26 / 09 / 2022 Abelian integrals in Holomorphic foliations

Let M be complex surface and f be a rational function on M.

Foliation $\mathcal{F}(df)$: its leaves are irreducible components of the fibers of $f: M \dashrightarrow \mathbb{P}^1$

$M = \mathbb{P}^2$, $f := \frac{F(x,y,z)}{G(x,y,z)}$, F, G homog. of degree d.

$G = z^d$ $f := \frac{F}{z^d}$, restriction to the affine chart

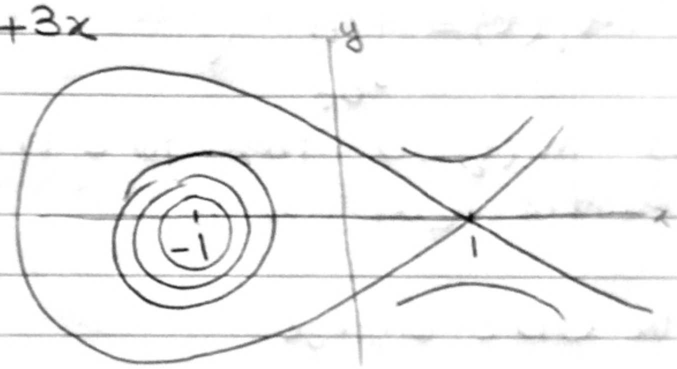
$$\mathbb{C}^2 := \{[x:y:z] \in \mathbb{P}^2 \mid z = 1\} \subseteq \mathbb{P}^2$$

Foliation $\mathcal{F}(df)$, $f: \mathbb{C}^2 \to \mathbb{C}$ polynomial of degree d

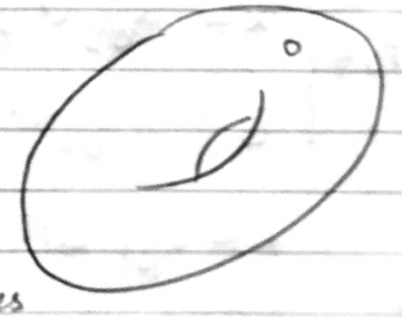
$f = y^2 - x^3 + 3x$

$\mathbb{R}^2$

$\cap$



$\mathbb{C}^2$ $L_t = f^{-1}(t) \overset{C^\infty}{\simeq}$



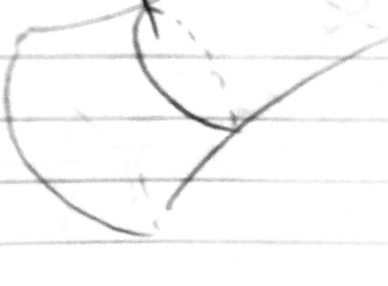
$t = 2, -2$.

Continuous family of cycles

Take a cycle δ_t in L_t with trivial holonomy,

$\Sigma = (\mathbb{R}^2, 0)$

$\Sigma = (\mathbb{C}, 0)$



$\omega_0 := df$ has poles along $D = \bar{f}^{-1}(\infty)$.

Let us perturb

$$\mathcal{F}_\varepsilon : \; df + \varepsilon\omega_1 + \cancel{\varepsilon^2\omega_2 + \cdots}$$

ω_i's differential forms with poles along D.

$f \in \mathbb{C}[x,y]$, $\omega_1 = P(x,y)\,dy - Q(x,y)\,dx$, $\omega_2 = \cdots$

Perturbed holonomy / Poincaré first return map

$$h_\varepsilon(t) : \Sigma \to \Sigma \qquad h_\varepsilon(t) = t + M_1(t)\varepsilon + M_2(t)\varepsilon^2 + \cdots$$

M_i's are called Melnikov functions / Σ is parametrized by the image of f

Prop 1: $M_1(t) = -\int_{\delta_t} \omega$

Prop 2: If $M_1(t_0) = 0$ then for ε enough small there is a fixed point of $h_\varepsilon(t)$ near t_0.

