $p$.

For 3-webs the situation is as good as for 2-webs. According to whether the sign of $\Delta$ is positive or negative at a given point $p$ the 3-web has one or three real leaves through $p$.

For $k$-webs with $k \geq 4$ the sign of $\Delta$ at $p$ does not determine the number of real leaves of $W$ through it but does tell that, see [81],

- when $k$ is odd, the number of real leaves through $p$ is congruent to 1 or 3 modulo 4 according as $\Delta(p) > 0$ or $\Delta(p) < 0$, and;
- when $k$ is even, the number of real leaves through $p$ is congruent to 0 or 2 modulo 4 according as $\Delta(p) < 0$ or $\Delta(p) > 0$.

It is tempting to claim that a planar $k$-web $W$ on $\mathbb{C}^2$ defined by a real $k$-symmetric 1-form with only one leaf through each point of a
given domain $U \subset \mathbb{R}^2$ is nothing more than an analytic foliation on $U$. Although trivially true if the discriminant of $W$ does not intersect $U$ this claim is far from being true in general.

Perhaps the simplest example comes from a variation on the classical Tait-Kneser Theorem presented in [103, 53].

Let $f \in \mathbb{R}[x]$ be a polynomial in one real variable of degree $k$. For fixed $n < k$ and $t \in \mathbb{R}$ define the $n$th osculating polynomial $g_t$ of $f$ as the polynomial of degree at most $n$ whose graph osculates the graph of $f$ at $(t, f(t))$ up to order $n$. From its definition follows that $g_t(x)$ is nothing than the truncation of the Taylor series of $f$ centered at $t$ at order $n + 1$, that is

$$g_t(x) = \sum_{i=0}^{n} \frac{f^{(i)}(t)}{i!}(x-t)^i.$$ 

Notice that for a fixed $t \in \mathbb{C}$ for which $f^{(n)}(t) \neq 0$ the graph of $g_t$, $G_t = \{y = g_t(x)\}$, is an irreducible plane curve of degree $n$. Moreover varying $t \in \mathbb{C}$ one obtains a family of degree $n$ curves which corresponds to a degree $k$ curve $\Gamma_f$ on the space of degree $n$ curves. The degree $n$ curves through a generic point $p \in \mathbb{C}^2$ determine a hyperplane $H$ in the space of degree $n$ curves. Because $H$ intersects $\Gamma_f$ in $k$ points, through a generic $p \in \mathbb{C}^2$ passes $k$ distinct curves of the family $\{G_t\}$. Therefore this family of curves determines a $k$-web $W_f$ on $\mathbb{C}^2$.

To obtain a polynomial $k$-symmetric 1-form defining $W_f$ it suffices to eliminate $t$ from the pair of equations

$$y - g_t(x) = 0, \quad dy - \partial_x g_t(x) dx = 0.$$ 

Such task can be performed by considering the resultant of $y - g_t(x)$ and $dy - \partial_x g_t(x) dx$ seen as degree $k$ polynomials in the variable $t$ with coefficients in $\mathbb{C}[x, y, dx, dy]$.

To investigate the real trace of $k$-web $W_f$ the following variant of the classical Tait-Kneser Theorem [53, 103] will be useful.

**Theorem 1.3.4.** If $n$ is even and $f^{(n+1)}(t) \neq 0$ for every real number $t$ in an interval $(a, b)$ then the curves $G_a$ and $G_b$ do not intersect in $\mathbb{R}^2$. 
Proof. On the one hand if \(a < b\) are real numbers for which \(G_a\) and \(G_b\) intersects in \(\mathbb{R}^2\) then there exists a real number \(x_0 \in \mathbb{R}\) such that \(g_a(x_0) - g_b(x_0) = 0\).

On the other hand the fundamental theorem of calculus implies

\[
g_a(x_0) - g_b(x_0) = \int_a^b \frac{\partial g_t}{\partial t}(x_0) dt = \\
= \int_a^b \left( \sum_{i=0}^{n} \frac{f^{(i+1)}(t)}{i!} (x_0 - t)^i - \sum_{i=1}^{n} \frac{f^{(i)}(t)}{(i-1)!} (x_0 - t)^{i-1} \right) dt = \\
\int_a^b \frac{f^{(n+1)}(t)}{n!} (x_0 - t)^n dt \neq 0.
\]

This contradiction concludes the proof.

Figure 1.9: The real trace of the 3-web \(\mathcal{W}_f\) for \(f = x(2x - 1)(2x + 1)\).

If \(f \in \mathbb{R}[x]\) is a function of odd degree \(k\) then the real trace of the \(k\)-web \(\mathcal{W}_f\) defined by the graph of the \((k-1)\)-th osculating functions of \(f\) is a continuous foliation in a neighborhood of \(\Gamma_f\), the real graph of \(f\), which is not differentiable since every point of \(\Gamma_f\) is tangent to some leaf of the foliation without being itself a leaf. To summarize: the real trace of a holomorphic (or even polynomial) web can be a non-differentiable, although continuous, foliation.
1.4 Examples

1.4.1 Global webs on projective spaces

Let $\mathcal{W} = [\omega] \in \mathbb{P}H^0(\mathbb{P}^n, \text{Sym}^k \Omega^1_{\mathbb{P}^n} \otimes \mathcal{N})$ be a $k$-web on $\mathbb{P}^n$. The **degree of** $\mathcal{W}$ is defined as the number of tangencies, counted with multiplicities, of $\mathcal{W}$ with a line not everywhere tangent to $\mathcal{W}$. More precisely, if $i : \mathbb{P}^1 \to \mathbb{P}^n$ is linear embedding of $\mathbb{P}^1$ into $\mathbb{P}^n$ then the points of tangency of the image line with $\mathcal{W}$ correspond to the zeros of $[i^*\omega] \in \mathbb{P}H^0(\mathbb{P}^1, \text{Sym}^k \Omega^1_{\mathbb{P}^1} \otimes i^*\mathcal{N})$. Notice that $i^*\omega$ vanishes identically if and only if the image of $i$ is everywhere tangent to $\mathcal{W}$.

Recall that every line-bundle $\mathcal{L}$ on $\mathbb{P}^n$ is an integral multiple of $\mathcal{O}_{\mathbb{P}^n}(1)$ and consequently one can write $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(\deg(\mathcal{L}))$. Because the embedding $i$ is linear, the identity $i^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^1}(1)$ holds true. Putting these two facts together with the identity $\text{Sym}^k \Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2k)$ yields

$$\deg(\mathcal{W}) = \deg(\mathcal{N}) - 2k.$$

**Characteristic numbers of projective webs**

Let $X \subset \mathbb{P}^n$ be an irreducible projective subvariety. The projectivized conormal variety of $X$, **conormal variety** of $X$ for short, is the unique closed subvariety $\text{Con}(X)$ of $\mathbb{P}T^*\mathbb{P}^n$ satisfying:

1. $\pi(\text{Con}(X)) = X$, where $\pi : \mathbb{P}T^*\mathbb{P}^n \to \mathbb{P}^n$ is the natural projection;
2. the fiber $\pi^{-1}(x) \cap \text{Con}(X)$ over any smooth point $x$ of $X$ is $\mathbb{P}T^*_x X \subset \mathbb{P}T^*\mathbb{P}^n$.

The conormal variety of $X \subset \mathbb{P}^n$ can succinctly be defined as

$$\text{Con}(X) = \overline{\mathbb{P}N^*X_{\text{sm}}},$$

with $X_{\text{sm}}$ denoting the smooth part of $X$ and $N^*X_{\text{sm}}$ its conormal bundle.

For example, the conormal of a point $x \in \mathbb{P}^n$ is all the fiber $\pi^{-1}(x) = \mathbb{P}T^*_x \mathbb{P}^n$. More generally the conormal variety of a linearly embedded $\mathbb{P}^i \subset \mathbb{P}^n$ is a trivial $\mathbb{P}^{n-i-1}$ bundle over $\mathbb{P}^i$. 
If $W \subset \mathbb{P}T^*\mathbb{P}^n$ is the natural lift of $W$ then the characteristic numbers of $W$ on $\mathbb{P}^n$ are, by definition, the $n$ integers

$$d_i(W) = W \cdot \text{Con}(\mathbb{P}^i),$$

with $i$ ranging from 0 to $n - 1$, and where $A \cdot B$ stands for the intersection product of $A$ and $B$.

Notice that $d_0(W)$ counts the number of leaves of $W$ through a generic point of $\mathbb{P}^n$, that is $W$ is a $d_0(W)$-web. The integer $d_1(W)$ counts the number of points over a generic line $\ell$ where the web has a leaf with tangent space containing $\ell$. Therefore $d_1(W)$ is nothing more than the previously defined degree of $W$.

1.4.2 Algebraic webs revisited

It is seems fair to say that the simplest $k$-webs on projective spaces are the ones of degree zero. Perhaps the best way to describe them is through projective duality.

Let $\mathbb{P}^n$ denote the projective space parametrizing hyperplanes in $\mathbb{P}^n$ and $I \subset \mathbb{P}^n \times \mathbb{P}^n$ be the incidence variety, that is

$$I = \{ (p, H) \in \mathbb{P}^n \times \mathbb{P}^n \mid p \in H \}.$$

The natural projections from $I$ to $\mathbb{P}^n$ and $\mathbb{P}^n$ will be respectively denoted by $\pi$ and $\hat{\pi}$.

**Proposition 1.4.1.** The incidence variety $I$ is naturally isomorphic to $\mathbb{P}T^*\mathbb{P}^n$ and also to $\mathbb{P}T^*\mathbb{P}^n$. Moreover, under these isomorphisms the natural projections $\pi$ and $\hat{\pi}$ from $I$ to $\mathbb{P}^n$ and $\mathbb{P}^n$, coincide with the projections from $\mathbb{P}T^*\mathbb{P}^n$ to $\mathbb{P}^n$ and from $\mathbb{P}T^*\mathbb{P}^n$ to $\mathbb{P}^n$ respectively.

**Proof.** If one identifies $\mathbb{P}^n \times \mathbb{P}^n$ with $\mathbb{P}(V) \times \mathbb{P}(V^*)$ where $V$ is a vector space of dimension $n + 1$ then the incidence variety can be identified with the projectivization of the locus defined on $V \times V^*$ through the vanishing of the natural pairing. Combining this with the natural isomorphism between $V$ and $V^{**}$ the proposition follows. For details see [51, Chapter 1, Section 3.A] □

Using this identification of $I$ with $\mathbb{P}T^*\mathbb{P}^n$ one defines for every projective curve $C \subset \mathbb{P}^n$ its dual web $W_C$ as the one defined by variety
\( \pi^{-1}(C) \subset \mathbb{P}T^* \mathbb{P}^n \) seen as a multi-section of \( \hat{\pi} : \mathbb{P}T^* \mathbb{P}^n \to \mathbb{P}^n \). At once one verifies that the germification of this global web at a generic point \( H_0 \in \mathbb{P}^n \) coincides with the germ of web \( W_C(H_0) \) defined in Section 1.1.3.

**Proposition 1.4.2.** If \( C \subset \mathbb{P}^n \) is a projective curve of degree \( k \) then \( W_C \) is a \( k \)-web of degree zero on \( \mathbb{P}^n \). Reciprocally, if \( W \) is a \( k \)-web of degree zero on \( \mathbb{P}^n \) then there exists \( C \subset \mathbb{P}^n \), a projective curve of degree \( k \), such that \( W = W_C \).

**Proof.** If \( C \subset \mathbb{P}^n \) is a projective curve then all the leaves of \( W_C \) are hyperplanes. Therefore a line tangent to \( W_C \) at a quasi-smooth point is automatically contained in the leaf through that point. This is sufficient to prove that \( W_C \) has degree 0. Alternatively one can compute directly

\[
d_0(W_C) = \pi^{-1}(C) \cdot \text{Con}(\mathbb{P}^0) = 0
\]

since \( \pi^{-1}(C) \) does not intersect the conormal variety of any point \( p = \mathbb{P}^0 \) outside the dual variety \( \check{C} \).

The proof of the reciprocal is similar and the reader is invited to work it out.

**The discriminant of \( W_C \)**

If \( C \) is a smooth projective curve then the discriminant of \( W_C \) is nothing more than the set of hyperplanes tangent to \( C \) at some point. Succinctly,

\[
\Delta(W_C) = \check{C} \quad \text{when } C \text{ is smooth.}
\]

For an arbitrary curve \( C \) the discriminant of \( W_C \) will also contain the hyperplanes on \( \mathbb{P}^n \) corresponding to singular points of \( C \), and the projective subspaces of codimension two dual to the lines contained in \( C \).

Because for a plane curve of degree \( k \), the normal bundle of the web \( W_C \) is \( \mathcal{O}_{\mathbb{P}^2}(2k) \), one has

\[
\deg(\Delta(W_C)) = \deg(K_{\mathbb{P}^2}^{k(k-1)} \otimes \mathcal{O}_{\mathbb{P}^2}(4k(k-1))) = -3k(k-1) + 4k(k-1) = k(k-1),
\]
according to Proposition 1.3.3. In particular, one recovers the classical Plücker formula for the degree of the dual of smooth curves:

\[ C \text{ smooth and } \deg(C) = k \implies \deg(\check{C}) = k(k - 1). \]

If \( C \) has singularities then the lines dual to the singular points will also be part of \( \Delta(\mathcal{W}_C) \). The multiplicity with which this line appears in the discriminant will vary according to the analytical type of the singularity. For instance the lines dual to ordinary nodes will appear with multiplicity two, while the lines dual to ordinary cusps will appear with multiplicity three. In particular for a degree \( k \) curve with at \( n \) ordinary nodes and \( c \) ordinary cusps as singularities one obtains another instance of Plücker formula

\[ \deg(\check{C}) = k(k - 1) - 2n - 3c. \]

The monodromy of \( \mathcal{W}_C \)

Recall that a subgroup \( G \subset \mathfrak{S}_k \) of the \( k \)-th symmetric group is 2-transitive if for any pair of pairs \((a, b), (c, d) \in \mathbb{F}^2\) there exists \( g \in G \) such that \( g(a) = c \) and \( g(b) = d \). To describe the monodromy of \( \mathcal{W}_C \) for irreducible curves \( C \) the simple lemma below will be useful.

**Lemma 1.4.3.** Let \( G \subset \mathfrak{S}_k \) be a subgroup. If \( G \) is 2-transitive and contains a transposition then \( G \) is the full symmetric group.

**Proof.** It is harmless to assume that \( G \) contains the transposition \((1 2)\). Since \( G \) is 2-transitive for every pair \((a, b) \in \mathbb{F}^2\) there exists \( g \in G \) such that \( g(a) = 1 \) and \( g(b) = 2 \). Therefore the transposition

\[ (a b) = g^{-1}(1 2)g \]

belongs to \( G \). Consequently, every transposition in \( \mathfrak{S}_k \) belongs to \( G \). Since \( \mathfrak{S}_k \) is generated by transpositions the lemma is proved.

**Proposition 1.4.4.** If \( C \) is an irreducible projective curve on \( \mathbb{P}^n \) of degree \( k \) then the monodromy group of \( \mathcal{W}_C \) is the full symmetric group.
Proof. It is harmless to assume that $n > 2$. Indeed, if $C \subset \mathbb{P}^2$ then just embed $\mathbb{P}^2$ linearly in $\mathbb{P}^3$ to obtain a projective curve $C' \subset \mathbb{P}^3$. Notice that $\mathcal{W}_C$ is the pull-back of $\mathcal{W}_{C'}$ under the linear projection dual to the embedding, and that both webs $\mathcal{W}_C$ and $\mathcal{W}_{C'}$ have isomorphic monodromy groups.

The irreducibility of $C$ implies that $\mathcal{W}_C$ is indecomposable and consequently its monodromy group is 1-transitive. Let $p \in C \subset \mathbb{P}^n$ be a generic point and consider the hyperplane $H_p$ in $\tilde{\mathbb{P}}^n$ determined by it. Since $n > 2$, the restriction of $\mathcal{W}_C$ at $H_p$ is still an algebraic web. If $C'$ is the curve in $\mathbb{P}^{n-1}$ image of the projection from $\mathbb{P}^n$ to $\mathbb{P}^{n-1}$ centered at $p$ then $(\mathcal{W}_C)_{|H_p}$ is projectively equivalent to $\mathcal{W}_{C'}$. Since the projection of irreducible curves are irreducible it follows that the monodromy of $\mathcal{W}_{C'}$ is also transitive. This suffices to show that the monodromy of $\mathcal{W}_C$ is 2-transitive. Indeed, given $a \in \bar{k}$ (where $\bar{k}$ is now identified with the set of leaves of $\mathcal{W}_C$ through a generic point and $\tilde{\mathbb{P}}^n$) by the transitivity of the monodromy group, one can send $a$ to an arbitrary $c \in \bar{k}$. If one now considers the restriction of $\mathcal{W}_C$ to the leaf corresponding to $c$ then the monodromy group of the restricted web is again transitive, but now on the set $\bar{k} - \{c\}$ of cardinality $k-1$, and for an arbitrary pair $b, d \in \bar{k} - \{c\}$ there exists an element that sends $b$ to $d$ while fixing $c$.

It remains to show that there exists a transposition on the monodromy group of $\mathcal{W}_C$. To do that one it suffices to consider the case of algebraic webs on $\mathbb{P}^2$ after restricting to a suitable intersection of leaves. Suppose now that $C$ is an irreducible plane curve and let $\ell$ be a simple tangent line of $C$, that is, $\ell$ is a tangent line of $C$ at a smooth, non-inflection point $p \in C$ and $\ell$ intersects $C$ transversely on the complement of $p$. In affine coordinates $(x, y)$ where $\ell = \{y = 0\}$ and $p$ is the origin the curve $C$ can be expressed as the zero locus of $y - x^2 + h.o.t.$ The intersection of $C$ with the line $y = \epsilon$ is therefore of the form $(\sqrt{\epsilon} + h.o.t., \epsilon)$. Notice that the intersections are exchanged when $\epsilon$ gives a turn around 0. In the dual plane this reads as the existence of a transposition for the dual web. \qed
Smoothness of $\mathcal{W}_C$

**Proposition 1.4.5.** Let $C$ be an irreducible non-degenerate projective curve in $\mathbb{P}^n$. If $H \in \mathbb{P}^n$ is a generic hyperplane then $\mathcal{W}_C(H)$ is a germ of smooth web.

By duality, the proposition is clearly equivalent to the so called uniform position principle for curves. The proof presented here follows closely [6, pages 109–113].

**Proposition 1.4.6.** If $C \subset \mathbb{P}^n$, $n \geq 2$, is an irreducible non-degenerate projective curve of degree $d \geq n$ then a generic hyperplane $H$ intersects $C$ at $d$ distinct points. Moreover, any $n$ among these $d$ points span $H$.

The restriction on the degree of $C$ is not really a hypothesis. Every non-degenerate curve on $\mathbb{P}^n$ have degree at least $n$ as will be shown in Proposition 2.3.11 of Chapter 2.

**Proof of Propositions 1.4.5 and 1.4.6.** Let $U = \mathbb{P}^n - \Delta(\mathcal{W}_C)$ and $I \subset C^n \times U$ be the locally closed variety defined by the relation

$$(p_1, \ldots, p_n, H) \in I \iff p_1, \ldots, p_n \text{ are distinct points in } H \cap C.$$ 

Because the monodromy group of $\mathcal{W}_C$ is the full symmetric group the variety $I$ is irreducible and in particular connected. Moreover the natural projection to $U$ is surjective and has finite fibers. Therefore $I$ has dimension $n = \dim U$.

Let now $J \subset I$ be the closed subset defined by

$$(p_1, \ldots, p_n, H) \in J \iff p_1, \ldots, p_n \text{ are contained in a } \mathbb{P}^{n-2}.$$ 

Since $C$ is non-degenerated, one can choose $n$ distinct points on it which span a $\mathbb{P}^{n-1}$. Thus $J$ is a proper subset of $I$. The irreducibility of $I$ implies $\dim J < \dim I = n$. Therefore the image of the projection to $U$ is a proper subset, with complement parametrizing hyperplanes intersecting $C$ with the wanted property. \qed
1.4.3 Projective duality

Given a global $k$-web $W$ on $\mathbb{P}^n$, and its natural lift $\tilde{W}$ to $\mathbb{P}T^*\mathbb{P}^n \cong \mathcal{I}$, it is natural to enquire which sort of object $W$ induces on $\tilde{\mathbb{P}}^n$ through the projection $\tilde{\pi}$.

To answer such question, assume for a moment that $W \subset \mathcal{I}$ is irreducible, or equivalently that the monodromy of $W$ is transitive.

If the map $\tilde{\pi}|_W : W \rightarrow \tilde{\mathbb{P}}^n$ is surjective, then there exists a web $\tilde{\mathcal{W}}$ on $\tilde{\mathbb{P}}^n$ with lift to $\tilde{\mathcal{I}} = \mathcal{I}$ equal to $W$. The order of $\tilde{\mathcal{W}}$ is precisely the degree of $\tilde{\pi}|_W$, that is $d_0(\tilde{\mathcal{W}}) = d_{n-1}(W)$.

In the two dimensional case the degree of $\tilde{\pi}|_W$ is nothing more than the degree of $W$. But beware that this is no longer true when the dimension is at least three. To determine the degree of $\tilde{\mathcal{W}}$ notice that $\tilde{\mathcal{W}}$ is tangent to a line $\ell$ at a point $p$ if and only if one of the tangent spaces of $\tilde{\mathcal{W}}$ at $p$ contains the line $\ell$. Therefore the number of tangencies of $\tilde{\mathcal{W}}$ and $\ell$ is the intersection of $W$ with the conormal variety of $\ell \subset \mathbb{P}T^*\mathbb{P}^n$. In other words $d_1(\tilde{W}) = d_{n-2}(W)$.

Arguing similarly, it follows that for $i$ ranging from 0 to $n - 1$ the identity $d_i(\tilde{W}) = d_{n-i-1}(W)$ holds true.

In order to deal with the case where $\tilde{\pi}|_W : W \rightarrow \mathbb{P}^n$ is not surjective it is convenient to extend the definition of characteristic numbers to pairs $(X, W)$, where $X \subset \mathbb{P}^n$ is an irreducible projective variety and $W$ is an irreducible web of codimension one on $X$. To repeat the same definition as before all that is needed is a definition of the lift of $(X, W)$ to $\mathbb{P}T^*\mathbb{P}^n$. Mimicking the definition of conormal variety for subvarieties of $\mathbb{P}^n$, define $\text{Con}(X, W)$, the conormal variety of the pair $(X, W)$, as the closed subvariety of $\mathbb{P}T^*\mathbb{P}^n$ characterized by the following conditions:

(a) $\text{Con}(X, W)$ is irreducible;

(b) $\pi(\text{Con}(X, W)) = X$;

(c) For a generic $x \in X$ the fiber $\pi^{-1}(x) \cap \text{Con}(X, W)$ is a union of linear subspaces corresponding to the projectivization of the conormal bundles in $\mathbb{P}^n$ of the leaves of $W$ through $x$.

---

*When $X$ is a singular variety, a web on $X$ is a web on its smooth locus which extends to a global web on any of its desingularizations.*
For every pair $(X, W)$, there exists a unique pair $D(X, W)$ on $\mathbb{P}^n$ with conormal variety in $\mathbb{P}T^*\mathbb{P}^n$ equal to the conormal variety of $(X, W)$, if the following conventions are adopted.

- an irreducible codimension one web $W$ on $\mathbb{P}^n$ is identified with the pair $(\mathbb{P}^n, W)$;
- on an irreducible projective curve $C$ there is only one irreducible web, the 1-web $\mathcal{P}$ which has as leaves the points of $C$, and;
- a projective curve $C$ is identified with the pair $(C, \mathcal{P})$.

In this terminology, Proposition 1.4.2 reads as
\[
D(C) = W_C \quad \text{and} \quad D(W_C) = C.
\]

Notice that the pairs $(X, W)$ come with naturally attached characteristic numbers
\[
d_i(X, W) = \text{Con}(X, W) \cdot \text{Con}(\mathbb{P}^i),
\]
and these generalize the characteristic numbers for a web $W$ on $\mathbb{P}^n$ as previously defined.

**Example 1.4.7.** Let $X$ be an irreducible subvariety of codimension $q \geq 1$ on $\mathbb{P}^n$ and $W$ an irreducible $k$-web on $X$. Since $X$ has codimension $q$, for $0 \leq i \leq q - 1$ a generic $\mathbb{P}^i$ linearly embedded in $\mathbb{P}^n$ does not intersect $X$. Therefore $d_i(X, W) = 0$ for $i = 0, \ldots, q - 1$. A generic $\mathbb{P}^q$ will intersect $X$ in $\deg(X)$ smooth points and over each one of these points $\text{Con}(\mathbb{P}^q)$ will intersect $\text{Con}(X, W)$ in $k$ points. Therefore $d_q(X, W) = \deg(X) \cdot k$.

With these definitions at hand it is possible to prove the following Biduality Theorem. Details will appear elsewhere.

**Theorem 1.4.8.** For any pair $(X, W)$, where $X \subset \mathbb{P}^n$ is an irreducible projective subvariety and $W$ is an irreducible codimension one web on $X$, the identity
\[
D(D(X, W)) = (X, W)
\]
holds true. Moreover, for $i = 0, \ldots, n - 1$, the characteristic numbers of $(X, W)$ and $D(X, W)$ satisfy
\[
d_i(X, W) = d_{n-1-i}(D(X, W)).
\]
Finally to deal with arbitrary pairs \((X, W)\), where \(X\) is not necessarily irreducible nor \(W\) has necessarily transitive monodromy, one writes \((X, W)\) as the superposition of irreducible pairs and applies \(D\) to each factor. Everything generalizes smoothly.

1.4.4 Webs attached to projective surfaces

One particularly rich source of examples of webs on surfaces is the classical projective differential geometry widely practiced until the early beginning of the XX\(^{th}\) century. The simplest example is perhaps the asymptotic webs on surfaces on \(\mathbb{P}^3\) that are now described.

Asymptotic webs

Let \(S \subset \mathbb{P}^3\) be a germ of smooth surface. As such it admits a parametrization \([\varphi] : (\mathbb{C}^2, 0) \rightarrow \mathbb{P}^3\), projectivization of a map \(\varphi : (\mathbb{C}^2, 0) \rightarrow \mathbb{C}^4 \setminus \{0\}\).

The tangent plane of \(S\) at the point \([\varphi(p)]\) is determined by the vector subspace of \(T_{\varphi(p)} \mathbb{C}^4\) generated by
\[
\varphi(p), \frac{\partial \varphi}{\partial x}(p), \frac{\partial \varphi}{\partial y}(p).
\]

A germ of smooth curve \(C\) on \(S\) admits a parametrization of the form \(\varphi \circ \gamma(t)\) where \(\gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)\) is an immersion. Its osculating plane at \(\varphi \circ \gamma(t)\) is determined by the vector space generated by \((\varphi \circ \gamma)(t), (\varphi \circ \gamma)'(t), (\varphi \circ \gamma)''(t)\). Although the vectors \((\varphi \circ \gamma)'(t)\) and \((\varphi \circ \gamma)''(t)\) do depend on the choice of the parametrization \(\varphi \circ \gamma\) of \(C\), the same is not true for the vector space generated by them and \((\varphi \circ \gamma)(t)\).

A curve \(C\) is an asymptotic curve of \(S\) if, at every point \(p\) of \(C\), its osculating plane is contained in the tangent space of \(S\) at \((\varphi \circ \gamma)(t)\), the determinant (where each entry represents a distinct row)
\[
\text{det} \left( (\varphi \circ \gamma)''(t), \varphi(\gamma(t)), \frac{\partial \varphi}{\partial x}(\gamma(t)), \frac{\partial \varphi}{\partial y}(\gamma(t)) \right)
\]
vanishes identically when \(\gamma\) parametrizes an asymptotic curve. But
\[
(\varphi \circ \gamma)''(t) = D^2 \varphi(\gamma(t)) \cdot \gamma'(t) \cdot \gamma'(t) + D \varphi(\gamma(t)) \cdot \gamma''(t)
\]
and the image of $D\varphi(\gamma(t))$ is always contained in the vector space generated by the last three rows of the above matrix. Hence the vanishing of (1.4) is equivalent to the vanishing of

$$\det \left( D^2 \varphi(\gamma(t)) \cdot \gamma'(t) \cdot \gamma'(t), \varphi(\gamma(t)), \frac{\partial \varphi}{\partial x}(\gamma(t)), \frac{\partial \varphi}{\partial y}(\gamma(t)) \right).$$

This last expression can be rewritten as

$$\gamma^*(adx^2 + 2bdxdy + cdy^2)$$

where

$$a = \det(\varphi_{xx}, \varphi_x, \varphi_y)$$
$$b = \det(\varphi_{xy}, \varphi_x, \varphi_y)$$
$$c = \det(\varphi_{yy}, \varphi_x, \varphi_y).$$

It may happen that $a, b, c$ are all identically zero. It is well-known that this is the case if and only if $S$ is contained in a hyperplane of $\mathbb{P}^3$. It may also happen that although non-zero the 2-symmetric differential form $adx^2 + 2bdxdy + cdy^2$ is proportional to the square of a differential 1-form. This is the case, if and only if, the surface $S$ is developable. Recall that a surface is developable if it is contained in a plane, a cone or the tangent surface of a curve.

In general for non-developable surfaces what one gets is a 2-symmetric differential form that induces an (eventually singular) 2-web on $S$: the asymptotic web of $S$.

The simplest example is the asymptotic web of a smooth quadric $Q$ on $\mathbb{P}^3$. Since it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and under these isomorphism the fibers of both natural projections to $\mathbb{P}^1$ are lines on $\mathbb{P}^3$, it is clear that the asymptotic web of $Q$ is formed by these two families of lines.

Asymptotic webs – Alternative take

When $S \subset \mathbb{P}^3$ is a smooth projective surface the definition of the asymptotic web of $S$ is amenable to a more intrinsic formulation. Suppose that $S$ is cutted out by an irreducible homogenous polynomial $F \in \mathbb{C}[x_0, \ldots, x_3]$. The Hessian matrix of $F$,

$$\text{Hess}(F) = \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right),$$
when evaluated at the tangent vectors of \( S \) gives rise to a morphism

\[
\text{Sym}^2 T S \rightarrow NS
\]

where \( NS \simeq O_S(\deg(F)) \) is the normal bundle of \( S \subset \mathbb{P}^3 \). This morphism is usually called the (projective) \textbf{second fundamental form} of \( S \). Dualizing it, and tensoring the result by \( NS \) one obtains a holomorphic section of \( \text{Sym}^2 \Omega^1_S \otimes NS \).

When \( S \) is not developable (which under the smoothness and projectiveness assumption on \( S \) is equivalent to \( S \) not being a plane) this section, after factoring eventual codimension one components of its zero set, defines a singular 2-web on \( S \). Its discriminant coincides, set theoretically, with the locus on \( S \) defined by the vanishing of \( \text{Hess}(F) \).

\textbf{The general philosophy}

One can abstract from the definition of asymptotic web the following procedure:

1. Take a linear system \( |V| \) on a surface \( S \);
2. Consider the elements of \( |V| \) with abnormal singularities at a generic point \( p \) of \( S \);
3. If there are only finitely many abnormal elements of \( V \) for a given generic point \( p \) consider the web with tangents at \( p \) determined by the tangent cone of these elements.

This kind of construction abounds in classical projective differential geometry.

\textbf{Darboux 3-web}

Let \( S \subset \mathbb{P}^3 \) be a surface and consider the restriction to \( S \) of the linear system of quadrics \( |\mathcal{O}_{\mathbb{P}^3}(2)| \).

\footnote{Recall that a linear system is the projectivization \( |V| \) of a finite dimensional vector subspace \( V \subset H^0(S, \mathcal{L}) \), where \( \mathcal{L} \) is a line-bundle on \( S \). In the case \( S \) is a surface germ, a linear system is nothing more than the projectivization of a finite dimensional vector space of germs of functions.}
If $S$ is generic enough then at a generic point $p \in S$ there are exactly three quadrics whose restriction at $S$ is a curve with first non-zero jet at $p$ of the form

$$\ell_i(x, y)^3 \quad i = 1, 2, 3$$

where $(x, y)$ are local coordinates of $S$ centered at $p$ and the $\ell_i$ are linear forms. These three quadrics are the quadrics of Darboux of $S$ at $p$. For more details see [72, pages 141–144].

In this way one defines a 3-web with tangents at $p$ given by $\ell_1, \ell_2, \ell_3$. This is the Darboux 3-web of $S$.

**Segre 5-web**

Let now $S$ be a surface on $\mathbb{P}^5$ and consider the restriction to $S$ of the linear system of hyperplanes $|O_{\mathbb{P}^5}(1)|$.

For a generic point $p$ in a generic surface $S$ there are exactly five hyperplanes which intersect $S$ along a curve which has a tacnode** singularity at $p$. The five directions determined by these tacnodes are Segre’s principal directions. By definition Segre’s 5-web is defined as the 5-web determined pointwise by Segre’s principal directions in the case where they are distinct at a generic point of $S$.

There are surfaces such that through every point there are infinitely many principal directions. For instance the developable surfaces — planes, cones and tangent of curves — do have this property and so do the degenerated surfaces, that is surfaces contained in a proper hyperplane of $\mathbb{P}^5$. A remarkable theorem of Corrado Segre says that besides these examples the only surfaces in $\mathbb{P}^5$ with infinitely many principal directions through every point are the ones contained in the Veronese surface obtained through the embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ given by the linear system $|O_{\mathbb{P}^2}(2)|$.

If $\varphi : (\mathbb{C}^2, 0) \to \mathbb{C}^6$ is a parametrization of the surface $S$, then

**An ordinary tacnode is a singularity of curve with exactly two branches, both of them smooth, having an ordinary tangency. Here tacnode refer to a curve cut out by a power series of the form**

$$\ell(x, y)^2 + \ell(x, y)P_2(x, y) + h.o.t.$$  

where $\ell$ is a linear form and $P_2$ is a homogeneous form of degree 2.
Segre’s 5-web of $S$ is induced by the 5-symmetric differential form

$$\omega_\varphi = \det \begin{pmatrix}
\varphi \\
\varphi_x \\
\varphi_y \\
\varphi_{xx} dx + \varphi_{xy} dy \\
\varphi_{xy} dx + \varphi_{yy} dy \\
\varphi_{xxx} dx^3 + 3 \varphi_{xxy} dx^2 dy + 3 \varphi_{xyy} dx dy^2 + \varphi_{yyy} dy^3
\end{pmatrix}.$$

It can be verified that once the parametrization $\varphi$ is changed by one of the form

$$\hat{\varphi}(x, y) = \lambda(x, y) \cdot \varphi(\psi(x, y))$$

where $\lambda$ in a unit in $\mathcal{O}(\mathbb{C}^2, 0)$ and $\psi \in \text{Diff}(\mathbb{C}^2, 0)$ is a germ of biholomorphism, then one has

$$\omega_{\hat{\varphi}} = \lambda^6 \cdot \det(D\psi)^2 \cdot \psi^* \omega_\varphi.$$

This identity implies that the collection $\omega_\varphi$, with $\varphi$ ranging over germs of parametrizations of $S$, defines a holomorphic section of

$$\text{Sym}^5 \Omega^1_S \otimes \mathcal{O}_S(6) \otimes K_S^{\otimes 2}.$$

A nice example is given by the cubic surface $S$ obtained as the image of the rational map from $\mathbb{P}^2$ to $\mathbb{P}^5$ determined by the linear system of cubics passing through four points $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$ in general position. At a generic point $p \in \mathbb{P}^2$, the five cubics in the linear system with a tacnode at $p$ are: the union of conic through $p_1, p_2, p_3, p_4$ and $p$ with its tangent line at $p$; and for every $i \in 4$, the union of the line $\overline{pp_i}$ with the conic through $p$ and all the $p_j$ with $j \neq i$ which is moreover tangent to $\overline{pp_i}$ at $p$. This geometric description makes evident the fact that Segre’s 5-web of $S$ is nothing more than Bol’s 5-web $\mathcal{B}_5$ presented in Section 1.2.3.
Chapter 2

Abelian relations

A central concept in this text is the one of abelian relation for a germ of quasi-smooth web $W$. Roughly speaking, abelian relations are additive functional equations among the first integrals of the defining foliations of $W$. More precisely, if $W = [\omega_1 \cdots \omega_k]$ is a germ of quasi-smooth $k$-web on $(\mathbb{C}^n, 0)$ then an abelian relation of $W$ is a $k$-uple of germs of 1-forms $\eta_1, \ldots, \eta_k$ satisfying the following three conditions:

(a) for every $i \in k$, the 1-form $\eta_i$ is closed, that is, $d\eta_i = 0$;

(b) for every $i \in k$, the 1-form $\eta_i$ defines $\mathcal{F}_i$, that is $\omega_i \wedge \eta_i = 0$;

(c) the 1-forms $\eta_i$ sum up to zero, that is $\sum_{i=1}^k \eta_i = 0$.

Notice that a primitive of $\eta_i$ exists, since $\eta_i$ is closed. Such primitive is a first integral of $\mathcal{F}_i$, because $\omega_i \wedge \eta_i = 0$. In particular, if $\mathcal{F}_i$ is defined through a submersion $u_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ then

$$\int_0^z \eta_i = g_i(u_i(z)),$$

for some germs of holomorphic functions $g_i : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. Condition (c) translates into

$$\sum_{i=1}^k g_i \circ u_i = 0.$$
which is the functional equation among the first integrals of $W$ mentioned at the beginning of the discussion. An abelian relation $\sum_{i=1}^{k} \eta_i = 0$ is **non-trivial** if at least one of the $\eta_i$ is not identically zero. If none of the 1-forms $\eta_i$ is identically zero then the abelian relation is called **complete**.

With the concept of abelian relation at hand Theorem 1.2.4 can be rephrased as the following equivalences for a germ of smooth 3-web on $(\mathbb{C}^2, 0)$.

$W$ is hexagonal $\iff K(W) = 0 \iff W$ has a non-trivial abelian relation.

To some extent, the main results of this text can be thought as generalizations of this equivalence to arbitrary webs of codimension one.

It is clear from the definition of abelian relation that for a given germ of quasi-smooth $k$-web $W$ the set of all abelian relations of $W$ forms a $\mathbb{C}$-vector space, the **space of abelian relations of** $W$, which will be denote by $\mathcal{A}(W)$. If $W = W(\omega_1, \ldots, \omega_k)$ then, one can write

$$\mathcal{A}(W) = \left\{ (\eta_1, \ldots, \eta_k) \in (\Omega^1(\mathbb{C}^n, 0))^k \mid \begin{array}{c} d\eta_i = 0 \\
\omega_i \wedge \eta_i = 0 \\
\sum_{i=1}^{k} \eta_i = 0 \end{array} \right\}.$$

Notice that a germ of diffeomorphism $\varphi$ establishing an equivalence between two germs of webs $W$ and $W'$, induces a natural isomorphism between their spaces of abelian relations.

One of the main goals of this chapter is to prove that $\mathcal{A}(W)$ is indeed a finite dimensional vector space and that its dimension — the **rank of** $W$, denoted by $\text{rank}(W)$ — is bounded by Castelnuovo’s number

$$\pi(n, k) = \sum_{j=1}^{\infty} \max(0, k - j(n - 1) - 1),$$

when $W$ is a germ of smooth $k$-web on $(\mathbb{C}^n, 0)$.

This bound, by the way, was proved by Bol when $n = 2$, and was generalized by S.-S. Chern in his PhD thesis under the direction of Blaschke. Before embarking in its proof, carried out in Section 2.2,
the determination of the space of abelian relations for planar webs in some particular cases is discussed in Section 2.1.

Of tantalizing importance for what is to come later in Chapter 5, is the content of Section 2.3. There Castelnuovo’s results on the geometry of point sets on projective space are proved, and from them are deduced constraints on the geometry of webs attaining Chern’s bound.

§

The space of abelian relations of a global web is no longer a vector space but a local system defined on an open subset containing the complement of the discriminant of the web. This can be inferred from the results by Pantazi-Hénaut expound in Section 6.3 of Chapter 6. For an elementary and simple argument see [90, Théorème 1.2.2]. Note that both approaches mentioned above deal a priori with webs on surfaces, but there is no real difficulty to deduce from them the general case.

2.1 Determining the abelian relations

If $W$ is a quasi-smooth $k$-web on $(\mathbb{C}^n, 0)$ and $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ is a generic holomorphic immersion then $\varphi^*W$ is a smooth $k$-web on $(\mathbb{C}^2, 0)$, and $\varphi$ induces naturally an injection of $\mathcal{A}(W)$ into $\mathcal{A}(\varphi^*W)$. Thus, the specialization to the two-dimensional case, in vogue up to the end of this section, is not seriously restrictive.

2.1.1 Abel’s method

Before the awaking of web geometry, Abel already studied functional equations of the form

$$\sum_{i=1}^{k} g_i \circ u_i = 0$$

for given functions $u_i$ depending on two variables. In his first published paper [1], he devised a method to determine the functions $g_i$ satisfying this functional equation. Abel’s method will not be presented in its full generality but instead the particular case where all
but one of the functions $u_i$ are homogenous polynomials of degree one will be carefully scrutinized following [93]. Under these additional assumptions, Abel’s method is remarkably simplified but still leads to interesting examples of functional equations and webs. For a comprehensive account and modern exposition of Abel’s method in the context of web geometry, the reader can consult [92].

For $i \in \mathbb{k}$, let $u_i(x, y) = a_i x + b_i y$ where $a_i, b_i \in \mathbb{C}$ are complex numbers satisfying $a_i b_j - a_j b_i \neq 0$ whenever $i \neq j$. These conditions imply the smoothness of the $k$-web $W(u_1, \ldots, u_k)$. It will be convenient to consider the vector fields $v_i = b_i \partial_x - a_i \partial_y$ which define the very same foliation as the submersions $u_i$.

Let $u : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic submersion satisfying $v_i(u) \neq 0$ for every $i \in \mathbb{k}$, and consider the smooth $(k + 1)$-web $W = W(u_1, \ldots, u_k, u)$ on $(\mathbb{C}^2, 0)$. To determine the rank of $W$, it suffices to look for holomorphic solutions $g, g_1, \ldots, g_k : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the equation

$$g \circ u = \sum_{i=1}^{k} g_i \circ u_i .$$

For that sake, apply the derivation $v_1$ to both sides of the equation above to obtain

$$g'(u) \cdot v_1(u) = \sum_{i=2}^{k} g_i'(u) \cdot v_1(u_i) .$$

Notice that $u_1$ no longer appears in the right hand-side.

Apply now the derivation $v_2$ to this new equation. Use the commutativity of $v_i$ and $v_2$, that is $[v_i, v_2] = 0$, to get

$$g''(u) \cdot v_2(u) \cdot v_1(u) + g'(u) \cdot v_2(v_1(u)) = \sum_{i=3}^{k} g''_i(u_i) \cdot v_2(u_i) \cdot v_1(u_i) + (g_i)'(u_i) \cdot v_2(v_1(u_i)) .$$

Iterating this procedure one arrives at a equation of the form

$$\left( \prod_{i=1}^{k} v_i(u) \right) g^{(k)}(u) + \cdots + v_k(v_{k-1}(\cdots v_1(u))) g'(u) = 0$$
which after dividing by the coefficient of $g^{(k)}(u)$ can be written as

$$g^{(k)}(u) = \sum_{i=1}^{k-1} h_i g^{(i)}(u)$$

where the $h_i$ are germs of meromorphic functions.

Let $v = u_y \partial_x - u_x \partial_y$ be the hamiltonian vector field of $u$. If for some $i$ the function $v(h_i)$ is not identically zero then one can apply the derivation $v$ to the above equation in order to reduce the order of it. Otherwise the functions $h_i$ are functions of $u$ only and not of $(x,y)$, that is $h_i = h_i(u)$.

Eventually one arrives at a linear differential equation of the form

$$g^{(\ell)}(u) = \sum_{i=1}^{\ell-1} h_i(u) g^{(i)}(u)$$

with $\ell \leq k$. Thus the possibilities for $g$ are reduced to a finite dimensional vector space: the space of solutions of (2.1).

After discarding the constant solutions of (2.1) one notices at this point that

$$\text{rank} \mathcal{W} \leq \text{rank} \mathcal{W}(u_1, \ldots, u_k) + k - 1$$

when $u_1, \ldots, u_k$ are linear homogeneous polynomials. Beware that this is no longer true, if the linear polynomials are replaced by arbitrary submersions. The point being that the hamiltonian vector fields $v_i$ no longer commute. One can still work his way out to deduce that an equation as (2.1) will still hold true, as done in [92], but it will be no longer true that $\ell$ is bounded by $k$.

**Example 2.1.1.** Let $u_1, \ldots, u_k \in \mathbb{C}[x,y]$ be homogeneous linear polynomials. Suppose that they are pairwise linearly independent and let $\mathcal{W} = \mathcal{W}(u_1, \ldots, u_k)$ be the induced $k$-web. Then

$$\text{rank}(\mathcal{W}) = \frac{(k-1)(k-2)}{2}$$

**Proof.** The proof goes by induction. For $k = 2$ there is no abelian relation. Suppose the result holds for $k \geq 2$. That is every parallel
$k$-web has rank $(k - 1)(k - 2)/2$. Looking for solutions of

$$g \circ u_{k+1} = \sum_{i=1}^{k} g_i \circ u_i$$

following the strategy explained above one arrives at the equation $g^{(k)}(u_{k+1}) = 0$. Thus $g$ must be a polynomial in $\mathbb{C}[t]$ of degree at most $(k - 1)$. Imposing that $g(0) = 0$ leaves a vector space of dimension $k - 2$ to choose $g$ from. Hence the rank of $\mathcal{W}_{k+1} = \mathcal{W}(u_1, \ldots, u_{k+1})$ is bounded by $(k - 1)(k - 2)/2 + (k - 2)$.

But for every positive integer $j \leq k - 1$, a dimension count shows that $(u_{k+1})^j = \sum_{i=1}^{k} \lambda_{i,j} \cdot (u_i)^j$ for suitable $\lambda_{i,j} \in \mathbb{C}$. It follows that the rank of $\mathcal{W}_{k+1}$ is $k(k-1)/2$ as wanted.

In the particular case under analysis one can make use of the following lemma.

**Lemma 2.1.2.** Suppose as above that the functions $u_i$ are linear homogenous and, still as above, let $v_i$ be the hamiltonian vector field of $u_i$. The following assertions are equivalent:

(a) the function $g(u)$ is of the form $\sum_{i=1}^{k} g_i(u_i)$;

(b) the identity $v_1 v_2 \cdots v_k(g(u)) = 0$ holds true.

**Proof.** Clearly (a) implies (b). The converse will be proved by induction. For $k = 1$ the result is evident. By induction hypothesis,

$$v_k(g(u)) = \sum_{i=1}^{k-1} h_i(u_i).$$

If $H_i$ is a primitive of $h_i$ then $v_k(H_i(u_i)) = h_i(u_i) \cdot v_k(u_i)$. Because $v_k(u_i)$ is a non-zero constant when $i < k$, one can write

$$v_k(g(u) - \sum_{i=1}^{k-1} v_k(u_i)^{-1} H_i(u_i)) = 0.$$

To conclude it suffices to apply the basis of the induction. □
SECTION 2.1: DETERMINING THE ABELIAN RELATIONS

**Proposition 2.1.3.** Let $W$ be as in Example 2.1.1. Let also $u : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a submersion and $\mathcal{F}$ be the induced foliation. If the $(k + 1)$-web $W \boxtimes \mathcal{F}$ is smooth then

$$\text{rank}(W \boxtimes \mathcal{F}) \leq \frac{k(k-1)}{2}.$$

Moreover equality holds if and only if $\ell = k$ in equation (2.1).

**Proof.** First notice that equation (2.1) has been derived by first developing formally

$$v_1v_2 \cdots v_k(g(u)) = \sum_{i=1}^k f_i(x,y)g^{(i)}(u),$$

then dividing by $f_k$, setting $h_i = f_i/f_k$ and deriving repeatedly with respect to the hamiltonian vector field of $u$ until arriving at an equation depending only on $u$. Therefore, Lemma 2.1.2 implies that any solution $g$ of

$$g(u) = \sum_{i=1}^k g_i(u_i)$$

will also be a solution of (2.1). But if $\text{rank}(W \boxtimes \mathcal{F}) = \frac{k(k-1)}{2}$ then equation (2.1) has to have at least $k-1$ non-constant solutions vanishing at zero. Consequently $k = \ell$.

Reciprocally if $k = \ell$ then there will be $k-1$ non-constant solutions vanishing at zero for $v_1v_2 \cdots v_k(g(u)) = 0$. Lemma 2.1.2 implies that $\text{rank}(W \boxtimes \mathcal{F}) = \text{rank}(W) + (k-1)$.

Webs as $W \boxtimes \mathcal{F}$ of the proposition above obtained from the superposition of a parallel web and one non-linear foliation will be called **quasi-parallel** webs.

**Example 2.1.4.** Let $u : (\mathbb{C}^2, 0) \to \mathbb{C}$ be a submersion of the form $u(x,y) = a(x) + b(y)$. Assume that the quasi-parallel 5-web $W(x,y,x-y,x+y,u(x,y))$ is smooth, that is $a_xb_y(a_x^2 - b_y^2)(0) \neq 0$.

A straightforward computation shows that

$$v_1v_2v_3v_4(g(u)) = g'''(u)a_xb_y(a_x^2 - b_y^2) + 3g''(u)a_xb_y(a_{xx} - b_{yy}) + g''(u)(b_ya_{xxx} - a_xb_{yyy}).$$
Proposition 2.1.3 implies that $W$ has rank equal to 6 if and only if

$$v_u \left( \frac{a_{xx} - b_{yy}}{a_x^2 - b_y^2} \right) = 0 \quad \text{and} \quad v_u \left( \frac{a_y b_{xxx} - a_x b_{yyy}}{a_x b_y (a_x^2 - b_y^2)} \right) = 0.$$ 

The simplest functions $u(x, y) = a(x) + b(y)$ satisfying this system of partial differential equations are $x^2 + y^2$, $x^2 - y^2$, $\exp x + \exp y$, $\log(\sin x \sin y)$ and $\log(\tanh x \tanh y)$. But these are not all. There is a continuous family of solutions that can be written with the help of theta functions of elliptic curves.

The 5-webs of the form $W(x, y, x - y, x + y, a(x) + b(y))$ with rank equal to 6, have been completely classified in [93] through a careful analysis of the above system of PDEs.

If nothing else, this example shows how involved can be the search for webs of high rank, even in considerably simple cases.

### 2.1.2 Webs with infinitesimal automorphisms

Let $F$ be a germ of smooth foliation on $(\mathbb{C}^2, 0)$ induced by a germ of 1-form $\omega$. A germ of vector field $v$ is an infinitesimal automorphism of $F$ if the foliation $F$ is preserved by the local flow of $v$. In algebraic terms: $L_v \omega \wedge \omega = 0$ where $L_v = i_v d + di_v$ is the Lie derivative with respect to the vector field $v$. Those not familiar with the Lie derivative can find its basic algebraic properties in [54, Chapter IV].

When the infinitesimal automorphism $v$ is transverse to $F$, that is $\omega(v) \neq 0$, then an elementary computation (see [89, Corollary 2] or [28, Chapter III Section 2] ) shows that the 1-form

$$\alpha = \frac{\omega}{i_v \omega}$$

is closed and satisfies $L_v \alpha = 0$. By definition, the integral

$$u(z) = \int_0^z \alpha$$

is the canonical first integral of $F$ with respect to $v$. Clearly $u(0) = 0$ and $L_v(u) = 1$. In particular the latter equality implies that $u$ is a germ of submersion.
SECTION 2.1: DETERMINING THE ABELIAN RELATIONS

The canonical first integral admits a nice physical interpretation: its value at \( z \) measures the time which the local flow of \( v \) takes to move the leaf through zero to the leaf through \( z \).

Now let \( W = W(\omega_1, \ldots, \omega_k) \) be a germ of smooth \( k \)-web on \((\mathbb{C}^2, 0)\) and let \( v \) be an infinitesimal automorphism of \( W \), in the sense that \( v \) is an infinitesimal automorphism of all the foliations defining \( W \).

By hypothesis, one has \( L_v \omega_i \wedge \omega_k = 0 \) for every \( i \in \mathbb{K} \). Because \( L_v \) commutes with \( d \), it induces a linear map \( L_v : A(W) \rightarrow A(W) \) (2.2)

\[
(\eta_1, \ldots, \eta_k) \mapsto (L_v \eta_1, \ldots, L_v \eta_k).
\]

A simple analysis of the \( L_v \)-invariants subspaces of \( A(W) \) will provide valuable information about the abelian relations of webs admitting infinitesimal automorphisms.

