

Project: Gauss-Marin Connection in disguise

Book 1: GMCD - Calabi-Yau modular forms

Book 2: Modular and Automorphic Forms & beyond  
GMCD - CY3 Alim-Monaschi-Scheidegger-Yau (AMSY)

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Chinese university of Hong Kong

Mathematician = mice touching the elephant

Mirror symmetry = A big elephant in the darkness

Compact CY3, symplectic geometry  
enumerative algebraic geometry  
counting rational curves

compact CY3,  
periods, Hodge structures  
mirror map,



HMS  
SYZ  
Computing GW invariants

string propagates and form a worldsheet  
Holomorphic worldsheet are special. Picard-Lefschetz theory

$N_{g,d}$  := GW of genus  $g$ , degree  $d$  after YY-AMSY

Conjecture: The field generated by  $F_g$ 's has transc. degree

$$F_g = \sum_{\beta \in H_2(X, \mathbb{Z})} N_{g,\beta} q^\beta \quad g \geq 2$$

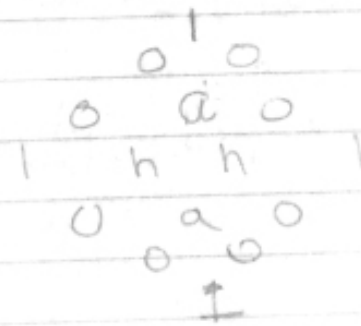
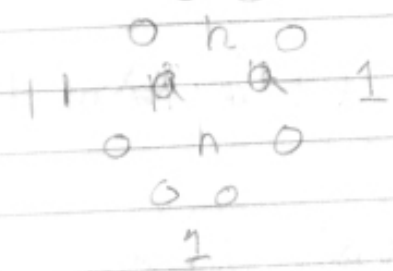
$$q^\beta = q_1^{a_1} q_2^{a_2} \dots q_h^{a_h}$$

$$\frac{3h^2 + 7h + 4}{2}$$

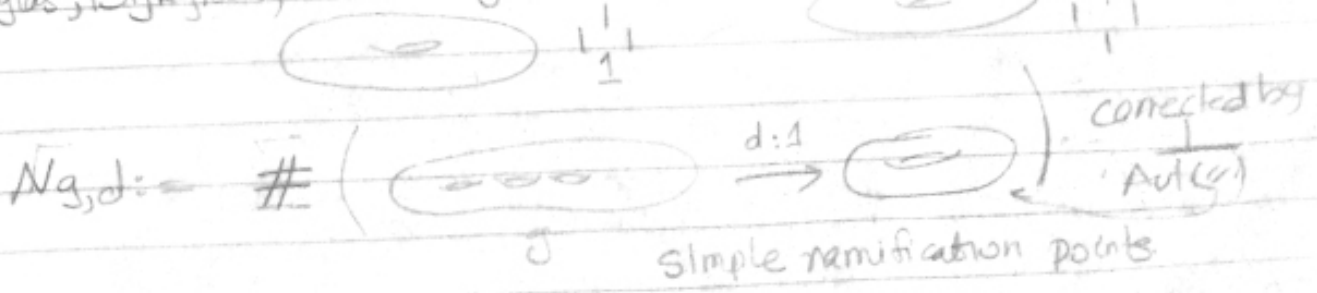
$$q_i \frac{\partial}{\partial q_i} F$$

$$C_{ijk} = q_i \frac{\partial}{\partial q_i} q_j \frac{\partial}{\partial q_j} q_k \frac{\partial}{\partial q_k} F$$

JunLi et al  $\rightarrow$  only mirror quantum



Douglas, Dijkgraaf, Kaneko Zagier



Transc degree  $\mathbb{C}(F_g, g=1, 2, \dots) = 3$

$F_1 = E_2$ ,  $F_g$  poly of degree  $6g-6$  in  $\mathbb{C}[E_2, E_4, E_6]$   
 $F_2 = \frac{1}{103680} (10E_2^3 - 6E_2 E_4 + E_6)$  credeal

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

$$\sigma_i(n) = \sum_{d|n} d^i$$

$$B_{2k} = \frac{1}{2}, \frac{1}{30}, \frac{1}{42}$$

$\mathbb{Q}[E_2, E_4, E_6]$  Theory of quasi-modular forms for  $SL(2, \mathbb{Z})$   
 $\mathbb{Q}[E_4, E_6]$  modular forms

Ibiporanga: (Pretty land): A moduli space of enhanced varieties.

