Katz-Grothendieck Conjecture: (02/02/2024) IMPA

$f: U \to \mathbb{C}$ a hol. function, $\overset{z_0}{\overset{\parallel}{0}} \in U \subseteq \mathbb{C}$, $f(z) = \sum_{n=0}^{\infty} f_n z^n$

its Talyor series. ~~$R \subseteq \mathbb{C}$ a finitely generated $\mathbb{Z}$-algebra~~

~~$R = \mathbb{Z}[a_1, a_2, \dots a_n]$, $a_1, a_2, \dots a_i \in \mathbb{C}$, Take $R = \mathbb{Z}$.~~

Def: $f(z)$ is called algebraic if there is a polynomial $P \in \mathbb{C}[X,Y]$ such that $P(z, f(z)) = 0$.

Thm (G. Eisenstein, 1823-1852). If $f \in \mathbb{Q}[[z]]$, that is $f_n \in \mathbb{Q}$, and f is algebraic, then there is $N \in \mathbb{N}$ suchthat $f(Nz) \in \mathbb{Z}[[z]]$, that is $f_n \cdot N^n \in \mathbb{N}$.

Cor. e^z, $\log(1-z)$, is not algebraic

Proof: $e^z = \sum \frac{z^n}{n!}$, all the prime numbers appear in the denominators of $\frac{1}{n!}$!

obs: Eisenstein's thm is not if and only if.

Counterexample: Eisenstein series

$$E_2(q) = 1 - 24 \left(\sum_{n=1}^{\infty} \sigma(n) q^n \right)$$

$$\sigma(n) = \sum_{d \mid n} d$$

Thm: (Herman Schwarz 1843-1921): Fix parameters $a, b, c \in \mathbb{C}$. All the solutions of the linear d.e

$$Y' = \left(\frac{1}{z} \begin{pmatrix} c-1, & -b \\ 0, & 0 \end{pmatrix} + \frac{1}{z-1} \begin{pmatrix} 0 & 0 \\ a & c-a-b-1 \end{pmatrix} \right) Y$$

are algebraic if and only if (a,b,c) belongs <u>essentially</u> to a list of 15 examples:

$\lambda = |1-c|$, $\mu = |c-a-b|$, $\nu = |a-b|$, $0 \leq \lambda, \mu, \nu < 1$

$0 \leq \lambda + \mu, \lambda + \nu, \mu + \nu \leq 1$

$(\lambda,\mu,\nu)=(\frac{1}{2},\frac{1}{2},\frac{1}{n}),(\frac{1}{2},\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{1}{3},\frac{2}{3}),(\frac{1}{2},\frac{1}{3},\frac{1}{4})$
$(\frac{2}{3},\frac{1}{4},\frac{1}{4}),(\frac{1}{2},\frac{1}{3},\frac{1}{5}),(\frac{1}{2},\frac{1}{3},\frac{2}{5}),(\frac{1}{2},\frac{1}{5},\frac{2}{5}),(\frac{1}{3},\frac{2}{3},\frac{1}{5}).$
$(\frac{1}{3},\frac{2}{3},\frac{1}{5}),(\frac{2}{3},\frac{1}{5},\frac{1}{5}),(\frac{1}{3},\frac{2}{5},\frac{3}{5}),(\frac{1}{3},\frac{1}{5},\frac{3}{5}),(\frac{1}{5},\frac{1}{5},\frac{4}{5}),(\frac{2}{5},\frac{2}{5},\frac{2}{5})$
E. landau (1877-1938). Apply Eisenstein to get Schwarz,
Linear differential Equations:

$$Y'=M(z)Y \qquad '=\frac{\partial}{\partial z}$$

$$M=\frac{A}{z},\ \boxed{\frac{A}{z}+\frac{B}{z-1}} \qquad Y=\begin{bmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix},\ y_{ij} \text{ hol. functions}$$

A, B, 2x2 constant matrices with coef. in $\mathbb{C}$. $\Delta:=z$ and $\Delta:=z(z-1)$ — in an open set $U\subseteq \mathbb{C}\setminus\{0,1\}$.

Proposition: Fix $z_0\in\mathbb{C}\setminus\{0,1\}$, if you want $z_0\in\mathbb{Z}$,
There is a unique power series

$$Y=\sum Y_n\cdot(z-z_0)^n \quad Y_n\ 2\times 2 \text{ matrices.}$$

$$Y'=\left(\frac{1}{z}A+\frac{1}{z-1}B\right)Y,\quad Y_0=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Proof:
$$\left(Y_1+2Y_2(z-z_0)+\cdots+nY_n(z-z_0)^{n-1}+\cdots\right)=$$
$$\left(C_0+C_1(z-z_0)+\cdots+C_{n-1}(z-z_0)^{n-1}+\cdots\right)\cdot$$
$$\left(Y_0+Y_1\cdot(z-z_0)+\cdots+Y_{n-1}(z-z_0)^{n-1}+\cdots\right)$$

coef. of $(z-z_0)^{n-1}$:

$$nY_n = \text{In terms of } Y_0, Y_1, \dots Y_{n-1} \quad n\geq 1.$$

In n-th step we divide over $\underline{n}$.
In total we are dividing over $n!$.

