Detecting Gauss-Manin and Calabi-Yau differential equations

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Abstract:

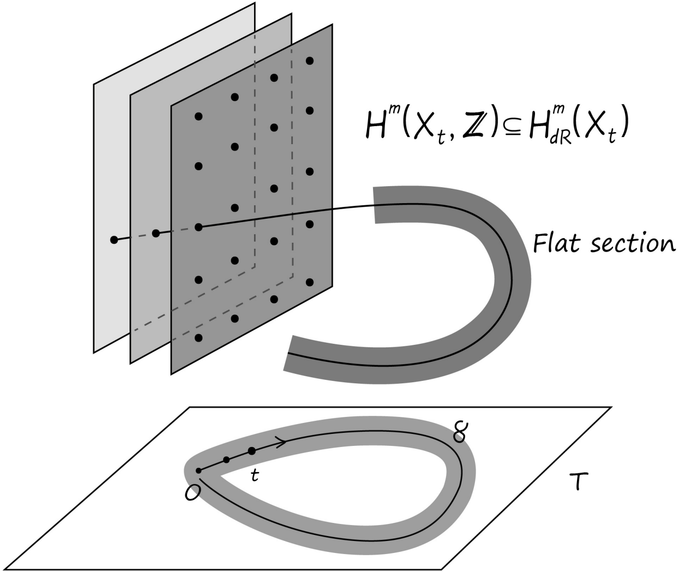
In this talk I will review few conjectures which aim to detect which linear differential equations come from Gauss-Manin connections, that is, they are satisfied by periods of families of algebraic varieties. This includes conjectures due to Katz-Grothendieck, André and Bombieri-Dwork. I will discuss another finer criterion to detect differential equations coming from families of hypergeometric Calabi-Yau varieties. Finally, I will explain a classification list in the case of Heun and Painlevé VI equations (joint works with S. Reiter).

Gauss-Manin connection:

Let $X \to \mathrm{T}$ be a family of smooth projective varieties over a field of arbitrary characteristic. We have a natural connection on the cohomology bundle:

$$\nabla : H^n_{\mathrm{dR}}(X/\mathrm{T}) \to \Omega^1_{\mathrm{T}} \otimes_{\mathcal{O}_{\mathrm{T}}} H^n_{\mathrm{dR}}(X/\mathrm{T}).$$

Over $\mathbb{C}$, this can be easily defined either by its flat sections or integrals.


$H^m(X_t, \mathbb{Z}) \subseteq H^m_{dR}(X_t)$
Flat section
δ
t
0
T

What is Gauss-Manin connection for Gauss?

Let $P(x) := 4(x-t_1)^3 + t_2(x-t_1) + t_3$. We have

$$\begin{pmatrix} d\left(\int \frac{dx}{\sqrt{P(x)}}\right) \\ d\left(\int \frac{xdx}{\sqrt{P(x)}}\right) \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}t_1\frac{\alpha}{\Delta} - \frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ dt_1 - \frac{1}{6}t_1\frac{d\Delta}{\Delta} - (\frac{3}{2}t_1^2 + \frac{1}{8}t_2)\frac{\alpha}{\Delta}, & \frac{3}{2}t_1\frac{\alpha}{\Delta} + \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} \begin{pmatrix} \int \frac{dx}{\sqrt{P(x)}} \\ \int \frac{xdx}{\sqrt{P(x)}} \end{pmatrix}$$

where

$$\Delta := 27t_3^2 - t_2^3, \ \alpha := 3t_3dt_2 - 2t_2dt_3.$$

The above data is the Gauss-Manin connection of the family of elliptic curves $y^2 = P(x)$ before the invention of cohomology theories (before 1900).

Let $\mathrm{T} := \mathbb{A}^1_{\mathbb{F}_p} \backslash \{\Delta = 0\} = \mathrm{Spec}(\mathbb{F}_p[z, \frac{1}{\Delta}])$.