Description of \( A(W) \)

Suppose that \( W = F_1 \boxtimes \cdots \boxtimes F_k \) is a smooth \( k \)-web on \((\mathbb{C}^2, 0)\) which admits an infinitesimal automorphism \( v \), regular and transverse to all the foliations \( F_i \).

Let \( i \in \mathbb{K} \) be fixed. Set \( A_i(W) \) as the vector subspace of \( \Omega^1(\mathbb{C}^2, 0) \) spanned by the \( i \)-th components \( \eta_i \) of abelian relations \( (\eta_1, \ldots, \eta_k) \in A(W) \). In other words, if \( p_i : \Omega^1(\mathbb{C}^2, 0)^k \rightarrow \Omega^1(\mathbb{C}^2, 0) \) is the projection to the \( i \)-th factor then

\[
A_i(W) = p_i(\mathcal{A}(W)).
\]

If \( u_i = \int \alpha_i \) is the canonical first integral of \( F_i \) with respect to \( v \), then for \( \eta_i \in A_i(W) \), there exists a germ \( f_i \in \mathbb{C}\{t\} \) for which \( \eta_i = f_i(u_i) \, du_i \).

Assume now that \( A_i(W) \) is not the zero vector space and let

\[
\{ \eta^\nu_i = f_{\nu}(u_i) \, du_i \mid \nu \in n_i \}
\]

be a basis of it, consequently \( n_i = \dim A_i(W) \). Since \( L_v : A_i(W) \rightarrow A_i(W) \) is a linear map, there exist complex constants \( c_{\nu \mu} \) such that

\[
L_v(\eta^\nu_i) = \sum_{\mu=1}^{n_i} c_{\nu \mu} \eta^\mu_i, \quad \nu \in n_i. \tag{2.3}
\]
But for any $\nu \in n_i$, the identity below holds true,

$$L_v(\eta_\nu^i) = L_v(f_\nu(u_i) \, du_i)$$

$$= v(f_\nu(u_i)) \, du_i + f_\nu(u_i) \, L_v(du_i) = f'_\nu(u_i) \, du_i.$$ 

Thus the relations (2.3) are equivalent to the following

$$f'_\nu = \sum_{\mu=1}^{n_i} c_{\nu \mu} \, f_\mu, \quad \nu \in n_i. \quad (2.4)$$

Now let $\lambda_1, \ldots, \lambda_\tau \in \mathbb{C}$ be the eigenvalues of the map $L_v$ acting on $A(W)$ corresponding to minimal eigenspaces of respective dimensions $\sigma_1, \ldots, \sigma_\tau$. The system of linear differential equations (2.4) provides the following description of $A(W)$.

**Proposition 2.1.5.** The abelian relations of $W$ are of the form

$$P_1(u_1) e^{\lambda_1 u_1} \, du_1 + \cdots + P_k(u_k) e^{\lambda_k u_k} \, du_k = 0$$

where $P_1, \ldots, P_k$ are polynomials of degree less or equal to $\sigma_i$.

Proposition 2.1.5 suggests an approach to effectively determine $A(W)$. Once the possible non-zero eigenvalues of the map (2.2) are restricted to a finite set then the abelian relations can be found by simple linear algebra.

To restrict the possible eigenvalues first notice that 0 is an eigenvalue of (2.2) if and only if for every germ of vector field $w$ the Wronskian determinant

$$\det \begin{pmatrix} u_1 & \cdots & u_k \\ w(u_1) & \cdots & w(u_k) \\ \vdots & \ddots & \vdots \\ w^{k-1}(u_1) & \cdots & w^{k-1}(u_k) \end{pmatrix} \quad (2.5)$$

is identically zero. In fact, if this is the case then there are two possibilities: the functions $u_1, \ldots, u_k$ are $\mathbb{C}$-linearly dependent or all the orbits of $w$ are cutted out by some element of the linear system generated by $u_1, \ldots, u_k$, see [86, Theorem 4]. In particular if $w$ is a vector field of the form $w = \mu x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, with $\mu \in \mathbb{C} \setminus \mathbb{Q}$, then the
leaves of $w$ accumulate at 0 and are cutted out by no holomorphic function. Therefore the vanishing of (2.5) implies the existence of an abelian relation of the form

$$\sum c_i u_i = 0,$$

where the $c_i$'s are complex constants.

To determine the possible complex numbers $\lambda$ which are eigenvalues of the map (2.2) first notice that these corresponds to a functional equation of the form $c_1 e^{\lambda u_1} + \cdots + c_k e^{\lambda u_k} = \text{cst.}$ where, as before, the $c_i$'s are complex constants. In the same spirit of what has just been made for the zero eigenvalue case consider the holomorphic function given by

$$\begin{pmatrix}
    \exp(\lambda u_1) & \cdots & \exp(\lambda u_k) \\
    w(\exp(\lambda u_1)) & \cdots & w(\exp(\lambda u_k)) \\
    \vdots & \ddots & \vdots \\
    w^{k-1}(\exp(\lambda u_1)) & \cdots & w^{k-1}(\exp(\lambda u_k))
\end{pmatrix}$$

(2.6)

for an arbitrary germ of vector field $w$.

The Wronskian determinant (2.6) is of the form

$$\exp(\lambda(u_1 + \cdots + u_k)) \lambda^{k-1} P_w(\lambda),$$

where $P_w$ is a polynomial in $\lambda$, of degree at most $\frac{(k-1)(k-2)}{2}$, with germs of holomorphic functions as coefficients. The common constant roots of these polynomials, when $w$ varies, are exactly the eigenvalues of the map (2.2).

**Example 2.1.6.** The $k$-web $\mathcal{W}$ induced by the functions $f_i(x, y) = y + x^i$, $i = 1, \ldots, k$, has no abelian relations.

**Proof.** Notice that the vector field $v = \frac{\partial}{\partial y}$ is an infinitesimal automorphism of $\mathcal{W}$ and $v(df_i) = 1$, for every $i \in \mathbb{K}$. It follows that $u_i = f_i$ are the canonical first integrals of $\mathcal{W}$. On the other hand for the vector field $w = \frac{\partial}{\partial x}$, $P_w(\lambda)|_{x=y=0} = (-1)^{k-1} \prod_{n=1}^{k-1} n!$. Consequently, the only candidate for an eigenvalue of the map (2.2) is $\lambda = 0$. Because the functions $f_i$ are linearly independent over $\mathbb{C}$ the web $\mathcal{W}$ carries no abelian relations at all. $\blacksquare$
CHAPTER 2: ABELIAN RELATIONS

The next example determines the abelian relations of one of the 5-webs of rank 6 discussed in Example 2.1.4.

**Example 2.1.7.** The radial vector field \( R = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \) is an infinitesimal automorphism of the 5-web \( W = W(x, y, x + y, x - y, x^2 + y^2) \). The canonical first integrals are \( u_1 = \log x, \ u_2 = \log y, \ u_3 = \log(x + y), \ u_4 = \log(x - y) \) and \( u_5 = \frac{1}{2} \log(x^2 + y^2) \).

If \( w = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \) then \( P_w \) is a complex multiple of

\[
 x^7 y^7 \lambda(\lambda - 1)^2(\lambda - 2)^2(\lambda - 4)(\lambda - 6).
\]

According to Proposition 2.1.5, it suffices to look for abelian relations of the form

\[
 \sum_{i=1}^{5} P_{\lambda i}(\log f_i) f_i^\lambda df_i = 0, \text{ for } \lambda = 0, 1, 2, 4, 6, \text{ where } P_{\lambda i} \text{ are polynomials and } f_i = \exp(u_i).
\]

Looking first for abelian relations where the polynomials \( P_{\lambda i} \) are constant polynomials one has to find linear dependencies between the linear polynomials \( f_1, \ldots, f_4 \) and the degree \( \lambda \) polynomials \( f_1^\lambda, \ldots, f_5^\lambda \) for \( \lambda = 2, 4, 6 \).

For \( \lambda = 1 \) there are two linearly independent abelian relations

\[
 f_1 + f_2 - f_3 = 0, \quad f_1 - f_2 - f_4 = 0.
\]

For \( \lambda = 2 \) there are another two:

\[
 f_1^2 + f_2^2 - f_5 = 0, \quad 2f_1^2 + 2f_2^2 - f_3^2 - f_4^2 = 0.
\]

Finally, there is one abelian relation for each \( \lambda \in \{4, 6\} \):

\[
 5f_1^4 + 5f_2^4 + f_3^4 + f_4^4 - 6f_5^4 = 0, \quad 8f_1^6 + 8f_2^6 + f_3^6 + f_4^6 - 10f_5^6 = 0.
\]

According to Proposition 2.1.3, \( \text{rank}(W) \leq 6 \). Hence the abelian relations above generate \( \mathcal{A}(W) \).

### 2.2 Bounds for the rank

Let \( W = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k = W(\omega_1 \cdots \omega_k) \) be a germ of quasi-smooth \( k \)-web on \( (\mathbb{C}^n, 0) \).
SECTION 2.2: BOUNDS FOR THE RANK

For every positive integer $j$, define $L^j(W)$ as the vector subspace of the $\mathbb{C}$-vector space $\text{Sym}^j(\Omega_1^0(C^n,0))$ generated by $\{\omega_i^j(0); i \in \mathcal{K}\}$, the $j$-th symmetric powers of the differential forms $\omega_i(0)$. Set

$$\ell^j(W) = \dim L^j(W).$$

Equivalently, one can define $\ell^j(W)$ in terms of the linear parts of the submersions $u_i : (\mathbb{C}^n,0) \rightarrow (\mathbb{C},0)$ defining $W$. If $h_i$ is the linear part at the origin of $u_i$ then

$$\ell^j(W) = \dim \left( \mathbb{C} h^j_1 + \cdots + \mathbb{C} h^j_k \right).$$

2.2.1 Bounds for $\ell^j(W)$

The integers $\ell^j(W)$ are bounded from above by the dimension of the space of degree $j$ homogeneous polynomials in $n$ variables, that is

$$\ell^j(W) \leq \min \left( k, \binom{n+j-1}{n-1} \right). \tag{2.7}$$

A good lower bound is more delicate to obtain. For smooth webs there is the following proposition.

**Proposition 2.2.1.** If $W$ is a germ of smooth $k$-web on $(\mathbb{C}^n,0)$ then

$$\ell^j(W) \geq \min(k, j(n-1) + 1).$$

The key point is next lemma which translates questions about the dimension of vector spaces generated by powers of linear forms to questions about the codimension of space of hypersurfaces containing finite sets of points.

**Lemma 2.2.2.** Let $h_1, \ldots, h_k \in \mathbb{C}[x_1, \ldots, x_n]$ be linear forms and let $\mathcal{P} = \{[h_1], \ldots, [h_k]\}$ be the corresponding set of points of $\mathbb{P}^{n-1} = \mathbb{P} \mathbb{C}[x_1, \ldots, x_n]$. If $V(j) \subset |\mathcal{O}_{\mathbb{P}^{n-1}}(j)|$ is the linear system of degree $j$ hypersurfaces through $\mathcal{P}$ then

$$\dim(\mathbb{C} h^j_1 + \cdots + \mathbb{C} h^j_k) = \dim |\mathcal{O}_{\mathbb{P}^{n-1}}(j)| - \dim V(j).$$
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**Proof.** Set \( n_j \) equal to \( h^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(j)) - 1 \), and consider the \( j \)-th Veronese embedding

\[
\nu_j : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n_j},
\]

\[
[h] \mapsto [h^j].
\]

On the one hand the projective dimension of \( \mathbb{C}h_1^j + \cdots + \mathbb{C}h_k^j \) is equal to the dimension of the linear span of the image of \( \mathcal{P} \). On the other hand the codimension of this linear span is equal to the dimension of the linear system of hyperplanes containing it. But the pull-back under \( \nu_j \) of these hyperplanes are exactly the elements of \( |V(j)| \), the degree \( j \) hypersurfaces in \( \mathbb{P}^{n-1} \) containing \( \mathcal{P} \). The lemma follows. \( \square \)

**Proof of Proposition 2.2.1.** Let, as above, \( h_i \) be the linear terms of the submersions defining \( \mathcal{W} \) and let \( \mathcal{P} \subset \mathbb{P}^{n-1} \) be the set of \( k \) points in general position determined by the linear forms \( h_1, \ldots, h_k \). According to the above lemma all that is needed to prove is that \( \mathcal{P} \) imposes \( m = \min(k, j(n-1) + 1) \) independent conditions on the space of degree \( j \) hypersurfaces in \( \mathbb{P}^{n-1} \). For that sake, it suffices to show that for a subset \( \mathcal{Q} \subset \mathcal{P} \) of cardinality \( m \) one can construct for each \( q \in \mathcal{Q} \) a degree \( j \) hypersurface that passes through all the points of \( \mathcal{Q} \) but \( q \).

The set \( \mathcal{Q} - \{q\} \) can be written as a disjoint union of \( j \) subsets of cardinality at most \( (n-1) \). Any of these subsets can be supposed contained in a hyperplane that does not contain \( q \). Thus there exists a union of \( j \) hyperplanes that contains \( \mathcal{Q} - \{q\} \) and avoids \( q \). \( \square \)

**Remark 2.2.3.** Notice that the proof of Proposition 2.2.1 shows that when \( k \le j(n-1) + 1 \) any \( k \) points in general position impose \( k \) independent conditions on the linear system of degree \( j \) hypersurfaces on \( \mathbb{P}^{n-1} \).

For an essentially equivalent proof of Proposition 2.2.1, but with a more analytic flavor, see [107, Lemme 2.1].

**Corollary 2.2.4.** If \( \mathcal{W} \) is a germ of smooth \( k \)-web on \( (\mathbb{C}^2, 0) \) then

\[
\ell^j(\mathcal{W}) = \min(k, j + 1).
\]
Proof. One has just to observe that the space of homogenous polynomials of degree \( j \) in two variables has dimension \( j + 1 \) and use Proposition 2.2.1.

For arbitrary webs, without any further restriction on the relative position of the tangent spaces at the origin besides pairwise transversality, it is not possible to improve the bound beyond the specialization of the above to \( n = 2 \). That is for an arbitrary quasi-smooth \( k \)-web

\[
\ell^j(W) \geq \min(k, j + 1).
\]

(2.8)

Remark 2.2.5. Recently Cavalier and Lehmann have drawn special attention to \( k \)-webs on \((\mathbb{C}^n, 0)\) for which the upper bounds (2.7) are sharp, see [26]. These have been labeled by them ordinary webs.

2.2.2 Bounds for the rank

There is a natural decreasing filtration \( F^\bullet A(W) \) on the vector space \( A(W) \). The first term is, of course, \( F^0 A(W) = A(W) \) and for \( j \geq 0 \) the \( j \)-th piece of the filtration is defined as

\[
F^j A(W) = \ker \left\{ A(W) \longrightarrow \left( \frac{\Omega^1(\mathbb{C}^n, 0)}{m^j \cdot \Omega^1(\mathbb{C}^n, 0)} \right)^k \right\},
\]

with \( m \) being the maximal ideal of \( \mathcal{O}(\mathbb{C}^n, 0) \).

Lemma 2.2.6. If \( W \) is a germ of quasi-smooth \( k \)-web on \((\mathbb{C}^n, 0)\) then

\[
\dim \frac{F^j A(W)}{F^{j+1} A(W)} \leq \max(0, k - \ell^{j+1}(W)).
\]

Proof. Let, as above, \( h_1, \ldots, h_k \) be the linear terms at the origin of the submersions defining \( W \). Consider the linear map

\[
\varphi : \mathbb{C}^k \longrightarrow \mathbb{C}_{j+1}[x_1, \ldots, x_n]
\]

\[
(c_1, \ldots, c_k) \longrightarrow \sum c_i (h_i)^{j+1}.
\]

From the definition of the space \( \mathcal{L}^j(W) \), it is clear that the image of \( \varphi \) coincides with it. In particular,

\[
\dim \ker \varphi = \max(0, k - \ell^{j+1}(W)).
\]
If \((\eta_1, \ldots, \eta_k)\) is an abelian relation in \(F^j\mathcal{A}(\mathcal{W})\) then for suitable complex numbers \(\mu_1, \ldots, \mu_k\), the following identity holds true
\[
(\eta_1, \ldots, \eta_k) = (\mu_1(h_1)^jdh_1, \ldots, \mu_k(h_k)^jdh_k) \mod F^{j+1}\mathcal{A}(\mathcal{W})
\]
Consider the linear map taking \((\eta_1, \ldots, \eta_k)\) \(\in F^j\mathcal{A}(\mathcal{W})\) to the \(k\)-uple of complex numbers \((\mu_1, \ldots, \mu_k)\) \(\in \mathbb{C}^k\). Since \(\sum \eta_i = 0\) it follows that this map induces an injection of \(F^j\mathcal{A}(\mathcal{W})/F^{j+1}\mathcal{A}(\mathcal{W})\) into the kernel of \(\varphi\). The lemma follows.

**Corollary 2.2.7.** If \(\mathcal{W}\) is a quasi-smooth \(k\)-web then for \(j \geq k-2\)
\[F^j\mathcal{A}(\mathcal{W}) = 0.\]

**Proof.** For \(j \geq k-2\), equation (2.8) reads as \(\ell^{j+1}(\mathcal{W}) = k\). Therefore Lemma 2.2.6 implies
\[F^j\mathcal{A}(\mathcal{W}) = F^{j+1}\mathcal{A}(\mathcal{W}).\]
Thus an element of \(F^{k-2}\mathcal{A}(\mathcal{W})\) has a zero of infinite order at the origin. Since it is a \(k\)-uple of holomorphic 1-forms it has to be identically zero.

With what have been done so far, Bol’s, respectively Chern’s, bound for the rank of smooth \(k\)-webs on \((\mathbb{C}^2, 0)\), respectively \((\mathbb{C}^n, 0)\), can be easily proved.

**Theorem 2.2.8.** If \(\mathcal{W}\) is a germ of quasi-smooth \(k\)-web on \((\mathbb{C}^n, 0)\) then
\[
\text{rank}(\mathcal{W}) \leq \sum_{j=0}^{k-3} \max(0, k - \ell^{j+1}(\mathcal{W})).
\]
Moreover, if \(\mathcal{W}\) is smooth then
\[
\text{rank}(\mathcal{W}) \leq \pi(n, k) = \sum_{j=0}^{k-3} \max(0, k - (j+1)(n-1) - 1).
\]

**Proof.** It follows from the corollary above that \(\mathcal{A}(\mathcal{W})\) is isomorphic as a vector space to
\[
\bigoplus_{j=0}^{k-3} \frac{F^j\mathcal{A}(\mathcal{W})}{F^{j+1}\mathcal{A}(\mathcal{W})}.
\]
SECTION 2.2: BOUNDS FOR THE RANK

If \( W \) is quasi-smooth the result follows promptly from Lemma 2.2.6. If moreover \( W \) is smooth one can invoke Proposition 2.2.1 to conclude.

The number \( \pi(n, k) \) appearing in the bound for the rank of smooth webs is **Castelnuovo number**. It is the bound for the arithmetical genus of non-degenerate irreducible curves in \( \mathbb{P}^n \) according to a classical result by Castelnuovo. In Chapter 3 Castelnuovo result will be recovered from Theorem 2.2.8 combined with Abel’s addition Theorem.

**Remark 2.2.9.** Following [60], let \( m = \left\lfloor \frac{k - 1}{n - 1} \right\rfloor \) and \( \epsilon \) be the remainder of the division of \( k - 1 \) by \( n - 1 \). Thus \( k - 1 = m(n - 1) + \epsilon \) with \( 0 \leq \epsilon \leq n - 2 \). Using this notation Castelnuovo’s numbers can be expressed as

\[
\pi(n, k) = \binom{m}{2}(n - 1) + m\epsilon .
\]

In this way one obtains a family of closed formulas for the bound of the rank of a \( k \)-web on \((\mathbb{C}^n, 0)\) according to the residue \( \epsilon \) of \( k - 1 \) modulo \( n - 1 \).

**Remark 2.2.10.** Alternatively, one can set \( \rho = \left\lfloor \frac{k - n - 1}{n - 1} \right\rfloor \) and \( \epsilon \) equal to the remainder of the division of \( k - n - 1 \) by \( n - 1 \). Hence \( k - n - 1 = \rho(n - 1) + \epsilon \) with \( 0 \leq \epsilon \leq n - 2 \). Castelnuovo’s numbers admit the following alternative presentation

\[
\pi(n, k) = (\epsilon + 1)\binom{\rho + 2}{2} + (n - 2 - \epsilon)\binom{\rho + 1}{2} .
\]

The two distinct presentations are given here because the former is the usual one found in the literature, while the latter seems to be better adapted to some constructions that will be carried out in Section 4.3.4 of Chapter 4.

Notice that for smooth webs the bound for the rank is attained if and only if the partial bounds provided by the combination of Proposition 2.2.1 with Lemma 2.2.6 are also attained. For further use, this remark is stated below as a corollary.
Corollary 2.2.11. Let \( \mathcal{W} \) be a germ of smooth \( k \)-web on \( (\mathbb{C}^n, 0) \). If \( \text{rank}(\mathcal{W}) = \pi(n, k) \) then
\[
\dim \frac{F^j \mathcal{A}(\mathcal{W})}{F^{j+1} \mathcal{A}(\mathcal{W})} = \max (0, k - (j + 1)(n - 1) - 1).
\]
for every \( j \geq 0 \).

2.2.3 Webs of maximal rank

One of the central problems in web geometry, and the central theme of this text, is the characterization of germs of smooth \( k \)-webs on \( (\mathbb{C}^n, 0) \) for which \( \text{rank}(\mathcal{W}) = \pi(n, k) \). They are called webs of maximal rank.

Theorem 2.2.8 recovers, and generalizes to arbitrary planar webs, the bound provided by Proposition 2.1.3 for germs of planar quasi-parallel webs. In particular the planar parallel webs are examples of webs of maximal rank, see Example 2.1.1. The 5-webs mentioned in Example 2.1.4 are also of maximal rank.

In dimension greater than two webs of maximal rank are harder to come by. In contrast with the planar case, not every parallel web is of maximal rank. In the next Section the parallel webs of maximal rank will be characterized in Proposition 2.3.3 and constraints on the distribution of conormals of maximal rank webs will be established.

2.3 Conormals of webs of maximal rank

If \( \mathcal{W} \) is a germ of smooth \( k \)-web of maximal rank then the lower bounds for \( \ell^j(\mathcal{W}) \) given by Proposition 2.2.1 are attained, that is,
\[
\ell^j(\mathcal{W}) = \min(k, j(n - 1) + 1)
\]
holds true for every positive integer \( j \).

When the ambient space has dimension two (\( n = 2 \)) these equalities do not impose any restriction on the web as Corollary 2.2.4 testifies. When \( n \) is at least three then the equalities above impose rather strong restrictions of the distributions of conormals of the web.
Indeed, in the next few pages the corresponding equality for \( j = 2 \) – that is, \( \ell^2(W) = \min(k, 2(n-1) + 1) \) – will be exploited and the following proposition will be proved.

**Proposition 2.3.1.** Let \( W \) be a germ of smooth \( k \)-web on \( (\mathbb{C}^n, 0) \). Suppose that \( n \geq 3 \) and \( k \geq 2n+1 \). If \( \ell^2(W) = 2n-1 \) then there exists a non-degenerate rational normal curve \( \Gamma \) in \( \mathbb{P}(T^*_0(\mathbb{C}^n, 0)) \) containing the conormals of the web \( W \) at the origin.

**A particular case**

For the sake of clarity the case \( n = 3 \) of Proposition 2.3.1 will be here presented. It is harmless to assume that \( k = 2n = 6 \) even if the hypothesis for \( n = 3 \) reads \( k \geq 7 \).

Let \( h_1, \ldots, h_6 \in \mathbb{C}_1[x_1, x_2, x_3] \) be six linear forms in general position and let \( L^2 \) be the vector space contained in \( \mathbb{C}_2[x_1, x_2, x_3] \) generated by theirs squares.

If \( \dim L^2 = 5 \), since \( \mathbb{C}_2[x_1, x_2, x_3] \) has dimension six, there is a hyperplane \( H \) through \( 0 \in \mathbb{C}_2[x_1, x_2, x_3] \) containing \( L^2 \). If one now interprets the linear forms \( h_i \) as points in \( \mathbb{P}^2 = \mathbb{P}\mathbb{C}_1[x_1, x_2, x_3] \) and consider the Veronese embedding of this \( \mathbb{P}^2 \) into \( \mathbb{P}^5 = \mathbb{P}\mathbb{C}_2[x_1, x_2, x_3] \), as in the proof of Lemma 2.2.2, then the pull-back of \([H]\) to \( \mathbb{P}^2 \) is a conic containing \([h_1], \ldots, [h_6]\). This is the sought rational normal curve.

**Dimension shift and reduction to Castelnuovo Lemma**

As suggested by its statement all the action in the proof of Proposition 2.3.1 will take place in \( \mathbb{P}T^*_0(\mathbb{C}^n, 0) = \mathbb{P}^{n-1} \). To avoid carrying over a \(-1\) throughout instead of working with a \( k \)-web on \( (\mathbb{C}^n, 0) \) it is convenient to consider a \( k \)-web on \( (\mathbb{C}^{n+1}, 0) \). Of course with this shift on the dimension the hypotheses of Proposition 2.3.1 now read as

\[
k \geq 2n + 3 \quad \text{and} \quad \ell^2(W) = 2n + 1.
\]

If \( h_1, \ldots, h_k \in \mathbb{C}[x_0, \ldots, x_n] \) are the linear forms defining the tangent space of the leaves of \( W \) through the origin then according to Lemma 2.2.2 the number of conditions imposed on quadrics of \( \mathbb{P}^n \)
by the corresponding set of points $P = \{[h_1], \ldots, [h_k]\} \subset \mathbb{P}^n$ is exactly $\ell^2(W)$. Therefore Proposition 2.3.1 is equivalent to the famous Castelnuovo Lemma.

**Proposition 2.3.2 (Castelnuovo Lemma).** Let $P \subset \mathbb{P}^n$ be a set of $k$ points in general position. Suppose that $n \geq 2$ and $k \geq 2n + 3$.

If $P$ imposes only $2n + 1$ conditions on the linear system of quadrics $|\mathcal{O}_{\mathbb{P}^n}(2)|$ then $P$ is contained in a rational normal curve $\Gamma$ of degree $n$.

Before dealing with the proof of Proposition 2.3.2 itself, which will follow [60, Chapter III], some basic properties of rational normal curves will be reviewed.

### 2.3.1 Rational normal curves

The rational normal curves on a projective space $\mathbb{P}^n$ are the ones that admit a parametrization of the form

$$
\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^n
$$

$$
[s : t] \longmapsto [a_0(s : t) : \cdots : a_n(s : t)].
$$

where $a_0, \ldots, a_n$ form a basis of the space $\mathbb{C}[s, t]$ of binary forms of degree $n$. In other words, a rational normal curve $\Gamma$ on $\mathbb{P}^n$ is the image of an embedding of $\mathbb{P}^1$ into $\mathbb{P}^n$ given by the linear system $|\mathcal{O}_{\mathbb{P}^1}(n)|$.

Since the hyperplanes in $\mathbb{P}^n$ are in one to one correspondence with the non-zero elements of $\mathbb{C}[s, t]$ modulo multiplication by $\mathbb{C}^*$, the intersection of $\Gamma$ with an hyperplane $H$ consists of at most $n$ points. If the hyperplane is generic then the intersection has exactly $n$ points, that is $\Gamma$ has degree $n$. It turns out that this is the minimal degree among the non-degenerated curves in $\mathbb{P}^n$. Moreover the rational normal curve is the unique irreducible non-degenerated curve of degree $n$, see Proposition 2.3.11 below.

Notice that any $k$ distinct points on a rational normal curve $\Gamma \subset \mathbb{P}^n$ are automatically in general position with respect to the linear system of hyperplanes. Indeed, if a subset of cardinality $a \leq n$ is contained in a $\mathbb{P}^{a-2}$, then by choosing other $n - a + 1$ points and considering a hyperplane containing all these $n + 1$ points one arrives
at a contradiction since a rational normal curve $\Gamma$ intersects every hyperplane in at most $\deg \Gamma = n$ points.

Parallel webs defined by points on a rational normal curve are the simplest examples of webs of maximal rank on $(\mathbb{C}^n,0)$, with $n \geq 3$. More precisely,

**Proposition 2.3.3.** Let $n \geq 2$ and $k \geq 2n + 3$ be integers. Let also $h_1, \ldots, h_k \in \mathbb{C}_1[x_0, x_1, \ldots, x_n]$ be pairwise distinct linear forms and $W = W(h_1, \ldots, h_k)$ be the corresponding parallel $k$-web on $(\mathbb{C}^{n+1},0)$. Then $\mathcal{P} = \{[h_1], \ldots, [h_k]\}$, the corresponding set of points of $\mathbb{P}^n$, lies in a rational normal curve $\Gamma$ if and only if $W$ is smooth and of maximal rank.

**Proof.** If $W$ is smooth and of maximal rank then Proposition 2.3.1 implies the result.

Reciprocally, if $\mathcal{P}$ is contained in a rational normal curve then $W$ is smooth because the points $[h_i]$ are in general position, see discussion preceding the statement of the Proposition. To prove that $W$ is of maximal rank notice that the kernel of the restriction map

$$H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(j)) \rightarrow H^0(\Gamma, O_{\Gamma}(j)) \simeq H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(jn))$$

has codimension at least $h^0(\mathbb{P}^1, O_{\mathbb{P}^1}(jn)) = jn + 1$. Therefore, by Lemma 2.2.2,

$$\ell^j(W) = \dim(\mathbb{C}h_1^j + \cdots + \mathbb{C}h_k^j) \leq jn + 1.$$ 

Hence Proposition 2.2.1 implies $\ell^j(W) = \min(k, jn + 1)$.

Because $W$ is a parallel web, $F^jA(W)/F^{j+1}A(W)$ not just embeds into the kernel of the map $\mathbb{C}^k \to \mathcal{L}^j(W)$, but is indeed isomorphic to it. Therefore

$$\text{rank}(W) = \sum_{j=0}^{k-3} \max(0, k - (j+1)n - 1)$$

as wanted.  

**Steiner’s synthetic construction**

The rational normal curves admit a nice geometric description: the so called **Steiner construction.** Let $p_1, \ldots, p_{n+3} \in \mathbb{P}^n$ be $n + 3$
points in general position. For each $i$ ranging from 1 to $n$, let $\Pi_i$ be the $\mathbb{P}^{n-2}$ spanned by $p_1, \ldots, \hat{p}_i, \ldots, p_n$. The hyperplanes containing $\Pi_i$ form a family $H_i((s : t))$ with $(s : t) \in \mathbb{P}^1$. One can choose the parametrizations in order to have

$$p_{n+1} = \bigcap_{i=1}^{n} H_i(0 : 1), \quad p_{n+2} = \bigcap_{i=1}^{n} H_i(1 : 0) \quad \text{and} \quad p_{n+3} = \bigcap_{i=1}^{n} H_i(1 : 1).$$

**Proposition 2.3.4.** The set

$$\Gamma = \bigcup_{[s : t] \in \mathbb{P}^1} \left( \bigcap_{i=1}^{n} H_i(s : t) \right)$$

is the unique rational normal curve through the points $p_1, \ldots, p_{n+3}$.

**Proof.** Because the points are in general position the expression under parentheses defines for each $(s : t) \in \mathbb{P}^1$ a unique point of $\mathbb{P}^n$. Consequently $\Gamma$ is a curve parametrized by $\mathbb{P}^1$.

Clearly it contains $p_{n+1}, p_{n+2}$ and $p_{n+3}$. To see that it contains $p_1, \ldots, p_n$ notice that $p_i \in H_j((s : t))$ for every $[s : t] \in \mathbb{P}^1$ when $j \neq i$ and there exists an $(s_i : t_i)$ such that $p_i \in H_i(s_i : t_i)$. It remains to show that the linear system defining $\Gamma$ is $|\mathcal{O}_{\mathbb{P}^1}(n)|$.

Using an automorphism of $\mathbb{P}^n$ the points $p_1, \ldots, p_{n+1}$ can normalized as

$$p_i = [0 : \ldots : 0 : \underbrace{1}_{i\text{-th entry}} : 0 : \ldots : 0] \quad i \in n+1.$$

If $p_{n+2} = [a_0 : a_1 : \ldots : a_n]$ and $p_{n+3} = [b_0 : b_1 : \ldots : b_n]$ then it is a simple matter to verify that

$$\mathbb{P}^1 \quad \mapsto \quad \mathbb{P}^n$$

$$(s : t) \quad \mapsto \quad [(a_1^{-1}s - b_1^{-1}t)^{-1} : \ldots : (a_n^{-1}s - b_n^{-1}t)^{-1}]$$

is a parametrization of $\Gamma$ in the normalization above. Multiplying all the entries by $\prod_i (a_i^{-1}s - b_i^{-1}t)$ one ends up with $n+1$ binary forms of degree $n$. Since $p_1, \ldots, p_{n+3}$ are not contained in any hyperplane these must generate the space of binary forms of degree $n$. \qed
2.3.2 Proof of Castelnuovo Lemma

A variant of the synthetic construction presented above allows to construct rank three quadrics – these are quadrics which in a suitable system of coordinates are cut out by a polynomial of the form \( x_0^2 + x_1^2 + x_2^2 \) containing \( \Gamma \). This construction will be the key to prove Proposition 2.3.2.

Let \( \mathcal{P} = \{p_1, \ldots, p_{2n+2}, p_{2n+3}\} \) be a set of \( 2n + 3 \) distinct points of \( \mathbb{P}^n \) in general position, and \( \Lambda \simeq \mathbb{P}^{n-2} \) be the linear span of \( p_1, \ldots, p_{n-1} \).

**Lemma 2.3.5.** If \( \mathcal{P} \) imposes at most \( 2n + 1 \) independent conditions on the space of quadrics then there are at least \( n-1 \) linearly independent quadrics containing \( \Lambda \cup \mathcal{P} \).

**Proof.** Let \( F_0 \) be a linear form (unique up to multiplication by \( \mathbb{C}^* \)) vanishing at the span of \( p_1, \ldots, p_n \) and \( G_0 \) be the one vanishing on \( p_1, \ldots, p_{n-1}, p_{n+1} \). Any quadric containing \( \Lambda \) can be written in the form \( F_0 G - G_0 F \) for suitable linear forms \( F, G \in \mathbb{C}[x_0, \ldots, x_n] \). Such pair \((F, G)\) is not unique, but is well defined modulo the addition of a multiple of \((G_0, F_0)\). Hence the vector space of quadrics containing \( \Lambda \) has dimension \( 2n + 1 \).

Further imposing that the quadrics contain the \( n + 2 \) points \( p_n, p_{n+1}, \ldots, p_{2n+1} \) one sees that there are at least \( 2n+1-(n+2) \), that is \( n-1 \), linearly independent quadrics containing \( \Lambda \cup \{p_1, \ldots, p_{2n+1}\} \). By hypothesis the space of quadrics containing \( \{p_1, \ldots, p_{2n+1}\} \) coincides with the space of quadrics containing \( \mathcal{P} \).

Keeping the notation from the lemma above one can write down the \( n-1 \) linearly independent quadrics \( Q_1, \ldots, Q_{n-1} \) containing \( \Lambda \cup \mathcal{P} \) in the form

\[
Q_i = \det \begin{pmatrix} F_0 & F_i \\ G_0 & G_i \end{pmatrix} = F_0 G_i - G_0 F_i
\]

for suitable linear forms \( F_i, G_i \in \mathbb{C}[x_0, \ldots, x_n], i \in n-1 \).

It is possible to recover a rational normal curve \( \Gamma \) from the quadrics just constructed. It will turn out that the variety \( X \) defined through the determinantal formula below

\[
X = \left\{ p \in \mathbb{P}^n \mid \text{rank} \begin{pmatrix} F_0(p) & \ldots & F_{n-1}(p) \\ G_0(p) & \ldots & G_{n-1}(p) \end{pmatrix} \leq 1 \right\} \quad (2.9)
\]
is a rational normal curve. To prove it, a couple of preliminary results is needed.

**Lemma 2.3.6.** For any pair \((\lambda, \mu) \in \mathbb{C}^2\) distinct from \((0, 0)\) the linear forms \(\{\lambda F_i + \mu G_i; i = 0, \ldots, n - 1\}\) are linearly independent.

**Proof.** Because the quadrics \(Q_1, \ldots, Q_{n-1}\) are linearly independent for any \(\alpha = (\alpha_1, \ldots, \alpha_{n-1})\) distinct from \((0, \ldots, 0)\) the quadric \(Q_\alpha = \sum_{i=1}^{n-1} \alpha_i Q_i\) cut out by

\[
\det \begin{pmatrix} F_0 & \sum_{i=1}^{n-1} \alpha_i F_i \\ G_0 & \sum_{i=1}^{n-1} \alpha_i G_i \end{pmatrix}
\]

is non-zero and still contains \(P\). If the linear forms \(\lambda F_i + \mu G_i, i = 0, \ldots, n - 1\) are linearly dependent, then there exists \((\alpha_0, \ldots, \alpha_n) \in \mathbb{C}^{n+1} \setminus \{0\}\) such that

\[
\alpha_0(\lambda F_0 + \mu G_0) = \sum_{i=1}^{n-1} \alpha_i (\lambda F_i + \mu G_i).
\]

Consequently the matrix \(Q_\alpha\) appearing in Equation (2.10) has rank one. Thus the quadric \(Q_\alpha\) has rank at most two. Since \(P\) is not contained in the union of two hyperplanes, the lemma follows.

**Lemma 2.3.7.** The restriction of any linear combination of the linear forms \(F_1, \ldots, F_{n-1}\) to \(\Lambda\) is non-zero. Consequently, it can be assumed that for every \(i = 1, \ldots, n - 1\), the linear form \(F_i\) satisfies

\[
p_1, \ldots, p_{i-1}, \hat{p}_i, p_{i+1}, \ldots, p_{n-1} \in \{F_i = 0\}.
\]

**Proof.** Since \(F_0(p_n) = 0\), \(G_0(p_n) \neq 0\) and the quadrics \(Q_i\) contain \(p_n\), the linear form \(F_i\) must vanish on \(p_n\) for every \(i \geq 1\). If some linear combination of \(F_1, \ldots, F_{n-1}\) vanishes on \(\Lambda\) then it would have to be a complex multiple of \(F_0\) because the span of \(\Lambda\) and \(p_n\) is the hyperplane cut out by \(F_0\). This contradicts the linear independence of \(F_0, \ldots, F_{n-1}\) established in the previous lemma and proves the first claim. The second claim follows immediately from plain linear algebra.
Castelnuovo Lemma follows from the following proposition.

**Proposition 2.3.8.** The variety $X$ is the unique rational normal curve through $p_1, \ldots, p_k$.

*Proof.* If, for $i = 0, \ldots, n-1$, $H_{n-i}(s : t)$ is the pencil of hyperplanes \( \{sF_i + tG_i = 0\} \) then $X$ can be described as below

$$X = \bigcup_{(s:t) \in \mathbb{P}^1} \left( \bigcap_{i=1}^n H_i(s : t) \right).$$

But this has exactly the same form as the presentation of a rational normal curve through Steiner’s construction, see Proposition 2.3.4. Consequently, $X$ is a rational normal curve.

By construction, when $l > n$, $Q_{\alpha}(p_l) = 0$ but $(F_0(p_l), G_0(p_l)) \neq 0$. Therefore

$$\det \begin{pmatrix} F_i(p_l) & \cdots & F_j(p_l) \\ G_i(p_l) & \cdots & G_j(p_l) \end{pmatrix} = 0$$

for every pair $i, j$ and every $l > n$. Thus $X$ contains the points $p_{n+1}, \ldots, p_k$.

The careful reader probably noticed that the inequality $k \geq 2n+3$ have not been used so far, only the weaker $k \geq 2n+1$ played a role. To prove that $p_1, \ldots, p_n$ belong to $X$ the stronger inequality enters the stage. Observe that the quadric $Q_{ij} = F_iG_j - F_jG_i$ contains the $k - 2 \geq 2n + 1$ points $P - \{p_i, p_j\}$. Remark 2.2.3 implies that these points impose at least $2n + 1$ conditions on the space quadrics. But, by hypothesis, the same holds true for $P$. Thus $Q_{ij}$ also contains $p_i$ and $p_j$. \qed

### 2.3.3 Normal forms for webs of maximal rank

For a quasi-smooth $k$-web $\mathcal{W} = [\omega_1, \ldots, \omega_k]$ on $(\mathbb{C}^n, 0)$ it is natural to consider $\ell(\mathcal{W})$ not just as an integer but as a germ of integer-valued function defined on $(\mathbb{C}^n, 0)$. The value at $x$ is given by the dimension of the span of $\{\omega_i(x) ; i \in \mathbb{N}\}$ in $\text{Sym}^2 \Omega^1_0(\mathbb{C}^n, 0)$.

A priori this function does not need to be continuous but just lower-semicontinuous. Nevertheless, as the reader can easily verify, when $\mathcal{W}$ is smooth and $F^{j-1}\mathcal{A}(\mathcal{W})/F^j\mathcal{A}(\mathcal{W})$ has maximal dimension
then $\ell^j(W)$ is constant. To ease further reference this fact is stated below as a lemma.

**Lemma 2.3.9.** Let $W$ be a smooth $k$-web. If

$$\dim \frac{F^{j-1}A(W)}{F^jA(W)} = \min(0, k - j(n - 1) - 1)$$

then the integer-valued function $\ell^j(W) : (\mathbb{C}^n, 0) \to \mathbb{N}$ is constant and equal to $j(n - 1) + 1$.

Combined with Castelnuovo Lemma, or rather with Proposition 2.3.1, the Lemma above yields the following normal forms for webs of maximal rank up to second order.

**Proposition 2.3.10.** Let $W = F_1 \boxtimes \cdots \boxtimes F_k$ be a germ of smooth $k$-web on $(\mathbb{C}^n, 0)$. Suppose that $n \geq 3$ and $k \geq 2n + 1$. If

$$\frac{F^1A(W)}{F^2A(W)} = k - 2n + 1$$

then there exist a coframe $\varpi = (\varpi_0, \ldots, \varpi_{n-1})$ on $(\mathbb{C}^n, 0)$ and $k$ germs of holomorphic functions $\theta_1, \ldots, \theta_k$ such that for every $i \in \mathbb{K}$

$$F_i = \left[ \sum_{q=0}^{n-1} (\theta_i)^q \varpi_q \right].$$

**Proof.** According to Lemma 2.3.9 the function $\ell^2(W)$ is constant and equal to $2(n - 1) + 1$. Proposition 2.3.1 implies the existence, for every $x \in (\mathbb{C}^n, 0)$, of a rational normal curve in $\mathbb{P}T_x^*(\mathbb{C}^n, 0)$ containing the conormals of the defining foliations of the web. Therefore it is possible to choose holomorphic 1-forms $\varpi_0, \ldots, \varpi_{n-1} \in \Omega^1(\mathbb{C}^n, 0)$ such that at every $x \in (\mathbb{C}^n, 0)$ the rational normal curve given by Proposition 2.3.1 is parameterized by

$$t \mapsto \sum_{q=0}^{n-1} t^q \varpi_q(x).$$

This parametrization can be chosen in such a way that none of the foliations $F_i$ have conormal corresponding to $t = \infty$. Thus, for every $i \in \mathbb{K}$, the foliation $F_i$ will be induced by $\sum_{q=0}^{n-1} (\theta_i)^q \varpi_q$ for a suitable germ of holomorphic function $\theta_i$. □
2.3.4 A generalization of Castelnuovo Lemma

Proposition 2.3.10 can be seen as the starting of the proof of the algebraization of webs of maximal rank to be presented in Chapter 5. As it has been made clear above, Proposition 2.3.10 is an easy consequence of Castelnuovo Lemma. Loosely phrased Castelnuovo Lemma says that if sufficiently many points in general position impose the minimal number of conditions on the space of quadrics then they must lie on particularly simple curves: the rational normal curves. A testimony of the simplicity of rational normal curves is the following proposition.

Proposition 2.3.11. If $C$ is a non-degenerate irreducible projective curve in $\mathbb{P}^n$ then $\deg C \geq n$. Moreover, if the equality holds then $C$ is a rational normal curve.

Proof. If $C$ is non-degenerate then there exists $n$ points in $C$ that are in general position, otherwise $C$ would be contained in a hyperplane. Intersecting $C$ with the hyperplane $H$ determined by $n$ of these points shows that the degree of $C$ is at least $n$.

To prove the second part let $p_1, \ldots, p_{n-1}$ be $n-1$ general points of $C$ and let $\Sigma$ be the $\mathbb{P}^{n-2}$ determined by them. By hypothesis each generic hyperplane containing $\Sigma$ intersects $C$ in exactly one point away from $\Sigma$. Therefore there is an injective map from the set of hyperplanes containing $\Sigma$, nothing else than a $\mathbb{P}^1$, to $C$. Thus $C$ is rational. Therefore $C$ is parametrized by $n+1$ homogenous binary forms of degree equal to $\deg C = n$. Since $C$ is non-degenerated these $n+1$ binary forms must generate the space of degree $n$ binary forms. In other words, $C$ is a rational normal curve. □

It is natural to enquire what can be said about sufficiently many points imposing a number of conditions on the space of quadrics close to minimal. For instance one can ask if they lie on simple varieties.

Of course to be more precise the meaning of simple varieties must be spelled out. One possibility is to look for non-degenerate irreducible varieties of minimal degree. For that sake it is important to generalize Proposition 2.3.11 for irreducible non-degenerated varieties of $\mathbb{P}^n$ of arbitrary dimension. The first part of the statement generalizes promptly as shown below.
Proposition 2.3.12. If $X$ is a non-degenerate irreducible subvariety of $\mathbb{P}^n$ then $\deg X \geq \text{codim } X + 1$.

Proof. Take $m + 1 = \text{codim}(X) + 1$ generic points on $X$. Because $X$ is non-degenerate they span a $\mathbb{P}^m$ intersecting $X$ in at least $m + 1$ points. To conclude, it remains to verify that for a generic choice of $m + 1$ points there are no positive dimensional component in the corresponding intersection $\mathbb{P}^m \cap X$.

For that sake let $k$ be the dimension of the intersection of $X$ with a generic $\mathbb{P}^m$. If $p_1, \ldots, p_{m-k+1} \in X$ are $m-k+1$ generic points then their linear span $\Sigma \simeq \mathbb{P}^{m-k}$ intersects $X$ in a finite number of points. If $\Lambda$ is the set of all the projective spaces $\mathbb{P}^{m-k+1}$ contained in $\mathbb{P}^n$ and containing $\Sigma$ then $\Lambda \simeq \mathbb{P}^{n-m+k}$.

On the one hand $\dim X = n - m$, while on the other hand

$$X - \Sigma = \bigcup_{\mathbb{P}^{m-k+1} \in \Lambda} (\mathbb{P}^{m-k+1} - \Sigma) \cap X,$$

implies that $\dim X = n - m + k$. Thus $k = 0$, that is, $X$ intersects a generic $\mathbb{P}^m$ in a finite number of points. \hfill $\square$

The second part also does generalize but the generalization, which can be traced back at least to Bertini, is by no means evident.

Theorem 2.3.13. If $V$ is an irreducible non-degenerated projective subvariety of $\mathbb{P}^n$ with $\deg V = \text{codim } V + 1$ then

1. $V$ is $\mathbb{P}^n$, or;
2. $V$ is a rational normal scroll, or;
3. $V$ is a cone over the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$, or ;
4. $V$ is a hyperquadric.

The proof of this theorem would take the exposition to far afield, and therefore will not be presented here. For a modern exposition see for instance [48].

A rational normal scroll of dimension $m$ in $\mathbb{P}^n$ is characterized, up to an automorphism of $\mathbb{P}^n$, by $m$ positive integers $a_1, \ldots, a_m$ summing up to $n - m + 1$ and can be described as follows. Decompose...
$\mathbb{C}^{n+1}$ as $\oplus \mathbb{C}^{a_i+1}$ and consider parametrizations $\varphi_i : \mathbb{P}^1 \to \mathbb{P}^{a_i} \subset \mathbb{P}^n$ of rational normal curves on the corresponding projective subspaces*. If $\Sigma(p)$ is the $\mathbb{P}^{m-1}$ spanned by $\varphi_1(p), \ldots, \varphi_m(p)$ then the associated rational normal scroll is

$$S_{a_1, \ldots, a_m} = \bigcup_{p \in \mathbb{P}^1} \Sigma(p).$$

Notice that the rational normal curves are rational normal scrolls, with $m = 1$ according to the definition above.

It is natural to consider the rational normal scroll as higher-dimensional analogues of rational normal curves. The analogies between rational normal curves and scrolls do not reduce to similar definitions, and to both being varieties of minimal degree. They encompass many other aspects. For instance, the rational normal scrolls of dimension $m$ in $\mathbb{P}^n$ admit determinantal presentations, similar to (2.9) used for rational normal curves. More precisely, if $F_0, \ldots, F_{n-m}, G_0, \ldots, G_{n-m}$ are linear forms such that for any $(\lambda, \mu) \neq (0, 0)$, the linear forms $\{\lambda F_i + \mu G_i\}_{i=0, \ldots, n-m}$ are linearly independent (compare with Lemma 2.3.6) then

$$X = \left\{ \text{rank} \begin{pmatrix} F_0 & \cdots & F_{n-m} \\ G_0 & \cdots & G_{n-m} \end{pmatrix} \leq 1 \right\}$$

is a rational normal scroll of dimension $m$. Moreover, any rational normal scroll can be presented in this way.

Another testimony of the similarity between rational normal curves and scrolls, is the following generalization of Castelnuovo Lemma.

**Proposition 2.3.14 (Generalized Castelnuovo Lemma).** Let $\mathcal{P} \subset \mathbb{P}^n$ be a set of $k$ points in general position. Suppose that $n \geq 2$ and $k \geq 2n + 1 + 2m$. If $\mathcal{P}$ imposes $2n + m$ conditions on the linear system of quadrics then $\mathcal{P}$ is contained in a rational normal scroll of dimension $m$.

*When $a_i = 0$, the linear subspace $\mathbb{P}^{a_i}$ is nothing but a point $p_i \in \mathbb{P}^n$. In this case, the following convention is adopted: the rational normal curve in $\mathbb{P}^{a_i}$ is not a curve, but the point $p_i$. 
The proof of Castelnuovo Lemma presented in Section 2.3.2 is the specialization to $m = 1$ of the Eisenbud-Harris proof of the generalized Castelnuovo Lemma. Having at hand the determinantal presentation of a rational normal scroll given above the reader should not have difficulties to recover the original proof as found in [60, pages 103-106].

While Castelnuovo Lemma is essential in the proof of Trépreau’s algebraization theorem, to be carry out in Chapter 5, the implications of the generalized Castelnuovo Lemma to web geometry, if any, remain to be unfolded.
Chapter 3

Abel’s addition Theorem

So far, not many examples of abelian relations for webs appeared in this text. Besides the abelian relations for hexagonal 3-webs, the polynomial abelian relations for parallel webs (see Example 2.1.1), and the abelian relations for the planar quasi-parallel webs discussed in 2.1.4, which are by the way also polynomial, no other example was studied.

The main result of this chapter, Abel’s addition Theorem, repairs this unpleasant state of affairs. It implies, and is essentially equivalent to, an injection of the space of global abelian differentials – also known as Rosenlicht’s or regular differentials – of a reduced projective curve $C$ into the space of abelian relations of the dual web $W_C$.

The exposition that follows renounces conciseness in favor of clarity. First the result is proved for smooth projective curves avoiding the technical difficulties inherent to the singular case. Only then the case of an arbitrary reduced projective curve is dealt with.

The readers familiar with Castelnuovo’s bound for the arithmetical genus of irreducible non-degenerate projective curves will promptly realize that Castelnuovo curves (briefly described in Section 3.3) give rise to webs of maximal rank. Those that are not, will get acquainted with Castelnuovo’s bound since it can be seen as a joint
corollary of the bound for the rank proved in Chapter 2, and Abel’s addition Theorem. While most of the arguments laid down in Chapter 2 to bound \( A(W) \) can be found in the modern textbooks dealing with Castelnuovo Theory, the same cannot be said about the use of Abel’s addition Theorem. These days, the textbook proof of Castelnuovo’s bound makes use of some basic results about the cohomology of projective varieties besides Castelnuovo Lemma.

Not deprived of weaknesses when compared to the modern approach, the path to Castelnuovo’s bound through Abel’s addition Theorem has its own strong points. For instance, it is more or less simple to obtain bounds for reduced curves on other projective varieties as explained in Section 3.4.

3.1 Abel’s Theorem I: smooth curves

3.1.1 Trace under ramified coverings

Let \( X \) and \( Y \) be two smooth connected complex curves. A holomorphic map \( f : X \to Y \) is a finite ramified covering if it is surjective and proper. The degree of a finite ramified covering is defined as the cardinality of the pre-image of any of its regular values.