Back to Schwarz: Y can be written in terms of Gauss hyp. function.

$$F(a,b,c|z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n, \qquad (a)_n=(a)(a-1)\cdots(a-n+1).$$

$Y^{(n)} = M_n \cdot Y$

$Y^{(n+1)} = M_{n+1} Y = (M_n Y)' = M_n' Y + M_n Y' = (M_n' + M_n M) Y$

$\Rightarrow M_{n+1} = M_n' + M_n M \quad M_1 := M.$

$R = \mathbb{Z}[\text{entries of } A, B] = \mathbb{Z}[\frac{1}{N}]$ if the entries of $A, B \in \mathbb{Q}$ → common denominator.

Proposition ⊗: If the solution Y is algebraic then for all except a finite number of prime numbers p, we have

$$M_p = 0 \text{ in } R[z, \tfrac{1}{\Delta}]/pR[z, \tfrac{1}{\Delta}]$$

Assume that the entries of $A, B \in \mathbb{Q}$

Set $\Delta = z \cdot (z-1) \cdot$ common denominator of the entries of A, B

$\Delta^n M_n$ has entries in $\mathbb{Z}[z]$ and so we can do mod p.

Katz-Grothendieck conjecture: If for $Y' = MY$, and we have for all except a finite number of primes p

$M_p \equiv_p 0$ then Y is algebraic.

M_p is called p-curvature.

Kronecker's criterion:

$$Y' = \frac{A}{z} Y, \quad Y, A, 1\times1 \text{ matrix}, A \in \mathbb{C}.$$

$$Y^{(n)} = \frac{A(A-1)\cdots(A-n+1)}{z^n} Y$$

$M_p = A(A-1)\cdots(A-p+1) \equiv_p 0$

A cannot be a transendental number.

$\Uparrow$

$A(A-1)\cdots(A-p+1) = 0$ in $\mathbb{Z}[A]/p\mathbb{Z}[A]$

Let $P(x) \in \mathbb{Z}[x]$ be the minimal poly. of A $/\mathbb{Z}$

$x(x-1)\cdots(x-p+1) - p\,Q(x) = P(x) R(x)$

Kronecker (1823-1891): Let $P(x) \in \mathbb{Z}[x]$ be irreducible of degree d if $P(x) = 0$ has always d roots in $\mathbb{F}_p$ for all except a finite number of primes then $P(x) = ax + b$, $a, b \in \mathbb{Z}$.

$\Longrightarrow$ A is a rational number.

Legendre symbole; P prime, $d \in \mathbb{N}$, $(d,P)=1$

$$\left(\frac{d}{P}\right) = \begin{cases} +1 & x^2 \equiv_p d \text{ has solutions} \equiv \text{the } p\text{-curvature of } Y' = \frac{\sqrt{d}}{z} Y \text{ is zero} \\ -1 & x^2 \equiv_p d \text{ has no solution} - \quad " \quad " \quad \text{is not zero} \end{cases}$$

Euler, Legendre conjectured, Gauss proved

Quadratic reciprocity: For P, q distinct odd prime

$$\left(\frac{P}{q}\right)\left(\frac{q}{P}\right) = (-1)^{\frac{P-1}{2} \cdot \frac{q-1}{2}}$$

Moral: $y' = \frac{\sqrt{P}}{z} y \mod q$ is related

$y' = \frac{\sqrt{q}}{z} y \mod P.$

Idea of the proof of Prop ⊛

$$\frac{\partial}{\partial z} : \mathbb{Z}[z] \longrightarrow \mathbb{Z}[z]$$

$$0 = \frac{\partial^P}{\partial z^P} : \mathbb{F}_p[z] \longrightarrow \mathbb{F}_p[z]$$

$$\frac{\partial^P}{\partial z^P} z^m = m(m-1) \cdots (m-P+1) \, z^{m-P}$$

$$P! \mid m(m-1) \cdots (m-P+1).$$

Proposition: If for all but a finite number of primes p the P-curvature of $Y' = \left(\frac{A}{z} + \frac{B}{z-1}\right) Y$ is zero then $A, B, -A-C$ are diagonalizable over $\mathbb{C}$ and have rational eigen-values.

Idea of the proof: Observe this for $Y' = \frac{A}{z} Y$ and apply at to the sing $z = 0, 1, \infty$ of the D.E.