$\mathcal{M}$ : a classical moduli space of  $X \subseteq \mathbb{P}^N$  smooth, complete,

$$\mathcal{M} = \text{Hilb}_p / G$$

quintic threefolds in  $\mathbb{P}^4$ ,  $G = GL(5, \mathbb{C})$ .

$X_1, X_2 \in \mathcal{M}$ ,  $X_1 \xrightarrow{\mathbb{C}^\infty} X_2$  but not analytically or algebraically.

$$X \in \mathcal{M} \quad X/k \quad H_{\text{dR}}^x(X/k) \times H_{\text{dR}}^x(X/k) \xrightarrow{\omega_F} H_{\text{dR}}^x(X/k)$$

$$0 = F^{m+1} \subseteq F^m \subseteq \dots = F^0 = H_{\text{dR}}^m(X/k)$$

$\theta =$  generator of  $H_{\text{dR}}^2(\mathbb{P}^N) |_{X \in F^1} H_{\text{dR}}^2(X/k)$ .

Hodge decomposition cannot be defined over  $k$ .

Prop 2.4 (Book) Let  $X_t, t \in T$  be a family of smooth projective varieties. Then

$$(H_{\text{dR}}^x(X), F^*, U, \theta) \simeq (H_{\text{dR}}^x(X_0), F_0^*, U_0)$$

Fix  $X_0$ :

$T$

Ibiporanga: Moduli of  $(X, \alpha)$ ,  $X \in \mathcal{M}$ .

Elliptic curve =  $\dim T = 3$

CB  $h = h^2$

$$\dim T = \frac{3h^2 + 7h + 4}{2}$$

Conjecture (AMSY):  $T$  is an affine variety

$F_g$ 's can be interpreted as functions on  $T$ .

Thm (M. 2012) For elliptic curves

$$T_0 = \text{Spec}(\mathbb{C}[t_1, t_2, t_3, \frac{1}{27t_3^2 - t_2^3}])$$

and we have a universal family

$$y^2 = 4(x - t_1)^3 - t_2(x - t_1) - t_3 \quad \left[ \frac{dx}{y} \right], \left[ \frac{x dx}{y} \right]$$

GICD, CY modular forms  $\rightarrow$  Mirror quintic CY3.

Differential equation of Fg's: BCOV  $\uparrow$

For CY3

Enhancement  $(X, \alpha_1, \alpha_2, \dots, \alpha_{2h+2})$ ,  $\alpha_i \in H^3_{dR}(X)$

compatible with Hodge filtration

$$[\langle \alpha_i, \alpha_j \rangle] = \Phi \text{ constant matrix} = \begin{pmatrix} 0 & I_{h,h} \\ -I_{h,h} & 0 \end{pmatrix}$$

$$G := \{ g \in GL(2h+2, \mathbb{C}) \mid g \text{ block upper triangular} \}$$

$$g^0 \Phi g = \Phi$$

$G \curvearrowright T$  base change

$X/T$  universal family

Gauss-Manin connection

$$\nabla: H^3_{dR}(X/T) \rightarrow \Omega^1_T \otimes_{\mathcal{O}_T} H^3(X/T)$$

Thm (AMSY): There are unique vector fields  $R_k$ ,  $k=1,2$ ,

with and  $C_{ijk} \in \mathcal{O}_T$

$$AR_k = \begin{pmatrix} & \delta_k^1 & & \\ & & C_{ij} & \\ & & & \delta_k^i \\ \text{sym in } i,j,k. & & & \end{pmatrix}$$

$$R_{i_1} C_{i_2 i_3 i_4} = R_{i_2} C_{i_1 i_3 i_4}$$

For  $g \in \text{Lie}(G)$  there is a unique vector field  $R_g$  on  $T$

$$AR_g = g^{\text{tr}}$$

vector field = derivation, the most important part of BCOV

$$R_{t^{ab}} F_g = \frac{1}{2} \sum_{n=1}^{g-1} R_a F_n R_b F_{g-n} + \frac{1}{2} R_a R_b F_{g-1}$$

$a, b = 1, 2, \dots, h$ .