Theorem (P. Deligne, N. Katz 1970)

Let $X \to \mathrm{T}$ be a family of smooth projective varieties over a field of characteristic p and

$$m + 1 = \#\{(p, q) | p + q = n,\ h^{p,q}(X_t) \neq 0\}.$$

Then

$$\nabla^{p(m+1)}_{\frac{\partial}{\partial z}} : H^n(X/\mathrm{T}) \to H^n(X/\mathrm{T})$$

is identically zero.

Since $m \leq n$, a well-known version of this theorem uses n in its announcement.

Let $V = \mathsf{T} \times \mathbb{A}^{\mathsf{h}}_{\mathbb{F}_p} \to \mathsf{T}$ be the trivial vector bundle over T. The data of a connection in V is equivalent to

$$\frac{\partial y}{\partial z} = \mathsf{B}(z)y \tag{1}$$

It is easy to see that $y^{(n)} = \mathsf{B}_n y$, where B_n are recursively computed by

$$\mathsf{B}_1 = \mathsf{B}, \quad \mathsf{B}_{n+1} = \frac{\partial \mathsf{B}_n}{\partial z} + \mathsf{B}_n \mathsf{B}.$$

Theorem

If (1) *comes from the Gauss-Manin connection then*

$$\mathsf{B}_p^m \equiv_p 0$$

.

Conjecture (Deligne, Katz, André)

If for a differential equation $\frac{\partial y}{\partial z} = \mathsf{B}(z)y$ *defined over a finitely generated* $\mathbb{Z}$ *sub algebra* $\mathfrak{R} \subset \mathbb{C}$*, for some* $m \in \mathbb{N}$ *and for almost all primes p we have* $\mathsf{B}_p^m \equiv_p 0$ *then* B *must come from geometry (must be a factor of Gauss-Manin connection).*

$m = 1$

Conjecture (Katz-Grothendieck)

If for a differential equation $L : \frac{\partial y}{\partial z} = \mathrm{B}(z)y$ *defined over* $\mathfrak{K} \subset \mathbb{C}$ *and for almost all primes p we have* $\mathrm{B}_p \equiv_p 0$ *then all the solutions of L are algebraic (L has finite monodromy).*

Definition

A power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is called a G-function if its coefficients are algebraic numbers and there exists a constant* M *such that:*

1. *We have* $|a_n| \leq M^n$ *for all* $n \in \mathbb{N}_0$.
2. *There exists a sequence of positive integers* d_n *with* $d_n \leq M^n$ *such that* $d_n a_m$ *is an algebraic integer for all* $m \leq n$.
3. $f(z)$ *satisfies a linear differential equation with coefficients in* $\bar{\mathbb{Q}}(z)$.

Conjecture (Bombieri-Dwork)

A G-function f *is period, that is, there is a family of algebraic varieties* $X \to \mathrm{T}$, *a section* ω *of* $H^n_{\mathrm{dR}}(\mathrm{X}/\mathrm{T})$ *(all defined over* $\bar{\mathbb{Q}}$*) and continuous family of cycles* $\delta_z \in H_n(X_z, \mathbb{Z}) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$ *such that* $f = \int_{\delta_z} \omega$.

Heun equations:

$$y'' + \left(\frac{1-\theta_1}{z-t} + \frac{1-\theta_2}{z} + \frac{1-\theta_3}{z-1}\right)y' + \left(\frac{\theta_{41}\theta_{42}z - q}{z(z-1)(z-t)}\right)y = 0 \quad (2)$$

with

$$\theta_{41} = -\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - 2 + \theta_4),\ \theta_{42} = -\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - 2 - \theta_4).$$

If it comes from geometry then the exponents $\theta_i,\ i = 1, 2, \ldots, 4$, are rational numbers.

Problem

For which rational numbers $\theta_i,\ i = 1, \ldots, 4$, and complex numbers $t, q \in \mathbb{C}$ does the corresponding Heun equation come from geometry?

