Computing the lines of a smooth cubic surface

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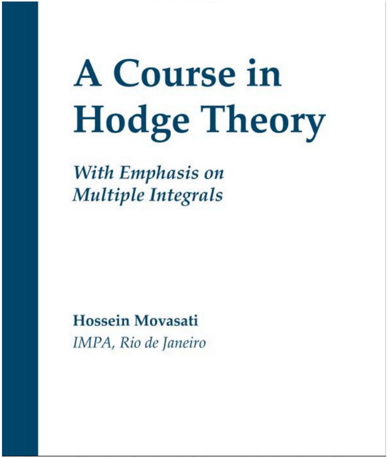
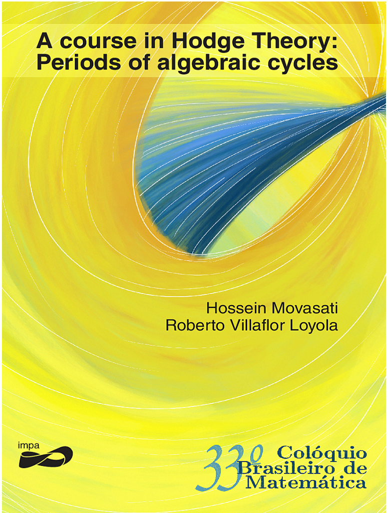
Abstract:

We give an explicit formula for the 27 lines of a smooth cubic surface near the Fermat surface. Our formula involves convergent power series with coefficients in the extension of rational numbers with the sixth root of unity. Our main tool is the Artinian Gorenstein ring of socle two attached to such lines. If times permits, I will also describe how one can use similar tools to construct a conjectural counterexample to a conjecture of Harris on special components of Noether-Lefschetz loci for degree eight surfaces.





Articles: Computing the lines of a smooth cubic surface, Special Components of Noether-Lefschetz loci

27 lines of a smooth cubic surface

Any smooth cubic surface in $\mathbb{P}^3$ has 27 lines.

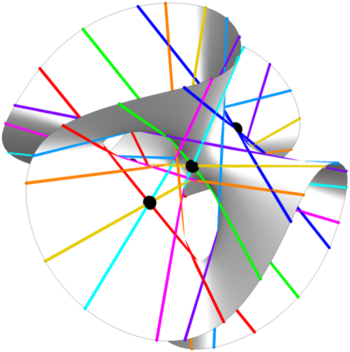


Figure: Clebsch surface[1]

$$x_0^3 + x_1^3 + x_2^3 + x_4^3 = (x_0 + x_1 + x_2 + x_3)^3$$

[1] blogs.ams.org/visualinsight/2016/02/15/27-lines-on-a-cubic-surface/

27 lines of the Fermat cubic surface

$$X_0: \quad x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

$$\mathbb{P}^1 : \begin{cases} x_0 - \zeta_1 x_1 = 0, \\ x_2 - \zeta_2 x_3 = 0, \end{cases} \qquad \zeta_1^3 = \zeta_2^3 = -1,$$

We write down a cubic surface in the format

$$X_t : \quad F_t := x_0^3 + x_1^3 + x_2^3 + x_3^3 - \sum_{i \in I} t_i x^i = 0, \qquad (1)$$

$$t := (t_i,\ i \in I) \in \mathsf{T} := \mathbb{C}^{20} \backslash \{\Delta = 0\},$$

where I is the set of exponents of monomials of degree 3 in four variables x_0, x_1, x_2, x_3 and $\Delta = 0$ is the loci of singular cubic surfaces.

Notation

1. For $\beta \in \mathbb{N}_0^4$ we denote by β_i its $(i+1)$-th coordinate, that is, $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)$,
2. for $n \in \mathbb{Z}$, $\bar{n} \in \mathbb{N}_0$ is defined by the rules $0 \leq \bar{n} \leq 2, \quad n \equiv_3 \bar{n}$.
3. For a positive rational number r, $[r]$ is the integer part of r, that is $[r] \leq r < [r] + 1$, $\{r\} := r - [r]$,

$$\langle r \rangle = (r-1)(r-2)\cdots(r-[r])$$

4. We consider the set of $(\mathbf{m}, \mathbf{n}, \mathbf{l}, \zeta_1, \zeta_2)$, where $(\mathbf{m}, \mathbf{n}, \mathbf{l}) = (1,2,3), (2,1,3), (3,1,2)$ and ζ_1, ζ_2 are roots of -1, that is, $\zeta_1^3 = \zeta_2^3 = -1$. This set consists of 27 elements.