For \( X \) and \( Y \) as above, let \( f : X \to Y \) be a finite ramified covering of degree \( k \). For a regular value \( q \in Y \) of \( f \) and a meromorphic 1-form \( \omega \) defined at a neighborhood of \( f^{-1}(y) \), define the trace of \( \omega \) at \( q \) as the germ of meromorphic 1-form

\[
\text{tr}_{f,q} \omega = \sum_{i=1}^{k} g_{*}^{i} \omega \in \Omega^{1}(Y,q),
\]

where \( g_1, \ldots, g_k : (Y,q) \to X \) are the local inverses of \( f \) at \( q \).

**Proposition 3.1.1.** Let \( X \) and \( Y \) be two smooth, compact and connected complex curves. If \( f : X \to Y \) is a finite ramified covering, \( \omega \) is a meromorphic 1-form globally defined on \( X \), and \( q \) is a regular value of \( f \) then \( \text{tr}_{f,q}(\omega) \) extends to a unique meromorphic 1-form \( \text{tr}_{f}(\omega) \), which does not depend on \( q \), and is globally defined on \( Y \). Moreover, if \( \omega \) is holomorphic then \( \text{tr}_{f}(\omega) \) is also holomorphic.
SECTION 3.1: ABEL’S THEOREM I: SMOOTH CURVES

The meromorphic 1-form \( \text{tr}_f(\omega) \) globally defined on \( Y \) is, by definition, the trace of \( \omega \) relative to \( f \).

**Proof of Proposition 3.1.1.** For \( q \) varying among the regular values of \( f \), the meromorphic 1-forms \( \text{tr}_{f,q} \) patch together to a meromorphic 1-form \( \eta \) defined on the whole complement of the critical values of \( f \). Furthermore, if \( \omega \) is holomorphic then the same will be true for \( \eta \).

Now, if \( q \in Y \) is a critical value of \( f \) then some point \( p \) in the fiber \( f^{-1}(q) \) is a critical point. Although it is not possible to consider a local inverse \( g : (Y,q) \to (X,p) \), the map \( f : (X,p) \to (Y,q) \) is, in suitable coordinates, the monomial map \( f(x) = x^n = y \) for some positive integer \( n \). Because \( X \) is compact, the set of critical values of \( f \) is discrete. Therefore it suffices to consider the trace of the monomial maps from the disc \( D \) to itself to prove that \( \eta \) extends through the critical set of \( f \).

For a point distinct from the origin, there are exactly \( n \) local inverses for \( f(x) = x^n \): the distinct branches of \( n\sqrt[n]{x} \). One passes from one to another, via multiplication by powers of \( \xi_n \), a primitive \( n \)-th root of the unity. Hence

\[
\text{tr}_f(x^m dx) = \sum \xi_n^{m+1}x^m dx.
\]

Consequently,

\[
\text{tr}_f(x^m dx) = \begin{cases} 
    y^{m+1} - 1 dy & \text{if } m + 1 = 0 \mod n, \\
    0 & \text{otherwise}.
\end{cases} \tag{3.1}
\]

Therefore the trace of \( x^m dx \) is meromorphic at the origin. It follows that \( \eta \) extends to the whole \( Y \). Moreover, when the 1-form \( \omega \) is holomorphic, \( m \geq 0 \) and, according to Equation (3.1), the trace of \( x^m dx \) is also holomorphic.

 Beware that there are meromorphic 1-forms with holomorphic trace as one can promptly infer from (3.1).

**Remark 3.1.2.** The algebraically inclined reader familiar with Kähler differentials and field extensions, might prefer to define the trace as follows. If \( X \) and \( Y \) are algebraic curves and \( f : X \to Y \) is a finite ramified covering, then there is an induced finite field extension
$f^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ of the corresponding function fields. In this case, a rational function $\phi \in \mathbb{C}(X)$ has trace $\text{tr}_f(\phi) \in \mathbb{C}(Y)$ equal to the trace of the endomorphism $\psi \mapsto \phi \psi$ of the finite dimensional $\mathbb{C}(Y)$-vector space $\mathbb{C}(X)$. Let $t \in \mathbb{C}(Y)$ be such that $dt$ generates $\Omega_{\mathbb{C}(Y)}$ as a $\mathbb{C}(Y)$-module. Therefore $f^*(dt) = df^*(t)$ generates the $\mathbb{C}(X)$-module of Kähler differentials on $X$. Hence, $\omega = \varphi df^*(t)$ with $\varphi$ meromorphic on $X$. The trace of $\omega$ relative to $f$ is algebraically defined as $\text{tr}_f(\varphi)dt$.

### 3.1.2 Trace relative to the family of hyperplanes

Let now $C \subset \mathbb{P}^n$ be a smooth and irreducible projective curve of degree $k$, $H_0$ a hyperplane intersecting $C$ transversely, and $\omega$ be a meromorphic 1-form defined on a neighborhood $V \subset C$ of $H_0 \cap C$. Consider the germs of holomorphic maps $p_1, \ldots, p_k : (\mathbb{P}^n, H_0) \to V \subset C$ verifying

$$H \cdot C = p_1(H) + \cdots + p_k(H)$$

for every $H \in (\mathbb{P}^n, H_0)$.

The trace of $\omega$ at $H_0$ relative to the family of hyperplanes, denoted by $\text{Tr}_{H_0}(\omega)$, is defined through the formula

$$\text{Tr}_{H_0}(\omega) = \sum_{i=1}^{k} p_i^* (\omega).$$

It is clearly a germ of meromorphic differential 1-form. As the trace relative to a ramified covering, it extends meromorphically to the whole projective space $\mathbb{P}^n$ as proved in the next section. This is essentially the content of Abel’s addition Theorem.

### 3.1.3 Abel’s Theorem for smooth curves

The next result is the version for smooth curves of what is called by web geometers Abel’s addition Theorem, or just Abel’s Theorem. The readers are warned that authors with other backgrounds might call a different, but essentially equivalent, statement by the same name. For a thorough discussion about the original version(s) of Abel’s theorem see [71].
Theorem 3.1.3 (Abel’s addition Theorem for smooth curves). If ω is a meromorphic 1-form on a smooth projective curve $C \subset \mathbb{P}^n$, then the germ $\text{Tr}_{H_0}(\omega)$ extends to a unique meromorphic 1-form $\text{Tr}(\omega)$ globally defined on $\mathbb{P}^n$ which does not depend on $H_0$. Moreover, $\omega$ is a holomorphic 1-form if and only if $\text{Tr}(\omega) = 0$.

To prove Theorem 3.1.3, let $\mathring{U}_C \subset \mathring{\mathbb{P}}^n$ be the Zariski open subset formed by the hyperplanes $H \subset \mathbb{P}^n$ which intersect $C$ at $k = \deg C$ distinct points. In other words, $\mathring{U}_C$ is the complement in $\mathring{\mathbb{P}}^n$ of the discriminant of the dual web $W_C$. The construction of $\text{Tr}_{H_0}(\omega)$ made above can be done for any hyperplane $H \in \mathring{U}_C$. The results patch together to define a meromorphic 1-form $\text{Tr}(\omega)$ on $\mathring{U}_C$.

To extend $\text{Tr}(\omega)$ through the discriminant of $W_C$, it will be used a relation between the trace under a ramified covering and the trace relative to the hyperplanes. To draw this relation, let $\ell$ be a line on $\mathring{\mathbb{P}}^n$. It corresponds to a pencil of hyperplanes in $\mathbb{P}^n$ which has base locus equal to $\Pi = \mathring{\ell} \subset \mathbb{P}^n$, the $\mathbb{P}^{n-2}$ dual to $\ell$. It will be convenient, although not strictly necessary, to assume that $\Pi$ does not intersect $C$. Define

$$\pi_{\ell} : C \to \ell \simeq \mathbb{P}^1$$

as the morphism that associates to a point $x \in C$ the hyperplane in $\ell$ containing it. Clearly it is a ramified covering, thus the trace $\text{tr}_{\pi_{\ell}}(\omega)$ makes sense for any meromorphic 1-form on $C$.

Lemma 3.1.4. For every meromorphic 1-form $\omega$ on $C$, the trace of $\omega$ under $\pi_{\ell}$ coincides with the pull-back to $\ell$ of the trace of $\omega$ relative to the family of hyperplanes, that is

$$\text{tr}_{\pi_{\ell}}(\omega) = i^* \text{Tr}(\omega),$$

where $i : \ell \to \mathring{\mathbb{P}}^n$ is the natural inclusion.

Proof. Consider the composition $\varphi = i \circ \pi_{\ell} : C \to \mathbb{P}^n$. The image of a point $q \in C$ is the hyperplane $H$ containing both the point $q$ and the linear space $\Pi = \mathring{\ell}$.

The hyperplane $H$ intersects $C$ at $q$, and at other $k - 1$ points of $C$ with multiplicities taken into account. All these other points are also mapped to $H$ by $\varphi$. Thus, the functions $p_i \circ i : \ell \cap \mathring{U} \to C$ are
local inverses of $\varphi$ at any $H \in \hat{U}_C$. Hence

$$\text{tr}_{\pi}(\omega) = \text{tr}_{10\pi}(\omega) = \sum_{i=1}^{k} (p_i \circ i)^* \omega = i^* \text{Tr}(\omega)$$

at the generic point of $\ell$. The lemma follows.

Back to the proof of Abel’s Theorem, recall that $\text{Tr}(\omega)$ is defined all over the Zariski open set $\hat{U}_C$. If it does not extend meromorphically to the whole $\mathbb{P}^n$, then its pull-back to a generic line $\ell \subset \mathbb{P}^n$ has an essential singularity at one of the points of $\ell \cap \Delta(W_C)$. Lemma 3.1.4 implies the existence of an essential singularity for $\text{tr}_{\pi}(\omega)$. But this cannot be the case according to Proposition 3.1.1.

To prove that $\omega$ holomorphic implies $\text{Tr}(\omega) = 0$, start by noticing that there are no non-zero holomorphic differential forms on $\mathbb{P}^n$. If $\omega$ is holomorphic and $\text{Tr}(\omega)$ is non-zero then $\text{Tr}(\omega)$ has non-empty polar set. Therefore $\text{Tr}(\omega)$ pulls-back to a generic line $\ell \subset \mathbb{P}^n$ as a meromorphic, but not holomorphic, differential. As above, Lemma 3.1.4 and Proposition 3.1.1 lead to a contradiction.

It remains to establish the converse implication. To prove the contrapositive, suppose $\omega$ is not holomorphic. If $x \in C$ is a pole of $\omega$ then the generic hyperplane $H \subset \mathbb{P}^n$ through $x$ intersects $C$ transversely and avoids all the other poles of $\omega$. Thus, in a neighborhood of $H$ in $\mathbb{P}^n$, the trace of $\omega$ is the sum of the pull-back by a holomorphic map of a meromorphic, but not holomorphic, 1-form with other $\deg(C) - 1$ holomorphic 1-forms. Hence $\text{Tr}(\omega)$ has non-empty polar set and, in particular, is not zero.

3.1.4 Abelian relations for algebraic webs

Theorem 3.1.3 can be interpreted in terms of webs/abelian relations instead of projective curves/holomorphic 1-forms. More precisely,

**Theorem 3.1.5.** If $C$ is a smooth projective curve of degree $k$ and $H_0$ is a hyperplane intersecting it transversely, then the space of holomorphic 1-forms on $C$ injects into the space of abelian relations of the dual web $W_C(H_0)$. 
Proof. Let $H_0$ be a hyperplane intersecting $C$ transversely in $k$ points, and $p_i : (\mathbb{P}^n, H_0) \to C$ be germs of holomorphic functions such that $H \cdot C = \sum_i p_i(H)$ for all $H \in (\mathbb{P}^n, H_0)$. Recall from Chapter 1 that the $k$-web $\mathcal{W}_C(H_0)$ is defined by the submersions $p_1, \ldots, p_k$. That is, $\mathcal{W}_C(H_0) = \mathcal{W}(p_1, \ldots, p_k)$. 

If $\omega$ is a holomorphic 1-form on $C$ then it is automatically closed, for dimensional reasons. Since the exterior differential commutes with pull-backs, the 1-forms $p_i^* \omega$ are also closed. Moreover, the 1-form $p_i^* \omega$ defines the very same foliation as the submersion $p_i$. Abel’s addition theorem, in its turn, implies that 

$$\text{Tr} (\omega) = p_1^*(\omega) + \cdots + p_k^*(\omega) = 0$$

holds identically on $(\mathbb{P}^n, H_0)$. Therefore $(p_1^* \omega, \ldots, p_k^* \omega)$ is an abelian relation of $\mathcal{W}_C(H_0)$. It follows that the injective linear map

$$H^0(C, \Omega^1_C) \longrightarrow (\Omega^1(\mathbb{P}^n, H_0))^k$$

$$\omega \longmapsto (p_1^* \omega, \ldots, p_k^* \omega)$$

factors through $\mathcal{A}(\mathcal{W}_C) \subset \Omega^1((\mathbb{P}^n, H_0))^k$. \qed

Recall that $g(C)$ – the genus of $C$ – coincides with $h^0(C, \Omega^1_C)$, the dimension of the vector space of holomorphic 1-forms on $C$.

**Corollary 3.1.6.** If $C$ is a smooth projective curve then, for any hyperplane $H_0$ intersecting it transversely,

$$\text{rank}(\mathcal{W}_C(H_0)) \geq g(C).$$

In Chapter 4 it will be seen that this lower bound is in fact an equality.

### 3.1.5 Castelnuovo’s bound

Corollary 3.1.6 read backwards yields the celebrated Castelnuovo’s bound for the genus of projective curves. More precisely,

**Theorem 3.1.7** (Castelnuovo’s bound). If $C$ is a smooth connected non-degenerate projective curve on $\mathbb{P}^n$ of degree $k$ then

$$g(C) \leq \pi(n, k).$$
Proof. For a generic $H_0$, the web $W_C(H_0)$ is smooth according to Proposition 1.4.5. Chern’s bound on the rank of smooth webs, see Theorem 2.2.8, combined with Corollary 3.1.6 implies the result.

It is instructive to compare this proof of Castelnuovo’s bound, with the usual textbook proof. The first step of both proofs, relies on the bounds for the number of conditions imposed by points on the complete linear systems of hypersurfaces on the relevant projective space. While the former proof uses Abel’s addition theorem to conclude, the latter instead appeals to Riemann-Roch Theorem. For thorough discussion on this matter see [30].

3.2 Abel’s Theorem II: arbitrary curves

When studying germs of smooth algebraic webs $W_C(H_0)$, it is hard to tell whether the curve $C$ is smooth or not. At first sight the web only exhibits properties of $C$ valid at a neighborhood of the transversal hyperplane $H_0$. For smooth curves, it has just been explained how the holomorphic differentials give rise to abelian relations for the dual web. It is then natural to enquire:

(a) are there another abelian relations for algebraic webs?

(b) what kind of differentials on a singular curve give rise to abelian relations for the dual web?

Question (a) will be treated in Chapter 4, while question (b) will be the subject of the present section. Before dwelling with it, some conventions about singular curves are settled below.

Conventions

Given a curve $X$, the desingularization of $X$ will denoted by $\nu = \nu_X : \overline{X} \to X$.

A meromorphic 1-form on $X$ is nothing more than a meromorphic 1-form $\omega$ on the smooth part of $X$ such that $\nu^*\omega$, its pull-back to $\overline{X}$, extends to the whole $\overline{X}$ as a meromorphic 1-form. The sheaf of meromorphic differentials on a curve $X$, singular or not, will be denoted by $\mathcal{M}_X$. 
For $X$ an arbitrary curve and $Y$ a smooth irreducible curve, a morphism $f : X \to Y$ will be called a **finite ramified covering**, if the restriction of $\tilde{f} = f \circ \nu_X$ to each of the irreducible components of $X$ is a finite ramified covering as defined in Section 3.1.1.

### 3.2.1 Residues and traces

Assume that $p$ is a smooth point of a curve $X$, and $x$ is a local holomorphic coordinate on $X$ centered at it. If $\omega$ is a germ of meromorphic differential at $p$ then

$$\omega = \sum_{i=i_0}^{\infty} a_i x^i dx$$

for some $i_0 \in \mathbb{Z}$ and suitable complex numbers $a_i$. The **residue of $\omega$ at $p$** is the complex number

$$\text{Res}_p(\omega) = \begin{cases} a_{i_0} & \text{if } i_0 \leq -1; \\ 0 & \text{otherwise.} \end{cases}$$

It is a simple matter to verify that this definition does not depend on the local coordinate $x$. One possibility is to notice that the residue can be determined through the integral formula

$$\text{Res}_p(\omega) = \frac{1}{2i\pi} \int_{\gamma} \omega,$$

for any sufficiently small (positively oriented) loop $\gamma$ around $p$.

The following properties can be easily verified:

1. $\text{Res}_p : \mathcal{M}_{X,p} \to \mathbb{C}$ is $\mathbb{C}$-linear;
2. $\text{Res}_p(\omega) = 0$ when $\omega$ is holomorphic at $p$;
3. $\text{Res}_p(f^n df) = 0$ for all $f \in \mathcal{O}_{X,p}$ when $n \neq -1$;
4. $\text{Res}_p(f^{-1} df) = \nu_p(f)$ for all meromorphic germs $f$ at $p$ (where $\nu_p$ is the valuation associated to $p$).
CHAPTER 3: ABEL’S ADDITION THEOREM

Residues at singular points

Assume now that $X$ is singular and let $\omega$ be a meromorphic differential 1-form on it. Let $\nu : X \rightarrow X$ be the desingularization of $X$. The residue of $\omega$ at a singular point $p \in X_{\text{sing}}$ is defined as

$$\text{Res}_p(\omega) = \sum_{q \in \nu^{-1}(p)} \text{Res}_q(\nu^*(\omega)).$$

(3.2)

It is completely determined by the germ of $\omega$ at $p$.

Given a ramified covering $f : X \rightarrow Y$ between an eventually singular curve $X$ and a smooth and irreducible curve $Y$ then the trace under $f$ of any meromorphic 1-form $\omega$ on $X$ is defined by the relation

$$\text{tr}_f(\omega) = \text{tr}_f(\nu_X^*(\omega)).$$

Proposition 3.2.1. Let $X$ and $Y$ be curves with $Y$ smooth and irreducible. If $f : X \rightarrow Y$ is a ramified covering then for every $\omega \in H^0(X, \mathcal{M}_X)$ and every $p \in Y$,

$$\text{Res}_p(\text{tr}_f(\omega)) = \sum_{q \in f^{-1}(p)} \text{Res}_q(\omega).$$

Proof. Let $\nu_X : X \rightarrow X$ be the normalizations of $X$. Let also $f : X \rightarrow Y$ be the natural lifting of $f$, that is $f = f \circ \nu_X$.

Since $f^{-1}(p) = (f \circ \nu_X)^{-1}(p)$ and because of definition (3.2), one verifies that the proposition holds for $f : X \rightarrow Y$ if it holds for $f : X \rightarrow Y$. It is therefore harmless to assume smoothness for both $X$ and $Y$.

Let $q_1, \ldots, q_m$ be the pre-images of $p$ under $f$. For $i \in m$, let $f_i : (X, q_i) \rightarrow (Y, p)$ be the germ of analytic morphism induced by $f$, and $\omega_i \in \mathcal{M}_{X,x_i}$ be the germ of $\omega$ at $p_i$. Clearly, $\text{tr}_f(\omega) = \sum_{i=1}^m \text{tr}_{f_i}(\omega_i)$ as germs at $p$. Thus

$$\text{Res}_p(\text{tr}_f(\omega)) = \sum_{i=1}^m \text{Res}_p(\text{tr}_{f_i}(\omega_i))$$

by the additivity of the residue. In suitable coordinates, each of the functions $f_i$ can be written as $f_i(z_i) = z_i^{n_i}$, for suitable $n_i \in \mathbb{Z}$. A quick inspection of (3.1) leads to the identity
3.2.2 Abelian differentials

Let $X$ be a curve and $\nu : \overline{X} \to X$ be its desingularization. An abelian differential $\omega$ on $X$ is a meromorphic 1-form on $X$ which satisfies

$$\text{Res}_p (f \omega) = \sum_{q \in \nu^{-1}(p)} \text{Res}_q [\nu^*(f \omega)] = 0$$

for every $p \in X$ and every $f \in \mathcal{O}_{X,p}$.

For any open subset $U \subset X$, let $\omega_X(U)$ be the set of abelian differentials on $U$. Of course $\omega_X(U)$ inherits from $\mathcal{M}_X(U)$ a structure of $\mathcal{O}_X(U)$-module. Indeed even more is true, the subsheaf $\omega_X$ of $\mathcal{M}_X$ is coherent. Later, a proof that $\omega_X$ is coherent will be presented under the assumption that $X$ is contained in a smooth surface. The general case will not be treated in this text. In the algebraic category the result goes back to Rosenlicht. For a treatment of the analytic case see $[9]$.

Remark 3.2.2. It is possible to characterize abelian differentials in terms of currents. Indeed, a meromorphic 1-form $\omega$ on a curve $X$ is abelian, if and only if the current $[\omega]$ defined by it is $\partial \overline{\partial}$-closed. That is, if

$$\langle \partial \overline{\partial} [\omega], \theta \rangle = \langle [\omega], \partial \overline{\partial} \theta \rangle = \int_X \omega \wedge \partial \overline{\partial} \theta = 0,$$

for every smooth complex-valued function $\theta$ with compact support on $X$. The characterization of abelian differentials in terms of currents generalizes promptly, and allows to define a notion of abelian differential $k$-forms in arbitrary dimension. The interested reader can consult the references $[9, 70]$.

For an arbitrary projective curve $X$, the coherence of $\omega_X$ implies that $H^0(X, \omega_X)$ is a finite dimensional vector space. Its dimension is, by definition, the arithmetic genus $g_a(X)$ of $X$. The geometric genus $g(X)$ of $X$, in its turn, is defined as the dimension of $H^0(\overline{X}, \Omega^{1}_{\overline{X}})$, where $\overline{X}$ is the desingularization of $X$. 

$$\text{Res}_p [\text{tr}_f (\omega_i)] = \text{Res}_{q_i} [\omega_i] \quad \text{for every } i \in m.$$
When $X$ is smooth, the sheaf $\omega_X$ is nothing more than $\Omega^1_X$, hence $g(X)$ coincides with $g_a(X)$. For singular curves, the equality between $g(X)$ and $g_a(X)$ is the exception rather than the rule.

The sheaf $\omega_X$ is also called the dualizing sheaf of $X$. The terminology stems from Serre’s duality for projective curves: for any coherent sheaf $\mathcal{F}$ on a projective curve $X$, there are natural isomorphisms between $H^i(X, \mathcal{F})$ and $H^{1-i}(X, \mathcal{F}^* \otimes \omega_X)$ for $i = 0, 1$.

When $X$ is not just projective but also connected, Serre’s duality for projective curves is essentially equivalent to Riemann-Roch Theorem: for any line bundle $L$ on $X$, the identity $\chi(X, L) = \deg(L) - g_a(X) + 1$ holds true.

Applying Riemann-Roch Theorem to the dualizing sheaf itself, one obtains the genus formula for irreducible projective curves
\[ \deg(\omega_X) = 2g_a(X) - 2. \] (3.3)

Before proceeding toward the proof of Abel’s Theorem for arbitrary curves, a couple of examples will be considered in order to clarify the concept of abelian differential.

Example 3.2.3. Let $X = \{(x, y) \in \mathbb{C}^2, y^2 - x^3 = 0\}$. Clearly, it is a curve with the origin of $\mathbb{C}^2$ as its unique singular point. The stalk $\omega_{X_0}$ is a free $\mathcal{O}_{X,0}$-module generated by $y^{-1}dx$. Indeed, the normalization of $X$ is given by
\[
\nu : \mathbb{C} \longrightarrow X
\]
\[ t \longmapsto (t^2, t^3). \]

Therefore $\nu^*\mathcal{O}_{X,0} = \nu^*\mathcal{O}_{\mathbb{C}^2,0} = \nu^*\mathbb{C}(x, y) = \mathbb{C}(t^2, t^3)$. If $\omega$ is a meromorphic differential on $X$ then $\nu^*\omega = \sum_{i=-k}^{\infty} a_i t^i dt$. Moreover, if $\omega$ is abelian then not only $a_{-1} = \text{Res}_0(\nu^*\omega)$ must be zero but also $a_{2n+3m-1} = \text{Res}_0(\nu^*(x^ny^m\omega))$ for any pair of positive integers $(n, m)$. Hence every germ at 0 of abelian differential on $X$ can be written as
\[
\frac{dt}{t^2} \left( a_{-2} + a_0 t^2 + a_1 t^3 + \cdots \right) \in \frac{dt}{t^2} \cdot \mathbb{C}(t^2, t^3). 
\]

\[ ^* \text{In the formula, } \chi(X, \mathcal{L}) \text{ stands for the Euler-characteristic of the line-bundle } \mathcal{L} \text{ which, by definition, is } h^0(X, \mathcal{L}) - h^1(X, \mathcal{L}). \]
In a similar vein, a family of rational projective curves contained in projective spaces of dimension \( n \geq 2 \) is considered below.

**Example 3.2.4.** Fix \( n \geq 2 \) and let \( C \) be the rational curve of degree \( 2n \) in \( \mathbb{P}^n \) parametrized by
\[
\nu : \mathbb{P}^1 \rightarrow \mathbb{P}^n \\
(s : t) \mapsto [s^{2n} : s^n t^n : s^{n-1} t^{n+1} : \cdots : st^{2n-1} : t^{2n}].
\]
The curve \( C \) is singular, \( p = [1 : 0 : \cdots : 0] \) is its unique singular point, and the parametrization \( \nu \) is its desingularization.

Any rational 1-form on \( C \) writes \( \omega = f(t)dt \) in the coordinate \((1 : t)\), for a certain \( f(t) \in \mathbb{C}(t) \). Assume that \( \omega \) is abelian. Since \( \omega_C \) coincides with the sheaf of holomorphic differentials on \( C_{sm} = C \setminus \{p\} \), the differential \( \omega = f(t)dt \) must be holomorphic on \( \mathbb{C} \setminus \{0\} \) as well as at infinity. Therefore \( \omega = t^{-a}dt \) for a certain integer \( a > 1 \). It is an instructive exercise to show that
\[
H^0(C, \omega_C) \simeq \left\langle \frac{dt}{t^2}, \frac{dt}{t^3}, \ldots, \frac{dt}{t^n}, \frac{dt}{t^{n+1}}, \frac{dt}{t^{2n+2}} \right\rangle.
\]

### Abelian differentials and traces

The following proposition can be seen as a first evidence that the concept of abelian differential is the appropriate one to extend Abel’s addition Theorem to singular projective curves. Note that, as for Proposition 3.1.1, its converse does not hold true.

**Proposition 3.2.5.** Let \( \omega \) be an abelian differential on a curve \( X \). If \( f : X \rightarrow Y \) is a ramified covering onto a smooth curve \( Y \) then \( \text{tr}_f(\omega) \) is an abelian, that is a holomorphic, differential on \( Y \).

**Proof.** The proof follows the same lines of Proposition 3.1.1’s proof with some extra ingredients borrowed from the proof of Proposition 3.2.1. The reader is invited to fill in the details.

Besides the concept of abelian differential, the main extra ingredient to generalize Abel’s addition Theorem from smooth to arbitrary curves, is the following characterization of abelian differentials in terms of their traces under linear projections.
CHAPTER 3: ABEL’S ADDITION THEOREM

Proposition 3.2.6. Let \( X \subset (\mathbb{C}^n, 0) \) be a germ of curve and \( \omega \) be a meromorphic \( 1 \)-form on \( X \). The following assertions are equivalent

(a) \( \omega \) is abelian;

(b) the trace of \( \omega \) at \( p \), \( \text{tr}_p(\omega) \), is holomorphic for a generic linear projection \( p : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \).

Proof. From Proposition 3.2.5 it is clear that (a) implies (b). To prove that (b) implies (a) suppose that \( \omega \) is not holomorphic at 0. If this is the case, then there exists \( f \in \mathcal{O}_{X,0} \) such that
\[
\text{Res}_0(f\omega) \neq 0.
\]

By the additivity of the residue, the function \( f \) can be replaced by the restriction to \( X \) of a monomial function
\[
x^J = x_1^{j_1} \cdots x_n^{j_n} \in \mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{x_1, \ldots, x_n\}
\]
which still satisfies \( \text{Res}_0(x^J\omega) \neq 0 \).

For \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \in (\mathbb{C}, 1)^n \), define \( p_{\epsilon} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) as the linear projection
\[
p_{\epsilon}(x_1, \ldots, x_n) = \sum_{i=1}^n \epsilon_i x_i.
\]

Consider now the monomial \( t^{\vert J \vert} \in \mathcal{O}_{\mathbb{C},0} = \mathbb{C}\{t\} \), where \( \vert J \vert = \sum j_i \).

Notice that, for every \( \epsilon \in (\mathbb{C}, 1)^n \),
\[
t^{\vert J \vert} \text{tr}_{p_{\epsilon}}(\omega) = \text{tr}_{p_{\epsilon}}(p_{\epsilon}^*(t^{\vert J \vert})\omega).
\]

Consequently, Proposition 3.2.1 implies
\[
\text{Res}_0(t^{\vert J \vert} \text{tr}_{p_{\epsilon}}(\omega)) = \text{Res}_0(p_{\epsilon}^*(t^{\vert J \vert})\omega).
\]

But, using again the additivity of the residue,
\[
\text{Res}_0(p_{\epsilon}^*(t^{\vert J \vert})\omega) = \sum_{K \in \mathbb{N}^n \atop \vert K \vert = \vert J \vert} \binom{\vert K \vert}{K} \epsilon^K \text{Res}_0(x^K \omega),
\]

where \( \epsilon^K = \epsilon_1^{k_1} \cdots \epsilon_n^{k_n} \) and \( \binom{\vert K \vert}{K} = \binom{k_1}{k_1} \cdots \binom{k_n}{k_n} \).
Since $\text{Res}_0(x^{J}\omega) \neq 0$, the polynomial in the variables $\epsilon_1, \ldots, \epsilon_m$ on the righthand side is not zero. Consequently, for a generic $\epsilon$, the meromorphic 1-form $\text{tr}_{p_i}(p^*_i(\tau^{(i)})\omega)$ has a non-zero residue at the origin. This suffices to establish that (b) implies (a).

### 3.2.3 Abel’s addition Theorem

Having the concept of abelian differential as well as Proposition 3.2.6 at hand, there is no difficulty to adapt the proof of Theorem 3.1.3 to establish Abel’s addition Theorem for arbitrary projective curves.

**Theorem 3.2.7** (Abel’s addition Theorem). If $\omega$ is a meromorphic 1-form on a projective curve $C$ then $\text{Tr}(\omega)$ is a meromorphic 1-form on $\mathbb{P}^n$. Moreover, $\omega$ is abelian if and only if its trace $\text{Tr}(\omega)$ vanishes identically.

As a by product, Theorem 3.1.5 also generalizes to the following

**Theorem 3.2.8.** If $C$ is a projective curve of degree $k$ and $H_0$ is a hyperplane intersecting it transversely, then the space of abelian 1-forms on $C$ injects into the space of abelian relations of the dual web $W_C(H_0)$.

Using notation similar to the one of Section 3.1.4, the preceding result can be formulated as follows: the injective linear map

$$H^0(C,\omega_C) \longrightarrow (\Omega^1(\mathbb{P}^n, H_0))^k$$

$$\omega \longmapsto (p^*_1 \omega, \ldots, p^*_k \omega)$$

factors through $\mathcal{A}(W_C(H_0))$.

Of course there is also a corresponding version of Castelnuovo’s bound for arbitrary reduced curves which intersects a generic hyperplane in points in general position.

**Theorem 3.2.9.** Let $C$ be a reduced projective curve on $\mathbb{P}^n$ of degree $k$. If the dual web is generically smooth, for instance if $C$ is irreducible and non-degenerate, then $h^0(C,\omega_C) \leq \pi(n,k)$.

To ease further reference to projective curves with generically smooth dual web, these will be labeled $W$-generic curves. Notice that according to Proposition 1.4.5, an irreducible curve is $W$-generic if and only if it is non-degenerate.
3.2.4 Abelian differentials for curves on surfaces

Let now $X$ be a curve on a smooth compact connected surface $S$. The purpose of this section is to describe the sheaf $\omega_X$ in terms of sheaves over $S$. As usual, $K_S$ denotes the sheaf of holomorphic 2-forms on $S$ – the canonical sheaf of $S$ – and $K_S(X)$ is used as an abbreviation of $K_S \otimes \mathcal{O}_S(X)$.

If $U$ is a sufficiently small neighborhood of a point $p \in X$ then $X \cap U = \{ f = 0 \}$ for some $f \in \mathcal{O}_S(U)$ generating the ideal $\mathcal{I}_X(U)$. Notice that any section $\eta \in \Gamma(U, K_S(X))$ can be written as $\eta = F^{-1} h dx \wedge dy$, with $h \in \mathcal{O}_S(U)$.

If the coordinate functions $x$ and $y$ are not constant on any of the irreducible components of $X$ then $\text{Res}_X(\eta)$ – the residue of $\eta$ along $X$ – is, by definition, the restriction to $X$ of the meromorphic 1-form $(h/\partial_y f) dx$. Explicitly,

$$\text{Res}_X(\eta) = \left( \frac{h dx}{\partial_y f} \right)_{|X}.$$  

It is easy to verify that $\text{Res}_X(\cdot)$ does not depend on the choice of $f$ nor on the choice of local coordinates $x, y$. In the literature, $\text{Res}_X(\eta)$ also appears under the label of Poincaré’s, as well as Leray’s, residue of $\eta$ along $X$.

Notice that for every $g \in \mathcal{O}_S(U)$ and $\eta$ as above,

$$\text{Res}_X(g \eta) = g|_X \cdot \text{Res}_X(\eta).$$

Thus the map $\text{Res}_X$ can be interpreted as a morphism of $\mathcal{O}_S$-modules from $K_S(X)$ to $\mathcal{M}_X$. Of course, the structure of $\mathcal{O}_S$-module on the latter sheaf is the one induced by the inclusion of $X$ into $S$.

Clearly, the kernel of $\text{Res}_X : K_S(X) \to \mathcal{M}_X$ coincides with the natural inclusion of the canonical sheaf $K_S$ into $K_S(X)$. Therefore the sequence

$$0 \longrightarrow K_S \longrightarrow K_S(X) \xrightarrow{\text{Res}_X} \text{Im} \text{Res}_X \subset \mathcal{M}_X$$

is exact.
SECTION 3.2: ABEL’S THEOREM II: ARBITRARY CURVES

**Proposition 3.2.10.** The image of $\text{Res}_X$ is exactly $\omega_X$, the sheaf of abelian differentials on $X$. Consequently the sheaf $\omega_X$ is coherent and the adjunction formula

$$\left( K_S \otimes \mathcal{O}_S(X) \right) |_X \simeq \omega_X .$$

holds true.

The proof of Proposition 3.2.10 will make use of the following lemma.

**Lemma 3.2.11.** Let $X = \{ f = 0 \}$ be a reduced complex curve defined at a neighborhood of the origin of $\mathbb{C}^2$. Suppose, as above, that the coordinate functions $x, y$ are not constant on any of the irreducible components of $X$. If $S$ is a sufficiently small sphere centered at the origin, and transverse to $X$, then the identity

$$\frac{1}{2\pi i} \int_{S \cap X} \frac{hdx}{\partial_y f} = \lim_{\epsilon \to 0} \int_{S \cap \{|f| = \epsilon\}} \frac{hdx \wedge dy}{f}$$

is valid for any meromorphic function $h$ on $(\mathbb{C}^2, 0)$.

We refer to [11, page 52] for a proof.

**Proof of Proposition 3.2.10.** Clearly $\text{Im} \text{Res}_X = \omega_X$ is a local statement. Moreover, for a smooth point $p$ of $X$, there is no difficult to see that both $\omega_{X,p}$ and $(\text{Im} \text{Res}_X)_p$ are isomorphic to $\Omega^1_{X,p}$. Let $p \in X$ be a singular point and take local coordinates $(x, y)$ centered at $p$ satisfying the assumption of Lemma 3.2.11.

To prove that $(\text{Im} \text{Res}_X)_p$ is contained in $\omega_{X,p}$ notice that the former $\mathcal{O}_{S,p}$-module is generated by $(\partial_y f)^{-1} dx = \text{Res}_X(f^{-1} dx \wedge dy)$. Lemma 3.2.11 implies that for every $h \in \mathcal{O}_{S,p}$

$$\text{Res}_0 \left( \frac{hdx}{\partial_y f} \right) = \frac{1}{2\pi i} \int_{S \cap X} \frac{hdx}{\partial_y f} = \lim_{\epsilon \to 0} \int_{S \cap \{|f| = \epsilon\}} \frac{hdx \wedge dy}{f} .$$

But, by Stoke’s Theorem,

$$\int_{S \cap \{|f| = \epsilon\}} \frac{hdx \wedge dy}{f} = \int_{S \cap \{|f| \geq \epsilon\}} d \left( \frac{hdx \wedge dy}{f} \right) = 0 .$$
Therefore $(\partial_y f)^{-1} dx \in \omega_{X, p}$ as wanted.

Suppose now that the 1-form $\eta = h(\partial_y f)^{-1} dx$ is abelian for some meromorphic function $h$ on $X$. Let $n \in \mathbb{N}$ be the smallest integer for which the function $x^n h$ is holomorphic at $p$, that is belongs to $\mathcal{O}_{X, p}$. Let $h_n \in \mathcal{O}_{S, p}$ be a holomorphic function with restriction to $X$ equal to $x^n h$.

If $n = 0$ then the relation $\eta = \text{Res}_X(h_0 f^{-1} dx \wedge dy)$ with $h_0 \in \mathcal{O}_{S, p}$ shows that $\eta \in (\text{Im Res}_X)_p$ as wanted. Thus, from now on, $n$ will be assumed positive.

Since $\eta$ is abelian

$$0 = \text{Res}_0(gx^{n-1} \eta) = \text{Res}_0 \left( \frac{h_n}{x} \frac{dx}{\partial_y f} \right)$$

for every $g \in \mathcal{O}_{S, p}$. Applying Lemma 3.2.11, and then Stoke’s Theorem, one deduces that the right-hand side is, up to multiplication by $2\pi i$, equal to

$$\lim_{\epsilon \to 0} \int_{S \cap \{|f| = \epsilon\}} \left( \frac{g h_n}{x} \frac{dx \wedge dy}{f} \right) = -\lim_{\epsilon \to 0} \int_{S \cap \{|x| = \epsilon\}} \left( \frac{g h_n}{x} \frac{dx \wedge dy}{f} \right).$$

Applying Lemma 3.2.11 again, but now to the curve $Y = \{x = 0\}$, yields

$$\text{Res}_0 \left( \frac{gh_n}{f} dy \right) = 0$$

for every $g \in \mathcal{O}_{S, p}$. But this implies that $h_n/f$ is a holomorphic function on $Y$. Therefore

$$h_n = h_{n-1} x + af$$

where $a, h_{n-1} \in \mathcal{O}_{S, p}$ are holomorphic functions. Thus on $X$

$$x^{n-1} h = h_{n-1}$$

contradicting the minimality of $n$. The inclusion $\omega_{X, p} \subset (\text{Im Res}_X)_p$ follows. Therefore $\omega_X = (\text{Im Res}_X)$. The coherence and adjunction formula follow at once from the exact sequence

$$0 \to K_S \to K_S(X) \to \omega_X \to 0.$$
The preceding proof also shows the following

**Corollary 3.2.12.** For a curve $X$ embedded in a smooth surface the sheaf $\omega_X$ is locally free.

Curves for which $\omega_X$ is locally free are usually called **Gorenstein**. Corollary 3.2.12 can be succinctly rephrased as: *germs of planar curves are Gorenstein.*

This is no longer true for arbitrary singularities. The simplest example is perhaps the germ of curve $X$ on $(\mathbb{C}^3,0)$ having the three coordinate axis as irreducible components.

Corollary 3.2.12 and the genus formula (3.3) imply the following

**Corollary 3.2.13.** If $X$ is an irreducible projective curve embedded in a smooth compact surface $S$ then

$$g_a(X) = \frac{(K_S + X) \cdot X}{2} + 1. \quad (3.4)$$

**Abelian differentials on planar curves**

The results just presented for arbitrary smooth compact surfaces $S$ will be now specialized to the projective plane $\mathbb{P}^2$. Let $C \subset \mathbb{P}^2$ be a reduced algebraic curve of degree $k$, and $\mathcal{I}_C$ its ideal sheaf.

Since $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$ and $\mathcal{O}_{\mathbb{P}^2}(C) = \mathcal{O}_{\mathbb{P}^2}(k)$, the adjunction formula reads as $\omega_C = \mathcal{O}_{\mathbb{P}^2}(k-3)|_C$.

Because $\mathcal{I}_C(k-3)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-3)$, both cohomology groups $H^0(\mathbb{P}^2, \mathcal{I}_C(k-3))$ and $H^1(\mathbb{P}^2, \mathcal{I}_C(k-3))$ are trivial. Therefore, the restriction map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) \longrightarrow H^0(C, \mathcal{O}_C(k-3))$$

is an isomorphism. Combined with the adjunction formula, this yields

$$H^0(\mathbb{P}^2, K_{\mathbb{P}^2}(C)) \simeq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) \simeq H^0(C, \omega_C).$$

The isomorphism from the leftmost to the rightmost group is induced by $\text{Res}_C$.

The discussion above is summarized, and made more explicit, in the following
Corollary 3.2.14. Let \{f(x, y) = 0\} be a reduced equation for \(C\) in generic affine coordinates \(x, y\) on \(\mathbb{P}^2\). Then
\[
H^0(C, \omega_C) \simeq \left\langle \frac{p(x, y)}{\partial_y f} \ dx \mid p \in \mathbb{C}[x, y], \deg p \leq k - 3 \right\rangle.
\]
Consequently \(g_a(C) = \frac{(k-1)(k-2)}{2}\).

3.3 Algebraic webs of maximal rank

In view of Theorem 3.2.8, it suffices to consider \(W\)-generic curves \(C \subset \mathbb{P}^n\) with \(h^k(C, \omega_C)\) attaining Castelnuovo’s bound \(\pi(n, \deg(C))\) in order to have examples of webs of maximal rank. Classically, a degree \(k\) irreducible non-degenerate curve \(C \subset \mathbb{P}^n\) such that \(g_a(C) = \pi(n, k) > 0\) is called a Castelnuovo curve. Notice that the definition implies that a Castelnuovo curve has necessarily degree \(k \geq n + 1\), since otherwise \(g_a(C) = 0\) according to Theorem 3.1.7.

The simplest examples of Castelnuovo curves are the irreducible planar curves of degree at least three. Since the arithmetic genus of a reduced planar curve \(C\) is \(\pi(2, \deg(C))\), such curves are certainly Castelnuovo.

In sharp contrast, when the dimension is at least three, the Castelnuovo curves are the exception, rather than the rule. Indeed, the analysis carried out in Section 2.3 to control the geometry of the conormals of maximal rank webs was originally developed by Castelnuovo to control the geometry of hyperplane sections of Castelnuovo curves. Reformulating Proposition 2.3.10 in terms of curves, instead of webs, provides the following

**Proposition 3.3.1.** If \(C \subset \mathbb{P}^n\) is a Castelnuovo curve of degree \(k \geq 2n + 1\) then a generic hyperplane section of \(C\) is contained in a rational normal curve.

Indeed, Castelnuovo’s Theory goes further and says that a Castelnuovo curve \(C \subset \mathbb{P}^n\) is contained in a surface \(S\), which is cut out by \(|I_C(2)|\) the linear system of quadrics containing \(C\), which has rational normal curves as generic hyperplane sections. Because the degree of \(S\) is the same as the one of a generic hyperplane section, and rational
normal curves in $\mathbb{P}^{n-1}$ have degree $n-1$, the surface $S$ is of minimal degree. Theorem 2.3.13 applies, and implies that $S$ is either a plane, a Veronese surface in $\mathbb{P}^5$, or a rational normal scroll $S_{a,b}$ with $a + b = n - 1$.

The proof of these results will not be presented here†, but the rather easier determination of Castelnuovo curves in surfaces of the above list will be sketched below.

### 3.3.1 Curves on the Veronese surface

Let $S \subset \mathbb{P}^5$ be the Veronese surface. Recall that $S$ is the embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$. If $C$ is an irreducible curve contained in $S$ then, its degree $k$ as a curve in $\mathbb{P}^5$ is twice its degree $d$ as a curve in $\mathbb{P}^2$.

On the one hand,

$$h^0(C, \omega_C) = \frac{(d-1)(d-2)}{2} = \frac{(k-2)(k-4)}{8}.$$ 

On the other hand, Castelnuovo’s bound predicts

$$h^0(C, \omega_C) \leq \begin{cases} (e-1)(2e-1) & \text{if } k = 4e \\ e(2e-1) & \text{if } k = 4e + 2. \end{cases}$$

Therefore, both cases lead to Castelnuovo curves.

### 3.3.2 Curves on rational normal scrolls

Let $S$ be a rational normal scroll $S_{a,b}$ with $0 \leq a \leq b$ and $a + b = n - 1$. If $S$ is singular, that is $a = 0$, replace $S$ by its desingularization $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ which will still be denoted by $S$.

The Picard group of $S$ is the free rank two $\mathbb{Z}$-module generated by $H$, the class of a hyperplane section, and $L$, the class of a line of the ruling. Of course,

$$H^2 = n - 1, \quad H \cdot L = 1 \quad \text{and} \quad L^2 = 0.$$ 

†The interested reader may consult, for instance, [61], [6], or [60].
If one writes $K_S = \alpha H + \beta L$ then the coefficients $\alpha$ and $\beta$ can be determined using the genus formula (3.4). Since both $H$ and $L$ are smooth rational curves, one has

$$-2 = 2g(H) - 2 = H^2 + K_S \cdot H = (\alpha + 1)(n - 1) + \beta,$$

and

$$-2 = 2g(L) - 2 = L^2 + K_S \cdot L = \alpha.$$ 

Therefore $\alpha = -2$ and $\beta = n - 3$, that is, $K_S = -2H + (n - 3)L$.

Let $\alpha$ and $\beta$ be new constants distinct from the ones above. Let also $C$ be an irreducible curve contained in $S$, numerically equivalent to $\alpha H + \beta L$. Notice that $\deg(C) = C \cdot H = \alpha(n - 1) + \beta$. The genus formula (3.4) implies

$$g_a(C) = \frac{C^2 + K_S \cdot C}{2} + 1 = \frac{1}{2}(\alpha - 1)((n - 1)\alpha + 2(\beta - 1))$$

Suppose now that the degree of $C$ is equal to $k \geq n + 1$ and write

$$k - 1 = m(n - 1) + \epsilon,$$

as in Remark 2.2.9. If $C$ is Castelnuovo then

$$g_a(C) = \left(\frac{m}{2}\right)(n - 1) + m\epsilon.$$

Thus the Castelnuovo’s curves on $S$ provide solutions to the following system of equations

$$\deg(C) = m(n - 1) + \epsilon + 1 = \alpha(n - 1) + \beta,$$

$$g_a(C) = \left(\frac{m}{2}\right)(n - 1) + m\epsilon = \frac{1}{2}(\alpha - 1)((n - 1)\alpha + 2(\beta - 1)),$$

subject to the arithmetical constraints $m, n \geq 0$, and $0 \leq \epsilon \leq n - 2$; and the geometrical constraint $\alpha = C \cdot L > 0$.

It can be shown that the only possible solutions are

$$(\alpha, \beta) = (m + 1, -(n - 1 - \epsilon))$$

or, only when $\epsilon = 0$, $(\alpha, \beta) = (m, 1)$. 

If \( \mathcal{L} = \mathcal{O}_S(\alpha H + \beta L) \), with \((\alpha, \beta)\) being one of the solutions above, then Riemann-Roch’s Theorem for surfaces, see for instance [62, Theorem 1.6, Chapter V], implies

\[
\chi(\mathcal{L}) = \frac{1}{2}(\alpha H + \beta L)((\alpha - 2)H + (\beta + n - 3)L) + \chi(S) = \left(\frac{\alpha + 1}{2}\right)(n - 1) + (\alpha + 1)(\beta + 1).
\]

Notice that \( K_S \otimes \mathcal{L}^* = \mathcal{O}_S((-2 - \alpha)H + (n - 3 - \beta)L) \). Because \( \alpha > 0 \), the following inequality holds true

\[
(K_S \otimes \mathcal{L}^*) \cdot L < 0.
\]

Thus \( h^0(S, K_S \otimes \mathcal{L}^*) = 0 \). Serre’s duality for surfaces implies the same for \( h^2(S, \mathcal{L}) \), that is \( h^2(S, \mathcal{L}) = 0 \). Consequently \( h^0(S, \mathcal{L}) \geq \chi(\mathcal{L}) \). Moreover, if \( \alpha > 2 \) then \( h^0(\mathcal{L}) \geq \chi(\mathcal{L}) > 0 \).

Since \( \alpha \) is \( \lfloor \frac{k - 1}{n - 1} \rfloor \) or \( \lfloor \frac{k - 1}{n - 1} \rfloor + 1 \), the linear system \(|\mathcal{L}|\) is non-empty when \( k \geq 2n \).

One can push this analysis further and show that the linear system \(|\mathcal{L}|\) is base-point free whenever \( k \geq 2n \). As a consequence, the generic element of \(|\mathcal{L}|\) is an irreducible smooth curve according to Bertini’s Theorem. See [61] for details.

The discussion above is summarized in the following statement.

**Proposition 3.3.2.** For any integer \( n \geq 3 \), any pair \((a, b)\) of non-negative integers summing up to \( n - 1 \), and any \( k \geq 2n \), there exist Castelnuovo curves of degree \( k \) contained in \( S_{a,b} \subset \mathbb{P}^n \).

### 3.3.3 Webs of maximal rank

From all that have been said in the previous sections, the following result follows promptly.

**Proposition 3.3.3.** For any integers \( n \geq 2 \) and \( k \geq 2n \), there exist smooth \( k \)-webs of maximal rank on \((\mathbb{C}^n, 0)\). These are the algebraic webs of the form \( \mathcal{W}_C(H_0) \), where \( C \) is a degree \( k \) Castelnuovo curve in \( \mathbb{P}^n \) and \( H_0 \) a hyperplane intersecting it transversely.
It remains to discuss smooth \( k \)-webs of maximal rank on \((\mathbb{C}^n, 0)\) with \( k < 2n \).

For \( k \leq n \) there is not much to say inasmuch smooth \( k \)-webs on \((\mathbb{C}^n, 0)\) have always rank zero and are equivalent to \( W(x_1, \ldots, x_k) \).

For \( k = n + 1 \), a web of maximal rank carries exactly one non-zero abelian relation because \( \pi(n, n + 1) = 1 \). Thus, if

\[
u_1, \ldots, u_{n+1} : (\mathbb{C}^n, 0) \to \mathbb{C}
\]

are submersions defining \( W \) then its unique abelian relation takes the form \( f_1(u_1) + \cdots + f_{n+1}(u_{n+1}) = 0 \) for suitable holomorphic germs \( f_i : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \). It follows that \( W \) is equivalent to the parallel \((n + 1)\)-web \( W(x_1, \ldots, x_n, x_1 + \cdots + x_n) \).

For \( k \in \{n + 2, \ldots, 2n - 1\} \), it is fairly simple to construct smooth \( k \)-webs of maximal rank \( \pi(n, k) = k - n \). It suffices to consider submersions \( u_1, \ldots, u_k : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) of the form:

\[
u_i(x_1, \ldots, x_n) = x_i \quad \text{for} \ i \in \mathbb{N}
\]

and \( u_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} u_i^{(j)}(x_j) \quad \text{for} \ i \in \{n + 1, \ldots, k\} \).

Here \( u_i^{(j)} : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \) are germs of submersions in one variable. It is clear that for a generic choice of the submersions \( u_i^{(j)} \), the \( k \)-web \( W = W(u_1, \ldots, u_k) \) is smooth. Moreover, the definition of \( u_i \) when \( i > n \) can be interpreted as an abelian relation of \( W \). Therefore \( \text{rank}(W) \geq k - n \). But \( \pi(n, k) = k - n \), Therefore \( W \) is of maximal rank.