Table 1: Heun equations coming from geometry, $a, b, c \in \mathbb{Q}$

*	q	t	θ_1	θ_2	θ_3	θ_4	θ_{42}	θ_{41}
1	$\frac{1}{3}(3a-2)(6a-1)t_1$	$\frac{t_1^2}{3},\ t_1^2+3t_1+3=0$	$a-\frac{1}{2}$	$a-\frac{1}{2}$	$a-\frac{1}{2}$	$9a-\frac{9}{2}$	$3a-\frac{1}{2}$	$-6a+4$
2	0	-1	$b-\frac{1}{2}$	$2b-1$	$b-\frac{1}{2}$	$4a+4b-4$	$2a$	$-2a-4b+4$
3	$-2(a+2b-2)(6b-5)$	-8	$b-\frac{1}{2}$	$3b-\frac{3}{2}$	$a+b-1$	$3a+3b-3$	$a-b+1$	$-2a-4b+4$
4	$-3(10a-7)(3a-2)t_1$	$-t_1^2,\ t_1^2-11t_1-1=0$	$a-\frac{1}{2}$	$5a-\frac{5}{2}$	$a-\frac{1}{2}$	$5a-\frac{5}{2}$	$-a+\frac{3}{2}$	$-6a+4$
5	0	-1	$a+c-1$	$2a+2b-2$	$a+c-1$	$2b+2c-2$	$-2a+2$	$-2a-2b-2c+4$
6	$\frac{-1}{3}(6a-5)(3a-2)t_1$	$\frac{t_1^2}{3}, t_1^2+3t_1+3=0$	$3a-\frac{3}{2}$	$3a-\frac{3}{2}$	$3a-\frac{3}{2}$	$3a-\frac{3}{2}$	$-3a+\frac{5}{2}$	$-6a+4$
7	$\frac{-2}{243}(96a-25)(3a-2)t_1$	$\frac{t_1^2}{9}, 3t_1^2-14t_1+27=0$	$a-\frac{1}{2}$	$\frac{1}{3}$	$a-\frac{1}{2}$	$8a-4$	$3a-\frac{2}{3}$	$-5a+\frac{10}{3}$
8	$\frac{-1}{288}(3a-2)(1029a-149)$	$\frac{81}{32}$	$a-\frac{1}{2}$	$\frac{1}{3}$	$2a-1$	$7a-\frac{7}{2}$	$2a-\frac{1}{6}$	$-5a+\frac{10}{3}$
9	$\frac{-125}{6}(4a-3)(3a-2)$	-80	$a-\frac{1}{2}$	$4a-2$	$\frac{1}{3}$	$5a-\frac{5}{2}$	$\frac{5}{6}$	$-5a+\frac{10}{3}$
10	$\frac{-25}{18}(3a-2)(6a-5)$	$-\frac{27}{5}$	$\frac{1}{3}$	$3a-\frac{3}{2}$	$2a-1$	$5a-\frac{5}{2}$	$\frac{5}{6}$	$-5a+\frac{10}{3}$
11	$\frac{1}{128}(49a-12)(3a-2)t_1$	$\frac{t_1^2}{8}, 4t_1^2+13t_1+32=0$	$a-\frac{1}{2}$	$\frac{1}{2}$	$a-\frac{1}{2}$	$7a-\frac{7}{2}$	$\frac{5}{2}a-\frac{1}{2}$	$-\frac{9}{2}a+3$
12	$\frac{-9}{16}a(a+2b-2)$	$\frac{1}{4}$	$2b-1$	$\frac{1}{2}$	$b-\frac{1}{2}$	$3a+3b-3$	$\frac{3}{2}a$	$-\frac{3}{2}a-3b+3$
13	$\frac{39}{500}(3a-2)(6a-5)$	$-\frac{3}{125}$	$\frac{1}{2}$	$3a-\frac{3}{2}$	$a-\frac{1}{2}$	$5a-\frac{5}{2}$	$\frac{1}{2}a+\frac{1}{2}$	$-\frac{9}{2}a+3$
14	$\frac{-3}{4}(a+2b-2)(6b-5)$	-3	$\frac{1}{2}$	$3b-\frac{3}{2}$	$a+b-1$	$2a+2b-2$	$\frac{1}{2}a-b+1$	$-\frac{3}{2}a-3b+3$
15	0	-1	$a-\frac{1}{2}$	$\frac{2}{3}$	$a-\frac{1}{2}$	$6a-3$	$2a-\frac{1}{3}$	$-4a+\frac{8}{3}$
16	$-\frac{14}{3}a+\frac{28}{9}$	$\frac{27}{2}$	$a-\frac{1}{2}$	$\frac{2}{3}$	$2a-1$	$5a-\frac{5}{2}$	$a+\frac{1}{6}$	$-4a+\frac{8}{3}$
17	$\frac{-2}{9}(3a-2)(6a-5)$	-1	$\frac{2}{3}$	$3a-\frac{3}{2}$	$2a-1$	$3a-\frac{3}{2}$	$-a+\frac{7}{6}$	$-4a+\frac{8}{3}$
18	$\frac{-1}{147}(58a-15)(3a-2)t_1$	$\frac{t_1^2}{49},\ t_1^2-13t_1+49=0$	$\frac{1}{3}$	$a-\frac{1}{2}$	$\frac{1}{3}$	$7a-\frac{7}{2}$	$3a-\frac{5}{6}$	$-4a+\frac{8}{3}$
19	0	-1	$\frac{1}{3}$	$2a-1$	$\frac{1}{3}$	$6a-3$	$2a-\frac{1}{3}$	$-4a+\frac{8}{3}$
20	$\frac{-4}{3}(4a-3)(3a-2)t_1$	$-\frac{t_1^2}{2}, t_1^2-10t_1-2$	$4a-2$	$\frac{1}{3}$	$4a-2$	$\frac{1}{3}$	$-4a+3$	$-4a+\frac{8}{3}$