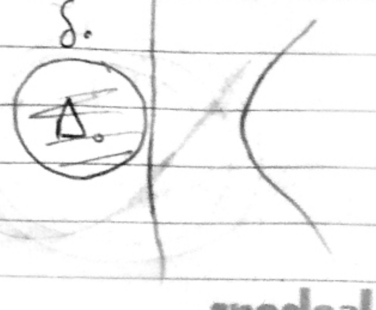
Write on the O.D.E language

$$\mathcal{F}_0 = \begin{cases} \dot{x} = \frac{\partial f}{\partial y}(x,y) \\ \dot{y} = -\frac{\partial f}{\partial x}(x,y) \end{cases} \qquad \begin{cases} \dot{x} = 2y \\ \dot{y} = 3x^2 - 3 \end{cases}$$

$$\mathcal{F}_\varepsilon \begin{cases} \dot{x} = f_y + \varepsilon P(x,y) \\ \dot{y} = -f_x + \varepsilon Q(x,y) \end{cases} \qquad \begin{cases} \dot{x} = 2y + \varepsilon \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + \varepsilon(sy) \end{cases}$$

$$M_1(0) = \int_{\delta_0} \frac{x^2}{2}dy - sy\,dx = \int_{\Delta_0} x\,dx \wedge dy + s\,dx\wedge dy = 0$$

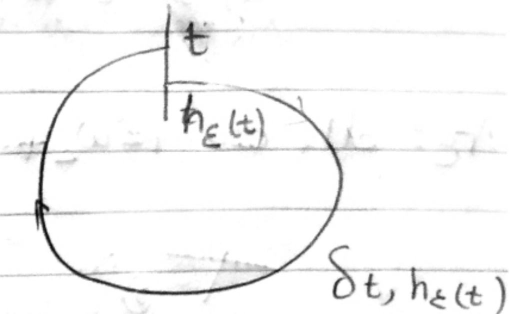
$$s := \frac{-\int_{\Delta_0} x\,dx\wedge dy}{\int_{\Delta_0} dx\wedge dy} = \frac{5}{7}\,\frac{\Gamma(\frac{5}{12})\Gamma(\frac{13}{12})}{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})} \sim 0.9025$$



We get the differential equation of the poster

$$S = 0.9 \qquad \varepsilon = 1.$$

Proof of prop. 1:



Restrict $df + \varepsilon\omega_1 + \varepsilon^2(\cdots)$ to the solution $\delta_{t,\, h_\varepsilon(t)}$ of $\mathcal{F}_\varepsilon$

$$df + \varepsilon\omega_1 + \varepsilon^2(\cdots)\Big|_{\delta_{t,\, h_\varepsilon(t)}} \equiv 0$$

$$f(h_\varepsilon(t)) - f(t) + \varepsilon \int_{\delta_{t,\, h_\varepsilon(t)}} \omega_1 + \varepsilon^2(\cdots) = 0$$

$\shortparallel$

$h_\varepsilon(t) - t$ because we have parametrized Σ by the image of f

∎

Proof of Prop. 2:

Apply Weierstrass preparation thm to

$$f(\varepsilon, t) \quad \frac{h_\varepsilon(t) - t}{\varepsilon} : (\mathbb{C}^2, 0) \longrightarrow \mathbb{C}$$

$$f(0, t) = M_1(t) = * t^N + * \, t^{N+1} + \cdots$$

$$f(\varepsilon, t) = g(\varepsilon, t)\left(t^N + g_1(\varepsilon, t)\, t^{N-1} + \cdots \quad g_N(\varepsilon, t)\right)$$

$$g(0,0) \neq 0 \qquad g_i(0,0) = 0.$$

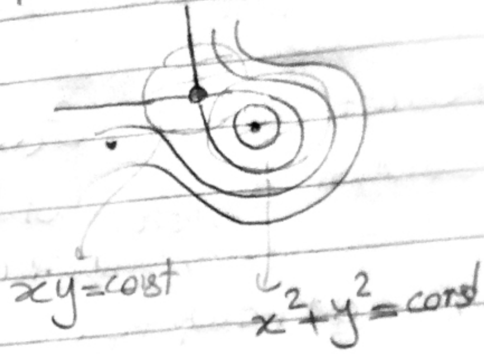
∎

my article abelian integrals in holomorphic foliations 2004.