§

The examples of \( k \)-webs of maximal rank with \( k = n + 1 \) or \( k \geq 2n \) are of different nature than the ones presented for \( k \in \{n + 2, \ldots, 2n - 1\} \). While the equivalence classes of the former examples belong to finite dimensional families, the ones of the latter belong to infinite dimensional families, as can be easily verified. Although logically this could be just a coincidence, the main result of this book says that this is not case at least when \( n \geq 3 \) and \( k \geq 2n \). A precise statement will be given in Chapter 5.
Remark 3.3.4. If $C \subset \mathbb{P}^n$ is a non-degenerate, but not \(W\)-generic, curve of degree \(k\) then it may happen that \(h^\theta(\omega_C) > \pi(n,k)\). According to Proposition 1.4.6, such a curve \(C\) cannot be irreducible. There are works (see [8, 63, 104]) showing the existence of a function \(\tilde{\pi}(n,k)\) which bounds the arithmetical genus of non-degenerate projective curves in \(\mathbb{P}^n\) of degree \(k\). Of course, \(\tilde{\pi}(n,k)\) is greater than Castelnuovo’s number \(\pi(n,k)\). Moreover, the curves \(C\) attaining this bound have been classified. They all have a plane curve among their irreducible components.

### 3.4 Webs and families of hypersurfaces

The construction of the dual web \(W_C\) of a projective curve \(C\), as well as the definition of the trace relative to the family of hyperplanes, make use of the incidence variety \(I \subset \mathbb{P}^n \times \mathbb{P}^n\). The interpretation of \(I\) as the family of hyperplanes in \(\mathbb{P}^n\), suggests the extension of both constructions to other families of hypersurfaces.

In this section, one such extension is described, and used in combination with Chern’s bound for the rank of smooth webs to obtain bounds for the geometric genus of curves on abelian varieties. The exposition is deliberately sketchy. A more detailed account will appear elsewhere.

#### 3.4.1 Dual webs with respect to a family

Let \(X\) and \(T\) be projective manifolds, and \(\pi_T : X \times T \to T\), \(\pi_X : X \times T \to X\) be the natural projections. Consider \(\mathcal{F}\), a family of hypersurfaces in \(X\) parametrized by \(T\). By definition, \(\mathcal{F}\) is an irreducible subvariety of \(X \times T\) for which \(\mathcal{F}_H = \pi_T^{-1}(H) \cap \mathcal{F}\) is a hypersurface of \(X \times \{H\}\) for every \(H \in T\) generic enough. It will be convenient, as has been done with the family of hyperplanes on \(\mathbb{P}^n\), to think of \(H\) as point of \(T\) (\(H \in T\)), as well as a hypersurface in \(X\) (\(H \subset X\)).

Let \(C \subset X\) be a reduced curve and \(H_0 \subset X\) be a hypersurface which belongs to the family \(\mathcal{F}\) – that is, \(H_0 \subset X\) is equal to \(\pi_X(\mathcal{F}_{H_0})\) with \(H_0 \in T\) being the corresponding point.
CHAPTER 3: ABEL’S ADDITION THEOREM

If \( H_0 \subset X \) intersects \( C \) transversely in \( k \) distinct points then, as for the family of hyperplanes in \( \mathbb{P}^n \), there are holomorphic maps

\[
p_i : (T, H_0) \to C, \quad \text{for } i \in \mathcal{K},
\]

implicitly defined by

\[
C \cap \pi_X(\mathcal{X}_H) = \sum_{i=1}^{k} p_i(H).
\]

It may happen that one of the points \( p_i(H_0) \) is common to all hypersurfaces in the family. It also may happen that for some pair \( i, j \in \mathcal{K} \), the functions \( p_i \) and \( p_j \) define the same foliation. But, if the maps \( p_i \) are non-constant and define pairwise distinct foliations then there is a naturally defined germ of (eventually singular) \( k \)-web on \( (T, H_0) \); \( \mathcal{W}(p_1, \ldots, p_k) \). It will called the \( \mathcal{X} \)-dual web of \( C \) at \( H_0 \in T \), and denoted by \( \mathcal{W}^\mathcal{X}_C(H_0) \).

If it is possible to define the germ of \( k \)-web \( \mathcal{W}^\mathcal{X}_C(H_0) \) at any generic \( H_0 \in T \) then there is no obstruction to define the global \( k \)-web \( \mathcal{W}^\mathcal{X}_C \), the \( \mathcal{X} \)-dual web of \( C \).

3.4.2 Trace relative to families of hypersurfaces

For a curve \( C \subset X \) and a meromorphic 1-form \( \omega \) on \( C \), it is possible to define the \( \mathcal{X} \)-trace of \( \omega \) relative to the family \( \mathcal{X} \) succinctly by the formula

\[
\text{Tr}_{\mathcal{X}}(\omega) = (\pi_T)_\ast (\pi_X)_\ast (\omega).
\]

To give a sense to this expression, it is necessary to assume that a generic hypersurface in the family intersects \( C \) in at most a finite number of points. In other words, no irreducible component of \( C \) is contained in the generic hypersurface of \( \mathcal{X} \). If this is the case, consider then \( \Sigma_0 \), one of the irreducible components of \( \pi_X^{-1}(C) \cap \mathcal{X} \) with dominant projection to \( T \). Because a generic \( H \in T \) intersects \( C \) in finitely many points, \( \Sigma_0 \) has the same dimension as \( T \).

Let \( \Sigma \to \Sigma_0 \) be a resolution of singularities, and still denote by \( \pi_X, \pi_T \) the compositions of the natural projections with the resolution morphism. Then \( \pi_X^\ast \omega \) is a meromorphic 1-form on \( \Sigma \), and for
a generic \( H \in T \) there are local inverses for \( \pi_T : \Sigma \to T \). Proceeding exactly as for the family of hyperplanes in \( \mathbb{P}^n \), one can define a meromorphic 1-form \( \eta_{\Sigma_0} \) at the complement of the critical values of \( \pi_T : \Sigma \to T \).

To extend \( \eta_{\Sigma_0} \) through the critical value set – or discriminant – of \( \pi_T \), observe that outside a codimension two subset, the discriminant is smooth; the fibers over points in it are finite; and locally, at each of its pre-images, \( \pi_T \) is conjugated to a map of the form \( (x_1, \ldots, x_n) \mapsto (x_1^r, x_2, \ldots, x_n) \) for some suitable positive integer \( r \). Hence, as for the family of hyperplanes, \( \eta_{\Sigma_0} \) can be extended through this set. But a meromorphic 1-form defined at the complement of a codimension two subset extends to the whole ambient space according to Hartog’s extension theorem.

The \( X \)-trace \( \text{Tr}_X(\omega) \) is then defined as the sum of the meromorphic 1-forms \( \eta_{\Sigma_0} \) for \( \Sigma_0 \) ranging over all the irreducible components of \( \pi_X^{-1}(C) \cap X \) dominating \( T \).

It is not hard to prove the following weak analogue of Abel’s addition Theorem.

**Proposition 3.4.1.** Let \( X \) be family of hypersurfaces of \( X \) over a smooth projective variety \( T \). If \( C \subset X \) is a curve intersecting a generic hypersurface of the family at finitely many points, and if \( \omega \) is a meromorphic 1-form on \( C \), then the \( X \)-trace of \( \omega \) is a meromorphic 1-form on \( T \). Moreover, if the pull-back of \( \omega \) to a desingularization of \( C \) is holomorphic then its \( X \)-trace is a holomorphic 1-form on \( T \).

If \( X \) is the family of all hypersurfaces of degree \( d \) of \( \mathbb{P}^n \) then a strong analogue of Abel’s Theorem is valid: the \( X \)-trace of \( \omega \) is holomorphic if and only \( \omega \) is abelian. For arbitrary families, the equivalence between holomorphic \( X \)-traces and abelian differentials is too much to be hoped for.

### 3.4.3 Bounds for rank and genus

From Proposition 3.4.1, one can obtain lower bounds for the rank of \( X \)-dual webs for curves in \( X \).
Proposition 3.4.2. Let $C \subset X$ be a curve with $\mathcal{X}$-degree $k$. If $\mathcal{W}_C^\mathcal{X}$ is a $k$-web then

$$\text{rank}(\mathcal{W}_C^\mathcal{X}) \geq g(C) - h^0(T, \Omega^1_T).$$

Proof. If $C$ is a desingularization of $C$ then, according to Proposition 3.4.1, the $\mathcal{X}$-trace of any holomorphic 1-form in $H^0(C, \Omega^1_C)$ is a holomorphic 1-form on $T$. Moreover, the map $\omega \mapsto \text{Tr}_{\mathcal{X}}(\omega)$ is a linear map from $H^0(C, \Omega^1_C)$ to $H^0(T, \Omega^1_T)$ whose kernel can be identified with a linear subspace of $\mathcal{A}(\mathcal{W}_C^\mathcal{X}(H_0))$ for a generic $H_0 \in T$. Since, by definition $g(C) = h^0(\Omega^1_C)$, the proposition follows. \hfill \square

In analogy with the standard case of families of hyperplanes, Proposition 3.4.2 read backwards provides a bound for the genus of curves $C \subset X$ as soon as the $k$-web $\mathcal{W}_C^\mathcal{X}$ is a generically smooth.

Proposition 3.4.3. Let $C \subset X$ be a curve with $\mathcal{X}$-degree $k$. If $\mathcal{W}_C^\mathcal{X}$ is a generically smooth $k$-web then

$$g(C) \leq \pi(\dim T, k) + h^0(T, \Omega^1_T).$$

3.4.4 Families of theta translates

Let $A$ be an abelian variety of dimension $n$, and $H_0 \subset A$ be an irreducible divisor in $A$. Recall that $A$ acts on itself by translations, and assume that $H_0$ has finite isotropy group under this action.

Consider a second copy of $A$ and denote it by $\hat{A}$. Let $\mathcal{X}$ be the family of translates of $H_0 \subset A$ by points of $\hat{A}$, that is

$$\mathcal{X} = \{(x, y) \in A \times \hat{A} | x - y \in H_0\}.$$

The natural projections from $\mathcal{X}$ to $A$ and $\hat{A}$ will be denoted by $\pi$ and $\hat{\pi}$ respectively.

Lemma 3.4.4. Let $A$, $H_0$ and $\mathcal{X}$ be as above. If $C \subset A$ is an irreducible curve not contained in a translate of $H_0$ then $\mathcal{W}_C^\mathcal{X}$ is generically smooth.
Proof. Let $H_0$ be a generic hypersurface in the family, intersecting $C$ transversely in $k$ points. At a neighborhood of $H_0$, write $C = C_1 \cup \cdots \cup C_k$. For simplicity, assume that $k = H_0 \cdot C \geq n$.

To prove that $W_X^C$ is generically smooth, first notice that

$$P = (p_1, p_2, \ldots, p_k) : (A, H_0) \rightarrow C^k$$

$$H \mapsto (H \cap C_1, \ldots, H \cap C_k)$$

has finite fibers. If not then the fiber over $P(H_0)$ has positive dimension. Consequently one of the irreducible components of its Zariski closure is an analytic subset $Z \subset \tilde{A}$ of positive dimension. Clearly for every $H \in Z \subset \tilde{A}$ the intersection $H \cap C \subset A$ contains $H_0 \cap C$. Since $H_0$ has finite isotropy group $\pi(\tilde{\pi}^{-1}(Z))$ is equal to $A$. Therefore given an arbitrary point $q \in C$, there exists $H_q \in Z$ containing $q$. If $C$ is not contained in $H_q$ then $H_q \cdot C > k$. But $H_q$ and $H_0$ are algebraically equivalent, thus $k > H_q \cdot C = H_0 \cdot C = k$. This contradiction shows that $P$ has finite fibers.

Using the irreducibility of $C$, one can show that the map $P_I = (p_{i_1}, \ldots, p_{i_n}) : (A, H_0) \rightarrow C^n$ also has finite fibers for any subset $I = \{i_1, \ldots, i_n\}$ of $\mathbb{K}$ having cardinality $n$. The reader is invited to fill in the details.

Since $P_I$ has finite fibers and $W(p_{i_1}, \ldots, p_{i_n})$ is an arbitrary $n$-subweb of $W_X^C$ at $H_0$, it follows that the web $W_X^C$ is generically smooth.

This lemma together with Proposition 3.4.3 promptly implies the following

**Theorem 3.4.5.** Let $C \subset A$ be a curve with $\mathcal{X}$-degree $k$. If $C$ is irreducible and is not contained in any translate of $H_0$ then

$$g(C) \leq \pi(n, k) + n.$$  
(3.5)

The most natural example of a pair $(A, H_0)$ satisfying the above conditions is an irreducible principally polarized abelian variety $(A, \Theta)$. There is a version of Castelnuovo’s Theory in this context, developed by Pareschi and Popa in [85]. In particular, they obtain the following bound for the genus of curves in $A$. 

Theorem 3.4.6. Let \((A, \Theta)\) be an irreducible principally polarized abelian variety of dimension \(n\). Let \(C \subset A\) be a non-degenerate\(^{\dagger}\) irreducible of degree \(k = C \cdot \Theta\) in \(A\). Let \(m = \lfloor \frac{k-1}{n} \rfloor\), so that \(k-1 = mn + \epsilon\), with \(0 \leq \epsilon < n\). Then

\[
g(C) \leq \left( \frac{m+1}{2} \right) n + (m+1)\epsilon + 1 \quad (3.6)
\]

Moreover, the inequality is strict for \(n \geq 3\) and \(k \geq n + 2\).

For a fixed \(n\), the bounds (3.5) and (3.6) are asymptotically equal to \(\frac{k^2}{2(n-1)}\) and \(\frac{k^2}{2n}\) respectively. The bound provided by Theorem 3.4.5 is asymptotically worse than the one provided by Theorem 3.4.6. But when \(n \geq 3\) and for comparably small values of \(k\), the former is sharper than the latter. Indeed, it can be verified that the bound of Theorem 3.4.5 is sharper than the one of Theorem 3.4.6, if and only if \(k\) is between \(n+2\) and \(2n^2 - 3n\).

\(^{\dagger}\)Here, non-degenerate means that the curve is not contained in any abelian subvariety. Although this differs from our assumption on \(C\), one of the first steps in the proof of this result is to establish that the maps \(P_t\) used in the proof of Lemma 3.4.4 contain open subsets of \(A^n\) in their images. Consequently, if \(C\) is non-degenerate then the \(\mathcal{X}\)–dual web is generically smooth.
Chapter 4

The ubiquitous algebraization tool

The main result of this chapter is a converse of Abel’s addition Theorem stated in Section 4.1. It assures the algebraicity of local datum satisfying Abel’s addition Theorem. Its first version was established by Sophus Lie in the context of double translation surfaces. Lie’s arguments consisted in a tour-de-force analysis of an overdetermined system of PDEs. Later Poincaré introduced geometrical methods to handle the problem solved analytically by Lie. Poincaré’s approach was later revisited, and made more precise by Darboux, to whom the approach presented in Section 4.2 can be traced back. By the way, those willing to take for granted the validity of the converse of Abel’s Theorem can safely skip Section 4.2.

Blaschke’s school reinterpreted the converse of Abel’s Theorem in the language of web geometry to obtain the algebraicity of germs of linear webs admitting complete abelian relations. This result turned out to be an ubiquitous tool for the algebraization problem of germs of webs. This dual version also provides a complete description of the space of abelian relations of algebraic webs. This will be treated in Section 4.1.2.

Web geometry also owes Poincaré a method to algebraize smooth, non necessarily linear, 2n-webs on $(\mathbb{C}^n,0)$ of maximal rank. The
strategy is based on the study of certain natural map from \((\mathbb{C}^0)\) to \(\mathbb{P}(\mathcal{A}(\mathcal{W}))\), here called Poincaré’s map of \(\mathcal{W}\). Closely related are the canonical maps of the web. Their definition mimics the one of canonical maps for projective curves. All this will be made precise in Section 4.3.

Poincaré’s original motivation had not much to do with web geometry, but instead focused on the relations between double-translation hypersurfaces and Theta divisors on Jacobian varieties of projective curves. In Section 4.4 these relations will be reviewed. Modern proofs of some of the theorems by Lie, Poincaré and Wirtinger on the subject are also laid out.

### 4.1 The converse of Abel’s Theorem

#### 4.1.1 Statement

Let \(H_0 \subset \mathbb{P}^n\), \(n \geq 2\), be a hyperplane, and \(k \geq 3\) be an integer. For \(i \in \mathbb{K}\), let \(C_i\) be a germ of complex curve in \(\mathbb{P}^n\) that intersects \(H_0\) transversely at the point \(p_i(H_0)\). Assume that the points \(p_i(H_0)\) are pairwise distinct. Let also \(p_i : (\mathbb{P}^n, H_0) \rightarrow C_i\) be the germs of holomorphic maps characterized by \(H \cap C_i = \{p_i(H)\}\) for every \(i \in \mathbb{K}\) and every \(H \in (\mathbb{P}^n, H_0)\). For a picture, see Figure 4.1 below.

Using the notation settled above, the converse of Abel’s Theorem can be phrased as follows.

**Theorem 4.1.1.** For \(i \in \mathbb{K}\), let \(\omega_i\) be a germ of non-zero holomorphic 1-form on \(C_i\). If

\[
p_1^*(\omega_1) + \cdots + p_d^*(\omega_d) \equiv 0 \quad (4.1)
\]

as a germ of 1-form at \((\mathbb{P}^n, H_0)\) then there exist an algebraic curve \(C \subset \mathbb{P}^n\) of degree \(k\), and an abelian differential \(\omega \in H^0(C, \omega_C)\) such that \(C_i \subset C\) and \(\omega|_{C_i} = \omega_i\) for all \(i \in \mathbb{K}\).

Starting from middle 1970’s, a number of generalizations of this remarkable result appeared in print. It has been generalized from curves to higher dimensional varieties in [57]; it has been shown in [70] that it suffices to have rational trace (4.1) to assure the algebraicity of
SECTION 4.1: THE CONVERSE OF ABEL’S THEOREM

Figure 4.1: Germs of analytic curves . . .

the data; and it has been considered in [49, 110] more general traces in the place of the one with respect to hyperplanes.

The proof of Theorem 4.1.1 is postponed to Section 4.2. Assuming its validity, a dual formulation in terms of webs is given and proved below.

4.1.2 Dual formulation

Let $H_0$ be a hyperplane in $\mathbb{P}^n$, $n \geq 2$. In Section 1.1.3 of Chapter 1 it was shown the existence of an equivalence between linear quasi-smooth $k$-webs on $(\mathbb{P}^n, H_0)$, and $k$ germs of curves in $\mathbb{P}^n$ intersecting $H_0$ transversely in $k$ distinct points. This equivalence implies the following variant of the converse of Abel’s Theorem.

**Theorem 4.1.2.** Let $W$ be a linear quasi-smooth web on $(\mathbb{P}^n, H_0)$. If it admits a complete abelian relation then it is algebraic. Accordingly, there exists a projective curve $C \subset \mathbb{P}^n$ such that $W$ coincides with the restriction of $\mathcal{W}_C(H_0)$. Furthermore, the space of abelian relations of $W$ is naturally isomorphic to $H^0(C, \omega_C)$. 
Proof. For reader’s convenience, the proof starts by detailing the equivalence between germs of linear webs and germs of curves. Let $W = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ be a quasi-smooth linear $k$-web on $(\mathbb{P}^n, H_0)$. For each of the foliations $\mathcal{F}_i$, consider its Gauss map $G_i : (\mathbb{P}^n, H_0) \to \mathbb{P}^n$, $H \mapsto T_H \mathcal{F}_i$.

Since the foliation $\mathcal{F}_i$ has linear leaves, the map $G_i$ is constant along them. Notice also that the restriction of $G_i$ to a line transversal to $H_0$ is injective. These two facts together imply that $G_i$ is a submersion defining $\mathcal{F}_i$ and its image is a germ of smooth curve $C_{\mathcal{F}_i} \subset \mathbb{P}^n$ intersecting $H_0$ transversely.

Notice that $G_i$ associates to a hyperplane $H \in (\mathbb{P}^n, H_0)$ the intersection of $H \subset \mathbb{P}^n$ with $C_{\mathcal{F}_i}$. In other words $G_i = p_i$ in the notation used in the converse of Abel’s Theorem.

If $\eta_i$ is a germ of closed 1-form defining $\mathcal{F}_i$ then there exists a germ of holomorphic 1-form $\omega_i$ in $C_{\mathcal{F}_i}$ such that $\eta_i = G_i^* \omega_i$. Therefore, if $\eta = (\eta_1, \ldots, \eta_k) \in \mathcal{A}(W)$ then there exist 1-forms...
\[ \omega_i \in \Omega^1(C_{F_i}, \mathcal{G}_H(H_0)) \text{ satisfying} \]
\[ \sum_{i=1}^{k} \mathcal{G}_i^*(\omega_i) = 0. \]

When \( \eta \) is complete, none of the 1-forms \( \omega_i \) vanishes identically. Thus the converse of Abel’s Theorem ensures the existence of a projective curve \( C \subset \mathbb{P}^n \) containing all the curves \( C_{F_i} \), and of an abelian differential \( \omega \in H^0(C, \omega_C) \) which pull-backs through \( \mathcal{G}_i \) to the \( i \)-th component of the abelian relation \( \eta \). It is then clear that \( \mathcal{W} \) is the restriction of \( \mathcal{W}_C \) at \( H_0 \). \( \square \)

**Corollary 4.1.3.** Let \( \mathcal{W} \) be a linear smooth \( k \)-web on \( (\mathbb{C}^n, 0) \). If \( \mathcal{W} \) has maximal rank then it is algebraic.

**Proof.** It suffices to show the existence of a complete abelian relation. If there is none then there exists a \( k' \)-subweb \( \mathcal{W}' \) of \( \mathcal{W} \) with \( k' < k \), and \( \text{rank}(\mathcal{W}') = \text{rank}(\mathcal{W}) = \pi(n, k) \). But \( \text{rank}(\mathcal{W}') \leq \pi(n, k') < \pi(n, k) \). This contradiction proves the corollary. \( \square \)

**Corollary 4.1.4.** A smooth web of maximal rank is algebraizable if and only if it is linearizable.

The latter corollary indicates the general strategy for the problem of algebraization of webs: *in order to prove that a web of maximal rank is algebraizable, it suffices to show that it is linearizable.* In fact a similar strategy also applies to webs of higher codimension. Most of the known algebraization results in web geometry are proved in this way. The simplest instance of this approach will be the subject of Section 4.3. A considerably more involved instance will occupy the whole Chapter 5.

### 4.2 Proof

The notation introduced in Section 4.1.1 is valid throughout this section.
4.2.1 Reduction to dimension two

This section is devoted to prove that the converse of Abel’s Theorem in dimension \( n \) follows from the case of dimension \( n - 1 \) when \( n > 2 \).

Assume \( n > 2 \) and consider a generic point \( p \in H_0 \subset \mathbb{P}^n \). The linear projection \( \pi : \mathbb{P}^n \to \mathbb{P}^{n-1} \) with center at \( p \) when restricted to the germs of curves \( C_i \), induces germs of biholomorphisms onto their images, which are denoted \( D_i \). Moreover, the dual inclusion \( \hat{\pi} : \mathbb{P}^{n-1} \to \mathbb{P}^n \) fits into the commutative diagram below.

\[
\begin{array}{c}
C_i \ar[d] \ar[r] & \mathbb{P}^n \ar[r]^-\pi & \mathbb{P}^{n-1} \ar[d] & D_i \\
\ar[r] \ar[u]_{p_i} & \mathbb{P}^{n-1} \ar[r]^-{\pi} & \mathbb{P}^n & \mathbb{P}^{n-1} \ar[r]^-{\hat{\pi}} & (\hat{\pi}(\mathbb{P}^{n-1}, \pi(H_0))) \\
\end{array}
\]

Since the restriction of \( \pi \) to \( C_i \) is a biholomorphism onto \( D_i \), the 1-forms \( \omega_i \) can be thought as a 1-form on \( D_i \). Under this identification, it is clear that

\[
\sum_{i=1}^{k} q_i^* \omega_i = \hat{\pi}^* \left( \sum_{i=1}^{k} p_i^* \omega_i \right) = 0.
\]

The converse of Abel’s Theorem in dimension \( n - 1 \) implies the existence of an algebraic curve \( D \subset \mathbb{P}^{n-1} \) containing all the curves \( D_i \), and of an abelian, thus rational, 1-form \( \omega \) on \( D \) such that \( \omega|_{D_i} = \omega_i \).

Notice that \( S = \overline{\pi^{-1}(D)} \) is the cone over \( D \) with vertex at \( p \). Notice also that it contains the curves \( C_i \) and has dimension two.

Let \( p' \in \mathbb{P}^n \) be another generic point of \( H_0 \). The same argument as above implies the existence of another surface \( S' \) containing the curves \( C_i \). It follows that the curves \( C_i \) are contained in the intersection \( S \cap S' \). This suffices to ensure the existence of a projective curve \( C \) in \( \mathbb{P}^n \) containing all the curves \( C_i \). Furthermore, the pull-back by \( \pi \) of \( \omega \) from \( D \) to \( C \) is a rational 1-form satisfying \( \omega|_{C_i} = \omega \) for every \( i \in \mathbb{L} \).

Thus, to establish Theorem 4.1.1 it suffices to consider the two-dimensional case. This will be done starting from the next section.
4.2.2 Preliminaries

To keep in mind that the ambient space has dimension two, the hyperplanes in the statement of Theorem 4.1.1 will be denoted by \( \ell_0 \) and \( \ell \), instead of \( H_0 \) and \( H \).

Assume that the main hypothesis of the converse of Abel’s Theorem is satisfied: for \( i \in k \), there are non-identically zero germs of holomorphic differentials \( \omega_i \in \Omega^1_{C_i} \) such that (4.1) holds.

Let \((x, y)\) be an affine system of coordinates on an affine chart \( C^2 \subset \mathbb{P}^2 \) where \( \ell_0 \cap C^2 = \{x = 0\} \), and none of the points \( p_i(H_0) \) belong to \( \ell_\infty = \mathbb{P}^2 \setminus C^2 \).

Since a generic line in the projective plane admits a unique affine equation of the form \( x = ay + b \), the variables \( a \) and \( b \) can be considered as affine coordinates on \((\mathbb{P}^2, \ell_0)\). If \( \ell_{a,b} \) denotes the line in \( \mathbb{P}^2 \) of affine equation \( x = ay + b \) then \( p_i(\ell_{a,b}) \) can be written as \((x_i(a, b), y_i(a, b))\), where \( x_i, y_i : (\mathbb{P}^2, \ell_0) \to \mathbb{C} \) are two germs of holomorphic functions satisfying \( x_i(a, b) = a y_i(a, b) + b \) identically on \((\mathbb{P}^2, \ell_0)\). It will be convenient to assume that for every \( i \in k \), the
function $y_i$ is non constant. Of course, this holds true for a generic choice of affine coordinates $(x, y)$ on $\mathbb{C}^2 \subset \mathbb{P}^2$.

Let also $\eta_i \in \Omega^1(\mathbb{P}^2, \ell_0)$ be the pull-back of $\omega_i$ by $p_i$ ($\eta_i = p_i^* \omega_i$), and $u_i : (\mathbb{P}^2, \ell_0) \to (\mathbb{C}, 0)$ be a primitive of $\eta_i$ with value at $\ell_0$ equal to zero, that is

$$u_i(a, b) = u_i(\ell_{a, b}) = \int^{(a, b)}_0 \eta_i.$$

**Differential identities**

One of the key ingredients of the proof of the converse of Abel's Theorem here presented is the following observation. It was first made in this context by Darboux.

**Lemma 4.2.1.** For every $i \in k$, the following differential equations are identically satisfied on $(\mathbb{P}^2, \ell_0)$

$$\frac{\partial y_i}{\partial a} = y_i \frac{\partial y_i}{\partial b}, \quad \frac{\partial x_i}{\partial a} = y_i \frac{\partial x_i}{\partial b} \quad \text{and} \quad \frac{\partial u_i}{\partial a} = y_i \frac{\partial u_i}{\partial b}. \quad (4.2)$$

**Proof.** First notice that, for a fixed $i \in k$, the functions $x_i, y_i,$ and $u_i$ define the very same foliation on $(\mathbb{P}^2, \ell_0)$. Consequently, the 1-forms $dx_i, dy_i,$ and $du_i$ are all proportional. Thus, it suffices to prove the identity

$$\frac{\partial y_i}{\partial a} = y_i \frac{\partial y_i}{\partial b} \quad (4.3)$$

to obtain the other two.

Since $C_i$ intersects $\ell_0 = \{x = 0\}$ transversely at $(0, y_0)$, there exists a germ $g \in \mathcal{O}_{\mathbb{C}, 0}$ for which $C_i = \{y = g(x)\}$. Therefore $y_i(a, b) = g(x_i(a, b)) = g(a y_i(a, b) + b)$ for all $(a, b) \in (\mathbb{P}^2, \ell_0)$. Differentiation of this identity implies

$$\frac{\partial y_i}{\partial a} \left(1 - a g'(x_i)\right) = y_i g'(x_i)$$

and

$$\frac{\partial y_i}{\partial b} \left(1 - a g'(x_i)\right) = g'(x_i).$$

The function $(a, b) \mapsto 1 - a g'(x_i(a, b))$ does not vanish identically, otherwise the holomorphic function $g(x_i(a, b))$ would be equal to $\log a$ at
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a neighborhood of \((a, b) = (0, 0)\). Therefore \(y_i\) verifies the differential equation (4.3). The lemma follows.

Notice that the relations

\[
\sum_{i=1}^{k} \frac{\partial u_i}{\partial a} = \sum_{i=1}^{k} y_i \frac{\partial u_i}{\partial b} \equiv 0 \quad \text{and} \quad \sum_{i=1}^{k} \frac{\partial u_i}{\partial b} \equiv 0 \quad (4.4)
\]

follow immediately from the hypothesis \(\sum_i p_i \omega_i = \sum_i u_i = 0\) combined with Lemma 4.2.1.

4.2.3 Lifting to the incidence variety

Let \(I \subset \mathbb{P}^2 \times \tilde{\mathbb{P}}^2\) be the incidence variety, that is

\[
I = \{ (p, \ell) \in \mathbb{P}^2 \times \tilde{\mathbb{P}}^2 \mid p \in \ell \}.
\]

As in Section 1.4.2 of Chapter 1, let \(\pi : I \to \mathbb{P}^2\) and \(\hat{\pi} : I \to \tilde{\mathbb{P}}^2\) be the natural projections.

There is an open affine subset \(V \subset I\) isomorphic to the closed subvariety of \(\mathbb{C}^2 \times \mathbb{C}^2\) defined by the equation \(x = ay + b\), where \((x, y)\) and \((a, b)\) are, respectively, the affine coordinates on \(\mathbb{C}^2 \subset \mathbb{P}^2\) and \((\tilde{\mathbb{P}}^2, \ell_0) \subset \mathbb{P}^2\) used above.

Notice that \(V\) is isomorphic to \(\mathbb{C}^3\) and \((y, a, b)\) is an affine coordinate system on it. Using these coordinates, define the germ of meromorphic 2-form \(\hat{\Psi}_0\) on \((\mathbb{C}^3, \{ay + b = 0\})^*\)

\[
\hat{\Psi}_0(a, b, y) = \sum_{i=1}^{d} \eta_i(a, b) \wedge \frac{dy}{y - y_i(a, b)}. \quad (4.5)
\]

Recall that the 1-forms \(\eta_i \in \Omega^1(\tilde{\mathbb{P}}^2, \ell_0)\) were introduced in Section 4.1.2 as the ones corresponding to the 1-forms \(\omega_i\) via projective duality. Notice that \(\hat{\Psi}_0\) is the restriction at \(V\) of a germ of meromorphic 2-form \(\Psi\) on \(\hat{\pi}^{-1}(\mathbb{P}^2, \ell_0) = \{\tilde{I}, \hat{\pi}^{-1}(\ell_0)\}\).

\*Here and throughout in the proof of the converse of Abel’s Theorem, the notation \((X, Y)\) means the germ of the variety \(X\) at \(Y\). One should think of open subsets of \(X\), arbitrarily small among the ones containing \(Y\).
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Remark 4.2.2. To understand the idea behind the definition of $\Psi$, imagine that the local datum \{$(C_i, \omega_i)$\} is indeed the germification at $\ell_0$ of a global curve $C$ and an abelian differential $\omega \in H^0(C, \omega_C)$. In this case, there exists a meromorphic 2-form $\Omega$ on $\mathbb{P}^2$ satisfying $\text{Res}_{C}\Omega = \omega$. Writing down the meromorphic 2-form $\pi^*\Omega$ in the coordinates $(a, b, y)$, one ends up with an expression exactly like (4.5).

Recall that the 1-form $\eta_i$ is equal to $du_i$, and that $u_i$ verifies (4.2). Therefore $\eta_i = (\partial u_i / \partial b)(y_i da + db)$ for every $i \in \mathbb{k}$. It is then easy to determine the expression $\Psi_0(x, y, a)$ of $\Psi$ in the coordinates $x, y, a$. Using (4.4) one obtains

$$\Psi_0(x, y, a) = \sum_{i=1}^{k} \frac{\partial u_i}{\partial b}(a, x - ay) \frac{y_i - y_i(a, x - ay)}{y_i} dx \wedge dy.$$  

If $F$ is the germ of meromorphic function on $\tilde{\pi}^{-1}(\mathbb{P}^2, \ell_0)$ which in the coordinates $(x, y, a)$ can be written as

$$F_0(x, y, a) = \sum_{i=1}^{k} \frac{\partial u_i}{\partial b}(a, x - ay) \frac{y_i - y_i(a, x - ay)}{y_i},$$

then

$$\Psi_0(x, y, a) = F_0(x, y, a) dx \wedge dy.$$  

4.2.4 Back to the projective plane

The next step of the proof consists in showing that the 2-form $\Psi$ defined above comes from a 2-form on the projective plane.

Lemma 4.2.3. There exists a germ of meromorphic function $f$ on $(\mathbb{P}^2, \ell_0)$ such that $F = \pi^*(f) = f \circ \pi$. Consequently, $\Psi = \pi^*\Omega$, where $\Omega$ is the meromorphic 2-form $f(x, y) dx \wedge dy$ on $(\mathbb{P}^2, \ell_0)$.

Proof. It suffices to prove that $\frac{\partial F_0}{\partial a}$ is identically zero. For that sake let $\tilde{F}_0$ be the expression for $F$ in the coordinate system $(a, b, y)$, that is,

$$\tilde{F}_0(a, b, y) = F_0(ay + b, y, a) = \sum_{i=1}^{k} \frac{\partial u_i}{\partial b}(a, b) \frac{y_i - y_i(a, b)}{y_i}. \tag{4.6}$$
Notice that
\[ y \hat{F}_0 = \sum_{i=1}^{k} \frac{y \partial u_i}{y - y_i} = \sum_{i=1}^{k} \frac{\partial u_i}{\partial b} + \sum_{i=1}^{k} \frac{y_i \partial u_i}{y - y_i} \]

Combining Equation (4.2) with the hypothesis \( \sum_{i=1}^{k} du_i = 0 \) yields
\[ y \hat{F}_0 = \sum_{i=1}^{d} \frac{\partial u_i}{y - y_i} \tag{4.7} \]

Differentiation of (4.6) and (4.7) with respect to \( a \) and \( b \) respectively give, in their turn, the identities
\[
\frac{\partial \hat{F}_0}{\partial a} = \sum_{i=1}^{k} \frac{\partial^2 u_i}{y - y_i} + \sum_{i=1}^{k} \frac{y_i \partial u_i}{(y - y_i)^2} \\
\text{and} \quad \frac{\partial \hat{F}_0}{\partial b} = \sum_{i=1}^{k} \frac{\partial^2 u_i}{y - y_i} + \sum_{i=1}^{k} \frac{y_i \partial u_i}{(y - y_i)^2}
\]
where \( y'_i = dy_i/du_i \).

Therefore \( \hat{F}_0 \) satisfies the equation
\[ \frac{\partial \hat{F}_0}{\partial a} - y \frac{\partial \hat{F}_0}{\partial b} = 0. \]

To conclude notice that
\[ \frac{\partial F_0}{\partial a}(x, y, a) = \left( \frac{\partial \hat{F}_0}{\partial a} - y \frac{\partial \hat{F}_0}{\partial b} \right)(a, x - ay, y) = 0. \]

\( \square \)

**Recovering the curves**

Now, notice that the polar set of the 2-form \( \Omega \) is nothing more than the union of the germs \( C_i \) with \( i \in \mathbb{k} \).

**Lemma 4.2.4.** If \( \Omega \) is the 2-form provided by Lemma 4.2.3 then
\[ (\Omega)_{\infty} = \bigcup_{i \in \mathbb{k}} C_i. \]
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Proof. Consider $\Psi = \pi^* \Omega$ in the coordinates $a, b, z = 1/y$. Performing the change of variable $y = 1/z$ in (4.5) gives

$$
\Psi(a, b, z) = -\sum_{i=1}^{k} \frac{\partial u_i}{\partial b}(y_i da + db) \frac{(y_i da + db)}{z(1 - z y_i)} \wedge dz
$$

$$
= -\sum_{r \geq 0} \sum_{i=1}^{k} \left( \frac{\partial u_i}{\partial b} y_i^{r+1} z^{r-1} \right) da + \left( \frac{\partial u_i}{\partial b} y_i^r z^{r-1} \right) db \wedge dz.
$$

Lemma 4.2.1 implies the vanishing of the coefficients of $z^{-1}da$ and $z^{-1}db$. Thus $\Psi(a, b, z)$ is holomorphic at an open neighborhood of $\{z = 0\}$. Therefore $\Omega$ is holomorphic at a neighborhood of $\ell_0 \cap \ell_\infty$.

For $(a, b) \in (\tilde{\mathbb{P}}^2, \ell_0)$, the restriction of $f$ at the line $\ell_{a,b} \subset (\mathbb{P}^2, \ell_0)$ is

$$
f_{a,b}(y) = F_0(a, b, y) = \sum_{i=1}^{k} \frac{\partial u_i}{\partial b}(a, b, y)\frac{y - y_i(a, b)}{y - y_i(a, b)}.
$$

Recall that $du_i = \eta_i$. According to the hypotheses, the partial derivative $\partial u_i/\partial b$ does not vanish identically on $C_i$ for $i \in \mathbb{K}$. Hence, for a generic $(a, b) \in (\tilde{\mathbb{P}}^2, \ell_0)$,

$$
C^2 \cap (f_{a,b})_{\infty} = p_1(a, b) + \cdots + p_k(a, b).
$$

This suffices to prove the lemma.

Recovering the 1-forms

It is also possible to extract from $\Omega$ the 1-forms $\omega_1, \ldots, \omega_k$ with the help of Poincaré’s residue.

Lemma 4.2.5. For every $i \in \mathbb{K}$, one has $\text{Res}_{C_i} \Omega = \omega_i$.

Proof. Fix $i \in \mathbb{K}$ and set $p = p_i(\ell_0) = (0, y_i(0, 0)) \in C_i$. Let $q$ be the point in $\pi^{-1}(\ell_0)$ which in the coordinate system $(a, b, y)$ is represented by $(0, 0, y_i(0, 0))$. If

$$
(D, q) = \{(a, b, y) \in \pi^{-1}(\mathbb{P}^2, \ell_0) | a = 0\}
$$

then the restriction of $\pi$ to $(D, q)$ is a germ of biholomorphism

$$
\rho : (D, q) \to (\mathbb{P}^2, p).
$$

In the coordinates $(b, y)$ on $D$, the 1-forms are

$$
\omega_i = \sum_{r \geq 0} \sum_{i=1}^{k} \left( \frac{\partial u_i}{\partial b} y_i^{r+1} z^{r-1} \right) da + \left( \frac{\partial u_i}{\partial b} y_i^r z^{r-1} \right) db \wedge dz.
$$
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a) the pull-back of $C_i$ to $D$ is $E_i = \rho^{-1}(C_i) = \{y - y_i(0, b) = 0\}$;

b) the 1-form $\rho^*(\omega_i)$ coincides with $\tilde{\pi}^*(\eta_i)_{|E_i}$; and

c) the pull-back of $\Omega$ to $(D, q)$ by $\rho$ can be written as

$$\rho^*(\Omega) = \sum_{i=1}^{k} \eta_i(0, b) \wedge dy_{y - y_i(0, b)}.$$  

Item a) implies that $\eta_j \wedge dy_{y - y_j}$ is holomorphic in a neighborhood of $E_i$ when $j \neq i$. Item b) and c), in their turn, imply

$$\text{Res}_{E_i}(\rho^*\Omega) = \text{Res}_{E_i}(\eta_i(0, b) \wedge dy_{y - y_i(0, b)}) = \eta_i(0, b)\Big|_{E_i} = \rho^*\omega_i.$$  

Since $\rho$ is an isomorphism, it follows that

$$\text{Res}_{C_i}(\Omega) = \omega_i$$

for every $i \in k$. The lemma is proved. \hfill \Box

4.2.5 Globalizing to conclude

At this point, to conclude the proof of Theorem 4.1.1, it suffices to prove that the 2-form $\Omega$ is the restriction at $(P^2, \ell_0)$ of a rational 2-form. Indeed, if this is the case then the polar set of $\Omega$ will be a projective curve $C$ containing the curves $C_i$, and its residue along $C$ will be an abelian differential $\omega \in H^0(C, \omega_C)$ according to Proposition 3.2.10.

The globalization of $\Omega$ follows from a particular case of a classical result of Remmert stated below as a lemma.

Lemma 4.2.6. Let $\ell \subset P^2$ be a line. Any germ of meromorphic function $g : (P^2, \ell) \to P^1$ extends to a rational function on $P^2$.

Proof. Let $(x, y)$ be affine coordinates on $C^2 \subset P^2$. Suppose that $\ell$ is the line at infinity. Fix an arbitrary representative of the germ $g$ defined in a neighborhood $U$ of the line at infinity. Still denoted it
by $g$. Notice that the restriction of $U$ to $\mathbb{C}^2$ contains the complement of a polydisc $\Delta$. Consider the Laurent expansion of $g$,

$$g(x, y) = \sum_{i,j \in \mathbb{Z}^2} a_{ij} x^i y^j.$$ 

Since it converges at a neighborhood of infinity to a meromorphic functions it suffices to show that

$$\Gamma = \{(i, j) \in \mathbb{Z}^2 | a_{ij} \neq 0\}$$

is contained in $(i_0, j_0) + \mathbb{N}^2$ for some $i_0, j_0 \in \mathbb{Z}$ in order to prove the lemma.

To prove this, rewrite the Laurent series of $g(x, y)$ as

$$g(x, y) = \sum_{i \in \mathbb{Z}} b_i(y)x^i,$$

and consider the function $I : \mathbb{C} \to \mathbb{Z} \cup \{-\infty\}$ defined by

$$I(t) = \inf\{i \in \mathbb{Z} | b_i(t) \neq 0\}.$$

If $|t| \gg 0$ then the function $g(x, t)$ is a global meromorphic function of $x$. Therefore $I(t) \in \mathbb{Z}$. Since $\mathbb{Z} \cup \{-\infty\}$ is a countable set, there exists an integer $i_0$, and an uncountable set $\Sigma \subset \mathbb{C}$ for which the restriction of $b_i$ to $\Sigma$ is zero whenever $i \leq i_0$. Therefore, for $i \leq i_0$, the functions $b_i$ are indeed zero all over $\mathbb{C}$. In other words, $\Gamma \subset (i_0 + \mathbb{N}) \times \mathbb{Z}$.

To conclude it suffices to change the roles of $x$ and $y$ in the above argument, and remind that a global meromorphic function on $\mathbb{P}^2$ is rational.

4.3 Algebraization of smooth $2n$-webs

As the title of this chapter indicates, Theorem 4.1.2 is an ubiquitous tool when the algebraization of webs comes to mind. It does not hurt to repeat that most of the known algebraization results use the abelian relations in order to linearize the web and then apply the converse of Abel’s Theorem in its dual formulation. As promised, the simplest instance where this strategy applies – the case of smooth $2n$-webs $\mathcal{W}$ on $(\mathbb{C}^{2n}, 0)$ of maximal rank – is treated below.
4.3.1 The Poincaré map

Let \( W = F_1 \boxtimes \cdots \boxtimes F_{2n} = W(\omega_1, \ldots, \omega_{2n}) \) be a smooth \( 2n \)-web on \((\mathbb{C}^n, 0)\). Assume that \( W \) has maximal rank. Since \( \pi(n, 2n) = n + 1 \), the space \( \mathcal{A}(W) \) is a complex vector space of dimension \( n + 1 \). In particular, according to Corollary 2.2.11,

\[
\dim F^1_1 \mathcal{A}(W) = 1.
\]

If \( F^*_x \mathcal{A}(W) \) denotes the corresponding filtration of \( \mathcal{A}(W) \) centered\(^1\) at \( x \) then one still has \( \dim F^1_1 \mathcal{A}(W) = 1 \). Poincaré's map of \( W \) is defined as

\[
P_W : (\mathbb{C}^n, 0) \to \mathbb{P} \mathcal{A}(W)
\]

\[
x \mapsto [F^1_1 \mathcal{A}(W)].
\]

It is a covariant of \( W \): if \( \varphi \in \text{Diff}(\mathbb{C}^n, 0) \) then

\[
P_{\varphi^* W} = \varphi^*(P_W) = P_W \circ \varphi.
\]

4.3.2 Linearization

For every \( i \in 2n \) and each \( x \in (\mathbb{C}^n, 0) \) consider the linear map

\[
ev_i(x) : \mathcal{A}(W) \to \Omega^1(\mathbb{C}^n, x)
\]

\[
(\eta_1, \ldots, \eta_{2n}) \mapsto \eta_i(x).
\]

The kernel of \( ev_i(x) \) corresponds to the abelian relations of \( W \) with \( i \)-component vanishing at \( x \).

**Lemma 4.3.1.** For every \( i \in 2n \), the linear map \( ev_i(x) \) has rank equal to one. In particular, \( x \mapsto \ker ev_i(x) \) is a sub-bundle of corank one of the trivial bundle over \((\mathbb{C}^n, 0)\) with fiber \( \mathcal{A}(W) \). Moreover, for every subset \( I \subset 2n \) of cardinality \( n \) the following identity holds true

\[
\bigcap_{i \in I} \ker ev_i(x) = F^1_x \mathcal{A}(W).
\]

\(^1\)Here and throughout, the convention about germs made in Section 1.1.1 is in use. If one wants to be more precise, then \( W \) has to be thought as a web defined on an open subset \( U \) of \( \mathbb{C}^n \) containing the origin and \( F^*_x \mathcal{A}(W) \) is the filtration of the germ of \( W \) at \( x \).
Proof. Since $\eta_i$ defines the foliation $F_i$, the rank of $ev_i(x)$ is at most one. By semi-continuity, if it is not constant and equal to one then it must be equal to 0 at $(\mathbb{C}^n, 0)$. In other words, the $i$-th component of every abelian relation $\eta \in \mathcal{A}(W)$ vanishes at the origin. Therefore

$$\dim \frac{F^1_x \mathcal{A}(W)}{F^0_x \mathcal{A}(W)} \leq (2n - 1) - f^i(W) = n - 1$$

according to the proof of Lemma 2.2.6. But then, according to Corollary 2.2.11, $\text{rank}(W) \leq (n - 1) + 1 = n < \pi(n, 2n)$ contradicting its maximality.

To prove the second part, notice that the smoothness of $W$ ensures the linear independence of $T_x F_i$ with $i \in 2n \setminus I$.

**Proposition 4.3.2.** If $L$ is a leaf of $W$ then $P_W(L)$, its image under Poincaré's map, is contained in a hyperplane.

**Proof.** At every point $x \in (\mathbb{C}^n, 0)$, one has

$$\bigcap_{i \in 2n} \ker ev_i(x) = F^1_x \mathcal{A}(W).$$

In particular, $F^1_x \mathcal{A}(W) \subset \ker ev_i(x)$ for every $i \in 2n$. Notice that for every $x \in (\mathbb{C}^n, 0)$, $\ker ev_i(x) \subset \mathcal{A}(W)$ is a hyperplane according to Lemma 4.3.1.

Fix $i \in 2n$. Suppose $L$ is a leaf of $F_i$ and let $u_i : (\mathbb{C}^n, 0) \to \mathbb{C}$ be a submersion defining $F_i$. Notice that $L$ is a level hypersurface of $u_i$. The $i$-component $\eta_i$ of an abelian relation $\eta \in \mathcal{A}(W)$ is of the form $g(u_i) du_i$. Therefore if $x, y \in L$ are two distinct points of $L$, then $\eta_i$ vanishes at $x$ if and only it vanishes at $y$. Thus $\ker ev_i(x) = \ker ev_i(y)$ for every $x, y \in L$. Hence, the image of $L$ by $P_W$ is contained in the hyperplane $\mathbb{P} \ker ev_i(x) \subset \mathbb{P} \mathcal{A}(W)$ determined by any $x \in L$.

**Proposition 4.3.3.** Poincaré's map $P_W : (\mathbb{C}^n, 0) \to \mathbb{P} \mathcal{A}(W)$ is a germ of biholomorphism.

**Proof.** For $i \in \eta_i$ let $L_i$ be the leaf of $F_i$ through zero. If $u_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a submersion defining $F_i$, then $L_i = u_i^{-1}(0)$. Let also $C_i$ be the curve in $(\mathbb{C}^n, 0)$ defined as

$$C_i = \bigcap_{j \in \mathbb{N} \setminus \{i\}} L_j.$$
Because \( \mathcal{W} \) is smooth, so is \( C_i \).

If \( \gamma_i : (\mathbb{C}, 0) \to C_i \) is a smooth parametrization of \( C_i \) then the image of \( P_W \circ \gamma_i \) is contained in the line \( \ell_i \) of \( \mathbb{P}A(\mathcal{W}) \) described by

\[
\bigcap_{j \in \mathbb{N}\setminus\{i\}} \mathbb{P}(\ker ev_i(0)),
\]

according to Lemma 4.3.1.

Since the tangent spaces of the lines \( \ell_1, \ldots, \ell_n \) at \( P_W(0) \) generate \( T_{P_W(0)}\mathbb{P}A(\mathcal{W}) \), it suffices to show that \( \Gamma_i = P_W \circ \gamma_i : (\mathbb{C}, 0) \to \ell_i \subset \mathbb{P}^n \) has non-zero derivative at \( 0 \in \mathbb{C} \), for every \( i \in \mathbb{N} \). But, \( \Gamma_i \) is a map between germs of smooth curves, hence everything boils down to the injectivity of \( \Gamma_i \) for a fixed \( i \in \mathbb{N} \).

If \( \Gamma_i \) is not injective then there are pairs of distinct points \( x, y \in C_i \) arbitrarily close to zero such that \( \ker ev_i(x) = \ker ev_i(y) \). Hence, if the \( i \)-th component of an abelian relation of \( \mathcal{W} \) vanishes at \( x \), then it also vanishes at \( y \). It follows that the \( i \)-th component of the abelian relation generating \( F_{10}^nA(\mathcal{W}) \) vanishes at the origin with multiplicity two. But this contradicts the equality \( \ell^2(\mathcal{W}) = 2n - 1 \) established in Proposition 2.2.1. Thus \( \Gamma_i \) is injective for any \( i \in \mathbb{N} \). Consequently, the differential of \( P_W \) at the origin is invertible. The proposition follows.

4.3.3 Algebraization

It is a simple matter to put the previous results together in order to prove the following algebraization result.

**Theorem 4.3.4.** A smooth \( 2n \)-web on \((\mathbb{C}^n, 0)\) of maximal rank is algebraizable. More precisely, its push-forward by its Poincaré’s map is a web \( \mathcal{W}_C \) where \( C \subset \mathbb{P}^n \) is a \( \mathcal{W} \)-generic projective of degree \( 2n \) and genus \( n + 1 \).

**Proof.** According to Proposition 4.3.3, \( P_W \) is a germ of biholomorphism. Hence \( (P_W)_*(\mathcal{W}) \) is a smooth \( 2n \)-web equivalent to \( \mathcal{W} \). In particular, its rank is also maximal. Proposition 4.3.2 implies that \( (P_W)_*(\mathcal{W}) \) is a linear web. To conclude apply Corollary 4.1.4. \( \square \)
4.3.4 Poincaré map for planar webs

It is possible to generalize Poincaré’s map for smooth \( k \)-webs on \((\mathbb{C}^n, 0)\) for all integers \( n \) and \( k \) such that \( 2n \leq k \). The idea is to consider the last non-trivial piece \( F^i \mathcal{A}(W) \neq 0 \) of the filtration \( F^* \mathcal{A}(W) \).

To be more precise, as in Remark 2.2.10,

\[ \rho = \left\lfloor \frac{k - n - 1}{n - 1} \right\rfloor \quad \text{and} \quad \epsilon = (k - n - 1) - \rho(n - 1). \]

In this notation, the last non-trivial piece of \( F^* \mathcal{A}(W) \) is \( F^{\rho+1} \mathcal{A}(W) \).