21	$(\frac{-27}{2}\zeta-\frac{29}{4})(a-\frac{10}{9589}\zeta-\frac{7442}{28767})(a-\frac{2}{3})$	$-\frac{2}{7}(3\zeta+1),\ \zeta^2+3=0$	$a-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$6a-3$	$\frac{5}{2}a-\frac{2}{3}$	$-\frac{7}{2}a+\frac{7}{3}$
22	$\frac{-14}{1125}(3a-2)(147a-22)$	$\frac{189}{125}$	$\frac{1}{2}$	$\frac{1}{3}$	$2a-1$	$5a-\frac{5}{2}$	$\frac{3}{2}a-\frac{1}{6}$	$-\frac{7}{2}a+\frac{7}{3}$
23	$\frac{77}{972}(3a-2)(6a-5)$	$-\frac{1}{27}$	$\frac{1}{2}$	$3a-\frac{3}{2}$	$\frac{1}{3}$	$4a-2$	$\frac{1}{2}a+\frac{1}{3}$	$-\frac{7}{2}a+\frac{7}{3}$
24	$-\frac{1}{6}a+\frac{1}{9}$	$-\frac{16}{9}$	$a-\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$5a-\frac{5}{2}$	$2a-\frac{1}{2}$	$-3a+2$
25	$-3a+2$	9	$\frac{1}{3}$	$\frac{2}{3}$	$2a-1$	$4a-2$	a	$-3a+2$
26	$\frac{-1}{125}(3a-2)(38a-9)t_1$	$\frac{4t_1^2}{125},\ t_1^2-11t_1+125/4=0$	$\frac{1}{2}$	$a-\frac{1}{2}$	$\frac{1}{2}$	$5a-\frac{5}{2}$	$2a-\frac{1}{2}$	$-3a+2$
27	0	-1	$\frac{1}{2}$	$2b-1$	$\frac{1}{2}$	$2a+2b-2$	a	$-a-2b+2$
28	$\frac{-1}{6}(6a-5)(3a-2)t_1$	$-\frac{t_1^2}{3},\ t_1^2-6t_1-3=0$	$3a-\frac{3}{2}$	$\frac{1}{2}$	$3a-\frac{3}{2}$	$\frac{1}{2}$	$-3a+\frac{5}{2}$	$-3a+2$
29	$\frac{5}{162}a-\frac{5}{243}$	$-\frac{5}{27}$	$\frac{1}{2}$	$\frac{2}{3}$	$a-\frac{1}{2}$	$4a-2$	$\frac{3}{2}a-\frac{1}{3}$	$-\frac{5}{2}a+\frac{5}{3}$
30	$-\frac{5}{3}a+\frac{10}{9}$	5	$\frac{1}{2}$	$\frac{2}{3}$	$2a-1$	$3a-\frac{3}{2}$	$\frac{1}{2}a+\frac{1}{6}$	$-\frac{5}{2}a+\frac{5}{3}$
31	0	-1	$\frac{2}{3}$	$2a-1$	$\frac{2}{3}$	$2a-1$	$\frac{1}{3}$	$-2a+\frac{4}{3}$
32	0	-1	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$2a-1$	$a-\frac{1}{3}$	$-a+\frac{2}{3}$
33	$\frac{1}{12}(3a-1)(3a-2)t_1$	$\frac{t_1^2}{3},\ t_1^2+3t_1+3=0$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$3a-\frac{3}{2}$	$\frac{3}{2}a-\frac{1}{2}$	$-\frac{3}{2}a+1$
34	0	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$3a-\frac{3}{2}$	$\frac{3}{2}a-\frac{1}{2}$	$-\frac{3}{2}a+1$
35	0	-1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$4a-2$	$2a-\frac{2}{3}$	$-2a+\frac{4}{3}$
36	$\frac{-16}{243}(3a-1)(3a-2)t_1$	$\frac{t_1^2}{27},\ t_1^2-10t_1+27=0$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$4a-2$	$2a-\frac{2}{3}$	$-2a+\frac{4}{3}$
37	$\frac{25}{768}(3a-2)(3a-1)t_1$	$\frac{t_1^2}{64}, t_1^2+11t_1+64=0$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$5a-\frac{5}{2}$	$\frac{5}{2}a-\frac{5}{6}$	$-\frac{5}{2}a+\frac{5}{3}$
38	$\frac{1}{3}(3a-2)(3a-1)t_1$	$\frac{t_1^2}{3},\ t_1^2+3t_1+3=0$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$6a-3$	$3a-1$	$-3a+2$