Theorem

For the twenty seven choice of $k = (\mathbf{m}, \mathbf{n}, \mathbf{l}, \zeta_1, \zeta_2)$ as above we have the following rational curve inside X_t:

$$\mathbb{P}^1_{k,t} : \begin{cases} c_{0212} \cdot x_0 - c_{0202} \cdot x_1 + c_{0201} \cdot x_2 + 0 \cdot x_3 = 0 \\ c_{0223} \cdot x_0 + 0 \cdot x_1 - c_{0203} \cdot x_2 + c_{0202} \cdot x_3 = 0 \end{cases} , \qquad (2)$$

where

$$c_{i_1 i_2 j_1 j_2} = \det \begin{bmatrix} p_{i_1 j_1}, & p_{i_1 j_2} \\ p_{i_2 j_1}, & p_{i_2 j_2} \end{bmatrix},$$

$$p_{ij} = \sum_{a: I \to \mathbb{N}_0} \frac{1}{a!} \zeta_1^{\overline{(\beta_{ij}+a^*)_0+1}} \cdot \zeta_2^{\overline{(\beta_{ij}+a^*)_{\mathbf{n}}+1}} \prod_{i=0}^{3} \left\langle \frac{(\beta_{ij} + a^*)_i + 1}{3} \right\rangle \cdot t^a,$$

the sum runs through all $\# I$-tuples $a = (a_\alpha, \ \ \alpha \in I)$ of non-negative integers such that

$$\left\{ \frac{(\beta_{ij} + a^*)_0 + 1}{3} \right\} + \left\{ \frac{(\beta_{ij} + a^*)_{\mathbf{m}} + 1}{3} \right\} = 1, \qquad (3)$$

$$\left\{ \frac{(\beta_{ij} + a^*)_{\mathbf{n}} + 1}{3} \right\} + \left\{ \frac{(\beta_{ij} + a^*)_{\mathbf{l}} + 1}{3} \right\} = 1,$$

and

$$t^a := \prod_{\alpha \in I} t_\alpha^{a_\alpha}, \qquad a! := \prod_{\alpha \in I} a_\alpha !, \quad a^* := \sum_\alpha a_\alpha \cdot \alpha. \qquad (4)$$

and $\beta_{ij} \in \mathbb{N}_0^4$ is the exponent vector of $x_i x_j$, that is $x^{\beta_{ij}} = x_i x_j$.

Let $X \subset \mathbb{P}^3$ be a smooth cubic surface given by the homogeneous polynomial F of degree 3. For a homogeneous polynomial P of degree 2 in x_0, x_1, x_2, x_3 define

$$\omega_P := \mathrm{Resi}\left(\frac{P \cdot \sum_{i=0}^{3}(-1)^i x_i \widehat{dx_i}}{f_t^2}\right) \in H^2_{\mathrm{dR}}(X),$$

where $\mathrm{Resi} : H^3(\mathbb{P}^3 \backslash X) \to H^2_{\mathrm{dR}}(X)$ is the Griffiths residue map.

Proposition

Let $\mathbb{P}^1$ be a line inside X. The 4×4 matrix

$$A = \left[\int_{\mathbb{P}^1} \omega_{x_i x_j} \right]_{0 \le i,j \le 3} \tag{5}$$

is of rank two and its kernel is generated by

$$a_i := (a_{1,i}, a_{2,i}, a_{3,i}, a_{4,i}), \ \ i = 1, 2$$

where

$$a_{1,i} x_0 + a_{2,i} x_1 + a_{3,i} x_2 + a_{4,i} x_3 = 0, \ \ i = 1, 2$$

are two linear equations of $\mathbb{P}^1$.

For the Fermat variety X_0, the twenty seven lines are given by

$$\begin{cases} x_0 - \zeta_1 x_{\mathbf{m}} = 0, \\ x_{\mathbf{n}} - \zeta_2 x_{\mathbf{l}} = 0, \end{cases} \qquad \zeta_1^3 = \zeta_2^3 = -1, \{0, \mathbf{m}, \mathbf{n}, \mathbf{l}\} = \{0, 1, 2, 3\}, \ \mathbf{n} < \mathbf{l}.$$

For $t \in \mathrm{T}$ near to the Fermat point 0, there is a unique rational curve $\mathbb{P}^1_{k,t}$ which is obtained by deformation of $\mathbb{P}^1_{k,0}$.
Its homology class $\delta_t = [\mathbb{P}^1_{k,t}] \in H_2(X_t, \mathbb{Z})$ is the monodromy (parallel transport) of the homology class of $\mathbb{P}^1_{k,0}$. We have

$$\frac{-6}{2\pi\sqrt{-1}} \int_{\delta_t} \omega_{x_i x_j} = p_{ij}, \quad i, j = 0, 1, 2, 3,$$