Then set \( e = \dim F^{\rho+1} \mathcal{A}(W) = \epsilon + 1 > 0 \) and define Poincaré’s map of \( W \) as

\[ P_W : (\mathbb{C}^n, 0) \longrightarrow \text{Grass}(\mathcal{A}(W), e) \]

\[ x \longmapsto F^{\rho+1}_x \mathcal{A}(W). \]

When \( e = 1 \), that is \( n = 2 \) or \( k \equiv 2 \mod (n - 1) \), then one still gets a map to a projective space with remarkable properties as shown in the next result for \( n = 2 \). Nevertheless, it does not linearize the web as when \( k = 2n \).

Proposition 4.3.5. If \( W \) is a smooth \( k \)-web of maximal rank on \((\mathbb{C}^n, 0)\) then \( P_W \) is an immersion. Moreover if \( S \) is the image of \( P_W \) and \( L \) is a leaf of \( W \) then the following assertions hold:

(a) the image of \( L \) by \( P_W(L) \) is contained in a projective space of codimension \( k - 3 \);

(b) the union of the projective tangent spaces of \( S \) along the points of the image of \( L \), that is

\[ \bigcup_{x \in P_W(L)} T_x S, \]

is contained in a projective space of codimension \( k - 4 \);

(c) if \( s \leq k - 4 \) then the union of the projective osculating spaces of \( S \) of order \( s \) along the points of the image of \( L \), that is

\[ \bigcup_{x \in P_W(L)} T_x^{(s)} S, \]

is contained in a projective space of codimension \( k - (3 + s) \).
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Proof. The proof is the natural generalization of the arguments used to prove Proposition 4.3.2 and Proposition 4.3.3. The reader is invited to fill in the details of the argument sketched below.

Instead of considering the evaluation morphism \( ev_i^0(x) = ev_i(x) : \mathcal{A}(W) \to \mathbb{C} \) one has to consider the higher order evaluation morphism \( ev_i^j(x) : \mathcal{A}(W) \to \mathbb{C}^{j+1} \) which sends the \( i \)-th component of an abelian relation \( \eta \) to its Taylor expansion truncated at order \( j \).

More precisely, if \( u_i \) is a submersion defining \( F_i \) and if the \( i \)-th component of \( \eta \) is \( \eta_i = f_i(u_i(x)) \), then

\[
ev_i^j(x)(\eta) = f_i(u_i(x)) + tf_i'(u_i(x)) + \cdots + t^j f_i^{(j)}(u_i(x))
\]

where \( 1, t, \ldots, t^j \) is thought as a basis of \( \mathbb{C}^{j+1} \).

It is a simple matter to adapt the arguments used to prove Proposition 4.3.3 in order to show that \( P_W \) is an immersion.

To prove item (a), notice that \( ev_i^{(k-4)}(x) \) has maximal rank. Consequently \( \dim \ker ev_i^{(k-4)}(x) = \pi(2, k) - (k-3) \). Furthermore, if \( L \) is the leaf of \( F_i \) through \( x \) then its image \( P_W(L) \) is contained in \( \mathbb{P} \ker ev_i^{(k-4)}(x) \). Putted together, these two remarks imply item (a).

To prove items (b) and (c), the key point is to notice the following. If \( f : (\mathbb{C}^2, 0) \to \mathcal{A}(W) \) is such that \( f(x) \in \ker ev_i^{(k-4)}(x) \) for every \( x \in (\mathbb{C}^2, 0) \) then the derivatives of \( f \) at \( x \) with order at most \( s \) lie in \( \ker ev_i^{(k-4-s)}(x) \).

The discussion of Poincaré’s map for planar webs of maximal rank just made leads naturally to the following characterization of algebraizable planar webs.

**Theorem 4.3.6.** If \( k \) is a integer greater than 4 and \( W \) is a smooth planar \( k \)-web of maximal rank then the image of Poincaré’s map \( P_W \) is contained in the \( (k-3) \)-th Veronese surface if and only if \( W \) is algebraizable.

**Proof.** Let \( S \) be the image of \( P_W \). Suppose first that \( S \) is contained in the \( (k-3) \)-th Veronese surface and let \( \nu = \nu_{k-3} : \mathbb{P}^2 \to S \subset \mathbb{P} \mathcal{A}(W) \) be the corresponding Veronese embedding. According to Proposition 4.3.5, for every \( i \in k \) and every \( x \in (\mathbb{C}^2, 0) \), the image of \( L \) (the leaf
of $F_i$ through $x$) is contained in a hyperplane $H$ that osculates $S$ up to order $k - 3$ along $P_V(L)$. But this means that $\nu^*H$ is a curve in $\mathbb{P}^2$ with an irreducible component $C$ for which every point is a singularity with algebraic multiplicity $k - 3$. Since $\nu^*H$ has degree $(k - 3)$, it follows that $\nu^*H = C = (k - 3)\ell$ for some line $\ell \subset \mathbb{P}^2$. Thus the composition $\nu^{-1} \circ P_W$ linearizes $W$. Corollary 4.1.3 implies that $(\nu^{-1} \circ P_W)_*W$ is an algebraic web.

Conversely, if $W = W_C(H_0)$ is algebraic then it is a simple matter to show that $P_W: (\mathbb{P}^2, H_0) \to \mathbb{P}A(W)$ is the germ at $H_0$ of the $(k-3)$-th Veronese embedding. For details see for instance [90].

For $k = 5$ the statement appears in [18, page 255], and the proof there presented involves the study of Poincaré-Blaschke’s map of the web, which will be introduced in Chapter 5. The result for arbitrary $k$ as stated above was proved in [66] using arguments very similar to the ones in the book just cited. The proof just presented uses slightly different arguments.

4.3.5 Canonical maps

For a $k$-web on $(\mathbb{C}^n, 0)$ of rank $r > 0$, it is possible to mimic the construction of canonical maps for projective curves as follows. Notice that the evaluation morphisms $ev_i(x)$ are linear functionals on $A(W)$. As such they can be thought as points of $A(W)^*$. For every $i \in \mathbb{K}$ consider the germ of meromorphic map

$$\kappa_{W,i}: (\mathbb{C}^n, 0) \to \mathbb{P}A(W)^* \simeq \mathbb{P}^{r-1}$$

$$x \mapsto [ev_i(x)].$$

By definition, $\kappa_{W,i}$ is the $i$-th canonical map of $W$.

Lemma 4.3.1, or more precisely its proof, shows that when $W$ is smooth and of maximal rank, $\kappa_{W,i}$ is regular at zero. Moreover, if $W = W_C(H_0)$ is an algebraic web on $(\mathbb{P}^n, H_0)$ dual to a projective curve $C$ then, after identifying $H^0(C, \omega_C)$ with $A(W)$, one can put together any one of the canonical maps of $W$ with the canonical map

\[\kappa_C: C \to PH^0(C, \omega_C)^* \simeq \mathbb{P}^{g_2(C) - 1}\]

\[x \mapsto \mathbb{P}(\ker{\omega \mapsto \omega(x)}).\]

\[\text{Recall that for projective curve, the canonical map is defined as}\]

\[\kappa_C: C \to PH^0(C, \omega_C)^* \simeq \mathbb{P}^{g_2(C) - 1}\]

\[x \mapsto \mathbb{P}(\ker{\omega \mapsto \omega(x)}).\]
of $C$ in the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{P}^n, H_0 & \xrightarrow{p_i} & C_i \\
& \searrow & \nearrow \\
& & \kappa_C \\
& & \searrow \\
& & \kappa_{W,i} \\
\end{array}
\]

For smooth $2n$-webs on $(\mathbb{C}^n, 0)$ of maximal rank, the Poincaré’s map and the canonical maps of $W$ are related through the formula

\[P_W(x) = \bigcap_{i \in 2n} \kappa_{W,i}(x)\]

where the points $\kappa_{W,i}(x)$ are interpreted as hyperplanes in $\mathbb{P}A(W)$.

Notice that the canonical maps, for no matter which $k$ and no matter which rank, always take values in $\mathbb{P}A(W)^*$. 

### 4.4 Double-translation hypersurfaces

By definition, a germ $S$ of smooth hypersurface at $(\mathbb{C}^{n+1}, 0)$ is a **translation hypersurface** if it is non-degenerate and admits a parametrization of the form

\[
\Phi : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n+1}, 0) \\
(x_1, \ldots, x_n) \longmapsto \phi_1(x_1) + \cdots + \phi_n(x_n)
\]

where $\phi_1, \ldots, \phi_{n+1} : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$ are germs of holomorphic maps. Notice that $\Phi$ induces naturally a $n$-web $W_\Phi = \Phi_* W(x_1, \ldots, x_n)$ on $S$.

To understand the logic behind the terminology, notice that a surface $S$ in $\mathbb{C}^3$ is a translation surface if it can be generated by translating a curve along another one (see the picture below).
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Figure 4.4: A translation surface in $\mathbb{C}^3$.

If

$$\Phi : (C^n, 0) \rightarrow (C^{n+1}, 0)$$

$$(y_1, \ldots, y_n) \mapsto \psi_1(y_1) + \cdots + \psi_n(y_n)$$

is another parametrization of $S$ as a translation hypersurface, then $\Psi$ is distinct from $\Phi$ if the superposition of the corresponding $n$-webs $W_\Phi$ and $W_\Psi$ is a quasi-smooth 2$n$-web on $S$.

A hypersurface $S$ is a double-translation hypersurface if it admits two distinct parametrizations in the above sense. The quasi-smooth 2$n$-web $W_S = W_\Phi \boxtimes W_\Psi$ will be denoted by $W_S$. Notice that there is a certain abuse of notation here since, a priori, a double-translation hypersurface may admit more than two distinct parametrizations as a translation hypersurface. Implicit in the notation $W_S = W_\Phi \boxtimes W_\Psi$, is the fact that the two distinct parametrizations of translation type are fixed in the definition of a double-translation hypersurface.

Example 4.4.1. The surface $S$ in $\mathbb{C}^3$ cut out by $4x + z^5 - 5zy^2 = 0$ is an example of an algebraic translation surface. In fact,

$$\Phi : (x_1, x_2) \mapsto \left( x_1^{-5} + x_2^{-5}, x_1^{-2} + x_2^{-2}, x_1^{-1} + x_2^{-1} \right)$$

and

$$\Psi : (y_1, y_2) \mapsto \left( -y_1^{-5} - y_2^{-5}, -y_1^{-2} - y_2^{-2}, -y_1^{-1} - y_2^{-1} \right)$$
are two parametrizations of $S$. Notice that $\Phi(x_1, x_2) = \Psi(y_1, y_2)$ if and only if $(x_1, x_2) = (-y_1, -y_2)$ or $y_1 = x_1 x_2 \zeta_+$ and $y_2 = x_1 x_2 \zeta_-$ where $\zeta_\pm = \zeta_\pm(x_1, x_2)$ are the two complex roots of the polynomial

$$(x_1^2 + x_1 x_2 + x_2^2) \zeta^2 + (x_1 + x_2) \zeta + 1 = 0.$$ 

If $p, q \in \mathbb{C}^2$ are points satisfying $\Phi(p) = \Psi(q)$ but $p \neq -q$, then $\Phi : (\mathbb{C}^2, p) \rightarrow S$ and $\Psi : (\mathbb{C}^2, q) \rightarrow S$ are two distinct parametrizations of $S$ at $\Phi(p) = \Psi(q)$. Hence $S$ is a double-translation hypersurface with

$$W_S \simeq W(x_1, x_2, x_1 x_2 \zeta_-, x_1 x_2 \zeta_+).$$

### 4.4.1 Examples

Example 4.4.1 is a particular instance of the general construction presented below.

Let $C \subset \mathbb{P}^n$ be a reduced non-degenerate projective curve of degree $2n$. Assume that $h^0(\omega_C) \geq n + 1$. If $C$ is $\mathcal{W}$-generic then $h^0(\omega_C) = n + 1$, but otherwise $h^0(\omega_C)$ can be larger.

Let $H_0$ be a hyperplane intersecting $C$ transversely at $2n$ points. As usual, consider the maps $p_1, \ldots, p_{2n} : (\mathbb{P}^n, H_0) \rightarrow C$ satisfying $C \cdot H = \sum_{i=1}^{2n} p_i(H)$ for every $H \in (\mathbb{P}^n, H_0)$. Since $C$ is non-degenerate, it is harmless to assume that the maps $p_i$ have been indexed in such a way that $p_1(H), \ldots, p_n(H)$ generate $H$ for every $H \in (\mathbb{P}^n, H_0)$.

If $\omega^1, \ldots, \omega^{n+1}$ are linearly independent abelian differentials on $C$ then for each $i \in \mathbb{N}$ consider the maps $\phi_i, \psi_i : (\mathbb{P}^n, H_0) \rightarrow \mathbb{C}^{n+1}$ defined as

$$\phi_i(H) = \left( \int_{p_i(H)} \omega^1, \ldots, \int_{p_i(H)} \omega^{n+1} \right)$$

and

$$\psi_i(H) = - \left( \int_{p_{i+n}(H)} \omega^1, \ldots, \int_{p_{i+n}(H)} \omega^{n+1} \right).$$

Notice that their sums, that is $\Phi = \sum_{i=1}^n \phi_i$ and $\Psi = \sum_{i=1}^n \psi_i$, satisfy $\Phi(H) = \Psi(H)$ for every $H \in (\mathbb{P}^n, H_0)$ according to Abel’s addition Theorem. Furthermore, they parametrize a non-degenerate hypersurface $S_C$, and $W_\Phi \boxtimes W_\Psi$ is a $2n$-web equivalent to $W_C$. From all
that have been said it is clear that $S_C$ is a double translation hypersurface. By definition, it is a **double-translation hypersurface associated to** $C$ at $H_0$. The use of the indefinite article *an* instead of the definite one *the* is due to the lack of uniqueness of the hypersurface for $C$ and $H_0$ fixed. It will depend on the choice of the 1-forms and of the points $p_i$ in general. When $C$ is irreducible, a monodromy argument shows it is possible to replace the *an* by a *the*.

### 4.4.2 Abel-Jacobi map

The double translation hypersurface associated to an irreducible non-degenerate projective curve $C \subset \mathbb{P}^n$ of degree $2n$ and arithmetic genus $n + 1$ admits a more intrinsic description which has the advantage of being global. It is defined in terms of the Abel-Jacobi map of $C$. Although much of the discussion can be carried out in greater generality, this will not be done here. The interested reader can consult [76].

If $C_{sm}$ stands for the smooth part of $C$ then there is a linear map from $H_1(C_{sm}, \mathbb{Z})$ to $H^0(C, \omega_C)^*$ defined as

$$\gamma \mapsto \left( \omega \mapsto \int_{\gamma} \omega \right).$$

It can be shown that its image $\Gamma$, is a discrete subgroup of $H^0(C, \omega_C)^*$. Consequently, the quotient of $H^0(C, \omega_C)^*$ by $\Gamma$ is a smooth complex variety $J(C)$: the **Jacobian** of $C$. When $C$ is smooth then $J(C)$ is indeed projective, as was shown by Riemann.

Once a point $p \in C_{sm}$ and a positive integer $k$ are fixed, one can consider the map

$$AJ_C^k : (C_{sm})^k \longrightarrow J(C)$$

$$(x_1, \ldots, x_k) \mapsto \left( \omega \mapsto \sum_{i=1}^{k} \int_{p}^{x_i} \omega \right)$$

where the integrations are performed along paths included in $C_{sm}$. Notice that $AJ_C^k$ is well-defined since all possible ambiguities disappear after taking the quotient by $\Gamma$. By definition, $AJ_C^k$ is the *k*-th **Abel-Jacobi map** of $C$. 
If $C$ is a smooth projective curve of genus $n + 1$ then a classical theorem of Riemann (see [6, Chap.I.§5]) asserts that the image of the $n$-th Abel-Jacobi map is a translate of the theta divisor $\Theta$ of $J(C)$, which is, by definition, the reduced divisor defined as the zero locus of Riemann’s theta function $\theta$ of the Jacobian $J(C)$\(^5\).

**Proposition 4.4.2.** If $C \subset \mathbb{P}^n$ is a smooth non-degenerate curve of degree $2n$ and genus $n + 1$ then the double-translation hypersurface $S_C$ associated to $C$ is nothing but the lift to $\mathbb{C}^{n+1}$ of (a translate of) the theta divisor $\Theta \subset J(C)$.

**Proof.** From Riemann’s result referred above it suffices to show that $S_C$ can be identified with the image of the $n$-th Abel-Jacobi map.

Since $C$ is non-generated $P = (p_1, \ldots, p_n) : (\mathbb{P}^n, H_0) \to \mathbb{C}^n$ is a germ of biholomorphism. Moreover, $\Phi$ is, up to a suitable choice of affine coordinates, equal to $AJ^n_C \circ P$ (or rather, its natural lift to $H^0(C, \Omega^1_C) \cong \mathbb{C}^{n+1}$). The proposition follows. \hfill \Box

When $C$ is smooth, it is known that the lift of $\Theta$ in $\mathbb{C}^{n+1}$ is a transcendental hypersurface. When $C$ is singular, the hypersurface $S_C$ is not necessarily transcendent. Loosely speaking, the more $C$ is singular, the less $S_C$ is transcendent. For instance, we let the reader verify that the rational double-translation surface of Example 4.4.1 is nothing but the surface $S_C$ associated to the plane singular rational quartic parametrized by $\mathbb{P}^1 \ni [s : t] \mapsto [s^4 : st^3 : t^4] \in \mathbb{P}^2$ (see also Example 3.2.4 in Chapter III).

### 4.4.3 Classification

**Theorem 4.4.3.** Let $S \subset \mathbb{C}^{n+1}$ be a non-degenerate double-translation hypersurface such that $W_S$ is smooth. Then $S$ is the double-translation hypersurface associated to a $W$-generic projective curve in $\mathbb{P}^n$ of degree $2n$ and arithmetic genus $n + 1$.

**Proof.** Let $\Phi$ and $\Psi$ be two distinct translation-type parametrizations of a double-translation hypersurface $S \subset \mathbb{C}^{n+1}$. The coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ in which $\Phi$ and $\Psi$ are expressed can be

\(^5\)The Riemann’s theta function $\theta_A$ of a polarized abelian variety $A = \mathbb{C}^g/\Delta$ with $\Delta = (I_g, Z)$ (where $Z \in M_g(\mathbb{C})$ is such that $Z = Z^t$ and $\text{Im} Z > 0$) is defined by $\theta_A(z) = \sum_{m \in \mathbb{Z}^g} \exp (i\pi (m, Zm) + 2i\pi (m, z))$ for all $z \in \mathbb{C}^g$ (see [6, Chap.I]).
thought as holomorphic functions on $S$ defining the web $\mathcal{W}_S$. Hence the identity

$$\sum_{i=1}^{n} \Phi(x_i(p)) + \sum_{i=1}^{n} \Psi(y_i(p)) = 0 \in \mathbb{C}^{n+1}$$

holds at any point $p \in S$. Since $S$ is non-degenerate this equation provides $n + 1 = \pi(n, 2n)$ linearly independent abelian relations for $\mathcal{W}_S$. Theorem 4.3.4 implies the result.

When $\mathcal{W}_S$ is only assumed to be quasi-smooth, one has the following algebraization result which can be traced back to Wirtinger [111].

**Theorem 4.4.4.** Let $S \subset (\mathbb{C}^{n+1}, 0)$ be a double-translation hypersurface. Assume that its distinguished parametrizations $\Phi$ and $\Psi$ are such that

\begin{equation}
\text{(⋆) none of the vectors } \frac{d^2 \phi_i}{dx_i^2}(0), \frac{d^2 \psi_i}{dy_i^2}(0) \text{ is tangent to } S \text{ at the origin.}
\end{equation}

Then $S$ is the double-translation hypersurface associated to a non-degenerate curve $C \subset \mathbb{P}^n$ of degree $2n$ and such that $h^0(\omega_C) \geq n + 1$.

For a proof of the above result, the reader is redirected to [76] from where the formulation above has been borrowed. There he will also find the following application to the Schottky Problem: characterize the Jacobian of curves among the principally polarized abelian varieties.

**Theorem 4.4.5.** Let $(A, \Theta)$ be a principally polarized abelian variety of dimension $n$. Suppose that there exists a point $p \in \Theta$ such that the germ of $\Theta$ at $p$ is a double-translation hypersurface satisfying (⋆). Then $(A, \Theta)$ is the canonically polarized Jacobian of a smooth nonhyperelliptic curve of genus $n$.

For more information about the Schottky Problem the reader is urged to consult [12], [79, Appendix, Lecture IV] and the references therein.
Chapter 5

Algebraization of maximal rank webs

This chapter is devoted to the following result.

**Theorem.** Let $n \geq 3$ and $k \geq 2n$ be integers. If $W$ is a smooth $k$-web of maximal rank on $(\mathbb{C}^n, 0)$ then $W$ is algebraizable.

Its proof crowns the efforts spreaded over at least three generations of mathematicians. For $n = 3$, the theorem is due to Bol and is among the deepest results obtained by Blaschke’s school. For $n > 3$, Chern and Griffiths provided a proof in [30] which later on [31] revealed to be incomplete. The definitive version stated above is due to Trépreau [107]. He not just followed Chern-Griffiths’s general strategy, but also simplified it, to prove the general case.

While many of the concepts and ideas used in the proof — Poincaré’s map and canonical maps — have already been introduced in Chapter 4, it will be essential to consider also another map, here called Poincaré-Blaschke’s map, canonically attached to webs of maximal rank. This map was originally introduced by Blaschke in [16] to prove that 5-webs of maximal rank are algebraizable. Using Blaschke’s ideas introduced in this paper, Bol succeeded to establish the algebraization of smooth $k$-webs of maximal rank on $(\mathbb{C}^3, 0)$, for $k \geq 6$. Ironically, not much latter [21], he came up with $6 = \pi(2, 5)$
linearly independent abelian relations for his 5-web $B_5$, showing in this way that his source inspiration [16] was irremediably flawed. Although wrong, Blaschke’s paper contained not just the germ of Bol’s algebraization result for webs on $(\mathbb{C}^3, 0)$, but also the germ of Chern-Griffiths’s strategy.

The main novelty in Trépreau’s approach is based on ingenious and involved, albeit elementary, computations. It is still considerably more technical than the other results previously presented in this book. Nevertheless the authors believe that the understanding of this praiseworthy theorem, as well as of the ideas/techniques involved in its proof, will reward those who persevere through this chapter.

## 5.1 Trépreau’s Theorem

Below, a more precise version of the theorem stated in the introduction of this chapter is formulated. Then the heuristic behind its proof is explained.

Arguably, one could complain about the unfairness of the title of this section. It would be perhaps more righteous to call it Bol-Trépreau’s Theorem or even Blaschke-Bol-Chern-Griffiths-Trépreau’s Theorem. To justify the choice made above, one could invoke the right to typographical beauty and/or the usual mathematical practice.

### 5.1.1 Statement

**Theorem 5.1.1.** Let $W$ be a smooth $k$-web on $(\mathbb{C}^n, 0)$. If $n \geq 3$, $k \geq 2n$ and

$$\dim \frac{A(W)}{F^2 A(W)} = 2k - 3n + 1$$

then $W$ is algebraizable.

To explain the heuristic behind the proof of Theorem 5.1.1 some of the geometry of Castelnuovo curves will be recalled in the Section 5.1.2. Then in Section 5.1.3 it will be discussed how one could infer corresponding geometrical properties for webs of maximal rank using their spaces of abelian relations.
5.1.2 The geometry of Castelnuovo curves

Let $C \subset \mathbb{P}^n$ be a Castelnuovo curve of degree $k \geq 2n$. To avoid technicalities assume $C$ is smooth. By definition $K_C = \omega_C$ and $h^0(K_C) = \pi(n, k) = \pi$. Recall from Chapter 4 that the canonical map of $C$ is

$$\varphi = \varphi|_{K_C} : C \longrightarrow \mathbb{P}H^0(C, K_C)^* = \mathbb{P}^{\pi-1}$$

$$x \mapsto [\eta \mapsto \eta(x)].$$

Since $C$ is smooth of genus strictly greater than one, the canonical linear system $|K_C|$ has no base point (cf. [62, IV.5]) thus $\varphi = \varphi|_{K_C}$ is a birational morphism from $C$ onto its canonical model $C_{can} = \varphi(C)$.

Recall from Section 3.3 that the linear system $|I_C(2)|$ of quadrics containing $C$ cut out a non-degenerate surface $S \subset \mathbb{P}^n$ of minimal degree $n-1$. Thus, $S$ is a rational normal scroll, or the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$.

Still aiming at simplicity, assume $S$ is smooth. Because $S$ is rational, it does not have holomorphic differentials. More precisely, $h^0(S, \Omega^1_S) = h^0(S, \Omega^2_S) = 0$. Consequently,

$$h^1(S, K_S) = h^1(S, \mathcal{O}_S) = 0 \quad \text{by Serre duality [62, III.7]}$$

$$= h^0(S, \Omega^1_S) = 0 \quad \text{by Hodge theory}.$$

From the exact sequence

$$0 \rightarrow K_S \rightarrow K_S \otimes \mathcal{O}_S(C) \rightarrow K_C \rightarrow 0$$

one deduces an isomorphism

$$H^0(S, K_S \otimes \mathcal{O}_S(C)) \simeq H^0(C, K_C).$$

Thus the canonical map (5.1) extends to $S$: there is a rational map $\Phi : S \dashrightarrow \mathbb{P}^{\pi-1}$ fitting into the commutative diagram below.

\[
\begin{array}{ccc}
C & \xrightarrow{\varphi} & \mathbb{P}^{\pi-1} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\Phi} & \mathbb{P}^{\pi-1}
\end{array}
\]

*In fact, one can prove that $C$ is not hyperelliptic. Thus $|K_C|$ induces an isomorphism $C \simeq C_{can}$, see [62, IV.5].
Moreover, as it is explained in [61], $X_C$, the image of $\Phi$, is a non-degenerate algebraic surface in $\mathbb{P}^{n-1}$.

If $H$ is a generic hyperplane in $\mathbb{P}^n$, then the hyperplane section $C_H = S \cap H$ is irreducible, non-degenerate in $H$, and of degree $\deg C_H = \deg S = n - 1$. Hence $C_H$ is a curve of minimal degree in $H$, that is a rational normal curve of degree $n - 1$. Now let $p_1, \ldots, p_n$ be $n$ generic points on $S$. They span a hyperplane $H_p$ and the generic hyperplane is obtained in this way. Thus $C_H = S \cap H_p$ is a rational normal curve of degree $n - 1$ that contains the points $p_1, \ldots, p_n$ and is contained in $S$. Therefore, through $n$ general points of $S$ passes a rational normal curve of degree $n - 1$. Surfaces having this property are said to be $n$-covered by rational normal curves of degree $n - 1$.

Figure 5.1: The curves $C_H$. 
SECTION 5.1: TRÉPRAU’S THEOREM

It can be proved that the image under $\Phi$ of $C_H$ is a rational normal curve $\mathcal{C}_H$ in a projective subspace of $\mathbb{P}H^0(C, K_C)^*$ of dimension $k - n - 1$. Consequently, the surface $X_C = \text{Im } \Phi \subset \mathbb{P}^{n-1}$ is $n$-covered by rational normal curves of degree $k - n - 1$.

![Figure 5.2: The curves $C_H$ and their images $\mathcal{C}_H$ under $\Phi$.](image)

The preceding facts will now be interpreted in terms of the web $W_C$ dual to the curve $C$. As usual, let $H_0 \subset \mathbb{P}^n$ be a hyperplane transverse to $C$, and let $p_1, \ldots, p_d : (\mathbb{P}^n, H_0) \to C$ be the usual holomorphic maps describing the intersection of $H \in (\mathbb{P}^n, H_0)$ with $C$. For $i \in \mathbb{B}$, set

$$\varphi_i = \varphi|_{K_C} \circ p_i : (\mathbb{P}^n, H_0) \to \mathbb{P}H^0(C, K_C)^*.$$  

If $H \in (\mathbb{P}^n, H_0)$ then the points $p_1(H), \ldots, p_d(H)$ span the hyperplane $H \subset \mathbb{P}^n$. Thus they belong to $C$, and hence to $S$. Therefore each $p_i(H)$ belongs to the rational normal curve $C_H = C \cap H \subset S$. 
Consequently the points $\varphi_i(H), \, i \in k$, belong to the rational normal curve $C_H \subset \mathbb{P}H^0(C, K_C)^*$. Moreover, the curves $C_H$ for $H \in (\mathbb{P}^n, H_0)$ fill out a germ of surface along $C_{H_0}$.

5.1.3 On the geometry of maximal rank webs

This section explains how the constructions presented in the preceding section extend to webs carrying sufficiently many abelian relations. Since the case $k = 2n$ has already been treated in Chapter 4, it will be assumed that $k > 2n$.

Let $W$ be a smooth $k$-web on $(\mathbb{C}^n, 0)$. To simplify the discussion, it will be assumed that $W$ has maximal rank, even if the heuristic described below will be implemented under the weaker hypothesis of Theorem 5.1.1.

Using the hypothesis on the space of abelian relations of $W$, it can be proved that the images $\kappa_{W,1}(x), \ldots, \kappa_{W,k}(x)$ of every $x \in (\mathbb{C}^n, 0)$ by the canonical maps of $W$ generate a projective subspace $\mathbb{P}^{k-n-1}(x) \subset \mathbb{P}A(W)^*$ of dimension $k - n - 1$, and lie on a unique rational normal curve $C(x) \subset \mathbb{P}^{k-n-1}(x)$.

The most delicate point, and the main novelty, in Trépreau’s argument, is his proof that the family of rational normal curves $C_W = \{C(x)\}_{x \in (\mathbb{C}^n, 0)}$ fills out a germ of non-degenerate, smooth surface $X_W \subset \mathbb{P}A(W)^*$.

It is then comparably simple to show that any two curves $C(x), C(y)$ intersect in exactly $n - 1$ points. Therefore $C(0)^2 > 0$, and well known results about germs of surfaces containing curves of positive self-intersection by Andreotti (in the analytic category) and Hartshorne (in the algebraic category) imply that $X_W$ is contained in a projective surface $S_W$. As the image under $\Phi$ of the surface of minimal degree in $\mathbb{P}^n$ containing a Castelnuovo curve $C$ of degree $k$ (see previous section), the surface $S_W$ is a rational surface $n$-covered by rational normal curves of degree $k - n - 1$.

At this point, one can apply an argument by Chern-Griffiths to linearize $W$. Since $S_W$ is rational, every in the family $C_W$ belongs to one and only linear system $|C| = \mathbb{P}H^0(S_W, \mathcal{O}_{S_W}(C))$, which turns out to have dimension $n$. One then defines a map sending $x \in (\mathbb{C}^n, 0)$ to the point in $|C|$ corresponding to the rational normal curve $C(x)$. 
It is possible to prove that this map is a biholomorphism which linearizes $\mathcal{W}$. Finally, the converse of Abel’s Theorem presented in Chapter 4 allows to conclude that $\mathcal{W}$ is algebraizable.

### 5.2 Maps naturally attached to $\mathcal{W}$

Until the end of this chapter $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ is a smooth $k$-web on $(\mathbb{C}^n, 0)$ satisfying

$$\dim \mathcal{A}(\mathcal{W})/F^2\mathcal{A}(\mathcal{W}) = 2k - 3n + 1 = \pi.$$

According to this hypothesis there exist $\pi$ linearly independent abelian relation with classes in $\mathcal{A}(\mathcal{W})/F^2\mathcal{A}(\mathcal{W})$ generating the whole space. Fix, once and for all, $\pi$ abelian relations $\eta^{(1)}, \ldots, \eta^{(\pi)}$ with this property, and let $\mathcal{A}_2(\mathcal{W}) \subset \mathcal{A}(\mathcal{W})$ be the vector space generated by them.
CHAPTER 5: ALGEBRAIZATION

Notice that the inclusion $\mathcal{A}_2(W) \subset \mathcal{A}(W)$ induces a linear projection $\mathbb{P}\mathcal{A}(W)^* \rightarrow \mathbb{P}\mathcal{A}_2(W)^*$. Notice also that the intersection of the filtration $F^* \mathcal{A}(W)$ of $\mathcal{A}(W)$ with $\mathcal{A}_2(W)$ induces the filtration

$$F^* \mathcal{A}_2(W) = \mathcal{A}_2(W) \cap F^* \mathcal{A}(W).$$

From the choice of $\mathcal{A}_2(W)$ it is clear that

$$\dim F^1 \mathcal{A}_2(W) = k - 2n + 1 \quad \text{and} \quad \dim F^j \mathcal{A}_2(W) = 0 \quad \text{for} \quad j > 1.$$

### 5.2.1 Canonical maps

Fix $i \in \mathbb{k}$. If $\kappa_i = \kappa_{\mathcal{W},i} : (\mathbb{C}^n, 0) \rightarrow \mathbb{P}\mathcal{A}(W)^*$ is the $i$-th canonical map of $W$, see Section 4.3.5, then its image does not intersect the center of the natural projection $\mathbb{P}\mathcal{A}(W)^* \rightarrow \mathbb{P}\mathcal{A}_2(W)^*$. Indeed, as argued in the proof of Lemma 4.3.1, the equality $\dim \mathcal{A}_2(W)/F^1 \mathcal{A}_2(W) = k - n$ implies the existence of an abelian relation in $\mathcal{A}_2(W)$ with $i$-th component not vanishing at 0. Thus composing $\kappa_i$ with the linear projection $\mathbb{P}\mathcal{A}(W)^* \rightarrow \mathbb{P}\mathcal{A}_2(W)^*$ defines a morphism from $(\mathbb{C}^n, 0)$ to $\mathbb{P}\mathcal{A}_2(W)^*$. To keep the notation simple it will still be denoted by $\kappa_i$, and will be called the $i$-th canonical map of $W$. More explicitly, $\kappa_i$ is now the map

$$(\mathbb{C}^n, 0) \rightarrow \mathbb{P}\mathcal{A}_2(W)^*$$

$$x \mapsto [ev_i(x) : \mathcal{A}_2(W) \rightarrow \mathbb{C}]$$

The image of the $i$-th canonical map of $W$ will be denoted by $C_i$ and will be called the $i$-th canonical curve of $W$.

### 5.2.2 Poincaré’s map

Consider the natural analogue of Poincaré’s map

$$P_W : (\mathbb{C}^n, 0) \rightarrow \text{Grass}(\mathcal{A}_2(W), k - 2n + 1)$$

$$x \mapsto F^1_x \mathcal{A}_2(W).$$

As in the case of canonical maps, it seems unjustifiable to change the terminology. Therefore the map $P_W$ above will also be called the Poincaré’s map of $W$. 
For each \( x \in (\mathbb{C}^n, 0) \), the projective subspace of dimension \( k-n-1 \) in \( \mathbb{P}A_2(\mathcal{W})^* \) determined by \( P_W(x) \) through projective duality will be denoted by \( \mathbb{P}^{n-k-1}(x) \).

### 5.2.3 Properties

From the very definition of \( F_x^1A_2(\mathcal{W}) \) it follows that

\[
F_x^1A_2(\mathcal{W}) = \bigcap_{i \in \bar{B}} \ker ev_i(x)
\]

where \( ev_i(x) \) is considered as a linear form on \( A_2(\mathcal{W}) \).

This remark about subspaces of \( A_2(\mathcal{W}) \) translates into the following relation between the canonical maps and Poincaré’s map: \( \mathbb{P}^{n-k-1}(x) \) is the smallest projective space among the ones containing the set \( \{\kappa_i(x)\}_{i \in \bar{B}} \). The lemma below shows that it is possible to replace \( \bar{B} \), in the statement above, by any subset \( B \) with at least \( k - n \) elements.

**Lemma 5.2.1.** For every \( x \in (\mathbb{C}^n, 0) \) and every subset \( B \subset \bar{B} \) of cardinality \( k - n \), \( \mathbb{P}^{n-k-1}(x) \) is the smallest linear subspace of \( \mathbb{P}A_2(\mathcal{W})^* \) containing \( \{\kappa_i(x)\}_{i \in B} \).

**Proof.** Notice that the smallest linear subspace of \( \mathbb{P}A_2(\mathcal{W})^* \) containing \( \{\kappa_i(x)\}_{i \in B} \) is the dual of the intersection

\[
I_x = \bigcap_{i \in B} [\ker ev_i(x)] \subset \mathbb{P}A_2(\mathcal{W}) .
\]

If a non-trivial abelian relation, or rather its projectivization, is in \( I_x \) then it has at most \( k - (k - n) = n \) components not vanishing at \( x \). But the constant term of these components are linearly independent because \( \mathcal{W} \) is a smooth web on \( (\mathbb{C}^n, 0) \). Thus

\[
I_x = \bigcap_{i \in B} [\ker ev_i(x)] = [F_x^1A_2(\mathcal{W})] \subset \mathbb{P}A_2(\mathcal{W}) .
\]

In order to express intrinsically the differential of \( \kappa_i \) at a point \( x \in (\mathbb{C}^n, 0) \), observe that the tangent space of \( \mathbb{P}A_2(\mathcal{W})^* \) at the point...
$\kappa_i(x) = [ev_i(x)]$ is naturally isomorphic to the quotient of $A_2(W)^*$ by the the 1-dimensional subspace $Cev_i(x)$. This quotient in its turn is isomorphic to the dual of $\ker ev_i(x)$. Thus the differential of $\kappa_i$ at $x$ can be written as follows

$$d\kappa_i(x) : T_x(C^n,0) \to \ker ev_i(x)^*$$

$$v \mapsto \lim_{t \to 0} \frac{ev_i(x+tv) - ev_i(x)}{t} \{v \mapsto \}.$$

Therefore, inasmuch $\kappa_i(x)$ can be identified with abelian relations in $A_2(W)$ with $i$-component vanishing at $x$, the image of its differential at $x$ can be identified with the abelian relations in $A_2(W)$ with $i$-th component vanishing at $x$ with multiplicity two. In other words, if $ev_i^{(1)}(x) : A_2(W) \to \mathbb{C}^2$ is the evaluation morphism of order one and $V_i(x) \subset A_2(W)$ is its kernel, then the image of $d\kappa_i(x)$ is the quotient of $\ker(A_2(W)^* \to V_i(x)^*)$ by $Cev_i(x)$.

**Lemma 5.2.2.** Let $x \in (C^n,0)$ and $B \subset k$ be a subset of cardinality smaller than or equal to $k - 2n + 1$. If $Y \subset \mathbb{P}A_2(W)^*$ is the set

$$\mathbb{P}^{n-k-1}(x) \cup \left( \bigcup_{i \in B} T_{\kappa_i(x)} C_i \right)$$

then the smallest projective subspace of $\mathbb{P}A_2(W)^*$ containing $Y$ has codimension equal to $(k - 2n + 1) - \text{card}(B)$. In particular, none of the canonical curves $C_i$ is tangent to $\mathbb{P}^{n-k-1}(x)$ at $\kappa_i(x)$.

**Proof.** The proof is similar to the proof of the previous lemma. As in the discussion preceding the statement, let $V_i$ be the kernel of the evaluation morphism of order one $ev_i^{(1)}(x) : A_2(W) \to \mathbb{C}^2$.

Since $\bigcap_{i \in B} \ker ev_i(x) = F^1_x A_2(W)$, to prove the lemma it suffices to show that the dimension of

$$A_x = F^1_x A_2(W) \cap \left( \bigcap_{i \in B} V_i \right)$$

is equal to $a = k - 2n + 1 - \text{card}(B)$. Since $\dim F^1_x A_2(W) = k - 2n + 1$ and $\dim F^1_x A_2(W)/(F^1_x A_2(W) \cap V_i(x)) \geq 1$, the number $a$ is greater than or equal to $k - 2n + 1 - \text{card}(B)$. 
Notice that the elements of $A_x$ are abelian relations with $i$-th components, for every $i \in B$, having constant and linear terms at $x$ equal to zero. Therefore,

$$a \leq (k - \text{card}(B)) - \ell^2 \left( \bigotimes_{i \in k \setminus B} F_i \right) + \dim F^2 A_2(W).$$

But $\dim F^2 A_2(W) = 0$ and Proposition 2.2.1 implies

$$\ell^2 \left( \bigotimes_{i \in k \setminus B} F_i \right) \geq 2(n - 1) + 1.$$  

Thus

$$a \leq (k - \text{card}(B)) - 2(n - 1) - 1 = k - 2n + 1 - \text{card}(B),$$

as wanted.

**Remark 5.2.3.** The proof above shows slightly more than what is stated. Indeed, it was proved that not just $C_i$ is smooth and not tangent to $P^{n-k-1}(x)$ at $\kappa_i(x)$ but also that $\kappa_i$ is a submersion onto $C_i$. This fact will be used in the proof of the next proposition and later on.

Lemma 5.2.2 for subsets $B \subset k$ of cardinality one is, together with Lemma 5.2.1, the main ingredient in the proof of the next proposition.

**Proposition 5.2.4.** Poincaré’s map $P_W$ is an immersion.

**Proof.** Let $\gamma : (C, 0) \to (C^n, 0)$ be a holomorphic immersion. Since $W$ is smooth, $\gamma$ is tangent to at most $n - 1$ of the foliations $F_i$. Thus there exists a set $B \subset k$ of cardinality $k - n \leq k - (n - 1)$, such that the composition $\kappa_i \circ \gamma$ is an immersion for every $i \in B$. Moreover, Lemma 5.2.1 implies that for every $t \in (C, 0)$, the points $\{ (\kappa_i \circ \gamma)(t) \}_{i \in B}$ generate the projective subspace $\mathbb{P}^{n-k-1}(\gamma(t)) \subset \mathbb{P}A_2(W)^*$ determined by $P_W(\gamma(t))$.

If Grass($A_2(W), k - 2n + 1$) is identified with its Plücker’s embedding\(^{\dagger}\) of Grass($A_2(W)^*, k - n$) then one can write

$$(\tilde{P}_W \circ \gamma)(t) = \bigwedge_{i \in B} (\tilde{\kappa}_i \circ \gamma)(t),$$

\(^{\dagger}\)The Grassmannian Grass($V, r$) is isomorphic to the projectivization of the
where the hats indicate liftings to $\bigwedge^{k-n} A_2(W)^*$ and $A_2(W)^*$ respectively. Consequently, the identity
\[
(\hat{P}_W \circ \gamma)'(t) \wedge (\hat{\kappa}_j \circ \gamma)(t) = (\hat{\kappa}_j \circ \gamma)'(t) \wedge (\hat{P}_W \circ \gamma)(t)
\]
holds true for every $j \in B$. Lemma 5.2.2 (see also Remark 5.2.3) ensures the non-vanishing of this latter expression. Since $\gamma$ is an arbitrary immersion, the differential of $P_W$ is injective at the origin. The proposition follows.

**Corollary 5.2.5.** For every distinct pair of points $x, y \in (\mathbb{C}^n, 0)$, the intersection $P_{k-n-1}(x) \cap P_{k-n-1}(y)$ is a projective subspace of $\mathbb{P}A_2(W)^*$ of dimension $n - 2$.

**Proof.** It suffices to prove the claim for $x = 0$ and $y$ arbitrarily close to it. Since $2(k - n - 1) - (2k - 3n) = n - 2$, the claim is equivalent to the transversality of $P_{k-n-1}(0)$ and $P_{k-n-1}(y)$. The reader is invited to verify that the lack of transversality between $P_{k-n-1}(0)$ and $P_{k-n-1}(y)$, for $y$ arbitrarily close to 0, would imply that the differential of $P_W$ at the origin is not injective. This contradiction implies the corollary.

### 5.3 Poincaré-Blaschke’s map

In this section the Poincaré-Blaschke’s map for webs satisfying the assumptions of Theorem 5.1.1 are defined, and it is proved that they have rank two when in dimension at least three. Its content is considerably more technical, although rather elementary, than the remaining of the book. The arguments herein follow very closely [107].

image of the multilinear map
\[
\varphi : V^r \rightarrow \bigwedge^r V
\]
\[
(v_1, \ldots, v_r) \mapsto v_1 \wedge \cdots \wedge v_r.
\]

The isomorphism is given of course by associating to any $W \in \text{Grass}(V, r)$ the point $[\varphi(w_1, \ldots, w_r)] \in \mathbb{P} \bigwedge^r V$ where $w_1, \ldots, w_r$ is an arbitrary basis of $W$. Clearly, $[\varphi(w_1, \ldots, w_r)]$ does not depend on the basis chosen. The induced map $\text{Grass}(V, r) \rightarrow \mathbb{P} (\wedge^r V)$ is the so called **Plücker embedding** of $\text{Grass}(V, r)$.
SECTION 5.3: POINCARÉ-BLASCHKE'S MAP

Settling the notation

Let \( u_1, \ldots, u_k : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be submersions defining the foliations \( \mathcal{F}_1, \ldots, \mathcal{F}_k \). Recall that Proposition 2.3.10 settles the existence of a coframe \( \mathcal{W} = (\mathcal{W}_0, \ldots, \mathcal{W}_{n-1}) \) for \( \Omega^1(\mathbb{C}^n, 0) \), and \( k \) holomorphic functions \( \theta_1, \ldots, \theta_k \), such that the foliation \( \mathcal{F}_i \) is induced by the 1-form

\[
\omega_i = \sum_{q=0}^{n-1} (\theta_i)_q \mathcal{W}_q .
\]

Notice also the existence of holomorphic functions \( h_1, \ldots, h_k \) satisfying \( du_i = h_i \omega_i \) for every \( i \in \mathbb{K} \).

Until the end of Section 5.3 the submersions \( u_i \); the coframe \( \mathcal{W} \); the functions \( \theta_i \) and \( h_i \); and the 1-forms \( \omega_i \) will have the same meaning as above.

For an arbitrary 1-form \( \alpha \in \Omega^1(\mathbb{C}^n, 0) \) its \( q \)-th component in the coframe \( \mathcal{W} \) will be written as \( \{\alpha\}_q \). More precisely, the holomorphic functions \( \{\alpha\}_0, \ldots, \{\alpha\}_{n-1} \) are implicitly defined by the identity

\[
\alpha = \sum_{q=0}^{n-1} \{\alpha\}_q \mathcal{W}_q .
\]

To write down the canonical maps \( \kappa_i \) explicitly, identify the point \( (a_1, \ldots, a_\pi) \in \mathbb{P}^{\pi-1} \) with the hyperplane \( \{a_1 \eta_1^{(1)} + \cdots + a_\pi \eta_\pi^{(\pi)} = 0\} \) in \( \mathcal{A}_2(W) \). The \( i \)-th evaluation morphism at a point \( x \in (\mathbb{C}^n, 0) \) is nothing more than

\[
(a_1, \ldots, a_\pi) \mapsto a_1 \eta_1^{(1)}(x) + \cdots + a_\pi \eta_\pi^{(\pi)}(x).
\]

Notice that the 1-forms \( \eta_i^{(j)} \) for \( j \in \mathbb{p} \) are all proportional to \( \omega_i \). Hence there are holomorphic functions \( z_i^{(j)} \) such that

\[
\eta_i^{(j)} = z_i^{(j)} \omega_i.
\]

\footnote{Here the abelian relations \( \eta^{(j)} \) are thought as coordinate functions on \( \mathcal{A}_2(W) \), which is the same as thinking of them as elements of \( \mathcal{A}_2(W)^* \).}
for every \( i \in k \) and every \( j \in \pi \). Therefore, for a fixed \( i \in k \), the map

\[
Z_i : (C^n, 0) \longrightarrow C^n \\
x \longmapsto (z_i^{(1)}(x), \ldots, z_i^{(\pi)}(x))
\]

is a lift of \( \kappa_i \) to \( C^n \). More precisely, the diagram

\[
\begin{array}{ccc}
C^n & \xrightarrow{Z_i} & \mathbb{A}_2(W)^* \\
\downarrow & & \downarrow \\
(C^n, 0) & \xrightarrow{\kappa_i} & \mathbb{P}\mathbb{A}_2(W)^*
\end{array}
\]

commutes.

For further use, the translation of the conditions \( \sum \eta_i = 0 \) and \( d\eta_i = 0 \) to conditions on the functions \( z_i^{(j)} \) is stated below as a lemma. The proof is immediate.

**Lemma 5.3.1.** If \( z_1, \ldots, z_k : (C^n, 0) \rightarrow C \) are holomorphic functions on \((C^n, 0)\) then \( \eta = (z_1 \omega_1, \ldots, z_k \omega_k) \) is an abelian relation of \( W \) if and only if

\[
d(z_i \omega_i) = 0 \quad \text{and} \quad \sum_{i=1}^k z_i(\theta_i)^\sigma = 0 \quad (5.2)
\]

for every \( i \in k \) and every \( \sigma \in \{0, \ldots, n-1\} \).

5.3.1 Interpolation of the canonical maps

For \( i \in k \) consider the polynomials \( P_i \in \mathcal{O}(C^n, 0)[t] \) defined through the formula

\[
P_i(t) = \prod_{j=1}^k (t - \theta_j).
\]

The canonical maps, or more precisely their lifts \( Z_i \) defined above,
are interpolated by the map $Z_*$ defined below.

$$Z_* : (\mathbb{C}^n, 0) \times \mathbb{C} \longrightarrow \mathbb{C}^*$$

$$(x, t) \longmapsto \sum_{i=1}^{k} P_i(t) Z_i(x).$$

Indeed $Z_*(x, \theta_i(x))$ is proportional to $Z_i(x)$ since

$$Z_*(x, \theta_i(x)) = P_i(\theta_i(x)) Z_i(x).$$

Some properties of the map $Z_*$ are collected in the following lemma.

**Lemma 5.3.2.** The map $Z_*$ has the following properties:

(a) for every $x \in (\mathbb{C}, 0)$ and every $t \in \mathbb{C}$, $Z_*(x,t) \neq 0$;

(b) the entries of $Z_*(x,t)$, seen as polynomials in $\mathcal{O}(\mathbb{C}^n, 0)[t]$, have degree at most $k - n - 1$;

(c) the coefficient of $t^{k-n-1}$ in $Z_*(x,t)$ is non-zero and equal to

$$\sum_{i=1}^{k} \theta_i(x)^n Z_i(x).$$

**Proof.** To prove item (a), let $(x_0, t_0) \in (\mathbb{C}^n, 0) \times \mathbb{C}$ be a point where $Z_*$ vanishes. If $t_0 = \theta_i(x_0)$ for some $i \in \mathbb{K}$, then clearly $Z(x_0, t_0) = P_i(\theta_i(x_0)) Z_i(x_0) \neq 0$.

Assume now that $t_0$ belongs to $\mathbb{C} \setminus \{\theta_1(x_0), \ldots, \theta_k(x_0)\}$. Because $\ell^1(W) = k - n$, Lemma 5.3.1 implies that every relation of the form $\sum c_i Z_i(x_0) = 0$ is a linear combination of the relations $\sum_i^k(\theta_i(x_0))^{\sigma} Z_i(x_0)$ for $\sigma = 0, \ldots, n - 1$.

If $\sum_i(t - \theta_i(x_0))^{-1} Z_i(x_0) = 0$, then there exist $\mu_1, \ldots, \mu_n \in \mathbb{C}$ which satisfy

$$\frac{1}{t_0 - \theta_i(x_0)} \equiv \sum_{\sigma=1}^{n} \mu_\sigma \theta_i(x_0)^\sigma$$

for every $i \in \mathbb{K}$. But this is not possible, since $\theta_i(x_0) \neq \theta_j(x_0)$ whenever $i \neq j$. Hence (a) holds true.
To prove item (b), the dependence in $x$ will be dropped from the notation in order to keep it simple. Let $P(t) = \prod_{j=1}^{k}(t - \theta_j)$ and write

$$P_i(t) = \prod_{j=1}^{k-1}\sigma_j^{(i)} t^j \quad \text{and} \quad P(t) = \sum_{j=1}^{k} \sigma_j t^j. \tag{5.3}$$

Comparing coefficients in the identities $P(t) = (t - \theta_i)P_i(t)$, one deduces that

$$\sigma_{j+1} = \sigma_j^{(i)} - \theta_i \sigma_{k+1}. \quad \text{Consequently,} \quad \sigma_j^{(i)} = \sum_{s=0}^{k-j-1} (\theta_i)^s \sigma_{j+s+1}. \tag{5.4}$$

From Equation (5.4), it follows that

$$Z_\ast(t) = \sum_{j=0}^{k-1} \left( \sum_{i=1}^{k} \sigma_j^{(i)} Z_i \right) t^j$$

$$= \sum_{j=0}^{k-1} \left( \sum_{i=1}^{k} \sum_{s=0}^{k-j-1} (\theta_i)^s \sigma_{j+s+1} Z_i \right) t^j$$

$$= \sum_{j=0}^{k-1} \left( \sum_{s=0}^{k-j-1} \left( \sum_{i=1}^{k} (\theta_i)^s Z_i \right) \sigma_{j+s+1} \right) t^j. \tag{5.5}$$

According to Lemma 5.3.1, $\sum_i Z_i(\theta_i)^s$ is identically zero for any $s \in \{0, \ldots, n-1\}$. Thus the coefficient of $t^j$ in (5.5) is identically zero for every $j \geq k - n$. Item (b) follows.