Example 7:

$$q = \frac{-2}{243}(96a - 25)(3a - 2)t_1, \quad t = \frac{t_1^2}{9}, 3t_1^2 - 14t_1 + 27 = 0$$

$$\theta = (a - \frac{1}{2}, \frac{1}{3}, a - \frac{1}{2}, 8a - 4)$$

The geometry:

$$y = (4x^3 - g_2 x - g_3)^a,$$

$$g_2(z) = 12z(z^3 - 6z^2 + 15z - 12),$$

$$g_3(z) = 4z(2z^5 - 18z^4 + 72z^3 - 144z^2 + 135z - 27).$$

Conjecture

A linear differential equation is a factor of Gauss-Manin connection of families of Calabi-Yau n-folds if the mirror map has integral coefficients.

Let $a_i,\ i=1,2,\ldots,n$ be rational numbers, $0<a_i<1$,

$$F(a|z) := {}_nF_{n-1}(a_1,\ldots,a_n;1,1,\ldots,1|z) = \sum_{k=0}^{\infty} \frac{(a_1)_k\ldots(a_n)_k}{k!^n}\, z^k,$$

be the holomorphic solution of the generalized hypergeometric differential equation

$$\theta^n - z(\theta+a_1)(\theta+a_2)\cdots(\theta+a_n) = 0$$

where $(a_i)_k = a_i(a_i+1)(a_i+2)...(a_i+k-1)$, $(a_i)_0 = 1$, is the Pochhammer symbol and $\theta = z\frac{d}{dz}$. The logarithmic solution around $z=0$ has the form $G(a|z) + F(a|z)\log z$, where

$$G(a|z) = \sum_{k=1}^{\infty} \frac{(a_1)_k\cdots(a_n)_k}{(k!)^n}\Big[\sum_{j=1}^{n}\sum_{i=0}^{k-1}\Big(\frac{1}{a_j+i} - \frac{1}{1+i}\Big)\Big] z^k. \quad (3)$$

The mirror map

$$q(a|z) =: z \exp(\frac{G(a|z)}{F(a|z)}).$$

For a rational number x such that p does not divide the denominator of x, we define

$$\delta_p(x) := \frac{x + x_0}{p},$$

where $0 \leq x_0 \leq p - 1$ is the unique integer such that p does not divide the denominator of $\delta_p(x)$. We call δ_p the Dwork operator.