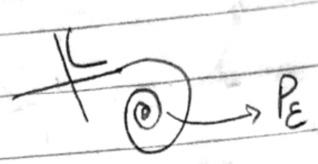
Ilyashenko 1969: Let f be a generic polynomial of degree $d+1$

generic = 1) least homog. piece of $f = (x-\alpha_1 y)(x-\alpha_2 y)\cdots(x-\alpha_{d+1} y)$, α_i's distinct

2) f has non-degenerated critical points with distinct values



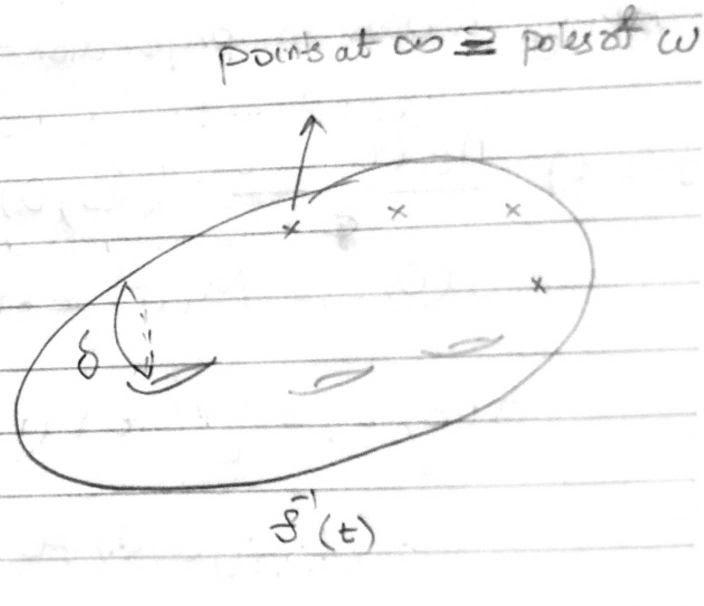
$\mathcal{F}_\varepsilon : df + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \qquad \deg(\omega_i) \leq d.$



Assume that P_ε is still a center singularity.

THEN $\mathcal{F}_\varepsilon$ is also Hamiltonian $\mathcal{F}_\varepsilon : df_\varepsilon$. $f_\varepsilon \in \mathbb{C}[x,y]$

$\deg f_\varepsilon = d+1.$



First lecture

Abelian integrals

$f \in \mathbb{C}[x,y]$

$\omega_1 = \omega = P\,dy - Q\,dx$ polynomial

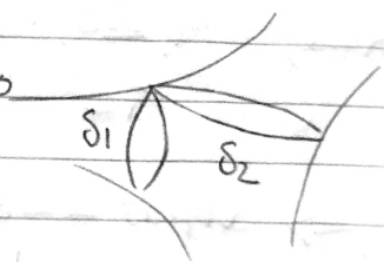
26 / 09 / 2022 Iterated integrals in holomorphic foliations.

$h_\varepsilon(t) - t = \varepsilon M_1(t) + \varepsilon^2 M_2(t) + \dots$ $\quad F_\varepsilon : df + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots$

$M_1 \equiv M_2 \equiv \dots \equiv M_k \equiv 0$ How to compute M_{k+1}

For example if $\delta = (\delta_1, \delta_2) = \delta_1 \delta_2 \delta_1^{-1} \delta_2^{-1}$ then $M_1 \equiv 0$.

Thm: (Françoise recursion) if $M_1 \equiv \dots \equiv M_k \equiv 0$ then



$$M_{k+1}(t) = -\int_{\delta_t} \sum_{i=0}^{k} P_i \omega_{k+1-i}$$

where P_i, g_i's holomorphic functions along δ defined by

$$P_0 \equiv 1$$

$$P_1 df + dg_1 = -P_0\omega_1$$

$$\vdots$$

$$P_i df + dg_i = -\sum_{j=0}^{i-1} P_j \omega_{i-j}$$

Gavrilov 2005: Melnikov functions can be written as iterated integrals

let $\omega_1, \omega_2, \dots \omega_r$ be $\underset{\text{hol.}}{\text{differential}}$ 1-forms in a Riemann surface L. and $\delta : [0,1] \to L$ be a path.