The coefficient of $t^{k-n-1}$ in $Z_\ast(t)$ is

$$\sum_{s=0}^{n} \left( \sum_{i=1}^{k} (\theta_i)^s Z_i \right) \sigma_{k-n+s} = \sigma_k \sum_{i=1}^{k} (\theta_i)^n Z_i$$

with the equality being obtained through the use of Lemma 5.3.1 exactly as above. To conclude the proof of Item (c) it suffices to notice that $\sigma_k = 1$ according to (5.3).
5.3.2 Definition of Poincaré-Blaschke’s map

The Poincaré-Blaschke’s map\(^5\) of \(\mathcal{W}\),
\[ \text{PB}_W : (\mathbb{C}^n, 0) \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{n-1}, \]
is defined as
\[
\text{PB}_W(x, t) = \begin{cases} 
\left[ Z_\ast(x, t) \right] & \text{for } t \in \mathbb{C} \\
\left[ \sum_{i=1}^k \theta_i(x)^n Z_i(x) \right] & \text{for } t = \infty.
\end{cases}
\]

Observe that the definition of \(Z_\ast\) does depend on the choice of: the subspace \(A_2(\mathcal{W}) \subset A(\mathcal{W})\); on the basis of \(A_2(\mathcal{W})\); and on the adapted coframe \(\varpi_0, \ldots, \varpi_{n-1}\). Nevertheless, modulo projective changes of coordinates on the target \(\mathbb{P}^{n-1}\) and on the \(\mathbb{P}^1\) factor of the source, \(\text{PB}_W\) only depends on the choice of the subspace \(A_2(\mathcal{W}) \subset A(\mathcal{W})\) as the reader can easily verify.

Notice that Poincaré-Blaschke’s map restricted at \(\{x\} \times \mathbb{P}^1\) parametrizes a rational curve which interpolates the canonical points \(\kappa_1(x), \ldots, \kappa_k(x)\). More precisely,

**Lemma 5.3.3.** For \(x \in (\mathbb{C}^n, 0)\) fixed, the map \(\varphi_x : t \in \mathbb{P}^1 \mapsto \text{PB}_W(x, t) \in \mathbb{P}^{n-1}\) is an isomorphism onto a rational normal curve \(\mathcal{C}(x) \subset \mathbb{P}^{n-k-1}(x)\) of degree \(k - n - 1\).

**Proof.** Clearly the map under scrutiny parametrizes a rational curve \(\mathcal{C}(x)\). Moreover \(\varphi_x(\theta_i(x)) = [Z_i(x)] = \kappa_i(x)\) for every \(i \in \mathbb{P}^1\), and
\[
\dim \langle \mathcal{C}(x) \rangle \geq \dim \langle Z_1(x), \ldots, Z_k(x) \rangle - 1.
\]
But the span of \(Z_1(x), \ldots, Z_k(x)\) has dimension \(\ell^1(\mathcal{W}) = k - n\).

Lemma 5.3.2 tells that the map \(Z_\ast\) has degree \(k - n - 1\) in \(t\). Hence \(\varphi_x\) parametrizes a non-degenerate rational curve in \(\mathbb{P}^{n-k-1}(x)\) of degree at most \(k - n - 1\). It follows from Proposition 2.3.11 that \(\mathcal{C}(x)\) is a rational normal curve of degree \(k - n - 1\).

---

\(^5\)Such map was first introduced by Blaschke extrapolating ideas of Poincaré [96]. In [16], Blaschke constructs and studies the Poincaré-Blaschke map of a maximal rank planar 5-web. He mistakenly asserted that its image lie in a surface of \(\mathbb{P}^5\).
5.3.3 The rank of Poincaré-Blaschke’s map

This section is devoted to the proof of the following proposition.

**Proposition 5.3.4.** If \( n > 2 \) then \( PB_W \) has rank 2 at every \((x,t) \in (\mathbb{C}^n,0) \times \mathbb{P}^1\).

**Lower bound for the rank**

It is not hard to show that the rank of \( PB_W \) is at least two as the result below shows.

**Lemma 5.3.5.** For every \((x,t) \in (\mathbb{C}^n,0) \times \mathbb{P}^1\), the rank of \( PB_W \) at \((x,t)\) is at least two. Moreover, if \( t = \theta_i(x) \) for some \( x \in (\mathbb{C}^n,0) \), then \( PB_W(x,t) \) has rank exactly two at \((x,t)\).

**Proof.** According to Lemma 5.3.3, \( PB_W \) restricted to the line \( \{x\} \times \mathbb{P}^1 \) is an isomorphism onto the rational normal curve \( \mathcal{C}(x) \subset \mathbb{P}^{n-k-1}(x) \). In particular, the tangent line to \( \mathcal{C}(x) \) at \( PB_W(x,t) \), namely

\[
\left\langle PB_W(x,t), \frac{\partial PB_W}{\partial t}(x,t) \right\rangle
\]

is contained in \( \mathbb{P}^{k-n-1}(x) \).

The restriction of \( PB_W \) to the hypersurface \( H_i = \{t = \theta_i(x)\} \) is a submersion onto the \( i \)-th canonical curve of \( C_i \). Thus the image of the differential of \( PB_W|_{H_i} \) at \((x,\theta_i(x))\) is \( T_{\nu_i(x)}C_i \). Lemma 5.2.2 implies the rank of \( PB_W \) is exactly two at any point of the hypersurface \( H_i \).

If \( P(x,t) \neq 0 \), that is if \((x,t) \notin \bigcup_{i \in \mathbb{Z}} H_i \), then one can deduce from Lemma 5.2.2 that the vectors

\[
\frac{\partial}{\partial x_j} \left( \sum_{i=1}^{k} P_i(x,t)Z_i(x) \right) \quad \text{with} \quad j \in \mathbb{Z}
\]

span the whole space \( \mathbb{C}^n \). Details are left to the reader. \( \square \)

To prove that the rank of \( PB_W \) is at most two is considerably more delicate as the next few pages testify.
SECTION 5.3: POINCARÉ-BLASCHKE’S MAP

The main technical point

The next result is essential in the proof of Proposition 5.3.4.

Proposition 5.3.6. There are germs of holomorphic functions $M^p$ determined by the coframe $\varpi$ such that for any abelian relation $(z_1 \omega_1, \ldots, z_k \omega_k) \in A(W)$, for every $p \in \{0, \ldots, n-2\}$, and every $i \in k$, the following identity holds true

$$\{dz_i\}^{p+1} - \theta_i \{dz_i\}^p - z_i \{d\theta_i\}^p = z_i \sum_{r=0}^{n-1} (\theta_i)^r M^p$$  \hspace{1cm} (5.6)

Moreover, if $n > 2$ then

$$\{d\theta_i\}^{p+1} - \theta_i \{d\theta_i\}^p = \sum_{\rho=0}^{n} (\theta_i)^\rho N^p$$  \hspace{1cm} (5.7)

where $N^p$ are holomorphic functions also determined by $\varpi$.

Remark 5.3.7. Let $\theta$ be any function on $(\mathbb{C}^n, 0)$ and set $\omega_\theta = \sum_q \theta^q \omega_q$. If $\omega_\theta$ is integrable then, after writing down the coefficients of $\omega_\theta \wedge \omega_\theta \wedge \omega_r$ in $\omega_\theta \wedge d\omega_\theta$ and imposing their vanishing, one deduces relations of the form

$$\{d\theta\}^{p+1} - \theta \{d\theta\}^p = \sum_{\rho=0}^{n+p} \theta^\rho N^p$$  \hspace{1cm} (5.8)

for $p = 0, \ldots, n-2$, where $N^p$ are certain holomorphic functions that do not depend on $\theta$ but only on the adapted coframe $\varpi$.

Similarly, one can prove that there are holomorphic functions $M^p$ depending only on $\varpi$ such that if $\omega_\theta$ is integrable, then any $z$ such that $d(z \omega_\theta) = 0$ necessarily verifies the relations (for any $p = 0, \ldots, n-2$):

$$\{dz\}^{p+1} - \theta \{dz\}^p - z \{d\theta\}^p = z \sum_{\rho=0}^{n+p} \theta^\rho M^p.$$  \hspace{1cm} (5.9)

Relations (5.8) and (5.9) are direct consequences of the integrability condition and have nothing to do with webs and/or their abelian relations. Proposition 5.3.6 improves these relations by lowering down the upper limit of both summations.
CHAPTER 5: ALGEBRAIZATION

Proof of the main technical point I – Preliminaries

For every $x \in (C^n, 0)$ and every $i \in \mathfrak{k}$, write the Taylor expansion of $u_i$ centered at the origin as

$$u_i(x) = \ell_i(x) + \frac{1}{2}q_i(x) + O_0(3)$$

where $\ell_i$ (resp. $q_i$) are linear (resp. quadratic) forms. Let $\xi = (\xi_1 du_1, \ldots, \xi_k du_k)$ be an abelian relation of $W$. Since $d\xi_i \wedge du_i(0) = 0$, the following identity holds

$$\xi_i(x) = a_i + b_i \ell_i(x) + O_0(2) \quad i \in \mathfrak{k}$$

for suitable complex numbers $a_i, b_i$. Looking at the order one jet at the origin of the relation $\sum_i \xi_i du_i = 0$, one deduces that

$$\sum_{i=1}^k a_i \ell_i = 0 \quad \text{and} \quad \sum_{i=1}^k a_i q_i + \sum_{i=1}^k b_i (\ell_i)^2 = 0. \quad (5.10)$$

Denote by $Q = \mathbb{C}_2[x_1, \ldots, x_n]$ the space of quadratic forms on $\mathbb{C}^n$. For $Q = \sum_{i \leq j} Q^{i,j} x_i x_j \in Q$ define the following differential operators

$$Q(\nabla F) = \sum_{i \leq j} Q^{i,j} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j} \quad \text{and} \quad Q_0(F) = \sum_{i \leq j} Q^{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j}$$

where $F$ is a germ of holomorphic function.

By hypothesis $F^0A(W)/F^1A(W)$ has dimension $k - n$. Therefore for every $a = (a_1, \ldots, a_d) \in \mathbb{C}^d$, the following implication holds true

$$\sum_{i=1}^k a_i \ell_i = 0 \implies \sum_{i=1}^k a_i q_i \in \text{Span}_\mathbb{C} \langle (\ell_1)^2, \ldots, (\ell_k)^2 \rangle. \quad (5.11)$$

To better understand this relation, notice that the smoothness of $W$ implies that $\ell_1, \ldots, \ell_n$ is a basis of $\mathbb{C}_1[x_1, \ldots, x_n]$. Thus for every $i \in \mathfrak{k}$, there is a decomposition $\ell_i = \sum_j l_i^j \ell_j$ with constants $l_i^j$ uniquely determined. Thus (5.11) translates into the more precise statement

$$q_i = \sum_{j=1}^n l_i^j q_j \in \text{Span}_\mathbb{C} \langle (\ell_1)^2, \ldots, (\ell_k)^2 \rangle. \quad (5.12)$$
for any \( i \in \mathbb{K} \).

If \( G \in \mathcal{Q} \) and \( \ell \) is a linear form then \( G_\partial(\ell^2) = 2G(\nabla \ell) \). Suppose that \( G(\nabla \ell_i) = 0 \) for every \( i \in \mathbb{K} \). Thus \( G_\partial(q) = 0 \) for every \( q \in \text{Span}_C (\ell_1)^2, \ldots, (\ell_k)^2 \). Using the relations (5.12) one deduces

\[
G_\partial(q_i) = \sum_{j=1}^{n} l_{ij} G_\partial(q_i)
\]

for every \( i \in \mathbb{K} \).

Set \( M_G \) as the vector field \( \sum_{i} G_i \partial \cdot \frac{\partial}{\partial x_i} \). Since \( G_\partial(\ell) = 0 \) for every linear form \( \ell \), one deduces that

\[
G_\partial(u_i)(0) = \langle du_i(0), M_G \rangle.
\]

for every \( i \in \mathbb{K} \).

By hypothesis, the implication (5.11) holds true for every \( x \in (\mathbb{C}^n, 0) \). The discussion above implies the following result.

**Lemma 5.3.8.** Let \( \mathcal{G} = \sum_{i,j} G_{ij}(x)x_i x_j \) be a field of quadratic forms. If \( G(\nabla u_i) \) vanishes identically for every \( i \in \mathbb{K} \) then there exits a vector field \( X_\mathcal{G} \) such that

\[
G_\partial(u_i) = \langle du_i, X_\mathcal{G} \rangle
\]

for every \( i \in \mathbb{K} \).

**Proof of the main technical point I – Conclusion**

Since \( \mathcal{w} = (w_0, \ldots, w_{n-1}) \) is a coframe on \( (\mathbb{C}^n, 0) \), there are basis change formulas (for \( j = 1, \ldots, n \) and \( q = 0, \ldots, n-1 \))

\[
du_j = \sum_{q=0}^{n-1} B_j^q w_q \quad \text{and} \quad w_q = \sum_{j=1}^{n} C_j^q du_j
\]

For \( l, m \in \mathbb{N} \), set

\[
u_{i,l} = \partial u_i / \partial x_l \quad \text{and} \quad u_{i,l,m} = \partial^2 u_i / \partial x_l \partial x_m.
\]

Thus

\[
du_i = h_i \sum_{q=0}^{n-1} (\theta_i)^q w_q = \sum_{j=1}^{n} u_{i,j} \, dx_j
\]
and consequently (for \( p = 0, \ldots, n - 1 \) and \( j = 1, \ldots, n \))

\[
h_i(\theta_i)^p = \sum_{j=1}^{n} B_j^p u_{i,j} \quad \text{and} \quad u_{i,j} = h_i \sum_{p=0}^{n-1} C_j^p(\theta_i)^p.
\]

(5.13)

Consider four integers \( p, p', q, q' \) in the interval \([0, n - 1]\) which satisfy \( p + q = p' + q' \). Because \( (\theta_i)^p(\theta_i)^q = (\theta_i)^{p'}(\theta_i)^{q'} \), the equations (5.13) imply the relation

\[
\sum_{j,j'=1}^{n} \left( B_j^p B_{j'}^{q'} - B_j^{p'} B_{j'}^q \right) u_{i,j} u_{i,j'} = 0
\]

holds true for every \( i \in k \).

Lemma 5.3.8 implies the existence of functions \( X^1, \ldots, X^n \), for which

\[
\sum_{j,j'=1}^{n} \left( B_j^p B_{j'}^{q'} - B_j^{p'} B_{j'}^q \right) u_{\alpha,j} u_{\alpha,j'} = \sum_{l=1}^{n} X_l^i u_{i,l}.
\]

for every \( i \in k \). It is important to observe that the functions \( X^1, \ldots, X^n \) do not depend on the function \( u_i \) but only on the integers \( p, q, p', q' \) and, of course, on the coframe \( \varpi \).

Notice that

\[
d(h_i(\theta_i)^p) = \sum_{j=1}^{n} \partial_{x_j}(h_i(\theta_i)^p) dx_j = \sum_{q=0}^{n-1} \left( \sum_{j=1}^{n} B_j^q \partial_{x_j}(h_i(\theta_i)^p) \right) \varpi_q
\]

and consequently \( \{d(h_i(\theta_i)^p)\}^q = \sum_{j,j'=1}^{n} B_j^q \partial_{x_j}(h_i(\theta_i)^p) \). Combining this last equation with (5.13) one obtains

\[
\{d(h_i(\theta_i)^p)\}^q = \sum_{j,j'=1}^{n} B_j^q \partial_{x_j}(B_j^p u_{i,j'})
\]

\[
= \sum_{j,j'=1}^{n} B_j^q \partial_{x_j}(B_j^p u_{i,j'}) + \sum_{j,j'=1}^{n} B_j^q B_{j'}^p u_{i,j'j'}.\]

Thus, one can write

\[
\{d(h_i(\theta_i)^p)\}^p - \{d(h_i(\theta_i)^p')\}^p' = \sum_{l=1}^{n} Y^l u_{i,l} \quad (5.14)
\]
with
\[ Y^l = X^l + \sum_{j,j' = 1}^n (B^p_j \partial_{x_j}(B_q^{j'}) - B_p^{j'} \partial_{x_j}(B_q^{j'})). \]

Once again one has to apply the relations (5.13). After setting
\[ M_r = \sum Y^l C^l_r \text{ for } r = 0, \ldots, n - 1, \]
it follows that
\[ \{d(h_i(\theta_i)^q)\}^p - \{d(h_i(\theta_i)^{q'})\}^{p'} = h_i \sum_{r=0}^{n-1} M_r (\theta_i)^r \]
for no matter which \( i \in \mathbb{k} \).

Note that the functions \( M_r \) depend only on the integers \( p, q, p', q' \), but not on \( i \). It suffices to take \( p' = p+1, q = 1 \) and \( q' = 0 \) to establish the existence of functions \( M^p \) satisfying
\[ \{d(h_i\theta_i)^p\} - \{dh_i\}^{p+1} = h_i \sum_{r=0}^{n-1} M^p (\theta_i)^r. \quad (5.15) \]

Let \( z = (z_1\omega_1, \ldots, z_k\omega_k) \) be an abelian relation of \( W \). The function \( z_i \) is such that \( d(z_i\omega_i) = 0 \) then \( d(z_i/h_i) \wedge du_i = 0 \). Therefore
\[ \sum_{p=0}^{n-1} \left( \{dz_i\}^p - z_i h_i^{-1}\{dh_i\}^p \right) \varpi_p \wedge \sum_{q=0}^{n-1} (\theta_i)^q \varpi_q = 0, \]
which implies
\[ \left( \{dz_i\}^p - z_i h_i^{-1}\{dh_i\}^p \right) (\theta_i)^q - \left( \{dz_i\}^q - z_i h_i^{-1}\{dh_i\}^q \right) (\theta_i)^p = 0 \quad (5.16) \]
for every \( p, q = 0, \ldots, n - 1 \). It suffices to set \( q = p + 1 \) in (5.16) and combine it with (5.15) to obtain the relations (5.6) of Proposition 5.3.6.

To obtain the relations (5.7), take \( q = 2, q' = 1 \) and \( p' = p + 1 \) in (5.14). Note that this is possible only because \( n \) is assumed to be at least 3. On the one hand, it follows from (5.14) the existence of holomorphic functions \( L_0, \ldots, L_{n-1} \) which satisfy for every \( i \in \mathbb{k} \) the following identity
\[ \{d(h_i(\theta_i)^q)\}^p - \{d(h_i\theta_i)^{p+1} = h_i \sum_{r=0}^{n-1} L_r (\theta_i)^r. \quad (5.17) \]
On the other hand,

\[
\{d(h_i(\theta_i^2))\}^p - \{d(h_i\theta_i)\}^{p+1} = \theta_i \left( \{d(h_i\theta_i)\}^p - \{dh_i\}^{p+1} \right) \\
+ h_i (\theta_i \{d\theta_i\}^p - \{d\theta_i\}^{p+1}).
\]

Plugging these formulae into (5.17) and using (5.15), one finally obtains (5.7) and concludes in this way the proof of Proposition 5.3.6.

Remark 5.3.9. Note that the condition \(n \geq 3\) is only used at the very end of the proof to obtain the relations (5.17) which imply rather straight-forwardly the relations (5.7). Except for these last lines, all the arguments above are valid in dimension two.

Upper bound for the rank I: technical lemmata

To prove that the rank of \(PB_W\) is two it suffices to show that

\[
\dim \text{Span} \left\langle Z_\ast(x,t), \frac{\partial Z_\ast(x,t)}{\partial t}, \frac{\partial Z_\ast(x,t)}{\partial x_1}, \ldots, \frac{\partial Z_\ast(x,t)}{\partial x_n} \right\rangle \leq 3.
\]

Since the focus now, after Lemma 5.3.5, is on points outside the hypersurfaces \(\{t = \theta_i(x)\}_{i \in \mathbb{K}}\), it is harmless to replace \(Z_\ast\) by

\[
Z: (\mathbb{C}^n, 0) \times \mathbb{C} \rightarrow \mathbb{C}^\pi \\
(x,t) \mapsto \sum_{i=1}^k \frac{Z_i(x)}{t - \theta_i(x)}.
\]

Notice that \(Z_\ast(x,t) = P(x,t) \cdot Z(x,t)\). Hence \(Z\) is also a lift of \(PB_W\). Notice also that the map \(Z\) has poles at the hypersurfaces \(\{t = \theta_i(x)\}_{i \in \mathbb{K}}\). That is why \(PB_W\) was not defined as the projectivization of \(Z\) from the beginning.

The following conventions will be used: the usual exterior derivative on \((\mathbb{C}^n, 0) \times \mathbb{P}^1\) will be denoted by \(d\), while the exterior differential on \((\mathbb{C}^n, 0)\) will denoted by \(d\). To clarify: \(dF = dF + (\frac{\partial F}{\partial t})dt\) for every germ of holomorphic function \(F\) on \((\mathbb{C}^n, 0) \times \mathbb{P}^1\).
For further reference, observe that the \( C^n \)-valued 1-form \( dZ \) can written as
\[
dZ = \sum_{i=1}^{k} (t - \theta_i)^{-1} dZ_i + \sum_{i=1}^{k} (t - \theta_i)^{-2} Z_i \, d\theta_i \quad (5.18)
\]

The following simple lemma will prove to be useful later.

**Lemma 5.3.10.** For \( l = 0, \ldots, n \) and \( L = 0, \ldots, n+1 \), respectively, the following identities hold true.
\[
\sum_{i=1}^{k} \frac{Z_i(\theta_i)^l}{t - \theta_i} = t^l Z ; \\
\sum_{i=1}^{k} \frac{Z_i(\theta_i)^L}{(t - \theta_i)^2} = L t^{L-1} Z - t^L (\partial Z / \partial t) .
\]

**Proof.** Both identities are proved by induction. Notice that they both are trivially true for \( l = 0 \) and \( L = 0 \).

Assume the first identity holds true for \( l < n \), and write
\[
\sum_{i=1}^{k} \frac{Z_i(\theta_i)^{l+1}}{t - \theta_i} = \sum_{i=1}^{k} \frac{Z_i(\theta_i)^l ((\theta_i - t) + t)}{(t - \theta_i)}
\]
\[
= - \sum_{i=1}^{k} Z_i(\theta_i)^l + t \sum_{i=1}^{k} \frac{Z_i(\theta_i)^l}{(t - \theta_i)}
\]

Observe that \( \sum_i Z_i(\theta_i)^l = 0 \) according to Equation (5.2). Using this observation together with the induction hypothesis, it follows that
\[
\sum_{i=1}^{k} (t - \theta_i)^{-1} Z_i \theta_i^{l+1} = t^{l+1} Z \]
as wanted.

Now, assume the second identity holds true for \( L \leq n \). Using the same trick as above, one obtains
\[
\sum_{i=1}^{k} \frac{Z_i(\theta_i)^{L+1}}{(t - \theta_i)^2} = \sum_{i=1}^{k} \frac{Z_i(\theta_i)^L}{(t - \theta_i)} + t \sum_{i=1}^{k} \frac{Z_i(\theta_i)^L}{(t - \theta_i)^2} .
\]
The first identity implies that the first summand of the right hand side is \( t^L Z \). The induction hypothesis implies that the second summand is \( t \left( L t^{L-1} Z - t^L (\partial Z / \partial t) \right) \). The lemma follows. \( \square \)

**Lemma 5.3.11.** If the identities (5.6) and (5.7) of Proposition 5.3.6 hold true (in particular if \( n > 2 \)) then, for \( p = 0, \ldots, n - 2 \), there are functions \( F_p, G_p \in \mathcal{O}(C^n, 0)[t] \) such that

\[
\{dZ\}^{p+1} - t\{dZ\}^p = F_p Z + G_p (\partial Z / \partial t).
\]

**Proof.** Let \( p \) be fixed and notice that equation (5.18) implies

\[
\{dZ\}^p = \sum_{i=1}^k \frac{\{dZ_i\}^p}{t - \theta_i} + \sum_{i=1}^k \frac{Z_i \{d\theta_i\}^p}{(t - \theta_i)^2}.
\]

Decompose \( I_p = \{dZ\}^{p+1} - t\{dZ\}^p \) as \( K_p + L_p \), where

\[
K_p = \sum_{i=1}^k \frac{\{dZ_i\}^{p+1} - t \{dZ_i\}^p}{t - \theta_i}
\]

and

\[
L_p = \sum_{i=1}^k \left( \{d\theta_i\}^{p+1} - \theta_i \{d\theta_i\}^p \right) Z_i (t - \theta_i)^2
\]

Replacing \( t \) by \( (t - \theta_\alpha) + \theta_\alpha \) in the numerator of \( K^p \) gives

\[
K_p = \sum_{i=1}^k (t - \theta_i)^{-1} \left( \{dZ_i\}^{p+1} - \theta_i \{dZ_i\}^p \right) + \sum_{i=1}^k \{dZ_i\}^p
\]

According to (5.2) \( \sum_i Z_i = 0 \), consequently

\[
K_p = \sum_{i=1}^k \frac{\{dZ_i\}^{p+1} - \theta_i \{dZ_i\}^p}{t - \theta_i} \quad (5.20)
\]

In exactly the same way, one proves that

\[
L_p = \sum_{i=1}^k \frac{\{d\theta_i\}^{p+1} - \theta_i \{d\theta_i\}^p} {(t - \theta_i)^2} Z_i + \sum_{i=1}^k \frac{Z_i \{d\theta_i\}^p}{t - \theta_i}.
\]
In what follows, $A \equiv B$ if and only if $A - B$ is equal to $FZ + G(\partial Z/\partial t)$ for suitable $F, G \in \mathcal{O}(\mathbb{C}^n, 0)[t]$. Notice that the lemma is equivalent to $I_p \equiv 0$.

The outcome of Proposition 5.3.6, more specifically equation (5.7), implies

$$
\sum_{i=1}^{k} \left( \frac{(d\theta_i)^{p+1} - \theta_i \{d\theta_i\}^p}{(t - \theta_i)^2} \right) Z_i = \sum_{i=1}^{k} \frac{Z_i}{(t - \theta_i)^2} \left( \sum_{\rho=0}^{n} (\theta_i)^{p} N_{\rho}^{p} \right).
$$

Lemma 5.3.10 in its turn, implies

$$
\sum_{i=1}^{k} \frac{Z_i}{(t - \theta_i)^2} \left( \sum_{\rho=0}^{n} (\theta_i)^{p} N_{\rho}^{p} \right) = \sum_{\rho=0}^{n} N_{\rho}^{p} \left( \sum_{i=1}^{k} \frac{Z_i (\theta_i)^{p}}{(t - \theta_i)^2} \right) \equiv 0.
$$

Therefore

$$
L_p \equiv \sum_{i=1}^{k} \frac{Z_i \{d\theta_i\}^p}{t - \theta_i}.
$$

Combining (5.20) and (5.21), one obtains

$$
I_p \equiv \sum_{i=1}^{k} \frac{(dZ_i)^{p+1} - \theta_i \{dZ_i\}^p - Z_i \{d\theta_i\}^p}{t - \theta_i}.
$$

Equation (5.6) from Proposition 5.3.6 together with equation (5.19) from Lemma 5.3.10, allow to conclude:

$$
I_p \equiv \sum_{i=1}^{k} \frac{\left( \sum_{r=0}^{n-1} (\theta_i)^{r} M_{\rho}^{p} \right) Z_i}{t - \theta_i} \\
\equiv \sum_{r=0}^{n-1} M_{\rho}^{p} \left( \sum_{i=1}^{k} \frac{Z_i (\theta_i)^{r}}{t - \theta_i} \right) \equiv \left( \sum_{r=0}^{n-1} M_{\rho}^{p} t^r \right) Z \equiv 0.
$$

Upper bound for the rank II: conclusion

**Proposition 5.3.12.** There are 1-forms $\Omega$ and $\Gamma$ on $(\mathbb{C}^n, 0) \times \mathbb{P}^1$ such that

$$
dZ = \{dZ\}^0 \left( \sum_{p=0}^{n-1} t^p \omega_p \right) + Z \Omega + (\partial Z/\partial t) (\Gamma + dt).
$$

(5.22)
Proof. Lemma 5.3.11 implies, for every $p = 0, \ldots, n - 1$,

$$\{dZ\}^p = t^p\{dZ\}^0 + \sum_{q=0}^{p} I_p t^{p-q-1}. $$

If $\Pi = \sum_{p=0}^{n-1} t^p\omega_p$ then

$$dZ = \sum_{p=0}^{n-1} \{dZ\}^p \omega_p = \sum_{p=0}^{n-1} \left( t^p \{dZ\}^0 + \sum_{q=0}^{p-1} I_q t^{p-q-1} \right) \omega_p$$

$$= \{dZ\}^0 \Pi + \sum_{p=0}^{n-1} \sum_{q=0}^{p-1} I_q t^{p-q-1} \omega_p. $$

Lemma 5.3.11 says that $I^q \equiv 0$ (see the definition of $\equiv$ in page 173) for every $q = 0, \ldots, n - 1$. The existence of two 1-forms $\Omega$ and $\Gamma$ satisfying

$$dZ = \{dZ\}^0 \Pi + Z \Omega + (\partial Z/\partial t) \Gamma$$

follows. Moreover, the coefficients of 1-forms $\Omega$ and $\Gamma$ in the basis $(\omega_0, \ldots, \omega_{n-1})$ are polynomials in $t$ with holomorphic functions on $(\mathbb{C}^n, 0)$ as coefficients. Since $dZ = dZ + (\partial Z/\partial t) dt$, the proposition follows.

Proposition 5.3.12 clearly implies that $PB_W$ has rank at most two at every $(x, t) \in (\mathbb{}{n}, 0) \times \mathbb{P}^1$. But Lemma 5.3.5 says it must be at least two. Hence Proposition 5.3.4 follows.

### 5.4 Poincaré-Blaschke’s surface

The Poincaré-Blaschke’s surface $X_W$ of $W$ is the image of its Poincaré-Blaschke’s map $PB_W$. That is,

$$X_W = \text{Im} \ PB_W \subset \mathbb{P}^{n-1}. $$

According to Proposition 5.3.4 it is a germ of smooth complex surface on $(\mathbb{P}^{n-1}, \mathcal{O}(0))$. It is clearly non-degenerate. The remarks laid down at the end of Section 5.3 imply that $X_W$ is canonically attached to the pair $(\mathcal{W}, A_2(\mathcal{W}))$ modulo projective transformations.
SECTION 5.4: POINCARÉ-BLASCHKE’S SURFACE

5.4.1 Rational normal curves everywhere

Notice that $X_W$ contains all the canonical curves $C_i$ of $W$. It also contains a lot of rational curves according to Lemma 5.3.3: the curves $\mathcal{C}(x)$. Exploiting the geometry of this family of rational curves, it will be possible to prove that $X_W$ is the germification at $\mathcal{C}(0)$ of a rational surface.

Lemma 5.4.1. The following assertions are verified:

(a) for every $i \in k$, the curve $\mathcal{C}(x)$ intersects the canonical curve $C_i$ transversely at $\kappa_i(x)$;

(b) for every $x, y \in (\mathbb{C}^n, 0)$, the curve $\mathcal{C}(x)$ coincides with $\mathcal{C}(y)$ if and only if $x = y$;

(c) for every subset $J \subset k$ of cardinality $n$, and every set $P = \{p_j \in C_j | j \in J\}$, there exists a unique $x \in (\mathbb{C}^n, 0)$ such that $\mathcal{C}(x)$ contains $P$.

Proof. The first assertion follows from the proof of Lemma 5.3.5. It is also a direct consequence of the expression (5.22) for $dZ$.

To prove the third assertion, notice that for every $j \in J$, $\kappa_j^{-1}(p_j)$ is a leaf $L_j$ of $\mathcal{F}_j$. Since $W$ is smooth, $\cap_{j \in J} L_j$ must reduce to a point $x \in (\mathbb{C}^n, 0)$. The assertion follows.

The second assertion follows immediately from the third. □

Lemma 5.4.1 implies that the surface $X_W$ is the union of rational normal curves of degree $k - n - 1$ belonging to the $n$-dimensional holomorphic family $\{\mathcal{C}(x)\}_{x \in (\mathbb{C}^n, 0)}$.

5.4.2 Algebraization I : $X_W$ is algebraic

Lemma 5.4.2. For every $x, y \in (\mathbb{C}^n, 0)$, the intersection number $\mathcal{C}(x) \cdot \mathcal{C}(y)$ is equal to $n - 1$.

Proof. Let $B \subset k$ be a subset of cardinality $n - 1$. Suppose $y$ belongs to $\cap_{i \in B} L_i(x)$, $L_i(x)$ being the leaf of $\mathcal{F}_i$ through $x$, but is not equal to $x$. After item (c) of Lemma 5.4.1, the curves $\mathcal{C}(x)$ and $\mathcal{C}(y)$ are distinct. Anyway, they share at least $n - 1$ points in common: $p_i = \kappa_i(x) = \kappa_i(y)$ for $i \in B$. Therefore $\mathcal{C}(x) \cdot \mathcal{C}(y) \geq n - 1$. 
If \( C(x) \cdot C(y) \geq n \) then either there exists \( p \notin \{ p_i \}_{i \in B} \) such that \( p \in C(x) \cap C(y) \); or there exists \( p \in \{ p_i \}_{i \in B} \) for which \( C(x) \) and \( C(y) \) are tangent at \( p \). But \( n \) points, or \( n - 1 \) points and one tangent, on a rational normal curve span a projective subspace of dimension \( n - 1 \). This contradicts Corollary 5.2.5.

To conclude it suffices to observe that any two curves in the family \( \{ C(x) \}_{x \in (\mathbb{C}^n,0)} \) are homologous.

**Proposition 5.4.3.** Let \( (S_0,C) \subset \mathbb{P}^N \) be a germ of smooth surface along a connected projective curve \( C \subset S_0 \). If \( C^2 > 0 \) then \( S_0 \) is contained in a projective surface \( S \subset \mathbb{P}^N \).

**Proof.** Let \( \mathbb{C}(S_0) \) be the field of meromorphic functions on \( S_0 \). To prove that \( S_0 \) is contained in a projective surface, it suffices to show that the transcendence degree of \( \mathbb{C}(S_0) \) over \( \mathbb{C} \) is two. Clearly, since the restriction at \( S_0 \) of any two generic rational functions on \( \mathbb{P}^N \) are algebraically independent, it suffices to assume that the polar set of both does not contain \( S_0 \) and that their level sets are generically transverse to obtain that \( \text{trdeg}[\mathbb{C}(S_0) : \mathbb{C}] \geq 2 \).

The hypothesis \( C^2 > 0 \) is equivalent to the ampleness of the normal bundle \( N_{C/S_0} \). Thus [64, Theorem 6.7] implies that \( \text{trdeg}[\mathbb{C}(S_0) : \mathbb{C}] \leq 2 \).

Alternatively, it is also possible to apply a Theorem of Andreotti: since any representative of \( S_0 \) contains a curve of positive self-intersection, it also contains a pseudo-concave open subset. Hence, [5, Théorème 6] implies \( \text{trdeg}[\mathbb{C}(S_0) : \mathbb{C}] \leq 2 \).

The preceding proposition together with Lemma 5.4.2 have the following consequence.

**Corollary 5.4.4.** Poincaré-Blaschke’s surface \( X_W \) is contained in an irreducible non-degenerate projective surface \( S_W \subset \mathbb{P}^{\pi-1} \).

**Remark 5.4.5.** Notice, that nothing is said about the smoothness of \( S_W \). A priori, it could even happen that the germ of \( S_W \) along \( C(0) \) is singular. Of course, this would happen if and only if the germ of \( S_W \) along \( C(0) \) has other irreducible components besides \( X_W \). The only thing clear is that \( S_W \) contains the smooth surface \( X_W \).
SECTION 5.4: POINCARÉ-BLASCHKE’S SURFACE

Remark 5.4.6. Those bewildered with the use of the results of Hartshorne or Andreotti in the proof of Proposition 5.4.3, might fell relieved by knowing that Corollary 5.4.4 can be proved by rather elementary means, which are sketched below.

Let $x_0$ be an arbitrary point of $\mathcal{C}(0)$ and consider the subset $X$ of $\text{Mor}_{k-n-1}(\mathbb{P}^1, \mathbb{P}^\pi-1)$ consisting of morphisms $\phi$ which map $(\mathbb{P}^1, (0 : 1))$ to $(X_W, x_0)$. Recall that the ring of formal power series in any number variables is Noetherian. If $I \subset \mathbb{C}[[x_1, \ldots, x_{\pi-1}]]$ is the ideal defining $(X_W, x_0)$ then, expanding formally $f(\phi(t : 1))$ for every defining equation $f \in I$ one deduces that $X$ is algebraic.

To conclude, one has just to prove that the natural projection from $\text{Mor}_{k-n-1}(\mathbb{P}^1, \mathbb{P}^\pi-1)$ to $\mathbb{P}^{\pi-1}$ — the evaluation morphism — sends $X$ onto a surface $S_W$ of $\mathbb{P}^{\pi-1}$ containing $X_W$.

5.4.3 Algebraization II and conclusion

Since the projective surface $S_W$ can be singular, it will be replaced by one of its desingularizations. It can be assumed that the chosen desingularization contains an isomorphic copy of $(X_W, \mathcal{C}(0))$. Notice also that every desingularization of a singular projective surface is still projective. To keep the notation simple, this desingularization will still be denoted by $S_W$.

Proposition 5.4.7. There are no holomorphic 1-forms on $S_W$, that is

$$h^0(S_W, \Omega^1_{S_W}) = 0.$$ 

Proof. Let $\xi$ be a holomorphic 1-form on $S_W$. If non-zero then it defines a foliation $\mathcal{F}_\xi$ on $S_W$. Since smooth rational curves have no holomorphic 1-forms, the pull-back of $\xi$ to $\mathcal{C}(\mathbb{C}^n, 0)$ must vanish for every $x \in (\mathbb{C}^n, 0)$. Therefore these curves are invariant by $\mathcal{F}_\xi$. But a foliation on a surface cannot have an $n$-dimensional family of pairwise distinct leaves. This contradiction shows that $\xi$ is identically zero. □

Hodge theory implies $H^1(S_W, \mathcal{O}_{S_W})$ is also trivial. Therefore from the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S^* \rightarrow 0$$
one deduces that the Chern class morphism

$$H^1(S_W, \mathcal{O}_{S_W}^*) \to H^2(S_W, \mathbb{Z})$$

is injective. Consequently, two projective curves in $S_W$ are linearly equivalent if and only if they are homologous.

Remark 5.4.8. The surface $S_W$ is rational. To see it, take $n - 1$ points in $\mathbb{C}(0)$, say $\kappa_1(0), \ldots, \kappa_{n-1}(0)$. If $L_i$ is a leaf of the foliation $\mathcal{F}_i$ through the origin then $\cap_{i \in \mathbb{N}} L_i$ is a curve $C$ in $(\mathbb{C}^n, 0)$. By construction, for every $x \in C$ the curve $\mathcal{C}(x)$ intersects $\mathcal{C}(0)$ at $\kappa_1(0), \ldots, \kappa_{n-1}(0)$. Thus blowing up at these $n - 1$ points, one obtains a surface $S$ containing a family parametrized by $(\mathbb{C}, 0)$, of rational curves of self-intersection zero: the strict transforms of $\mathcal{C}(x)$ for $x \in C$. Any two of these curves are linearly equivalent, therefore there exists a non-constant holomorphic map $F : S \to \mathbb{P}^1$ sending all of them to points. Thus $S$ is a rational fibration over $\mathbb{P}^1$, hence a rational surface.

Theorem 5.4.9. The web $\mathcal{W}$ is linearizable.

Proof. Recall that any two curves in the family $\{\mathcal{C}(x)\}$ are homologous, and consequently linearly equivalent. Thus they all belong to the complete linear system

$$|\mathcal{C}(0)| = \mathbb{P}H^0(S_W, \mathcal{O}_{S_W}(\mathcal{C}(0))).$$

Tensoring the standard exact sequence

$$0 \to \mathcal{O}_{S_W}(-\mathcal{C}(0)) \to \mathcal{O}_{S_W} \to \mathcal{O}_{\mathcal{C}(0)} \to 0$$

by $\mathcal{O}_{S_W}(\mathcal{C}(0))$, one obtains

$$0 \to \mathcal{O}_{S_W} \to \mathcal{O}_{S_W}(\mathcal{C}(0)) \to \mathcal{O}_{\mathcal{C}(0)}(\mathcal{C}(0)) \to 0.$$  \hspace{1cm} (5.23)

Notice that

$$\deg \mathcal{O}_{\mathcal{C}(0)}(\mathcal{C}(0)) = \mathcal{C}(0)^2 = n - 1.$$  \hspace{1cm} (5.24)

Consequently $\mathcal{O}_{\mathcal{C}(0)}(\mathcal{C}(0)) \cong \mathcal{O}_{\mathbb{P}^1}(n - 1)$. 

Since $H^1(S_W, \mathcal{O}_{S_W}) \simeq H^0(S_W, \Omega^1_{S_W}) = 0$, it follows from (5.23) that

$$h^0(S_W, \mathcal{O}_{S_W}(\mathcal{C}(0))) = h^0(S_W, \mathcal{O}_{S_W}) + h^0(S_W, \mathcal{O}_{\mathcal{C}(0)}(\mathcal{C}(0))) = 1 + n.$$ 

Therefore $|\mathcal{C}(0)| \simeq \mathbb{P}^n$. According to Lemma 5.4.1 item (b), the map

$$\mathcal{C} : (\mathbb{C}^n, 0) \rightarrow |\mathcal{C}(0)| \simeq \mathbb{P}^n$$

is injective. Moreover, there exists a factorization

$$\xymatrix{ (\mathbb{C}^n, 0) \ar[r]^\mathcal{C} \ar[rd]_{P_W} & |\mathcal{C}(0)| \simeq \mathbb{P}^n \ar[d]^\langle \rangle \\ & \text{Grass}(\mathbb{A}_\mathbb{C}(\mathcal{W})^*, k - n) }$$

where $\langle \rangle$ is the map that associates to the curve $\mathcal{C}(x)$ its projective span $\langle \mathcal{C}(x) \rangle$. Since $P_W$ is an immersion (Proposition 5.2.4), so is $\mathcal{C}$. The image by $\mathcal{C}$ of $L_i(x)$, the leaf of $\mathcal{F}_i$ through $x \in (\mathbb{C}^n, 0)$, is nothing more than the elements of $|\mathcal{C}(0)|$ passing through $\kappa_i(x) \in X_W \subset S_W$. Because $|\mathcal{C}(0)|$ is a linear system, $\mathcal{C}(L_i(x))$ is a hyperplane in $\mathbb{P}^n$. Therefore $\mathcal{C}$ is a germ of biholomorphism which linearizes $\mathcal{W}$. $\square$

It suffices to apply the algebraization theorem for linear webs to conclude that $\mathcal{W}$ is algebraizable.
Chapter 6

Exceptional Webs

Taking the risk of being anticlimactic, this last chapter is devoted to planar webs of maximal rank. More specifically, a survey of the currently state of the art concerning exceptional planar webs is presented.

In Section 6.1 a criterium to decide whether or not a given planar web is linearizable is presented. The criterium is phrased in terms of a projective connection adapted to the web under study. The lack of conciseness of the presentation carried out here is balanced by the geometric intuition it may help to build. As corollaries, the existence of exceptional webs and an algebraization result for planar webs, are obtained in Section 6.1.4 and Section 6.1.5 respectively.

Section 6.2 deals with planar webs with infinitesimal automorphisms following [77]. Using the result on the structure of the space of abelian relations of a web $W$ carrying a transverse infinitesimal automorphism $v$ laid down earlier in Section 2.1.2 of Chapter 2, the rank of the web obtained from the superposition of $W$ and the foliation $F_v$ determined by $v$, is computed as a function of the rank of $W$. As a corollary, the existence of exceptional $k$-webs for arbitrary $k \geq 5$ is settled. Some place is taken to recall some basic definitions from differential algebra, and make more precise a problem posed in [77] that the authors think has some interest.

Section 6.3 starts with basic facts from Cartan-Goldsmith-Spencer
theory on differential linear systems. This theory is applied to the
differential system which solutions are the abelian relations of a given
planar web, in order to obtain a computational criterion which de-
cides whether or not the web under study has maximal rank. The
approach followed there is a mix of the classical one by Pantazi with
the more recent by Hénaut. The necessary criterion for the maxi-
mality of the rank by Mihăileanu is briefly discussed without proof.

In Section 6.4 some recent classification results obtained by the
authors are stated, and the proof of one of them is outlined.

Finally in Section 6.5 all the exceptional webs known up-to-date,
to the best of the authors knowledge, are collected.

6.1 Criterium for linearization

Throughout this section $W = W(\omega_1, \ldots, \omega_k)$ is a germ of smooth
$k$-web on $(\mathbb{C}^2, 0)$. For $i \in k$, let $v_i \in T(\mathbb{C}^2, 0)$ be a germ of smooth
vector field defining the same foliation as $\omega_i$. Thus $\omega_i(v_i) = 0$ and
$v_i(0) \neq 0$.

6.1.1 Characterization of linear webs

Recall that a web $W$ is linear if all its leaves are contained in affine
lines of $\mathbb{C}^2$. Recall also that $W$ is linearizable if it is equivalent to
a linear web.

It is rather simple to characterize the linear webs. It suffices to
notice that a curve $C \subset (\mathbb{C}^2, 0)$ is contained in a line, if and only if
for any parametrization $\gamma : (\mathbb{C}, 0) \to C$ the following identity holds true

$$\gamma'(t) \wedge \gamma''(t) = 0, \quad \text{for every } t \in (\mathbb{C}, 0).$$

Therefore, every orbit of a vector field $v$ will be linear, if and only if,
the determinant

$$\det \begin{pmatrix} v(x) & v(y) \\ v^2(x) & v^2(y) \end{pmatrix}$$
vanishes identically. Indeed, if \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \) satisfies \( v(\gamma(t)) = \gamma'(t) \) then
\[
\det \begin{pmatrix} v(x) & v(y) \\ v^2(x) & v^2(y) \end{pmatrix} (\gamma(t)) = \det \begin{pmatrix} \gamma_1'(t) & \gamma_2'(t) \\ \gamma_1''(t) & \gamma_2''(t) \end{pmatrix}.
\]

Therefore \( \mathcal{W} \) is linear if and only if
\[
\det \begin{pmatrix} v_i(x) & v_i(y) \\ v_i^2(x) & v_i^2(y) \end{pmatrix} = 0
\]
for every \( i \in \mathbb{R} \).

§

To characterize linearizable webs one is naturally lead to less elementary considerations. Below, the approach laid down in Section §27 of Blaschke-Bol’s book [18] is presented in a modern language. For more recent references see [65], [3] and [94].

The inherent difficulty in obtaining an analytic criterium characterizing linearizable webs comes from the fact that the usual method to treat linearization problems comes from differential geometry and is well adapted to deal with one-dimensional families of foliations on \((\mathbb{C}^2,0)\). But a web is not builded from a continuous family of foliations but by a finite number of them. It is the contrast between the finiteness and continuity that makes the linearization of webs a non-trivial question. For instance, despite many efforts spreaded over time, Gronwall’s conjecture is still unsettled:

**Conjecture 1** (Gronwall’s conjecture). If \( \mathcal{W} \) and \( \mathcal{W}' \) are germs of linear 3-webs on \((\mathbb{C}^2,0)\) with non-vanishing curvature and \( \phi : (\mathbb{C}^2,0) \rightarrow (\mathbb{C}^2,0) \) is a germ of biholomorphism sending \( \mathcal{W} \) to \( \mathcal{W}' \) then \( \phi \) is the germification of a projective automorphism of \( \mathbb{P}^2 \). In other words, a non-hexagonal 3-web admits at most one linearization.

In sharp contrast, the equivalent statement for planar \( k \)-webs with \( k \geq 4 \), is true as will be explained below. The point is that it is possible interpolate the defining foliations by a unique second order differential equation cubic in the first derivative when \( k = 4 \). When \( k = 3 \), although possible to interpolate as for \( k = 4 \), the lack of uniqueness adds an additional layer of difficulty to the problem.
6.1.2 Affine and projective connections

A germ of holomorphic affine connection on \((\mathbb{C}^2, 0)\) is a map

\[ \nabla : T(\mathbb{C}^2, 0) \to \Omega^1(\mathbb{C}^2, 0) \otimes T(\mathbb{C}^2, 0) \]

satisfying

1. \( \nabla(\zeta + \zeta') = \nabla(\zeta) + \nabla(\zeta') \);
2. \( \nabla(f\zeta) = f\nabla(\zeta) + df \otimes \zeta \);

for every \( \zeta, \zeta' \in T(\mathbb{C}^2, 0) \) and every \( f \in \mathcal{O}(\mathbb{C}^2, 0) \).

Beware that the map \( \nabla \) is not \( \mathcal{O}(\mathbb{C}^2, 0) \)-linear. In particular, the image of a vector field \( \zeta \) is a tensor which at a given point \( p \in (\mathbb{C}^2, 0) \) is determined by the whole germ of \( \zeta \) at \( p \) and not only by its value at \( p \).

If \( \chi = (\chi_0, \chi_1) \) is a holomorphic frame on \((\mathbb{C}^2, 0)\) and \( \omega = (\omega_0, \omega_1) \) is the dual coframe then \( \nabla \) is completely determined by its Christoffel's symbols \( \Gamma^k_{ij} \) (relative to the frame \( \chi \)), defined by the relations

\[ \nabla(\chi_j) = \sum_{i,k=0}^1 \Gamma^k_{ij} \omega_i \otimes \chi_k \quad \text{for } j = 0, 1. \]

Although \( \nabla \) when evaluated at a vector field \( \zeta \) does depend on the germ of \( \zeta \) as explained above, if \( C \subset (\mathbb{C}^2, 0) \) is a curve and \( \zeta \) is a vector field tangent to it then the pullback of \( \nabla(\zeta) \) at \( C \) is completely determined by the restriction of \( \zeta \) to \( C \). More precisely, \( \nabla(\zeta) \) naturally determines a section of \( \Omega^1(C) \otimes T(\mathbb{C}^2, 0) \) which only depends on the restriction of \( \zeta \) to \( C \). Indeed, if \( \zeta, \zeta' \) and \( \zeta'' \) are vector fields such that \( \zeta - \zeta' = f\zeta'' \) where \( f \in \mathcal{O}(\mathbb{C}^2, 0) \) is a defining equation for \( C \), then

\[ \nabla(\zeta) - \nabla(\zeta') = f\nabla(\zeta'') + df \otimes \zeta'' \]

which clearly vanishes at \( C \) when contracted with vector fields tangent to \( C \).

A smooth curve \( C \subset (\mathbb{C}^2, 0) \) is a geodesic of \( \nabla \), if for every germ of vector field \( \zeta \in TC \subset T(\mathbb{C}^2, 0) \), the vector field \( \nabla_\zeta(\zeta) := \langle \nabla(\zeta), \zeta \rangle \) still belongs to \( TC \). To wit, if \( \gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0) \) is a parametrization
of \( C \) then \( C \) is a geodesic for \( \nabla \) if and only if there exists a holomorphic 1-form \( \eta \in \Omega^1(\mathbb{C}, 0) \) such that \( \nabla(\gamma') = \eta \otimes \gamma' \). If \( \nabla(\gamma') \) vanishes identically on \( (\mathbb{C}, 0) \) then \( \gamma \) is called a geodesic parametrization of \( C \).

Two affine connections on \( (\mathbb{C}^2, 0) \) are projectively equivalent if they have exactly the same curves as geodesics. This defines a equivalence relation on the space of affine connections on \( (\mathbb{C}^2, 0) \). By definition, a projective connection is an equivalence class of this equivalence relation. The class of an affine connection \( \nabla \) will be denoted by \( [\nabla] \). A smooth curve \( C \subset (\mathbb{C}^2, 0) \) is a geodesic of \( \Pi = [\nabla] \) if it is a geodesic of the affine connection \( \nabla \). Of course, the definition does not depend on the representative \( \nabla \) of \( \Pi \).

Two projective connections are equivalent if there exists a germ of biholomorphism \( \varphi \) sending the geodesics of one into the geodesics of the other. The trivial projective connection \( \Pi_0 \) is the global projective connection on \( \mathbb{P}^2 \) having as geodesics the projective lines. A projective connection \( \Pi \) is flat (or integrable) if it is equivalent to \( \Pi_0 \).

Projective connections and ordinary differential equations

**Lemma 6.1.1.** If \( \Pi \) is a projective connection on \( (\mathbb{C}^2, 0) \) then there exists a unique affine connection \( \nabla \) on \( (\mathbb{C}^2, 0) \) such that

1. the affine connection \( \nabla \) is a representative of \( \Pi \), that is \( [\nabla] = \Pi \);
2. the Christoffel’s symbols \( \Gamma^k_{ij} \) \( (i, j, k = 1, 2) \) of \( \nabla \) relative to the coframe \( (dx, dy) \) verify the relations
   \[
   \Gamma^k_{ij} = \Gamma^k_{ji} \quad \text{and} \quad \Gamma^1_{1j} + \Gamma^2_{2j} = 0 \quad (6.1)
   \]
   for every \( i, j, k = 1, 2 \).