Conjecture

The mirror map $q(a|z)$ is N-integral if and only if for any good prime

$$\{\delta_p(a_1), \delta_p(a_2)\} = \{a_1, a_2\},\ \textit{or}\ \{1 - a_1, 1 - a_2\}\ \textit{for} \quad n = 2 \quad (4)$$

and

$$\{\delta_p(a_1), \delta_p(a_2), \delta_p(a_3), \ldots, \delta_p(a_n)\} = \{a_1, a_2, a_3, \ldots, a_n\}, \quad \textit{for}\ n \neq 2. \quad (5)$$

$n = 2$
(1/2, 1/2), (2/3, 1/3), (3/4, 1/4), (5/6, 1/6),
(1/6, 1/6), (1/3, 1/6), (1/2, 1/6), (1/3, 1/3), (2/3, 2/3),
(1/4, 1/4), (1/2, 1/4), (3/4, 1/2), (3/4, 3/4), (1/2, 1/3),
(2/3, 1/6), (2/3, 1/2), (5/6, 1/3), (5/6, 1/2), (5/6, 2/3),
(5/6, 5/6), (3/8, 1/8), (5/8, 1/8), (7/8, 3/8), (7/8, 5/8),
(5/12, 1/12), (7/12, 1/12), (11/12, 5/12), (11/12, 7/12)
$n = 4$
(1/2, 1/2), (1/3, 2/3), (1/4, 1/2), (1/4, 1/4), (2/5, 1/5),
(3/8, 1/8), (3/10, 1/10), (1/2, 1/6), (1/2, 1/3), (1/3, 1/6),
(1/6, 1/6), (1/3, 1/4), (1/4, 1/6), (5/12, 1/12)
$n = 6$
(1/2, 1/2, 1/2), (1/3, 1/3, 1/3), (1/2, 1/2, 1/4), (1/2, 1/4, 1/4),
(1/4, 1/4, 1/4), (1/2, 1/2, 1/3), (1/2, 1/3, 1/3), (1/2, 1/2, 1/6),
(1/2, 1/3, 1/6), (1/3, 1/3, 1/6), (1/2, 1/6, 1/6), (1/3, 1/6, 1/6),
(1/6, 1/6, 1/6), (3/7, 2/7, 1/7), (1/2, 3/8, 1/8), (3/8, 1/4, 1/8),
(4/9, 2/9, 1/9), (1/2, 2/5, 1/5), (1/2, 3/10, 1/10)(1/2, 1/3, 1/4),
(1/3, 1/3, 1/4), (1/3, 1/4, 1/4), (1/2, 1/4, 1/6), (1/3, 1/4, 1/6),
(1/4, 1/4, 1/6), (1/4, 1/6, 1/6), (1/2, 5/12, 1/12), (5/12, 1/3, 1/12),
(5/12, 1/4, 1/12), (5/12, 1/6, 1/12), (5/14, 3/14, 1/14), (2/5, 1/3, 1/5),
(7/18, 5/18, 1/18), (2/5, 1/4, 1/5), (3/10, 1/4, 1/10), (3/8, 1/3, 1/8),
(3/8, 1/6, 1/8), (2/5, 1/5, 1/6), (1/3, 3/10, 1/10), (3/10, 1/6, 1/10)

Table 1: *N*-integral hypergeometric mirror maps.

Theorem (Lian-Yau, Zudilin, Krattenthaler-Rivoal, ..., Movasati-Shokri)

We have

1. *For an arbitrary n the only if part of the conjecture is true.*
2. *It is true for* $n = 1, 2, 3, 4$.

References

1. H. Movasati, S. Reiter, Heun equations coming from geometry. Bull. Braz. Math. Soc. 43(3), 423-442, 2012.
2. Appendix A with Khosro Shokri in the book: Gauss-Manin connection in disguise: Calabi-Yau modular forms, Surveys in Modern Mathematics, Vol 13, International Press, Boston.