Project: Gauss-Manin Connection in disguise
Book1: GMCD - Calabi-Yau modular forms
Book2: Modular and Automorphic Forms & beyond
GMCD_CY3 Alim-Movasati-Scheidegger-Yau (AMSY)

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Mathematician = mice toching the elefent

Mirror symmetry = A big elephant in the darkness

$\check{X}$ Compact CY3, symplectic geometry, enumerative algebraic geometry, Counting rational curves

X compact CY3, periods, Hodge structures, mirror map, Computing GW invariants

 → CY3

HMS
SYZ

world-Sheet

string propogets and form a worldsheet
Holomorphic worldsheet are special. (Picard-Lefschetz theory.)

$N_{g,d}$:= GW of genus g, degree d after YY-AMSY

$$F_g = \sum_{\beta \in H_2(\check{X},\mathbb{Z})} N_{g,\beta} q^\beta \quad g \geq 2$$

$$q^\beta = q_1^{\beta_1} q_2^{\beta_2} \cdots q_h^{\beta_h}$$

$$q_i \frac{\partial}{\partial q_i} F_1$$

$$C_{ijk} = q_i \frac{\partial}{\partial q_i} q_j \frac{\partial}{\partial q_j} q_k \frac{\partial}{\partial q_k}$$

Conjecture: The field generated by F_g's has transc. degree

$$\frac{3h_1^2 + 7h + 4}{2}$$

Jun Li et al → only mirror quintic

```
      1
     0 0
    0 h 0
   1 a a 1
    0 h 0
     0 0
      1
```

```
      1
     0 0
    0 a 0
   1 h h 1
    0 a 0
     0 0
      1
```

Douglas, Dijkgraaf, Kaneko Zagier

 1 1 1 1

 1 1 1

$N_{g,d}$:= # ($\xrightarrow{d:1}$ ) corrected by $\frac{1}{Aut(\ )}$

g

simple ramification points

Transc degree $\mathbb{C}(F_g, g=1,2,\dots) = 3$

$F_1 = E_2$, F_g poly of degree $6g-6$ in $\mathbb{Q}[E_2, E_4, E_6]$, $\deg E_i = 2i$

$$F_2 = \frac{1}{103680}(10E_2^3 - 6E_2E_4 \ldots$$

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(d) q^n$$

$$\sigma_i(n) = \sum_{d|n} d^i$$

$B_{2k} = \frac{1}{6}, \frac{1}{30}, \frac{1}{42}$ (2, 4, 6)

$\mathbb{Q}[E_2, E_4, E_6]$ Theory of quasi-modular forms for $SL(2,\mathbb{Z})$

$\mathbb{Q}[E_4, E_6]$ " modular " " "

Ibiporanga: (Pretty land): A moduli space of enhanced varieties.

$\mathcal{M}$: a classical moduli space of $X \subseteq \mathbb{P}^N$ smooth, complete,

$$\mathcal{M} = \mathrm{Hilb}_P / G$$

quintic threefolds in $\mathbb{P}^4$, $G = GL(5, \mathbb{C})$.

$X_1, X_2 \in \mathcal{M}$, $X_1 \overset{C^\infty}{\simeq} X_2$ but not analy or algebraically.

$X \in \mathcal{M}$ X/k $H^*_{dR}(X/k) \times H^*_{dR}(X/k) \xrightarrow[\cup p]{} H^*_{dR}(X/k)$

$$0 = F^{m+1} \subseteq F^m \subseteq \cdots = F^0 = H^m_{dR}(X/k)$$

θ = generator of $H^2_{dR}(\mathbb{P}^N)|_X \in F^1 H^2_{dR}(X/k)$.

Hodge decomposition cannot be defined over k.

Prop 2.4 (Book) Let $X_t, t \in T$ be a family of smooth projective varieties. Then

$$(H^*_{dR}(X), F^*, \cup, \theta) \simeq (H^*_{dR}(X_0), F_0^*, \cup, \theta_0)$$

Fix X_0:

T

Ipiporanga: Moduli of (X, α), $X \in \mathcal{M}$.

Elliptic curve = $\dim T = 3$

CY3 $h = h^{21}$ $\dim T = \frac{3h^2 + 7h + 4}{2}$

Conjecture (AMSY): T is an affine variety

F_g's can be interpreted as functions on T.

Thm (M. 2012) For elliptic curves

$$T := \mathrm{Spec}\left(\mathbb{C}\left[t_1, t_2, t_3, \frac{1}{27t_3^2 - t_2^3}\right]\right)$$

and we have a universal family

$$y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3 \quad \left[\frac{dx}{y}\right], \left[\frac{x\,dx}{y}\right]$$

GMCD, CY modular forms $\longrightarrow$ Mirror quintic CY3.

Differential equation of F_g's: BCOV

For CY3

Enhancement $(X, \alpha_1, \alpha_2, \ldots, \alpha_{2h+2})$, $\alpha_i \in H^3_{dR}(X)$

compatible with Hodge filtration

$$[\langle \alpha_i, \alpha_j \rangle] = \Phi \quad \text{constant matrix} = \begin{pmatrix} 0 & & & 1 \\ & 0 & I_{h\times h} & \\ & -I_{h\times h} & 0 & \\ -1 & & & 0 \end{pmatrix}$$

$G := \{ g \in GL(2h+2, \mathbb{C}) \mid g$ block upper triangular$\}$.

$$g^{tr} \Phi g = \Phi$$

$G \curvearrowright T$ base change

X/T universal family

Gauss-Manin connection

$$\nabla : H^3_{dR}(X/T) \to \Omega^1_T \otimes_{\mathcal{O}_T} H^3(X/T)$$

Thm (AMSY): There are unique vector fields R_k, $k = 1, 2, \ldots, h$ and $C_{ijk} \in \mathcal{O}_T$

$$A_{R_k} = \left(\begin{array}{c|c|c|c} & \delta^i_k & & \\ & & C_{kij} & \\ & & & \delta^i_k \end{array} \right)$$

sym in i, j, k.

$$R_{i_1} C_{i_2 i_3 i_4} = R_{i_2} C_{i_1 i_3 i_4}$$

For $g \in \mathrm{Lie}(G)$ there is a unique vector field R_g in T

$$A_{R_g} = g^{tr}.$$

vector field = derivation, the most important part of BCOV

$$R_{t^{ab}} F_g = \frac{1}{2} \sum_{h=1}^{g-1} R_a F_h R_b F_{g-h} + \frac{1}{2} R_a R_b F_{g-1}$$

$a, b = 1, 2, \ldots, h.$