**Proof.** Exercise for the reader. \( \square \)

From now on, a projective connection \( \Pi \), as well as its normalized representative \( \nabla \) provided by the lemma above, will be fixed.

Let \( \gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0) \) be a parametrization of a curve \( C \subset (\mathbb{C}^2, 0) \). If one writes \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \) for \( t \in (\mathbb{C}, 0) \) then it is a simple exercise to show the equivalence of the three assertions below:
1. \( C \) is a geodesic of \( \Pi \);

2. \( C \) is a geodesic of \( \nabla \);

3. there exists a function \( \varphi \) such that

\[
\frac{d^2\gamma_k}{dt^2} + \sum_{i,j=1}^{2} \Gamma^k_{ij} \left( \frac{d\gamma_i}{dt} \right) \left( \frac{d\gamma_j}{dt} \right) = \varphi \frac{d\gamma_k}{dt}.
\] (6.2)

for \( k = 1, 2 \).

Under the additional hypothesis that \( C \) is transverse to the vertical foliation \( \{x = \text{cst.}\} \), the inequality \( \frac{d\gamma_1}{dt}(0) \neq 0 \) holds true and, modulo a change of variables, one can assume that \( \gamma_1(t) = t \). It is then a simple matter to eliminate the function \( \varphi \) in (6.2) and deduce the following lemma which can be traced back to Beltrami [10].

**Lemma 6.1.2.** The geodesics of \( \Pi \) transverse to \( \{x = \text{cst.}\} \) can be identified with the solutions of the second order differential equation

\[
(\mathcal{E}_\Pi) \quad \frac{d^2y}{dx^2} = A \left( \frac{dy}{dx} \right)^3 + B \left( \frac{dy}{dx} \right)^2 + C \frac{dy}{dx} + D
\] (6.3)

where \( A, B, C, D \) are expressed in function of the Christoffel’s symbols \( \Gamma^k_{ij} \) of \( \nabla \) as follows

\[
A = \Gamma^1_{22}, \quad B = 2 \Gamma^1_{12} - \Gamma^2_{22}, \quad C = \Gamma^1_{11} - 2 \Gamma^2_{12}, \quad D = -\Gamma^2_{11}. \quad (6.4)
\]

Combining equations (6.1) and (6.4), it follows that the Christoffel’s symbols \( \Gamma^k_{ij} \) of the normalized affine connection \( \nabla \) can be expressed in terms of the functions \( A, B, C, D \). Taking into account Lemma 6.1.1, one deduces the following proposition.

**Proposition 6.1.3.** Once a coordinate system \( x, y \) is fixed, a projective connection can be identified with a second order differential equation of the form (6.3).

It is evident, in no matter which affine coordinate system \( x, y \) on \( \mathbb{C}^2 \), that the second order differential equation \( (\mathcal{E}_{\Pi_0}) \) associated to the trivial projective connection \( \Pi_0 \) is nothing more than \( d^2y/dx^2 = 0 \). Therefore, a projective connection is integrable if and
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only if the second order differential equation \( (\mathcal{E}_1) \) in transformed into the trivial one, \( \frac{d^2Y}{dX^2} = 0 \), through a point transformation \((x, y) \mapsto (X(x, y), Y(x, y))\).

The characterization of the second order differential equations equivalent to the trivial one stated below is due to Liouville [75] and Tresse [108].

Theorem 6.1.4. The second order differential equation \( y'' = f(x, y, y') \) is equivalent to the trivial equation \( \frac{d^2y}{dx^2} = 0 \) through a point transformation if and only if the function \( F = f(x, y, p) \) verifies

\[
\frac{\partial^4 F}{\partial p^4} = 0 \quad \text{and} \quad D^2 \left( \frac{F_{pp}}{F_{pp}} \right) - 4D \left( \frac{F_{yp}}{F_{pp}} \right) + 6F_{yy} + F \left( 4F_{yp} - D \left( \frac{F_{pp}}{F_{pp}} \right) \right) = 0
\]

where \( D = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + F \frac{\partial}{\partial p} \).

Once one restricts to equations of the form (6.3), which are exactly the ones satisfying the first condition \( \frac{\partial^4 F}{\partial p^4} = 0 \), the second condition can be rephrased in more explicit terms. If

\[
L_1 = 2C_{xy} - B_{xx} - 3D_{yy} - 6DA_x - 3AD_x
+ 3DB_y + 3DB_y + CB_x - 2CC_y
\]
and

\[
L_2 = 2B_{xy} - C_{yy} - 3A_{xx} + 6AD_y + 3DA_y
- 3AC_x - 3CA_y - BC_y + 2BB_x
\]

then, according to [22], Liouville has shown that the tensor

\[
L = (L_1dx + L_2dy) \otimes (dx \wedge dy)
\]

is invariant under point transformations and that the differential equation (6.3) is equivalent to the trivial one if and only if \( L \) is identically zero.

6.1.3 Linearization of planar webs

Let \( W \) be a smooth \( k \)-web on \((\mathbb{C}^2, 0)\). It is compatible with the projective connection \( \Pi \) if the leaves of \( W \) are geodesics of \( \Pi \). This is clearly a geometric property: if \( \varphi \) is a biholomorphism then
$W$ is compatible with $\Pi = [\nabla]$ if and only if $\varphi^*W$ is compatible with $\varphi^*\Pi = [\varphi^*\nabla]$.

**Lemma 6.1.5.** A web $W$ is linear if and only if it is compatible with the trivial projective connection $\Pi_0$. Consequently, $W$ is linearizable if and only if it is compatible with a flat projective connection.

It is on this tautology that the linearization criterion presented below is based.

**Proposition 6.1.6.** If $W$ is a smooth 4-web on $(\mathbb{C}^2, 0)$ then

(a) there is a unique projective connection $\Pi_W$ compatible with $W$;

(b) the web $W$ is linearizable if and only if $\Pi_W$ is flat.

**Proof.** It is clear that (b) follows from (a) combined with Lemma 6.1.5.

To prove (a), it is harmless to assume the leaves of $W$ transverse to the line $\{x = 0\}$. The foliation defining $W$ are thus defined by vector fields $v_i$ (for $i = 1, 2, 3, 4$) which can be written as $v_i = \frac{\partial}{\partial x} + e_i \frac{\partial}{\partial y}$ with $e_i \in \mathcal{O}(\mathbb{C}^2, 0)$. The orbits of the vector fields $v_i$ are solutions of the second order differential equation

$$y'' = A(y')^3 + B(y')^2 + Cy' + D$$

if and only if

$$A(e_i)^3 + B(e_i)^2 + Ce_i + D = v_i(e_i)$$

holds true on $(\mathbb{C}^2, 0)$ for $i = 1, \ldots, 4$.

Consider (6.6) as a system of linear equations in the variables $A, B, C, D$. The determinant of the associated homogeneous linear system is the Vandermonde determinant

$$\det \begin{bmatrix} 1 & e_1 & e_1^2 & e_1^3 \\ 1 & e_2 & e_2^2 & e_2^3 \\ 1 & e_3 & e_3^2 & e_3^3 \\ 1 & e_4 & e_4^2 & e_4^3 \end{bmatrix} = \prod_{1 \leq i < j \leq 4} (e_j - e_i).$$

Since $W$ is smooth, $e_i(0) \neq e_j(0)$ when $i \neq j$. Therefore the Vandermonde determinant above is non-zero and consequently there exists a
unique second order differential equation of the form (6.5) admitting
the orbits of $v_i$ for $i = 1, \ldots, 4$, as solutions. Item (b) follows from
Proposition 6.1.3.

Corollary 6.1.7. If $\mathcal{W}$ is a smooth $k$-web $\mathcal{W}$ on $(\mathbb{C}^2, 0)$ with $k \geq 4$
then the following assertions are equivalent

1. $\mathcal{W}$ is linearizable;

2. there exists a flat projective connection $\tilde{\Pi}$ such that $\tilde{\Pi} = \Pi_{W'}$
   for every 4-subweb $W'$ of $\mathcal{W}$.

Remark 6.1.8. If $v_i = \frac{\partial}{\partial x} + e_i \frac{\partial}{\partial y}$ (for $i \in k$) are vector fields defining
a smooth $k$-web $\mathcal{W}$ then mimicking the proof of Proposition 6.1.6 it
is possible to prove the existence of a unique differential equation of
the form $y'' = F(x, y, y')$ satisfying the following conditions:

1. the $p$-degree of $F(x, y, p)$ is at most $k - 1$; and

2. the leaves of $\mathcal{W}$ are solutions of $y'' = F(x, y, y')$.

The previous corollary can be rephrased as follows: $\mathcal{W}$ is linearizable
if and only if the degree of $F$ is at most three, and its coefficients
satisfy the conditions $L_1 = L_2 = 0$ of Theorem 6.1.4. Since the de-
teration of the differential equation is purely algebraic and can
be carried over rather easily this provides a nice computational test
to decide whether or not a given web is linearizable. The draw-back
is that, in contrast with the case $k = 4$, the differential equation ob-
tained does not behaves nicely under arbitrary change of coordinates
$(x, y) \mapsto (X(x, y), Y(x, y))$ as a simple computation shows. Indeed,
it can be verified that in the resulting equation $Y'' = G(X, Y, Y')$ the
function $G(x, y, p)$ may be no longer polynomial, but only rational,
in the variable $p$.

Corollary 6.1.9. Let $\mathcal{W}$ be a smooth linearizable $k$-web on $(\mathbb{C}^2, 0)$
with $k \geq 4$. Modulo projective transformations it admits a unique
linearization: if $\varphi, \psi$ are germs of biholomorphisms such that $\varphi^*W$
and $\psi^*W$ are linear webs then there exists a $g \in PGL_2(\mathbb{C})$ for which
$\psi = g \circ \varphi$. 

Proof. The hypothesis implies that $\mu = \varphi \circ \psi^{-1}$ verifies $\mu^* \Pi_0 = \Pi_0$. In other words, if $U \subset \mathbb{P}^2$ is an open subset where $\mu$ is defined then for every line $\ell \subset \mathbb{P}^2$ intersecting $U$, there exists a line $\ell_\mu$ such that $\mu(U \cap \ell) = \mu(U) \cap \ell_\mu$. The fundamental theorem of projective geometry implies that $\mu$ is the restriction at $U$ of a projective transformation.

6.1.4 First examples of exceptional webs

Corollary 6.1.9 is rather useful to prove that a given web is not linearizable. Most of the examples of exceptional webs known up-to-date are the superposition of an algebraic, in particular linear, $k$-web with one or more non-linear foliations. For example, Bol’s 5 web is the superposition of the algebraic 4-web dual to four lines in general position and of a non-linear foliation: the pencil of conics through the four dual points. It follows from Corollary 6.1.9 that $\mathcal{B}_5$ is non-algebraizable.

On the other hand, Bol realized that the rank of $\mathcal{B}_5$ is six. If $\mathcal{B}_5$ is presented as the web

$$\mathcal{B}_5 = \mathcal{W} \left( x, y, \frac{x}{y}, \frac{1 - y}{1 - x}, \frac{x(1 - y)}{y(1 - x)} \right)$$

then the abelian relations coming from its 3-subwebs generate a subspace of $\mathcal{A}(\mathcal{B}_5)$ of dimension 5. Moreover, Bol found one extra abelian relation, which can be written in integral form using the logarithm and Euler’s dilogarithm\textsuperscript{*}, which is essentially equivalent to Abel’s functional equation for the dilogarithm. Explicitly,

$$D_2(x) - D_2(y) - D_2 \left( \frac{x}{y} \right) - D_2 \left( \frac{1 - y}{1 - x} \right) + D_2 \left( \frac{x(1 - y)}{y(1 - x)} \right) = 0$$

where $D_2(z) = \text{Li}_2(z) + \frac{1}{2} \log(z) \log(1 - z) - \frac{z^2}{2}$.

Although $\mathcal{B}_5$ was the first exceptional web to appear in the literature, there are simpler examples. For instance the 5-webs presented

\textsuperscript{*}Euler’s dilogarithm is the function $\text{Li}_2(z) = \sum_{n=0}^{\infty} z^n/n^2$. The series converges for $|z| < 1$ and has analytic continuations along all paths contained in $\mathbb{C} \setminus \{0, 1\}$. 
in Example 2.1.4 of Chapter 2, are all the superposition of four pencils of lines and one non-linear foliation, thus non-algebraizable. Since they all have rank six, they are exceptional webs.

6.1.5 Algebraization of planar webs

Let now \( W = F_1 \sharp \cdots \sharp F_k \) be a smooth k-web on \((\mathbb{C}^2, 0)\). In contrast with the higher dimensional case, no hypothesis on \( A(W) \) is needed to assume that the foliation \( F_i \) is induced by \( \omega_i = \varpi_0 + \theta_i \varpi_1 \), where \( \varpi = (\varpi_0, \varpi_1) \) is a coframe and \( \theta_1, \ldots, \theta_k \) are functions on \((\mathbb{C}^2, 0)\).

**Proposition 6.1.10.** The assertions below are equivalent:

1. \( W \) is compatible with a projective connection \( \Pi \);
2. there exist functions \( N_0, N_1, N_2, N_3 \) such that

\[
\{d\theta_i\}^1 - \theta_i \{d\theta_i\}^0 = \sum_{\rho=0}^3 (\theta_i)^\rho N_\rho \quad (6.7)
\]

for every \( i \in k \).

**Proof.** Let \( \nabla \) be an affine connection representing \( \Pi \), and let \( \Gamma^k_{ij} \) (with \( i, j, k = 0, 1 \)) be its Christoffel's symbols relative to the coframe \( \chi = (\chi_0, \chi_1) \) dual to the coframe \( \varpi \). For \( i \in k \), set \( v_i = \theta_i \chi_0 - \chi_1 \); it is a nowhere vanishing section of \( T F_i \). If \( i \in k \) is fixed then the leaves of \( F_i \) are geodesics of \( \Pi \) if and only if there exists \( \zeta_i \in \Omega^1(\mathbb{C}^2, 0) \) satisfying \( \nabla(v_i) = \zeta_i \otimes v_i \). More explicitly

\[
d\theta_i \otimes \chi_0 + \theta_i \nabla(\chi_0) - \nabla(\chi_1) = \zeta_i \otimes (\theta_i \chi_0 - \chi_1).
\]

After decomposing this relation in the basis \( \chi_p \otimes \omega_q \) (with \( p, q = 0, 1 \)), it is a simple matter to deduce the following four scalar equations:

\[
\{d\theta_i\}^0 + \theta_i \Gamma^0_{00} - \Gamma^0_{01} = \theta_i \{\zeta_i\}^0
\]

\[
\{d\theta_i\}^1 + \theta_i \Gamma^1_{00} - \Gamma^1_{01} = \theta_i \{\zeta_i\}^1
\]

\[
\theta_i \Gamma^1_{00} - \Gamma^1_{01} = -\{\zeta_i\}^0
\]

\[
\theta_i \Gamma^1_{10} - \Gamma^1_{11} = -\{\zeta_i\}^1.
\]

(6.8)
Notice that the last two equations determine \( \{ \zeta_i \}_0 \) and \( \{ \zeta_i \}_1 \). Plugging them into the first two equations to deduce the following.

If the leaves of \( \mathcal{F}_i \) are geodesics for \( \Pi \) then \( \theta_i \) verifies

\[
\{ d\theta_i \}_1 - \theta_i \{ d\theta_i \}_0 = A + \theta_i B + (\theta_i)^2 C + (\theta_i)^3 D \tag{6.9}
\]

where

\[
A = \Gamma_{11}^0 \\
B = \Gamma_{11}^1 - \Gamma_{10}^0 - \Gamma_{01}^0 \\
C = \Gamma_{00}^0 - \Gamma_{10}^1 - \Gamma_{01}^1 \\
D = \Gamma_{00}^1.
\tag{6.10}
\]

Hence the first assertion does imply the second.

Reciprocally, if the second assertion holds true, – what is clearly equivalent to the validity of (6.9) – let \( \nabla \) be the affine connection with Christoffel’s symbols (in the coframe \( \pi \)) determined by (6.10) and the last two equations of (6.8). It is a simple matter to verify that the result is indeed an affine connection which represents a projective connection having the leaves of the web as geodesics.

The next result, when \( k = 5 \), is discussed in Section \( \S 30 \) of the book [18]. In its most general form, it has been obtained by Hénaut in [66]. He phrased it in a slightly different form and under the stronger assumption that the rank of \( \mathcal{W} \) is maximal, but his proof still works under the weaker assumption stated below.

**Theorem 6.1.11.** Let \( \mathcal{W} \) be a smooth \( k \)-web on \( (\mathbb{C}^2, 0) \). If

1. \( \dim A(\mathcal{W})/F^2 A(\mathcal{W}) = 2k - 5 \); and
2. \( \mathcal{W} \) is compatible with a projective connection

then \( \mathcal{W} \) is algebraizable.

**Proof.** If \( k \geq 4 \) and \( \dim A(\mathcal{W})/F^2 A(\mathcal{W}) = 2k - 5 \) then it is possible to construct a Poincaré-Blaschke map \( PB_{\mathcal{W}} : (\mathbb{C}^2, 0) \times \mathbb{P}^1 \rightarrow \mathbb{P}^{2k-6} \). If the rank of this map is two then the argument used at the end of the previous Chapter allows to conclude that \( \mathcal{W} \) is linearizable and, consequently, of maximal rank.

The key result to establish the bound on the rank of \( PB_{\mathcal{W}} \) is Lemma 5.3.11. Its proof is based on the relations (5.6) and (5.7) of
Proposition 5.3.6. Recall that relations (5.6) does hold true, in no matter which dimension. A careful reading of the proof of Lemma 5.3.11 reveals that to prove it, one just need to have relations similar to (5.7) but with the summation in the right hand-side being allowed to range from 0 to \( n + 1 \) instead of from 0 to \( n \).

But, according to Proposition 6.1.10, the existence of such relations is equivalent to the compatibility of \( \mathcal{W} \) with a projective connection. Thus the proof of Trépreau’s Theorem presented in the previous Chapter works as well in dimension two when \( \mathcal{W} \) is compatible with a projective connection.

6.2 Infinitesimal automorphisms

As already mentioned, exceptional planar webs and their abelian relations are still mysterious up-to-date. For instance, even for a web defined by rational submersions it is not known what kind of transcendency its abelian relations can have. To formulate this question more precisely it is useful to recall first some basic definitions of differential algebra.

6.2.1 Basics on differential algebra

Recall that a differential field\(^\dagger\) is a pair \((\mathbb{K}, \Delta)\), where \(\mathbb{K}\) is field containing \(\mathbb{C}\), and \(\Delta\) is a finite collection of \(\mathbb{C}\)-derivations of \(\mathbb{K}\) subject to the conditions

(a). any two derivations in \(\Delta\) commute;

(b). the field of constants of \(\Delta\), that is the intersection of the kernels of the derivations in \(\Delta\), is equal to \(\mathbb{C}\).

A differential extension of \((\mathbb{K}, \Delta)\) is a differential field \((\mathbb{K}_0, \Delta_0)\) such that \(\mathbb{K}_0\) is a field extension of \(\mathbb{K}\), and for every \(\partial_0 \in \Delta_0\) there exists a unique \(\partial \in \Delta\) satisfying \(\partial = \partial_0|_{\mathbb{K}}\). A differential extension \((\mathbb{K}_0, \Delta_0)\) of \((\mathbb{K}, \Delta)\) is said to be primitive if there exists an element \(h \in \mathbb{K}_0\) such that \(\mathbb{K}_0 = \mathbb{K}(h)\).

\(^\dagger\)Here only differential fields over \(\mathbb{C}\) will be considered. Of course, it is possible to deal with more general fields.
The simplest kind of differential extension is when $K_0$ is an algebraic field extension of $K$. These are called **algebraic extensions**.

Another particularly simple kind of differential extension are the so-called **Liouvillian extensions**. A differential field $(K', \Delta')$ is a Liouvillian extension of $(K, \Delta)$, if there exists a finite sequence of differential extensions

$$(K, \Delta) = (K_1, \Delta_1) \subset \cdots \subset (K_r, \Delta_r) \subset (K_{r+1}, \Delta_{r+1}) = (K', \Delta')$$

such that for each $i \in \mathbb{N}$, $K_{i+1} = K_i(h_i)$ for some $h_i \in K_{i+1}$ satisfying one of the following conditions

(a) $h_i$ is algebraic over $K_i$, or ;
(b) for every $\partial \in \Delta_{i+1}$, $\partial h_i$ belongs to $K_i$, or;
(c) for every $\partial \in \Delta_{i+1}$, $\frac{\partial h_i}{h_i}$ belongs to $K_i$.

If $(K, \Delta) = (C((x, y)), \{\partial_x, \partial_y\})$ is the differential field of germs of meromorphic functions at the origin of $\mathbb{C}^2$ endowed with the natural derivations $\partial_x, \partial_y$, then a primitive Liouvillian extension over it is obtained by taking (a) a primitive algebraic extension; or (b) the integral of a closed meromorphic 1-forms; or (c) the exponential of the integral of a closed meromorphic 1-form.

If $(K, \{\partial_x, \partial_y\})$ is a differential subfield of $(C((x, y)), \{\partial_x, \partial_y\})$ then, by definition, a web $W$ on $(C^2, 0)$ is **defined over** $K$ if there exists a $k$-symmetric 1-form with coefficients in $K$ defining $W$. More explicitly, there exists

$$\omega = \sum_{i+j=k} a_{ij}(x, y)dx^idy^j \in \text{Sym}^k \Omega^1(C^2, 0)$$

such that $W = W(\omega)$ and $a_{ij} \in K$ for every pair $(i, j) \in \mathbb{N}^2$ satisfying $i + j = k$. Similarly, an abelian relation of a given smooth web on $(C^2, 0)$ is defined over $K$, if its components are 1-forms with coefficients in $K$. If $W$ is a web defined over $K$ then the **field of definition** of its abelian relations is the differential extension of $K$ generated by all coefficients of all components of all abelian relations of $W$. 
Problem 1. Let \( W \) be a germ of smooth \( k \)-web on \((\mathbb{C}^2, 0)\). If \( W \) is defined over \( \mathbb{K} \), what can be said about the field of definition of its abelian relations? Is it a Liouvillian extension of \( \mathbb{K} \)?

If \( W \) is a hexagonal 3-web on \((\mathbb{C}^2, 0)\) then its unique abelian relation is defined over a Liouvillian extension of its field of definition. An argument has already been given in the proof of the implication (b) \( \implies \) (c) of Theorem 1.2.4.

An answer, but in a very particular case

Due to the apparent difficulty of Problem 1 one could propose to low down the dimension of the ambient space in order to obtain some progress. At first sight this seems to be pure non-sense, since it doesn’t appear to be reasonable to talk about webs on \((\mathbb{C}, 0)\). A way out, is to interpret less strictly the lowing down of the dimension. In the study of systems of differential equations, the usual setup where one is allowed to low down dimensions is when the system posses infinitesimal symmetries. Having this vague discourse in mind, it suffices to look back at Chapter 2 to realize that it has been formally implemented in the proof of Proposition 2.1.5. In particular, one obtains as a corollary the following

Proposition 6.2.1. Let \((\mathbb{K}, \{\partial_x, \partial_y\})\) be a differential subfield of \((\mathbb{C}(x, y)), \{\partial_x, \partial_y\}\), and let \( W \) be a germ of smooth \( k \)-web defined over \( \mathbb{K} \). If \( W \) admits a transverse infinitesimal automorphism, also defined over \( \mathbb{K} \), then there exists a Liouvillian extension of \( \mathbb{K} \) over which all the abelian relations of \( W \) are defined.

Proof. It follows immediately from Proposition 2.1.5.

6.2.2 Variation of the rank

Proposition 2.1.5 also allows to compare the rank of a web \( W \) admitting a transverse infinitesimal automorphism \( v \), with the rank of the web \( W \boxtimes \mathcal{F}_v \) obtained by the superposition of \( W \) and the foliation induced by \( v \).
Theorem 6.2.2. Let $\mathcal{W}$ be a smooth $k$–web which admits a transverse infinitesimal automorphism $v$. Then

$$\text{rank}(\mathcal{W} \boxtimes \mathcal{F}_v) = \text{rank}(\mathcal{W}) + (k - 1).$$

In particular, $\mathcal{W}$ is of maximal rank if and only if $\mathcal{W} \boxtimes \mathcal{F}_v$ is of maximal rank.

Proof. Let $\mathcal{W} = \mathcal{W}(\omega_1, \ldots, \omega_k) = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. Recall from Chapter 2 that the canonical first integral of $\mathcal{F}_i$ relative to $v$ is

$$u_i = \int \frac{\omega_i}{\omega_i(v)}.$$

In particular, its differential is $\eta_i = \frac{\omega_i}{\omega_i(v)} = du_i$.

Notice that when $j$ varies from 2 to $k$, the following identities hold

$$i_v(\eta_1 - \eta_j) = 0 \quad \text{and} \quad L_v(\eta_1 - \eta_j) = 0.$$

Consequently there exists $g_j \in \mathbb{C}\{t\}$ for which

$$du_1 - du_j - g_j(u_{k+1})du_{k+1} = 0,$$

where $u_{k+1}$ is a first integral of $\mathcal{F}_v$.

Clearly these are abelian relations for the web $\mathcal{W} \boxtimes \mathcal{F}_v$. If $\mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_v)$ stands for the maximal eigenspace of $L_v$ associated to the eigenvalue zero, then the abelian relations above span a vector subspace of it which will be denoted by $\mathcal{V}$. Notice that $\dim \mathcal{V} = k - 1$.

Observe that $\mathcal{V}$ fits into the sequence

$$0 \to \mathcal{V} \xrightarrow{i} \mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_v) \xrightarrow{L_v} \mathcal{A}_0(\mathcal{W}).$$

Notice that this sequence is exact. Indeed, $K = \ker \{L_v : \mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_v) \to \mathcal{A}_0(\mathcal{W})\}$ is generated by abelian relations of the form

$$\sum_{i=1}^k c_i du_i + h(u_{k+1})du_{k+1} = 0,$$

where $c_i \in \mathbb{C}$ and $h \in \mathbb{C}\{t\}$. Since $i_v du_i = 1$ for each $i \in \mathbb{C}$, it follows that the constants $c_i$ satisfy $\sum_{i=1}^k c_i = 0$. This implies that the abelian relations in the kernel of $L_v$ can be written as linear combinations of abelian relations of the form (6.11). Therefore

$$K = \mathcal{V}$$

(6.12)
and consequently \( \ker L_v \subset \text{Im } i \). The exactness of the above sequence follows easily.

From general principles one can deduce that the sequence

\[
0 \to \frac{\mathcal{V}}{\mathcal{A}_0(W) \cap \mathcal{V}} \xrightarrow{i} \frac{\mathcal{A}_0(W \boxtimes \mathcal{F}_v)}{\mathcal{A}_0(W)} \xrightarrow{L_v} \frac{\mathcal{A}_0(W)}{L_v \mathcal{A}_0(W)},
\]

is also exact. Thus to prove the theorem it suffices to verify that

(a) \( \mathcal{V} \) is isomorphic to \( \frac{\mathcal{V}}{\mathcal{A}_0(W) \cap \mathcal{V}} \oplus \frac{\mathcal{A}_0(W)}{L_v \mathcal{A}_0(W)} \);

(b) the morphism \( L_v : \mathcal{A}_0(W \boxtimes \mathcal{F}_v) \to \mathcal{A}_0(W) \) is surjective;

(c) the vector spaces \( \frac{\mathcal{A}_0(W \boxtimes \mathcal{F}_v)}{\mathcal{A}_0(W)} \) and \( \frac{\mathcal{A}(W \boxtimes \mathcal{F}_v)}{\mathcal{A}_0(W)} \) are isomorphic.

The key to verify (a) is the nilpotency of \( L_v \) on \( \mathcal{A}_0(W) \). It implies that \( \mathcal{A}_0(W) \) is isomorphic to \( \mathcal{A}_0(W) \cap K \). Combining this with (6.12) assertion (a) follows.

To prove assertion (b) it suffices to construct a map \( \Phi : \mathcal{A}_0(W) \to \mathcal{A}_0(W \boxtimes \mathcal{F}_v) \) such that \( L_v \circ \Phi = \text{Id} \). Proposition 2.1.5 implies that \( \mathcal{A}_0(W) \) is spanned by abelian relations of the form \( \sum_{i=1}^{k} c_i u_i^r d u_i = 0 \), where \( c_1, \ldots, c_k \) are complex numbers and \( r \) is a non-negative integer. Since

\[
\sum_{i=1}^{k} c_i u_i^r d u_i = \frac{1}{r+1} L_v \left( \sum_{i=1}^{k} c_i u_i^{r+1} d u_i \right) = 0
\]

there exists a unique function \( h \in \mathbb{C} \{t\} \) satisfying

\[
\sum_{i=1}^{k} c_i u_i^{r+1} d u_i + h(u_{k+1}) d u_{k+1} = 0.
\]

If one sets

\[
\Phi \left( \sum_{i=1}^{k} c_i u_i^r d u_i \right) = \frac{1}{r+1} \left( \sum_{i=1}^{k} c_i u_i^{r+1} d u_i + h(u_{k+1}) d u_{k+1} \right)
\]

then \( L_v \circ \Phi = \text{Id} \) and assertion (b) follows.

To prove assertion (c), first notice that

\[
\mathcal{A}(W \boxtimes \mathcal{F}_v) = \mathcal{A}_0(W \boxtimes \mathcal{F}_v) \oplus \mathcal{A}_* (W \boxtimes \mathcal{F}_v)
\]
where $\mathcal{A}_v(W \boxtimes \mathcal{F}_v)$ is the sum of eigenspaces corresponding to non-zero eigenvalues. Of course $\mathcal{A}_v(W \boxtimes \mathcal{F}_v)$ is invariant by $L_v$. Moreover the equality

$$L_v(\mathcal{A}_v(W \boxtimes \mathcal{F}_v)) = \mathcal{A}_v(W \boxtimes \mathcal{F}_v),$$

holds true. On the other hand $L_v$ kills the component of an abelian relation corresponding to the foliation $\mathcal{F}_v$. In particular

$$L_v(\mathcal{A}_v(W \boxtimes \mathcal{F}_v)) \subset \mathcal{A}_v(W).$$

This is sufficient to show that $\mathcal{A}_v(W \boxtimes \mathcal{F}_v) = \mathcal{A}_v(W)$ and deduce assertion (c).

Putting all together, it follows that

$$\text{rank}(W \boxtimes \mathcal{F}_v) = \text{rank}(W) + (k - 1).$$

To prove the last claim of the theorem, just remark that the $(k + 1)$-web $W \boxtimes \mathcal{F}_v$ is of maximal rank if and only if

$$\text{rank}(W \boxtimes \mathcal{F}_v) = \frac{k(k-1)}{2} = \frac{(k-1)(k-2)}{2} + (k - 1).$$

This result was obtained in [77] by David Marín together with the authors of this book. As a corollary, it was then obtained the existence of exceptional planar $k$-webs for every $k \geq 5$, as explained in the next section.

Of course, one can also deduce from Theorem 6.2.2 the following analogue of Proposition 6.2.1.

**Proposition 6.2.3.** Let $(\mathbb{K}, \{\partial_x, \partial_y\})$ be a differential subfield of $(\mathbb{C}((x, y)), \{\partial_x, \partial_y\})$, and let $W$ be a germ of smooth $k$-web defined over $\mathbb{K}$. If $W$ admits a transverse infinitesimal automorphism, also defined over $\mathbb{K}$, then there exists a Liouvillian extension of $\mathbb{K}$ over which all the abelian relations of $W \boxtimes \mathcal{F}_v$ are defined.

It should not be very hard to drop the hypothesis, in Proposition 6.2.1 as well as in Proposition 6.2.3, on the field of definition of the infinitesimal automorphism of $W$. More precisely, it should be true that the infinitesimal automorphisms of a web defined over $\mathbb{K}$ should also be defined over $\mathbb{K}$, or at least over a Liouvillian extension of $\mathbb{K}$.
6.2.3 Infinitely many families of exceptional webs

Let $C$ be a degree $k$ curve in $\mathbb{P}^2$ invariant by a $\mathbb{C}^*$-action $\varphi : \mathbb{C}^* \times \mathbb{P}^2 \to \mathbb{P}^2$. Notice that $\varphi$ induces a dual action $\tilde{\varphi} : \mathbb{C}^* \times \tilde{\mathbb{P}}^2 \to \tilde{\mathbb{P}}^2$ which is a one-parameter group of automorphisms of the dual $k$-web $\mathcal{W}_C$. Consequently, the web $\mathcal{W}_C(\ell_0)$, the germification of $\mathcal{W}_C$ at a generic point $\ell_0 \in \tilde{\mathbb{P}}^2$ admits an infinitesimal automorphism.

It is a simple matter to show that in a suitable projective coordinate system $[x : y : z]$, a plane curve $C$ invariant by a $\mathbb{C}^*$-action is cut out by an equation of the form

$$x^{\epsilon_1} \cdot y^{\epsilon_2} \cdot z^{\epsilon_3} \cdot \prod_{i=1}^{n} (x^a + \lambda_i y^b z^{a-b})$$  \hspace{1cm} (6.13)

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0,1\}$, $n, a, b \in \mathbb{N}$ are such that $n \geq 1$, $a \geq 2$, $1 \leq b \leq a/2$, $\gcd(a,b) = 1$ and the $\lambda_i$ are distinct non zero complex numbers. For a curve of this form the $\mathbb{C}^*$-action in question is

$$\varphi : \mathbb{C}^* \times \mathbb{P}^2 \to \mathbb{P}^2$$

$$(t, [x : y : z]) \mapsto [t^b(a-b)x : t^a(a-b)y : t^{ab}z].$$

Moreover once $\epsilon_1, \epsilon_2, \epsilon_3, n, a, b$ are fixed, one can always choose $\lambda_1 = 1$ and in this case the set of $n-1$ complex numbers $\{\lambda_2, \ldots, \lambda_n\}$ projectively characterizes the curve $C$. In particular, there exists a $(d-1)$-dimensional family of degree $2d$ (or $2d+1$) reduced plane curves all projectively distinct and invariant by the same $\mathbb{C}^*$-action: for a given $2d + \delta$ with $\delta \in \{0,1\}$ set $a = 2$, $b = 1$, $\epsilon_1 = \delta$ and $\epsilon_2 = \epsilon_3 = 0$.

If $C$ is a reduced curve of the form (6.13) then $\mathcal{W}_C$ is invariant by the $\mathbb{C}^*$-action $\tilde{\varphi}$ dual to $\varphi$. Denote by $v$ the infinitesimal generator of $\tilde{\varphi}$ and by $\mathcal{F}_v$ the corresponding foliation.

**Theorem 6.2.4.** If $\deg C \geq 4$ then $\mathcal{W}_C \boxtimes \mathcal{F}_v$ is exceptional. Moreover if $C'$ is another curve invariant by $\varphi$ then $\mathcal{W}_C \boxtimes \mathcal{F}_v$ is analytically equivalent to $\mathcal{W}_{C'} \boxtimes \mathcal{F}_{v'}$ if and only if the curve $C$ is projectively equivalent to $C'$.

**Proof.** Since $\mathcal{W}_C$ has maximal rank it follows from Theorem 6.2.2 that $\mathcal{W}_C \boxtimes \mathcal{F}_v$ is also of maximal rank. Suppose that its localization
at a point $\ell_0 \in \mathbb{P}^2$ is algebraizable and let $\psi : (\mathbb{P}^2, \ell_0) \to (\mathbb{C}^2, 0)$ be a holomorphic algebraization. Since both $\mathcal{W}_C$ and $\psi_*(\mathcal{W}_C)$ are linear webs of maximal rank it follows from Corollary 6.1.9 that $\psi$ is the localization of an automorphism of $\mathbb{P}^2$. But the generic leaf of $\mathcal{F}_v$ is not contained in any line of $\mathbb{P}^2$ and consequently $\psi_*(\mathcal{W} \boxtimes \mathcal{F}_v)$ is not linear. This concludes the proof of the theorem.

Remark 6.2.5. It is not known if the families of examples above are irreducible components of the space of exceptional webs in the sense that the generic element does not admit a deformation as an exceptional web that is not of the form $\mathcal{W}_C \boxtimes \mathcal{F}_v$. Due to the presence of infinitesimal automorphisms one could imagine that they are indeed degenerations of some other exceptional webs.

6.3 Pantazi-Hénaut criterium

If $\mathcal{W}$ is a 3-web on $(\mathbb{C}^2, 0)$ then $\mathcal{W}$ has maximal rank if and only if its curvature $K(\mathcal{W})$ vanishes identically. In this section, a generalization of this result to arbitrary planar webs will be presented. The strategy sketched below can be traced back to Pantazi [84]. Recently, unaware of Pantazi’s result, Hénaut [68] proved an essentially equivalent result but formulated in more intrinsic terms.

Even more recently, Cavalier and Lehmann proved that it is possible to extend Pantazi-Hénaut construction to ordinary codimension one webs on $(\mathbb{C}^n, 0)$ for arbitrary $n$. This text will not deal with this generalization. For details see [26].

Below, after presenting the basics of the theory of linear differential systems, the arguments of [84] are presented using the modern formalism introduced in this context by [68].

6.3.1 Linear differential systems

For a more detailed exposition the reader can consult [102] and references therein.
Jet spaces

Let $E$ be a rank $r$ vector bundle over $(\mathbb{C}^n, 0)$. Since the setup is local, $E$ is of course trivial. Thus $x, p$ with $x = (x_1, \ldots, x_n) \in (\mathbb{C}^n, 0)$ and $p = (p^1, \ldots, p^r) \in \mathbb{C}^r$ constitute a coordinate system on the total space of $E$. A section $\xi$ of $E$ will be identified with a map $(\xi^1, \ldots, \xi^r): (\mathbb{C}^n, 0) \to \mathbb{C}^r$.

The space $J^\ell(E)$ of $\ell$-jets of sections of $E$ is the vector bundle over $(\mathbb{C}^n, 0)$ with fiber $J^\ell_x(E)$ over a point $x \in (\mathbb{C}^n, 0)$ equal to the quotient of the space of germs of sections of $E$ at $x$ by the subspace of germs vanishing at $x$ up to order $\ell + 1$. Given a section $\xi$ of $E$ then its $\ell$-jet at $x$ will be denoted by $j^\ell_x(\xi)$. In order to make sense of the following map

$$j^\ell : E \longrightarrow J^\ell(E)$$

one has to think of it not as a map of vector bundles (derivations are not $\mathcal{O}(\mathbb{C}^n, 0)$-linear) but as a morphism of sheaves of $\mathbb{C}$-modules.

On $J^\ell_x(E)$ there is natural system of linear coordinates:

$$p^s_x(j^\ell_x(\xi)) = \partial_x(\xi^s)(x) = \frac{\partial^{|\sigma|} \xi^s}{\partial x^\sigma}(x)$$

for $s \in \mathbb{R}$ and $\sigma = (\sigma_a)_{a=1}^n \in \mathbb{N}^n$ with $|\sigma| = \sum_a \sigma_a \leq \ell$.

For every $\ell, \ell'$ with $\ell \geq \ell'$, there is a natural morphism of vector bundles (unlike $j^\ell$ this morphisms is $\mathcal{O}(\mathbb{C}^n, 0)$-linear)

$$\pi^\ell_{\ell'} : J^\ell(E) \longrightarrow J^{\ell'}(E)$$

$$(x, j^\ell_x(\xi)) \longmapsto (x, j^{\ell'}_x(\xi)).$$

By convention, $J^q(E)$ is the 0 vector bundle when $q < 0$.

Linear differential systems

A linear differential system of order $q$ in $r$ indeterminates is defined as the kernel $S = \text{Ker} \Phi \subset J^q(E)$ of a linear differential operator of order $q$ in $r$ indeterminates, that is a morphism of $\mathcal{O}(\mathbb{C}^n, 0)$-modules

$$\Phi : J^q(E) \longrightarrow F.$$
Explicitly, if \( F = (\mathbb{C}^n, 0) \times \mathbb{C}^m \) and \( \Phi = (\Phi^1, \ldots, \Phi^m) : J^q(E) \rightarrow F \) with
\[
\Phi^\kappa(x, p) = \sum_{s, |\sigma| \leq q} A_{s, \sigma}^\kappa(x) p_\sigma^s, \quad \kappa \in \mathbb{m}
\]
then the sections of \( S = \text{Ker} \Phi \) which are also images of sections of \( E \) through \( j^q : E \rightarrow J^q(E) \) correspond to solutions of the system of differential equations
\[
(S) \sum_{s, |\sigma| \leq q} A_{s, \sigma}^\kappa \partial_{\sigma}(\varphi^s) = 0 \quad \kappa \in \mathbb{m}
\]
in the unknowns \( \varphi^1, \ldots, \varphi^r : (\mathbb{C}^n, 0) \rightarrow \mathbb{C} \). Beware that in general \( S \subset J^q(E) \) is not a vector subbundle of \( J^q(E) \): the rank of the fibers of \( S \) may vary from point to point.

If the \( q \)-jet of \( (\varphi_1, \ldots, \varphi_r) \) is in \( S \) then its \( (q + 1) \)-jet will be in \( S^{(1)} \subset J^{q+1}(E) \), the homogeneous linear partial differential equation of order \( (q + 1) \) deduced from \( S \) through derivation with respect to the free variables \( x_a \):

\[
(S^{(1)}) \begin{cases} \\
\sum_{s, |\sigma| \leq q} A_{s, \sigma}^\kappa \frac{\partial |\sigma| \varphi^s}{\partial x_\sigma} = 0 & \kappa \in \mathbb{m}; \\
\sum_{s, |\sigma| \leq q} \frac{\partial A_{s, \sigma}^\kappa}{\partial x_a} |\sigma| \varphi^s + A_{s, \sigma}^\kappa \frac{\partial |\sigma|+1 \varphi^s}{\partial x_{\sigma(a)}} = 0 & a \in \mathbb{n}
\end{cases}
\]

with \( \sigma(a) = (\sigma_b)_{b=1}^n \) defined as follows: \( \sigma_a = \sigma_a + 1 \) and \( \sigma_b = \sigma_b \) if \( b \neq a \).

Of course, this operation can be iterated. Setting \( S = S^{(0)} \) and
\[
S^{(\ell+1)} = (S^{(\ell)})^{(1)}
\]
for every \( \ell \geq 0 \), one derives from \( S \) a family of linear systems of partial differential equations \( S^{(\ell)} \subset J^{q+\ell}(E) \). By definition, \( S^{(\ell)} \) is the \( \ell \)-th prolongation of \( S \). It is not hard to construct a morphism of vector bundles
\[
\Phi^{(\ell)} : J^{q+\ell}(E) \rightarrow J^\ell(F)
\]
such that \( S^{(\ell)} = \text{Ker} \Phi^{(\ell)} \). The morphism \( \Phi^{(\ell)} \) is the \( \ell \)-th prolongation of \( \Phi \).
SECTION 6.3: PANTAZI-HÉNAUT CRITERIUM

Formal integrability

A differential system $S$ is regular if $S^{(\ell)}$ is a vector subbundle of $J^{q+\ell}(E)$ for no matter which $\ell \geq 0$.

Let $\ell \geq 0$ be fixed. The restriction of the natural projection $J^{q+\ell+1}(E) \to J^{q+\ell}(E)$ to $S^{(\ell+1)}$ induces a natural morphism of $\mathcal{O}(\mathbb{C}^n, 0)$-modules

$$S^{(\ell+1)} \xrightarrow{\pi^{q+\ell+1}} S^{(\ell)}.$$ 

By definition, $S$ is formally integrable if for every $\ell \geq 0$ the morphism $\pi^{q+\ell+1}$ is surjective. In less precise terms, a system is formally integrable if given a $q$-jet in $S$ then there exists a $(q+\ell)$-jet in $S^{(\ell)} \subset J^{q+\ell}(E)$ coinciding with the original one up to order $q$ for no matter which $\ell \geq 0$.

A solution of $S$ is a section $\sigma$ of $E$ such that $j^q(\sigma)$ is a section of $S^{(\ell)}$. Consequently, $j^{q+\ell}(\sigma)$ is a section of $S^{(\ell+1)}$ for every $\ell \geq 0$. If $\text{Sol}(S)$ denotes the space of solutions of $S$, then the surjectivity of $j^q: \text{Sol}(S) \to S$ on the fibers is a sufficient condition for the formal integrability of $S$.

In the analytic category the formal integrability of a differential system is particularly meaningful: Cartan-Kähler theorem ensures the convergence of formal solutions. In the case of linear differential systems, the proof of this result boils down to a simple application of the method of majorants for formal power series.

For $\ell \geq 0$, let $f_1, \ldots, f_\ell \in \mathcal{O}(\mathbb{C}^n, 0)$§ be functions in the maximal ideal $\mathfrak{m}_x$ of $x$, that is $f_1(x) = \cdots = f_\ell(x) = 0$. Set $f = f_1 \cdots f_\ell$. Since $\text{Sym}^\ell(T^*_x(\mathbb{C}^n, 0))$ is generated, as a $\mathbb{C}$-vector space, by the $\ell$-symmetric forms $(df_1 \cdots df_\ell)(x)$, one can define a linear map

$$\varepsilon_x^\ell: \text{Sym}^\ell(T^*_x(\mathbb{C}^n, 0)) \otimes E \to J^\ell_x(E)$$

through the formula

$$\varepsilon_x^\ell(df_1 \cdots df_\ell \otimes e) = j^\ell(f_1 \cdots f_\ell e)(x).$$

§Recall the convention about germs used throughout. Here $(\mathbb{C}^n, 0)$ must be thought as a small open subset containing the origin.
Because \( f \in (\mathcal{M}_x)_{\ell} \), \( \pi_{\ell}^{-1}(j^{\ell}(fe)(x)) = 0 \) for every section \( e \) of \( E \). Varying \( x \in (\mathbb{C}^n, 0) \), one deduces an injective morphism of vector bundles

\[
\begin{array}{c}
\text{Sym}^\ell(T^*U) \otimes E \\
\downarrow^\varepsilon \quad \downarrow^{\pi_{\ell}} \quad \downarrow^j \quad \downarrow \pi_{\ell-1}
\end{array}
\rightarrow
\begin{array}{c}
J^\ell(E) \\
\downarrow
\end{array}
\]

which fits into the exact sequence

\[
0 \rightarrow \text{Sym}^{\ell+1}(T^*U) \otimes E \xrightarrow{\varepsilon^\ell} J^{\ell+1}(E) \xrightarrow{\pi_{\ell+1}} J^\ell(E) \rightarrow 0. \quad (6.14)
\]

For \( \ell \geq 0 \), the \( \ell \)-th symbol map of \( \Phi : J^q(E) \rightarrow F \) is defined as the composition

\[
\sigma^{(\ell)}(\Phi) = \Phi^{(\ell)} \circ \varepsilon^\ell : \text{Sym}^{q+\ell}(T^*U) \otimes E \longrightarrow J^\ell(F).
\]

The 0-th symbol map is, by convention, \( \sigma(\Phi) = \sigma^{(0)}(\Phi) \). By definition, \( \mathcal{S}^{(\ell)} = \ker \sigma^{(\ell)}(\Phi) \) is the \( \ell \)-th symbol of the system \( S \). It is completely determined by \( S \), that is it does not depend on the presentation of \( S \) as the kernel of \( \Phi \). From the exact sequence (6.14), it follows that \( \mathcal{S}^{(\ell)} \) fits into the exact sequence of \( \mathbb{C} \)-sheaves

\[
0 \rightarrow \mathcal{S}^{(\ell)} \xrightarrow{\varepsilon} S^{(\ell)} \xrightarrow{\pi} S^{(\ell-1)} \quad (6.15)
\]

for every \( \ell \geq 0 \), with the convention that \( S^{(-1)} = J^{q-1}(E) \).

Notice that \( \mathcal{S}^{(\ell)} \) is not a vector bundle a priori: it can be naively thought as a family of vector spaces \( \{ \mathcal{S}^{(\ell)}(x) \}_{x \in (\mathbb{C}^n, 0)} \) but the dimension may vary with \( x \).

It can be verified that \( \mathcal{S}^{(\ell)} \) is completely determined by \( \mathcal{S} \). In particular, if \( \mathcal{S} = 0 \) then \( \mathcal{S}^{(\ell)} = 0 \) for every \( \ell \geq 0 \).

**Theorem 6.3.1.** Let \( S \subset J^q(E) \) be a linear differential system with \( \mathcal{G}_S = 0 \). If the natural morphism \( S^{(1)} \rightarrow S \) is surjective then \( S \) is regular and formally integrable.

**Proof.** This is a particular case of a theorem by Goldschmidt, see [102, Theorem 1.5.1]. The proof is omitted. \( \square \)
SECTION 6.3: PANTAZI-HÉNAUT CRITERIUM

Spencer operator and connections

The **Spencer operator** (for \( s, q \geq 0 \))

\[
D : \Omega^s \otimes J^q(E) \to \Omega^{s+1} \otimes J^{q-1}(E)
\]

is characterized by the following properties: (1) for section \( \sigma \) of \( E \), \( D(j^q(\sigma)) = 0 \); (2) for every \( \omega \in \Omega^s \) and \( \eta \in \Omega^s \otimes J^q(E) \), \( D(\omega \wedge \eta) = d\omega \wedge \pi_q^{-1}(\eta) + (-1)^j \omega \wedge D(\eta) \).

It can be verified that \( D \circ D = 0 \). Consequently, there is following complex of \( \mathbb{C} \)-sheaves

\[
0 \to E \xrightarrow{j^q} J^q(E) \xrightarrow{D} \Omega^1 \otimes J^{q-1}(E) \xrightarrow{D} \cdots \xrightarrow{D} \Omega^n \otimes J^{q-n}(E) \to 0
\]

(6.16)

from which one can extract the exact sequence

\[
0 \to E \xrightarrow{j^q} J^q(E) \xrightarrow{D} \Omega^1 \otimes J^{q-1}(E).
\]

To simplify, suppose the dimension is two. If \( S \subset J^q(E) \) is a linear differential system over \((\mathbb{C}^2, 0)\) then the restriction of the complex (6.16) to \( S \) and its prolongations yields the **first Spencer complex** of \( S^{(\ell)} \):

\[
0 \to \text{Sol}(S) \xrightarrow{j^q} S^{(r)} \xrightarrow{D} \Omega^1 \otimes S^{(r-1)} \xrightarrow{D} \Omega^2 \otimes S^{(r-2)} \to 0
\]

where \( r = q + \ell \). Note that this is not a complex of \( \mathcal{O} \)-sheaves but only of \( \mathbb{C} \)-sheaves. It is clarifying to notice the resemblance with the usual de Rham complex

\[
0 \to \mathbb{C} \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega_1 \xrightarrow{d} \Omega^2 \to 0.
\]

There is the following commutative diagram

\[
\begin{array}{c}
0 \\
\Omega^2 \otimes \mathcal{G}_S \\
\text{Sol}(S) \xrightarrow{j^{q+2}} S^{(2)} \xrightarrow{D} \Omega^1 \otimes S^{(1)} \xrightarrow{D} \Omega^2 \otimes S \\
\text{Sol}(S) \xrightarrow{j^{q+1}} S^{(1)} \xrightarrow{D} \Omega^1 \otimes S \xrightarrow{D} \Omega^2 \otimes J^{q-1}(E)
\end{array}
\]
with columns and lines being complexes.

Suppose now that $S(1) = 0$ and that the natural morphism $S(1) \to S$ is surjective. Together, these two conditions imply the isomorphism $S(1) \simeq S$. Then one can define two operators $\nabla$ et $\nabla'$ of $\mathbb{C}$-sheaves such that the diagram below commutes.

![Diagram](image)

(6.17)

It is possible to associate to this diagram a connection

$$\nabla : S^{(1)} \to \Omega^1 \otimes S^{(1)}$$

with corresponding curvature given by the $\mathcal{O}$-linear operator

$$K_S = \nabla' \circ \nabla : S^{(1)} \to \Omega^2 \otimes \mathcal{G}_S.$$ 

Since the first line of (6.17) is a complex, it is clear that the surjectivity of $S^{(2)} \to S^{(1)}$ implies the vanishing of the curvature $K_S$. From Theorem 6.3.1 applied to $S^{(1)}$, one deduces that $K_S \equiv 0$ if $S \simeq S^{(1)}$ is formally integrable.

Conversely, one can show that $K_S \equiv 0$ implies that $S$ is regular and integrable. This non-trivial fact will not be shown here. Those interested might consult [102].

**Corollary 6.3.2.** If $\mathcal{G}_S^{(1)} = 0$ and the natural morphism $S^{(1)} \to S$ is surjective then $S$ is integrable if and only if $K_S \equiv 0$. 
6.3.2 The differential system $S_W$

The theory sketched above will now be applied to the differential system which has as solutions the integral forms of abelian relations of a smooth planar web.

Let $\mathcal{W} = \mathcal{W}(\omega_1, \ldots, \omega_k)$ be a smooth $k$-web on $(\mathbb{C}^2, 0)$. It is harmless to assume that

$$\omega_i = dx + \theta_i dy$$

for $i \in \mathbb{K}$ and suitable germs $\theta_1, \ldots, \theta_k$.

Let $v_i = \partial_y - \theta_i \partial_x$ and consider the following differential system

$$S_w : \left\{ \begin{array}{l}
\sum_{i=1}^{k} \varphi_i = 0, \\
v_i(\varphi_i) = 0 \quad i \in \mathbb{K}.
\end{array} \right.$$ 

The space of holomorphic solutions of $S_W$ will be denoted by $\text{Sol}(\mathcal{W})$. Notice that $\text{Sol}(\mathcal{W})$ fits into the exact sequence of vector spaces

$$0 \rightarrow A(\mathcal{W}) \xrightarrow{\delta} \text{Sol}(\mathcal{W}) \rightarrow \mathbb{C}^k \xrightarrow{\sum} \mathbb{C} \rightarrow 0,$$

where the arrow from $\text{Sol}(\mathcal{W})$ to $\mathbb{C}^k$ is given by evaluation at the origin. In particular the rank of $\mathcal{W}$ is maximal if and only if $\text{Sol}(\mathcal{W})$ has dimension $k(k - 1)/2$.

Notice that $S_W$ is a differential linear system of first order and as such can be defined as follows. If $E$ (resp. $F$) is the trivial bundle of rank $k$ (resp. $k + 3$) over $(\mathbb{C}^2, 0)$ then $\text{Sol}(\mathcal{W})$ can be identified with the sections of $E$ with first jet belonging to the kernel of the map

$$\Phi = (F_{00}, F_{10}, F_{01}, G_1, \ldots, G_k) : J^1(E) \rightarrow F$$

where

$$F_{\sigma} = \sum_{i=1}^{k} p_{\sigma}^i$$

and

$$G_i = p_{01}^i - \theta_i p_{10}^i$$

for $\sigma$ such that $|\sigma| \leq 1$ and $i \in \mathbb{K}$.

The smoothness of $\mathcal{W}$ promptly implies that $\Phi$ has constant rank. The subbundle $\text{Ker}(\Phi) \subset J^1(E)$ will be identified with $S_W$.
Prolongations of $S_W$

For every $\ell \geq 0$, denote by $S^{(\ell)}_W \subset J^{1+\ell}(E)$ the $\ell$-th prolongation of $S_W$. To write down $S^{(\ell)}_W$, explicitly, set $D^\tau = D^\tau_x \circ D^\tau_y$ for $\tau = (\tau_1, \tau_2) \in \mathbb{N}^2$, where $D_x$ and $D_y$ are the total derivatives

$$
D_x = \frac{\partial}{\partial x} + \sum_{i, \sigma} p^i_{\sigma(1)} \frac{\partial}{\partial p^i_{\sigma}}, \quad D_y = \frac{\partial}{\partial y} + \sum_{i, \sigma} p^i_{\sigma(2)} \frac{\partial}{\partial p^i_{\sigma}}.
$$

Using the notation just introduced it is not hard to check that $S^{(\ell)}_W$ is defined by the linear differential equations

$$
D^\tau(F_\sigma) = 0 \quad \sigma = 0, 1 \quad \text{and} \quad D^\tau(G_i) = 0 \quad i \in \mathbb{k}
$$

with $\tau$ satisfying $|\tau| \leq \ell$.

Notice that for every $\tau, \sigma \in \mathbb{N}^2$ and $i \in \mathbb{k}$, the identities

$$
D^\tau(F_\sigma) = \sum_{i=1}^{d} p^i_{\sigma+\tau} \quad \text{and} \quad D^\tau(G_i) = p^i_{\tau(2)} - D^\tau(\theta_i p^i_{10})
$$

hold true. Since $D_x$ and $D_y$ are derivations it follows that

$$
D^\tau(G_i) = p^i_{\tau(2)} - \sum_{\alpha + \beta = \tau} D^\alpha(\theta_i) D^\beta(p^i_{10})
$$

$$
= p^i_{\tau(2)} - \sum_{\alpha + \beta = \tau} D^\alpha(\theta_i) p^i_{\beta(1)}
$$

$$
= p^i_{\tau(2)} - \theta_i p^i_{\tau(1)} + \sum_{\kappa=1}^{|\tau|-1} L^\tau_\kappa(i)
$$

where

$$
L^\tau_\kappa(i) = \sum_{\substack{|\beta| = \kappa \\alpha + \beta = \tau}} D^\alpha(\theta_i) p^i_{\beta(1)}.
$$

If $\ell \geq 0$ and $(e_1, \ldots, e_k)$ is a basis of $E$, then $\mathcal{S}^{(\ell)}$, the $\ell$-th symbol of $S_W$, is generated by the elements

$$
\sum_{i=1, \ldots, k}^{k} \xi^i_\sigma (dx^{\sigma_1} \cdot dy^{\sigma_2}) \otimes e_i \in \text{Sym}^{1+\ell}(T^*(\mathbb{C}^2, 0)) \otimes E
$$

with $|\sigma| = 1+\ell$. 

subject to the conditions
\[(1)_{ab} \sum_{k=1}^{d} \xi^k_{ab} = 0 \quad \text{and} \quad (2)_{ab}^i \xi_{a,b+1}^i = \theta_i \xi_{a+1,b}^i\]
for every \(i \in k\) and every \((a, b) \in \mathbb{N}^2\) satisfying \(a + b = 1 + \ell\).

The equations \((2)_{ab}^i\) are equivalent to the equations below.
\[(2')_{ab}^i \xi_{ab}^i = (\theta_i)^b \xi_{\ell+1,0}^i\]

Consequently \(S^{(\ell)}\) is generated by elements of the form
\[\sum_{i=1}^{k} z^i \left( \sum_{a+b=1+\ell} (\theta_i)^b (dx^a \cdot dy^b) \right) \otimes e_i\]
where the \(z_i\) are subject to the conditions
\[(1')_s \sum_{i=1}^{k} z^i (\theta_i)^s = 0\]
for \(s \in \{0, \ldots, \ell + 1\}\). These equations can be written in matrix form:
\[
\begin{pmatrix}
1 & \ldots & 1 \\
\theta_1 & \ldots & \theta_k \\
\vdots & \ddots & \vdots \\
(\theta_1)^{\ell+1} & \ldots & (\theta_k)^{\ell+1}
\end{pmatrix}
\begin{pmatrix}
z^1 \\
z^2 \\
\vdots \\
z^k
\end{pmatrix}
= 0.
\]

Because \(W\) is smooth, the functions \(\theta_i\) have pairwise distinct values at every point of \((\mathbb{C}^2, 0)\). Thus \((V_{\theta}^{(\ell)})\) is a linear system of Vander Monde type, and there are only two possibilities.

- If \(\ell \geq k - 2\) then \((V_{\theta}^{(\ell)})\) has no non-trivial solution; in other terms \(S^{(\ell)} = \langle 0 \rangle\).

- If \(\ell \in \{0, \ldots, k - 3\}\) then \((V_{\theta}^{(\ell)})\) is a linear system of rank \(\ell + 2\), therefore \(S^{(\ell)}\) has dimension \(k - \ell - 2\), and can be parametrized explicitly via the VanderMonde matrix associated to functions \(\theta_i\). If \(M_\theta = (a_{ij})\) is the inverse of \(V_\theta = ((\theta_i)^{j-1})_{i,j=1}^{k}\) then
\[S^{(\ell)} = \left\langle \sum_{i=1}^{k} a_{ik} \left( \sum_{a+b=1+\ell} (\theta_i)^b dx^a \cdot dy^b \right) \otimes e_i \right| k = \ell + 3, \ldots, k \right\rangle.
\]
6.3.3 Pantazi-Hénaut criterium

Set $\mathcal{S}_W$ as the differential system $S^{(k-3)}_W$. According to Section 6.3.2 it has the following properties:

1. It is a subbundle of $J^{k-2}(E)$ of rank $k(k-2)/2$;
2. Its symbol $\sigma_W$ is a sub-line bundle of $\text{Sym}^{k-2}(T^*) \otimes E$;
3. The map $\mathcal{S}^{(1)}_W \rightarrow \mathcal{S}_W$ is an isomorphism (thus $\sigma^{(1)}_W = 0$).

The discussion laid down in Section 6.3.1 imply the existence of Hénaut’s connection of $\nabla_W$:

$$\nabla_W : \mathcal{S}^{(1)}_W \rightarrow \Omega^1 \otimes \mathcal{S}^{(1)}_W.$$ 

Its curvature is a $\mathcal{O}$-linear operator

$$\Theta_W : \mathcal{S}^{(1)}_W \rightarrow \Omega^2 \otimes \sigma_W.$$ 

called the Pantazi-Hénaut curvature of the web $W$.

**Theorem 6.3.3.** The following assertions are equivalent.

1. $W$ has maximal rank;
2. Pantazi-Hénaut curvature $\Theta_W$ vanishes identically.

The above theorem can be understood as a wide generalization from 3-webs to arbitrary $k$-webs of the equivalence between the items (b) and (c) of Theorem 1.2.4.

If the functions $\theta_1, \ldots, \theta_k$ are explicit, one can easily construct an effective algorithm (see [90, Appendice] for an implementation in MAPLE) which computes the curvature $\Theta_W$ as defined above. If instead of the functions $\theta_i$ one only knows an explicit expression of a $k$-symmetric 1-form defining $W$, then the implicit approach implemented by Hénaut shows the existence, but without exhibiting, of an algorithm to compute $\Theta_W$. Such algorithm is not as easy to derive as in the approach followed here. Albeit, Ripoll spelled out and implemented in MAPLE the corresponding algorithm for 3, 4 and 5-webs presented in implicit form.
An extensive study of Hénaut’s connection $\nabla_W$ remains to be done. The authors believe that a careful investigation of its properties may lead to an answer of Problem 1. Casale’s results [25] on the Galois-Malgrange groupoid of codimension one foliations seems to be useful in this context.

### 6.3.4 Mihăileanu criterium

Despite the relative easiness to implement the computation of $\Theta_W$, it seems very difficult to interpret the corresponding matrix of 2-forms. Nevertheless, in [78] Mihăileanu, based on Pantazi’s result and after ingenious computations, shows that

$$K(W) = \sum_{W_3 < W} K(W_3),$$

the sum of curvatures of all 3-subwebs of $W$, appears as a linear combination of the coefficients of Pantazi-Hénaut curvature. From this result he obtained from Theorem 6.3.3 the following necessary condition for rank maximality.

**Theorem 6.3.4 (Mihăileanu criterium).** If $W$ is a planar $k$-web of maximal rank then $K(W)$ vanishes identically.

By definition, the 2-form $K(W)$ is the **curvature** of $W$. An intrinsic interpretation of it has been recently provided in [97, Théorème 5.2] when $k \leq 6$, and stated for arbitrary $k$ in [69, p. 281],[98]. According to them the curvature $K(W)$ is nothing more than the trace of Pantazi-Hénaut curvature $\Theta_W$.

### 6.4 Classification of CDQL webs

Although Pantazi-Hénaut criterium together with the linearization criterium presented in Section 6.1 provide an algorithmic way to decide if a given planar web is exceptional or not, the classification problem for these objects is wide open up-to-date. The only classification results available so far consider the classification of (very) particular classes of webs. Even worse, only two classes have been studied so far. The first class are the germs of 5-webs on $(\mathbb{C}^2,0)$ of
the form $W(x, y, x+y, x-y, u(x) + v(y))$ mentioned in Example 2.1.4 of Chapter 2. The second class of exceptional webs so far classified are the CDQL\(^3\) webs on compact complex surfaces.

### 6.4.1 Definition

Linear webs are classically defined as the ones for which all the leaves are open subsets of lines. Here we will adopt the following global definition. A web $W$ on a compact complex surface $S$ is linear if (a) the universal covering of $S$ is an open subset $\tilde{S}$ of $\mathbb{P}^2$; (b) the group of deck transformations acts on $\tilde{S}$ by automorphisms of $\mathbb{P}^2$, and; (c) the pull-back of $W$ to $\tilde{S}$ is linear in the classical sense.

A CDQL $(k+1)$-web on a compact complex surface $S$ is, by definition, the superposition of $k$ linear foliations and one non-linear foliation. It can be verified (see \cite{88}) that the only compact complex surfaces carrying CDQL $(k+1)$-webs when $k \geq 2$ are the projective plane, the complex tori and the Hopf surfaces. Moreover the only Hopf surfaces admitting four distinct linear foliations are the primary Hopf surfaces $H_\alpha$, $|\alpha| > 1$. Here $H_\alpha$ is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the map $(x,y) \mapsto (\alpha x, \alpha y)$.

The linear foliations on complex tori are pencils of parallel lines on theirs universal coverings. The ones on Hopf surfaces are either pencils of parallels lines or the pencil of lines through the origin of $\mathbb{C}^2$. In particular a completely decomposable linear web on compact complex surface is algebraic\(^\dagger\) on its universal covering.

### 6.4.2 On the projective plane

The classification of exceptional CDQL webs on the projective plane is summarized in the following result.

**Theorem 6.4.1.** Up to projective automorphisms, there are exactly four infinite families and thirteen sporadic exceptional CDQL webs on $\mathbb{P}^2$.

\(^3\)The acronym CDQL stands for Completely Decomposable Quasi-Linear.

\(^\dagger\)Beware that algebraic here means that they are locally dual to plane curves.

In the cases under scrutiny they are dual to products of lines.
Figure 6.1: A sample of the real models for exceptional CDQL on $\mathbb{P}^2$. In the first and second rows the first three members of the infinite family $A_k^I$ and $A_k^II$ respectively. In the third row, from left to right, $A_k^III$, $A_k^IV$ and $A_k^V$. In the fourth row: $A_k^a$, $A_k^b$ and $A_k^c$. 
To describe the exceptional webs mentioned in Theorem 6.4.1, the notation of [88] will be adopted. If \( \omega \) is a rational \( k \)-symmetric differential 1-form then \( [\omega] \) denotes the \( k \)-web on \( \mathbb{P}^2 \) induced by it.

In suitable affine coordinates \((x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2\), the four infinite families are

\[
A^k_I = \left[ (dx^k - dy^k) \right] \boxtimes \left[ d(xy) \right] \quad (\text{where } k \geq 4);
\]

\[
A^k_{II} = \left[ (dx^k - dy^k) (x dy - y dx) \right] \boxtimes \left[ d(xy) \right] \quad (\text{where } k \geq 3);
\]

\[
A^k_{III} = \left[ (dx^k - dy^k) dx dy \right] \boxtimes \left[ d(xy) \right] \quad (\text{where } k \geq 2);
\]

\[
A^k_{IV} = \left[ (dx^k - dy^k) dx dx (x dy - y dx) \right] \boxtimes \left[ d(xy) \right] \quad (\text{where } k \geq 1).
\]

The diagram below shows how these webs relate to each other in terms of inclusions for a fixed \( k \). Moreover if \( k \) divides \( k' \) then \( A^k_I, A^k_{II}, A^k_{III}, A^k_{IV} \) are subwebs of \( A^{k'}_I, A^{k'}_{II}, A^{k'}_{III}, A^{k'}_{IV} \) respectively.

![Diagram](image)

All the webs above are invariant by the action \( t \cdot (x, y) = (tx, ty) \) of \( \mathbb{C}^* \) on \( \mathbb{P}^2 \). Among the thirteen sporadic examples of exceptional CDQL webs on the projective plane, seven (four 5-webs, two 6-webs and one 7-web) are also invariant by the same \( \mathbb{C}^* \)-action. They are:

\[
A^5_3 = \left[ dx \ dy (dx + dy) (x dy - y dx) \right] \boxtimes \left[ d(xy(x + y)) \right];
\]

\[
A^5_5 = \left[ dx \ dy (dx + dy) (x dy - y dx) \right] \boxtimes \left[ d\left(\frac{xy}{x+y}\right) \right];
\]

\[
A^5_6 = \left[ dx \ dy (dx + dy) (x dy - y dx) \right] \boxtimes \left[ d\left(\frac{x^2 + xy + y^2}{x+y}\right) \right];
\]

\[
A^5_7 = \left[ dx (dx^3 + dy^3) \right] \boxtimes \left[ d(x(x^3 + y^3)) \right];
\]

\[
A^6_3 = \left[ dx (dx^3 + dy^3) (x dy - y dx) \right] \boxtimes \left[ d(x(x^3 + y^3)) \right];
\]

\[
A^6_5 = \left[ dx dy (dx^3 + dy^3) \right] \boxtimes \left[ d(x(x^3 + y^3)) \right];
\]

\[
A^7 = \left[ dx dy (dx^3 + dy^3) (x dy - y dx) \right] \boxtimes \left[ d(x^3 + y^3) \right].
\]
Four of the remaining six sporadic exceptional CDQL webs (one k-web for each $k \in \{5, 6, 7, 8\}$) share the same non-linear foliation $F$: the pencil of conics through four points in general position. They all have been previously known (see [100]).

$$B_5 = \left[ dx \, dy \, d\left( \frac{x}{1-y} \right) \, d\left( \frac{y}{1-x} \right) \right] \boxtimes \left[ d\left( \frac{xy}{(1-x)(1-y)} \right) \right];$$

$$B_6 = B_5 \boxtimes \left[ d(x+y) \right];$$

$$B_7 = B_6 \boxtimes \left[ d\left( \frac{x}{y} \right) \right];$$

$$B_8 = B_7 \boxtimes \left[ d\left( \frac{1-x}{1-y} \right) \right].$$

The last two sporadic CDQL exceptional webs (the 5-web $H_5$ and the 10-web $H_{10}$) of Theorem 6.2.2 also share the same non-linear foliation: the Hesse pencil of cubics. Recall that this pencil is the one generated by a smooth cubic and its Hessian and that it is unique up to automorphisms of $\mathbb{P}^2$. Explicitly (with $\xi_3 = \exp(2i\pi/3)$):

$$H_5 = \left[ (dx^3 + dy^3) \left( \frac{x}{y} \right) \right] \boxtimes \left[ d\left( \frac{x^3 + y^3 + 1}{xy} \right) \right];$$

$$H_{10} = \left[ (dx^3 + dy^3) \prod_{k=0}^{2} \left( d\left( \frac{y - \xi^k}{x} \right) \cdot d\left( \frac{x - \xi^k}{y} \right) \right) \right] \boxtimes \left[ d\left( \frac{x^3 + y^3 + 1}{xy} \right) \right].$$

The 10-web $H_{10}$ is better described synthetically: it is the superposition of Hesse pencil of cubics and of the nine pencil of lines with base points at the base points of Hesse pencil. It shares a number of features with Bol’s web $B_5$. For instance, they both have a huge group of birational automorphisms (the symmetric group $S_5$ for $B_5$ and Hesse’s group $G_{216}$ for $H_{10}$), and their abelian relations can be expressed in terms of logarithms and dilogarithms.

Because they have parallel 4-subwebs whose slopes have non real cross-ratio the webs $A_{k11}$ for $k \geq 3$, $A_{k1v}$ for $k \geq 3$, $A_5$, $A_6$, $A_7$ and $A_8$ do not admit real models. The web $H_{10}$ also does not admit a real model. There are number of ways to verify this fact. One possibility is to observe that the lines passing through two of the nine base points always contains a third and notice that this contradicts Sylvester-Gallai Theorem [38]: for every finite set of non collinear points in $\mathbb{P}^2$ there exists a line containing exactly two points of the set. All
the other exceptional CDQL webs on the projective plane admit real models. For a sample see Figure 6.1.

6.4.3 On Hopf surfaces

The classification of exceptional CDQL webs on Hopf surfaces reduces to the one on \( \mathbb{P}^2 \). The result is far from interesting. One has just to remark that a foliation on a Hopf surface of type \( H_\alpha \) when lifted to \( \mathbb{C}^2 \setminus \{0\} \) gives rise to an algebraic foliation on \( \mathbb{C}^2 \) invariant by the \( \mathbb{C}^* \)-action \( t \cdot (x, y) = (tx, ty) \), and then use the classification of exceptional CDQL webs on \( \mathbb{P}^2 \) to obtain the following result.

**Corollary 6.4.2.** Up to automorphisms, the only exceptional CDQL webs on Hopf surfaces are quotients of the restrictions of the webs \( A^*_\alpha \) to \( \mathbb{C}^2 \setminus \{0\} \) by the group of deck transformations.

6.4.4 On abelian surfaces

The CDQL webs on tori are the superposition of a non-linear foliation with a product of foliations induced by global holomorphic 1-forms. Since \( \text{étale} \) coverings between complex tori abound and because the pull-back of exceptional CDQL webs under these are still exceptional CDQL webs, it is natural to extend the notion of isogenies between complex tori to isogenies between webs on tori. Two webs \( \mathcal{W}_1, \mathcal{W}_2 \) on complex tori \( T_1, T_2 \) are isogeneous if there exist a complex torus \( T \) and \( \text{étale} \) morphisms \( \pi_i : T \to T_i \) for \( i = 1, 2 \), such that \( \pi_1^*(\mathcal{W}_1) = \pi_2^*(\mathcal{W}_2) \).

**Theorem 6.4.3.** Up to isogenies, there are exactly three sporadic exceptional CDQL \( k \)-webs (one for each \( k \in \{5, 6, 7\} \)) and one continuous family of exceptional CDQL 5-webs on complex tori.

The elements of the continuous family are

\[
E_\tau = \left[ dx \, dy \, (dx^2 - dy^2) \right] \boxtimes \left[ d \left( \frac{\partial_1(x, \tau)\partial_1(y, \tau)}{\partial_4(x, \tau)\partial_4(y, \tau)} \right)^2 \right],
\]

respectively defined on the square \((E_\tau)^2\) of the elliptic curve \( E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \) for arbitrary \( \tau \in \mathbb{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \). The functions \( \partial_i \) involved in the definition are the classical **Jacobi theta functions**, whose definition is now recalled.
For \((\mu, \nu) \in \{0, 1\}^2\) and \(\tau \in \mathbb{H}\) fixed, let \(\vartheta_{\mu, \nu}(\cdot, \tau)\) be the entire function on \(\mathbb{C}\)
\[
\vartheta_{\mu, \nu}(x, \tau) = \sum_{n = -\infty}^{+\infty} (-1)^n \exp \left( i \pi (n + \frac{\mu}{2})^2 \tau + 2 i \pi (n + \frac{\mu}{2}) x \right).
\]

These are usually called the \textbf{theta functions with characteristic}. The Jacobi theta functions \(\vartheta_i\) are nothing more than \(\vartheta_1 = -i \vartheta_{1, 0}, \vartheta_2 = \vartheta_{1, 0}, \vartheta_3 = \vartheta_{0, 0}\) and \(\vartheta_4 = \vartheta_{0, 1}\).

The webs \(\mathcal{E}_\tau\) first appeared in Buzano’s work [23] but their rank was not determined at that time. They were later rediscovered\(^\text{1}\) in [93] where it is proved that they are all exceptional and that \(\mathcal{E}_\tau\) is isogeneous to \(\mathcal{E}_{\tau'}\) if and only if \(\tau\) and \(\tau'\) belong to the same orbit under the natural action of the \(\mathbb{Z}/2\mathbb{Z}\) extension of \(\Gamma_0(2) \subset \text{PSL}(2, \mathbb{Z})\) generated by \(\tau \mapsto -2\tau^{-1}\). Thus the continuous family of exceptional CDQL webs on tori is parameterized by a \(\mathbb{Z}/2\mathbb{Z}\)-quotient of the modular curve \(X_0(2)\).

The sporadic CDQL 7-web \(\mathcal{E}_7\) is strictly related to a particular element of the previous family. Indeed \(\mathcal{E}_7\) is the 7-web on \((E_{1+i})^2\)
\[
\mathcal{E}_7 = [dx^2 + dy^2] \boxtimes \mathcal{E}_{1+i}.
\]

The sporadic CDQL 5-web \(\mathcal{E}_5\) lives naturally on \((E_{\xi^3})^2\) and can be described as the superposition of the linear web
\[
[dx \, dy \, (dx - dy) \,(dx + \xi^2 \, dy)]
\]
and of the non-linear foliation
\[
\left[ d\left( \vartheta_1(x, \xi_3) \vartheta_3(y, \xi_3) \vartheta_3(x - y, \xi_3) \vartheta_4(x + \xi^2 \, y, \xi_3) \right) \right].
\]

The sporadic CDQL 6-web \(\mathcal{E}_6\) also lives in \((E_{\xi^3})^2\) and is best described in terms of Weierstrass \(\wp\)-function.
\[
\mathcal{E}_6 = \left[ dx \, dy \, (dx^3 + dy^3) \right] \boxtimes \left[ \wp(x, \xi_3)^{-1} \, dx + \wp(y, \xi_3)^{-1} \, dy \right].
\]

For a more geometric description of these exceptional \textit{elliptic webs} the reader is invited to consult [88, Section 4].

\(^\text{1}\)These are the 5-webs mentioned in Example 2.1.4.
6.4.5 Outline of the proof

In the remaining of this section, the proof of Theorem 6.4.1 will be sketched. It makes use of Mihăileanu’s criterium in an essential way. Its starting point is the following trivial observation: if $K(W)$, the curvature of $W$, is zero then it must be, in particular, a holomorphic 2-form.

Regularity of the curvature

The tautology just spelled out, makes clear the relevance of obtaining criterium to ensure the absence of poles of $K(W)$. The result obtained for that in [88] is best stated in term of $\beta_F(W)$ — the $F$-barycenter of a web $W$. If $W$ is a k-web and $F$ is a foliation not contained in $W$, both defined on a surface $S$ then at a generic point $p \in S$ the tangents of $F$ and $W$ determine $k + 1$ points in $\mathbb{P}(T_pS)$. The complement of the point $[T_pF]$ in $\mathbb{P}(T_pS)$ is clearly isomorphic to $\mathbb{C}$, and any two distinct isomorphisms differ by an affine map. The $F$-barycenter of $W$ is then defined as the foliation on $S$ with tangent at a generic point $p$ of $S$ determined by the barycenter of the directions determined by $W$ at $p$ in the affine structure on $\mathbb{C} \simeq \mathbb{P}(T_pS) \setminus [T_pF]$ determined by $F$.

**Theorem 6.4.4.** Let $F$ be a foliation and $W = F_1 \boxtimes \cdots \boxtimes F_k$ be a k-web, $k \geq 2$, both defined on the same domain $U \subset \mathbb{C}^2$. Suppose that $C$ is an irreducible component of $\text{tang}(F, F_1)$ that is not contained in $\Delta(W)$. The curvature $K(F \boxtimes W)$ is holomorphic over a generic point of $C$ if and only if the curve $C$ is $F$-invariant or $\beta_F(W')$-invariant, where $W' = F_2 \boxtimes \cdots \boxtimes F_k$.

This result is the key tool used in [88] to achieve the classification. Having it at hand, the next step is to the describe the $L$-barycenters of completely decomposable linear (CDL) webs with respect to a linear foliation $L$.

$L$-barycenters of CDL webs

A linear foliation $L$ on $\mathbb{P}^2$ is nothing more than a pencil of lines. Thus, it is determined by its unique singular point. So, let $p_0 \in \mathbb{P}^2$ be the point determining $L$ and $\{p_1, \ldots, p_k\} \subset \mathbb{P}^2$ be the set of points
determining a CDL \( k \)-web \( \mathcal{W} \). In order to describe the \( \mathcal{L} \)-barycenter of \( \mathcal{W} \), let \( \Pi : S \to \mathbb{P}^2 \) be the blow-up of \( \mathbb{P}^2 \) at \( p_0 \); \( E \) its exceptional divisor; \( \pi : S \to \mathbb{P}^1 \) be the fibration on \( S \) induced by the lines through \( p_0 \); \( \mathcal{G} \) be the foliation \( \Pi^* \beta_\mathcal{L}(\mathcal{W}) \); and \( \ell_i \) be the strict transform of the line \( \overline{p_0p_i} \) under \( \Pi \) for \( i = 1, \ldots, k \).

![Figure 6.2: The \( \mathcal{L} \)-barycenter of a CDL web \( \mathcal{W} \).](image)

If the points \( \{p_0, \ldots, p_k\} \) are aligned then \( \beta_\mathcal{F}(\mathcal{W}) \) is also a pencil of lines with base point at the \( p_0 \)-barycenter of \( \{p_1, \ldots, p_k\} \). If instead the points \( \{p_0, \ldots, p_k\} \) are not aligned then a simple computation shows that the foliation \( \mathcal{G} \) is a Riccati foliation with respect to \( \pi \), that is, \( \mathcal{G} \) has no tangencies with the generic fiber of \( \pi \). In fact, a much more precise description of \( \mathcal{G} \) can be obtained. In [88, Lemma 6.1] it is shown that \( \mathcal{G} \) has the following properties:

1. the exceptional divisor \( E \) of \( \Pi \) is \( \mathcal{G} \)-invariant;
2. the only \( \mathcal{G} \)-invariant fibers of \( \pi \) are the lines \( \ell_i \), for \( i \in k \);
3. the singular set of \( \mathcal{G} \) is contained in the union \( \bigcup_{i \in k} \ell_i \);
4. over each line \( \ell_i \), the foliation \( \mathcal{G} \) has two singularities. One is a complex saddle at the intersection of \( \ell_i \) with \( E \), the other is a complex node at the \( p_0 \)-barycenter of \( \mathcal{P}_i = \{p_1, \ldots, p_k\} \cap \ell_i \). Moreover, if \( r_i \) is the cardinality of \( \mathcal{P}_i \) then the quotient of eigenvalues of the saddle (resp. node) over \( \ell_i \) is \(-r_i/k\) (resp. \( r_i/k \)).
5. the monodromy of $G$ around $\ell_i$ is finite of order $k/ \gcd(k, r_i)$;

6. the separatrices of $\beta_{\mathcal{F}}(\mathcal{W})$ through $p_0$ are the lines $\overline{p_0 p_i}$, $i \in \mathbb{K}$.

It is interesting to notice that the generic leaf of $\beta_{\mathcal{F}}(\mathcal{W})$ is transcendental in general. Indeed, the cases when there are more algebraic leaves than the obvious ones (the lines $\overline{p_0 p_i}$) are conveniently characterized by [88, Proposition 6.1], which says that the foliation $\beta_{\mathcal{F}}(\mathcal{W})$ has an algebraic leaf distinct from the lines $\overline{p_0 p_i}$ if and only if all its singularities distinct from $p_0$ are aligned. Moreover if this is the case all its leaves are algebraic.

The $\ell$-polar map and bounds for the degree of $\mathcal{F}$

For a set $\mathcal{P}$ of $k$ distinct points in $\mathbb{P}^2$, let $\mathcal{W}(\mathcal{P})$ be the CDL $k$-web on the projective plane formed by the pencils of lines with base points at the points of $\mathcal{P}$.

Once the description of the $\mathcal{L}$-barycenters of CDL webs have been laid down, the next step is to use it to obtain constraints on the non-linear foliation $\mathcal{F}$ and on the position of the points $\mathcal{P}$ in case $\mathcal{W}(\mathcal{P}) \boxtimes \mathcal{F}$ has zero curvature.

It is not hard to show (see [88, Section 8]) that when the cardinality of $\mathcal{P}$ is at least 4, either (a) there are three aligned points in $\mathcal{P}$; or (b) $\mathcal{P}$ is a set of 4 points in general position and $\mathcal{F}$ is the pencil of conics through them.

When in case (b) there is not much left to do, since $\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})$ is nothing more than Bol’s 5 web; in case (a) one is naturally lead to consider a line $\ell$ containing $k_\ell$ points of $\mathcal{P}$, with $k_\ell \geq 3$; and the pencil $V = \{\text{tang}(\mathcal{F}, \mathcal{L}_p)\}_{p \in \ell}$ of polar curves of $\mathcal{F}$ centered at $\ell$. It can be shown that $\ell$ is a fixed component of $V$ (in other words $\ell$ is $\mathcal{F}$-invariant); and the restriction of $V - \ell$ to $\ell$ defines a non-constant rational map $f : \ell \simeq \mathbb{P}^1 \to \mathbb{P}^1$. The map $f$ is characterized by the following equalities between divisors on $\ell$

$$f^{-1}(p) = \left(\text{tang}(\mathcal{F}, \mathcal{L}_p) - \ell\right)\big|_{\ell},$$

with $p \in \ell$ arbitrary. The map $f$ is called the $\ell$-polar map of $\mathcal{F}$.

Once all these properties of $f$ are settled, it follows from a simple application of Riemann-Hurwitz formula that the degree of $\mathcal{F}$ is at most four. Moreover, if $\deg(\mathcal{F}) \geq 2$ then $k_\ell \leq 7 - \deg(\mathcal{F})$. 
The final steps

At this point the proof has a two-fold ramification. In one branch one is lead to consider foliations of degree one and put to a good use the acquired knowledge on the structure of the space of abelian relations of web admitting infinitesimal automorphisms, see [88, Section 9]. In the other branch, one first derive from the structure of the $L$-barycenter of CDL webs, the normal forms for the $\ell$-polar map of $F$ presented in [88, Table 1]. Then, the proof goes by a case by case analysis, see [88, Section 10]. While the arguments can be considered as elementary, they are too involved to be detailed here.

6.5 Further examples

In this last section, different examples of exceptional webs are listed. Apart from the fact that they are all exceptional webs, there is no general directrix. The reason behind this chaotic exposition is the lack of a general framework encompassing all known the exceptional webs.

6.5.1 Polylogarithmic webs

It is well known that the polylogarithms

$$\text{Li}_n(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^n}$$

satisfy two variables functional equations at least when $n$ is small, see [73].

For instance, Spence (1809) and Kummer (1840) have independently established some variants of the following functional equation, nowadays called Spence-Kummer equation, satisfied by the
trilogarithm $\text{Li}_3$

\[
2\text{Li}_3(x) + 2\text{Li}_3(y) - \text{Li}_3\left(\frac{x}{y}\right) + 2\text{Li}_3\left(\frac{1-x}{1-y}\right) + 2\text{Li}_3\left(\frac{x(1-y)}{y(1-x)}\right)
\]

\[
-\text{Li}_3(xy) + 2\text{Li}_3\left(\frac{y(1-x)}{x(1-y)}\right) + 2\text{Li}_3\left(\frac{y(1-y)}{y(1-x)}\right) - \text{Li}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right)
\]

\[
= 2\text{Li}_3(1) - \log(y)^2 \log\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3} \log(y) + \frac{1}{3} \log(y)^3
\]

when $x, y$ are real numbers subject to the constraint $0 < x < y < 1$.

Kummer proved that the tetralogarithm and the pentalogarithm verify similar equations. If $\zeta = 1-x$ and $\eta = 1-y$ with $0 < x < y < 1$, $x, y \in \mathbb{R}$ then the tetralogarithm $\text{Li}_4$ satisfies the equation $K(4)$, written down below:

\[
\text{Li}_4\left(-\frac{x^2y\eta}{\zeta}\right) + \text{Li}_4\left(-\frac{y^2x\zeta}{\eta}\right) + \text{Li}_4\left(\frac{x^2y}{\eta^2\zeta}\right) + \text{Li}_4\left(\frac{y^2x}{\zeta^2\eta}\right) - 6\text{Li}_4(xy) - 6\text{Li}_4\left(\frac{x\eta}{\zeta}\right) - 6\text{Li}_4\left(\frac{y\zeta}{\eta}\right) - 3\text{Li}_4\left(\frac{x}{\eta}\right) - 3\text{Li}_4\left(\frac{y}{\zeta}\right) - 3\text{Li}_4\left(\frac{y\zeta}{\eta}\right) - 3\text{Li}_4\left(\frac{x\eta}{\zeta}\right) + 6\text{Li}_4(x) + 6\text{Li}_4(y) + 6\text{Li}_4\left(\frac{x}{\zeta}\right) + 6\text{Li}_4\left(\frac{y}{\zeta}\right) - \frac{3}{2} \log^2(\zeta) \log^2(\eta).
\]

The pentalogarithm $\text{Li}_5$ satisfies an equation of the same type, which will be referred as $K(5)$. It involves more than thirty terms and will not be written down to save space.

All the known functional equations, in two variables, satisfied by the classical polylogarithms $\text{Li}_n$ are of the form

\[
\sum_{i=1}^{N} c_i \text{ Li}_n(U_i) \equiv \text{ Elem}_n
\]

(6.18)

where $c_1, \ldots, c_N$ are integers; $U_1, \ldots, U_N$ are rational functions; $\text{ Elem}_n$ is of the form $P(\text{Li}_{k_1} \circ V_1, \ldots, \text{Li}_{k_m} \circ V_m)$ with $P$ being a polynomial.
and $V_1, \ldots, V_m$ being rational functions; and $k_1, \ldots, k_m$ are integers satisfying $1 \leq k_i < n$ for every $i \in \mathbb{m}$.

Of relevance for web geometry are the webs defined by the functions $U_i$ appearing in (6.18). The presence of a non-vanishing right-hand side $\text{Elem}_n$ is an apparent obstruction to interpret (6.18) as an abelian relation of the web defined by the functions $U_i$. This difficulty can be bypassed because the classical polylogarithms have univalued “cousins”, denoted by $\mathcal{L}_n$, globally defined on $\mathbb{P}^1$ which satisfy, globally on $\mathbb{P}^2$, homogeneous analogues of every equation of the form (6.18) locally satisfied by $\text{Li}_n$.

For $n \geq 2$, the \textbf{n-th modified polylogarithm} is the function

$$L_n(z) = \Re_m \left( \sum_{k=0}^{n-1} \frac{2k}{k!} B_k \log |z|^k \text{Li}_{n-k}(z) \right)$$

for $z \in \mathbb{C} \setminus \{0, 1\}$. It can be shown that these functions are well defined real analytic functions on $\mathbb{C} \setminus \{0, 1\}$. They can be continuously extended to the whole projective line $\mathbb{P}^1$ by setting $L_n(0) = 0$, $L_n(\infty) = 0$, and $L_n(1) = \zeta(n)$ for $n$ odd, $L_n(1) = 0$ for $n$ even.

The existence of univalued versions of polylogarithms have been established by several authors. The ones introduced here have the peculiarity of satisfying \emph{clean} versions of the functional equations for the classical polylogarithms present above.

\textbf{Theorem 6.5.1.} Let $n \geq 2$. The following assertions are equivalent:

(a) there exists a simply connected open subset of $\mathbb{P}^2$ where the functional equation $\sum_{i=1}^{N} c_i \text{Li}_n(U_i) = \text{Elem}_n$ holds true;

(b) the expression $\sum_{i=1}^{N} c_i \log |U_i|^{n-k} L_k(U_i)$ is constant on $\mathbb{P}^2$ for $k = 2, \ldots, n$.

\[ \text{In this definition, } \Re_m \text{ stands for the real part if } n \text{ is odd otherwise it is the imaginary part; } B_k \text{ is } k\text{-th Bernoulli number: } B_0 = 1, B_1 = \frac{-1}{2}, B_2 = \frac{1}{6}, \text{ etc.} \]
For a proof of this result the reader is redirected to [82] and [37, pages 45–46]. It implies that the web $W_{e_n}$ associated to a polylogarithmic relation $e_n$ as (6.18) admits polylogarithmic abelian relations and hence are susceptible of having high rank.

For instance, according to Theorem 6.5.1, the function $L_3$ verifies on $\mathbb{P}^2$ the homogeneous version of Spence-Kummer equation. For every $x, y \in \mathbb{R}$, the following identity holds true.

$$2L_3(x) + 2L_3(y) - L_3\left(\frac{x}{y}\right) + 2L_3\left(\frac{1 - y}{1 - x}\right) + 2L_3\left(\frac{x(1 - y)}{y(1 - x)}\right) - L_3(xy) + 2L_3\left(-\frac{x(1 - y)}{1 - x}\right) + 2L_3\left(-\frac{1 - y}{y(1 - x)}\right) - L_3\left(\frac{x(1 - y)^2}{y(1 - x)^2}\right) = \zeta(3)\frac{2}{2}.$$

This equation can be complexified, and the result after differentiation gives rise to an abelian relation for the \textbf{Spence-Kummer web} $W_{SK}$ defined as

$$W\left(x, y, xy, \frac{x}{y}, \frac{1 - x}{1 - y}, \frac{x(1 - y)}{y(1 - x)}, \frac{1 - y}{y(1 - x)}, \frac{x(1 - y)}{y(1 - x)}, \frac{(1 - y)}{y(1 - x)}, \frac{x(1 - y)^2}{y(1 - x)^2}\right).$$

This web seems to be to Spence-Kummer equation for the trilogarithm, what Bol’s web is to Abel’s equation for the dilogarithm. It was Hénaut in [67], that recognized it as a good candidate for being an exceptional 9-web. This was later settled independently by the second author [92] and Robert and [100]. It has to be emphasized that back then in 2001, $W_{SK}$ was the first example of planar exceptional web to come to light after Bol’s exceptional 5-web. Between the appearance of the two examples a hiatus of more or less 70 years took place.

One might think that all the webs naturally associated to the equations of the form (6.18) satisfied by the polylogarithms are all exceptional (see [74, pages 196-197]). Although these webs are certainly of high rank, they are not necessarily of the highest rank. For example, using Mihăileanu criterium, one can show by brute force computation that the webs associated to Kummer equations $K(4)$ and $K(5)$ are not of maximal rank (for details, see [90, Chap. VII]).
Nevertheless, it seems to exist a (large?) class of global exceptional webs with abelian relations expressed in terms of a natural generalization of the classical polylogarithms: the iterated integrals of logarithmic 1-forms on $\mathbb{P}^1$. For instance all the elements of the family of 10-webs with parameters $a, b \in \mathbb{C} \setminus \{0, 1\}$

$$W_{a,b} = W\left( x, y, \frac{x - y}{1 - x}, \frac{x(1 - y)}{a - x}, \frac{x(b - y)}{y(1 - x)}, \frac{(1 - x)(b - y)}{(1 - y)(a - x)}, \frac{(bx - ay)(1 - y)}{(y - x)(b - y)}, \frac{(bx - ay)(1 - x)}{(y - x)(a - x)} \right)$$

are exceptional webs. Through a method proposed by Robert in [100], it is possible to determine $A(W_{a,b})$ for no matter which $a$ and $b$ (see [95]) and deduce the maximality of the rank.

### 6.5.2 A very simple series of exceptional webs

It is hard to imagine examples of webs simpler than the ones presented below.

- $W_1 = W(x, y, x + y, x - y, xy)$
- $W_2 = W(x, y, x + y, x - y, xy, x/y)$
- $W_3 = W(x, y, x + y, x - y, x/y, x^2 + y^2)$
- $W_4 = W(x, y, x + y, x - y, xy, x^2 + y^2)$
- $W_5 = W(x, y, x + y, x - y, xy, x^2 - y^2)$
- $W_6 = W(x, y, x + y, x - y, xy, x/y, x^2 - y^2)$
- $W_7 = W(x, y, x + y, x - y, xy, x/y, x^2 + y^2)$
- $W_8 = W(x, y, x + y, x - y, xy, x^2 - y^2, x^2 + y^2)$
- $W_9 = W(x, y, x + y, x - y, xy, x/y, x^2 - y^2, x^2 + y^2)$

It turns out that they are all exceptional as proved in [90, Appendice]. Notice that the webs $W_1$ and $W_2$ above are nothing more than the webs $A_{II}^2$ and $A_{IV}^2$ from Section 6.4.2. Moreover, $W_3$ is equivalent to $A_{II}^1$ under a linear change of coordinates. In the graph below the inclusions between them are schematically represented.
6.5.3 An exceptional 11-web

Let $\mathcal{F}_2$ be the degree two foliation on $\mathbb{P}^2$ induced by the rational 1-form 
$$\tilde{y}(\tilde{y} - 1)d\tilde{x} - \tilde{x}(\tilde{x} - 1)d\tilde{y}.$$  

It is nothing more than the pencil of conics $\frac{\tilde{x}(\tilde{y} - 1)}{\tilde{y}(\tilde{x} - 1)} = \text{cte}$. Let $C$ be the completely decomposable curve of degree nine in $\mathbb{P}^2$ defined by the homogeneous polynomial 
$$\tilde{x}\tilde{y}\tilde{z}(\tilde{x} - \tilde{z})(\tilde{y} - \tilde{z})(\tilde{x} + \tilde{y})(\tilde{x} - \tilde{y})(\tilde{x} + \tilde{y} - \tilde{z})(\tilde{x} - \tilde{y} + \tilde{z}) = 0.$$  

As can be seen above, $C$ is the reunion of six lines invariant by $\mathcal{F}_2$ with three extra lines synthetically described as the lines joining the three singular points of the fibers of the pencil: these latter are cut out by $\tilde{x} + \tilde{y}$, $\tilde{x} - \tilde{y} - \tilde{z}$ and $\tilde{x} - \tilde{y} + \tilde{z}$.

The algebraic web $\mathcal{W}_C$ is formed by nine pencil of lines. If $\mathcal{W}_{\mathcal{F}_2}$ is the dual web of $\mathcal{F}_2$, in the sense of Section 1.4.3 of Chapter 1 then $\mathcal{W}_{\mathcal{F}_2} \boxtimes \mathcal{W}_C$ is an exceptional 11-web on $\mathbb{P}^2$. After a two-fold ramified covering it can be written as the completely decomposable web 
$$\mathcal{W} = \mathcal{W}(F_1, \ldots, F_{11})$$.
where $F_1, \ldots, F_{11}$ are the rational functions below:

\[
\begin{align*}
F_1 &= \frac{(x - 1)y}{(y - 1)x} \\
F_2 &= F_1 \left(\frac{y - x - 1}{y - x + 1}\right)^2 \\
F_3 &= \frac{(y - 1)y}{(x - 1)x} \\
F_4 &= \frac{(y - x)y}{x - 1} \\
F_5 &= \frac{(x - 1)y}{(y - 1)x} \\
F_6 &= \frac{(x - y + 1)y}{x} \\
F_7 &= \frac{x + y - 1}{xy} \\
F_8 &= \frac{(y - x - 1)x}{y} \\
F_9 &= F_1 \left(\frac{x - y + 1}{y - x + 1}\right) \\
F_{10} &= \frac{y(x - 1)(x - y + 1)}{x(y^2 - xy - x + 1)} \\
F_{11} &= \frac{x(y - 1)(y - x + 1)}{y(x^2 - xy - y + 1)}.
\end{align*}
\]

Using Abel’s method, the abelian relations of $\mathcal{W}$ can be explicitly determined. As a by-product, it follows that not only $\mathcal{W}$ is exceptional, but also a certain number of its subwebs. A partial list is provided by the following

**Proposition 6.5.2.** The following ascending chain of subwebs of $\mathcal{W}$

\[
\mathcal{W}(F_1, \ldots, F_5) \subset \mathcal{W}(F_1, \ldots, F_6) \subset \ldots \subset \mathcal{W}(F_1, \ldots, F_{11}) = \mathcal{W}
\]

is formed by exceptional webs.

It was David Marín together with the first author who guessed that this 11-web was interesting in what concerns its rank. The second author confirmed this intuition, proving the proposition above using Abel’s method.

### 6.5.4 Terracini and Buzano’s webs

As explained in Section 4.3.4 of Chapter 4, there is a germ of smooth surface $S_\mathcal{W} \subset \mathbb{P}^5$ attached to every exceptional 5-web $\mathcal{W}$: the image of its Poincaré’s map. Moreover, the geometry of $S_\mathcal{W}$ has rather special geometrical features as recalled below
1. At a generic point the second osculating space of $S_W \subset \mathbb{P}^5$ coincides with the whole $\mathbb{P}^5$;

2. The image of $W$ by Poincaré’s map of $W$ is Segre’s web of $S_W$;

3. The union of the tangent planes of $S_W$ along one of the leaves of its Segre’s web is included in a hyperplane.

If $S \subset \mathbb{P}^5$ is a germ of surface on $\mathbb{P}^5$ satisfying the above three conditions, it is natural to ask if its Segre’s web, as defined in Section 1.4.4 of Chapter 1, is of maximal rank or not. A positive answer would establish the equivalence between the classification problem for exceptional 5-webs with a problem of projective differential geometry: the classification of surfaces subject to the constraints enumerated above. It is the latter problem which motivated Terracini and subsequently Buzano toward the results recalled below.

A surface $S \subset \mathbb{P}^5$ will be called exceptional surface if it is not included in a Veronese surface; its Segre’s 5-web $W_S$ is generically smooth; and conditions 1. and 3. above are satisfied. Under this assumption, one proves the existence of five germs of curve $C_{S,i} \subset \mathbb{P}^5$ called Poincaré-Blaschke’s curves of $S$, satisfying

$$S = \bigcap_{i=1}^{5} (C_{S,i})^*$$

where $C^* \subset \mathbb{P}^5$ stands for the dual variety of a germ of curve $C \subset \mathbb{P}^5$. In other words, $C^*$ is the subset of $\mathbb{P}^5$ corresponding to the hyperplanes $H \in \mathbb{P}^5$ tangent to $C$.

In [105], Terracini obtained a characterization of exceptional surfaces as solutions of a certain non-linear differential system. Under additional simplifying hypothesis, he succeeded to integrate explicitly the resulting system, and in this way proved the following result.

**Theorem 6.5.3.** Up to projective automorphism, there are exactly four exceptional surfaces $S \subset \mathbb{P}^5$ for which three of its Poincaré-Blaschke curves – say $C_{S,i}$ for $i = 1, 2, 3$ – are plane and the three planes $\langle C_{S,i} \rangle \subset \mathbb{P}^5$ have one point in common. One of these surface
is the image of Poincaré’s map of Bol’s web, and the other three are the image of Poincaré’s map of the following webs:

\[ \text{Terr}(b) = W(x, y, x + y, x - y, x^2 - y^2) \quad (6.19) \]

\[ \text{Terr}(c) = W\left(x, y, \frac{(x + y)^2}{1 + y^2}, \frac{y(x^2 - 2x - y)}{1 + y^2}, \frac{x^2 y - 2x - y}{x^2 + 2xy - 1}\right) \]

\[ \text{Terr}(d) = W\left(x, y, x + y, \frac{x}{y}, \frac{y}{x}(x + y)\right). \]

Using Terracini’s approach, Buzano [24] proved the following result.

**Theorem 6.5.4.** Up to projective automorphism, there are exactly two exceptional surfaces for which three of its Poincaré-Blaschke curves – say \( C_{S,i} \) for \( i = 1, 2, 3 \) – are planar and satisfy

(a). for every distinct \( i, j \in \mathbb{Z}_3 \), \( \langle C_{S,i}, C_{S,j} \rangle \subset \mathbb{P}^5 \) is a hyperplane;

(b). the intersection \( \langle C_{S,1} \rangle \cap \langle C_{S,2} \rangle \cap \langle C_{S,3} \rangle \) is empty.

They are the image of Poincaré’s map of the following webs:

\[ \text{Buz}(a) = W(x, y, x + y, x - y, \tanh(x) \tanh(y)) \quad (6.20) \]

\[ \text{Buz}(b) = W(x, y, x + y, x - y, e^x + e^y). \]

It turns out that the webs (6.19) and (6.20) are all exceptional. Curiously, this was not proved by Terracini nor by Buzano. They focused on the differential-geometric problem. The exceptionality has been established just recently in [91, 93] (see also [90]), using Abel’s method.

Certain exceptional surfaces are transcendent, as for example the image of Poincaré’s map of Bol’s 5web, while other are algebraic and even rational as the one associate to Terr\((b)\). The image of Poincaré’s map of Terr\((b)\) can be described as the Zariski closure of the image of the map

\[
(x, y) \mapsto \left[ 1 : x^3 + y^3 : x^3 - y^3 : x^2 + y^2 : x^2 - y^2 : (x^2 - y^2)^2 \right].
\]

A toy problem that might shed some light on the subject, consists in determining the linear systems \( \mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^3}(q)| \), for small \( q \),
of dimension 5 for which the Zariski closure of the image of the associated rational map $\mathbb{P}^2 \rightarrow \mathbb{P}^5$ are exceptional surfaces. Notice that for $q = 4$, $\text{Terr}(b)$ is an example, and that 4 is the minimal $q$ which can happen. Indeed for $q = 2$ one obtains a Veronese surface, and for $q = 3$ the hyperplane containing the tangent spaces of leaves of Segre’s web would pull-back to a cubic containing an irreducible component with multiplicity two. This implies that the pull-back of Segre’s web to $\mathbb{P}^2$ is a linear, and consequently, algebraic web.
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