$$\int_\delta \omega_1\omega_2\dots\omega_r := \int_\delta \left(\int_{\delta_r} \omega_1 \dots \omega_{r-1} \right) \omega_r$$

recursive definition

$$\delta_r : [0, t_r] \to L \quad \delta_r = \delta|_{[0,t_r]}$$

$$= \int \cdots \int_{0 < t_1 < t_2 < \dots t_r < 1} f_1(t_1) f_2(t_2) \dots f_r(t_r)\, dt_1 \dots dt_r$$

$$\delta^* \omega_i = f_i(t)\, dt.$$

Chen 1977, Hain 87: We can capture homotopy groups by integration.

For example:

if $\delta \in \pi_1(L_i, b)$ and $\int_\delta \omega_1\omega_2\cdots\omega_r = 0 \ \forall\, \omega_1,\omega_2,\dots$ hol. 1-forms then $\delta = 0$, that is δ is homotopic to zero

De Rham theorem on the duality of homology and de Rham cohomology
Chen's π_1-de Rham thm.

$k=1$, $M_1 = 0$, $M_2 = -\int \omega_2 + P_1\omega_1$

$$P_1\, df + dg_1 = -\omega_1$$

$$\Rightarrow dP_1 \cdot df = -d\omega_1 \Rightarrow dP_1 = \frac{d\omega_1}{df} = \omega_1^0$$

well-defined on the fibers of of f

$$P_1 = \int_\delta \omega_1^0 \qquad M_2 := -\int_{\delta_t} \omega_2 + \omega_1^0\omega_1$$

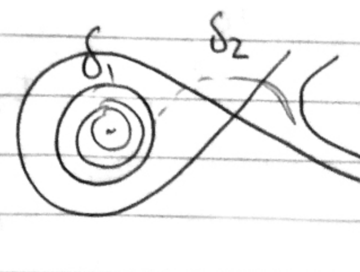
$\omega_1^0 = \frac{d\omega_1}{df}$ Gauss-Manin connection of $f - t = 0$, t parameter.

$$M_3(t) = -\int \omega_3 + \omega_1^0\omega_2 + \omega_2^0\omega_1 + \frac{\omega_1^0\wedge\omega_1}{df}\omega_1 + (\omega_1^0\omega_1)^0\omega_1$$

Gavrilov + Movasati $\xrightarrow{\text{unpublished}}$ general formulas for M_k.

Movasati + Nakai (2008):
$f \in \mathbb{R}[x,y]$ as in Ilyashenko's theorem

$\mathcal{F}(df)$ the holonomy $h_i, i=1,2$ along δ_i is identity



$$\mathcal{F}_\varepsilon : df + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$$

perturbed holonomies $h_{i,\varepsilon} : \Sigma \to \Sigma$

IF $h_{1,\varepsilon}\circ h_{2,\varepsilon} = h_{2,\varepsilon}\circ h_{1,\varepsilon}$ then $\mathcal{F}_\varepsilon := \mathcal{F}_\varepsilon(df_\varepsilon)$ is again Hamiltonian.

(δ_1, δ_2)

Idea of the proof: $\delta = \delta_1 \circ \delta_2 \circ \delta_1^{-1} \circ \delta_2$ $\quad h_\varepsilon(t)$ holonomy along δ

$$M_1 = \int_{\delta_t} \omega_1 \equiv 0 \quad \text{automatically.}$$

$$M_2 = \int_{(\delta_1, \delta_2)} \omega_1^0 \omega_1 = \begin{vmatrix} \int_{\delta_1} \omega_1^0 & \int_{\delta_2} \omega_1^0 \\ \int_{\delta_1} \omega_1 & \int_{\delta_2} \omega_1 \end{vmatrix} \equiv 0$$

$\hookrightarrow$ hypothesis of the thm.

$$\cdots \Rightarrow \quad \omega_1 = dg, \; g \in \mathbb{C}[x,y]_{\leq d}$$

Proof of Françoise recursion:

Induction by k, $M_1 = M_2 = \dots = M_{k-1} \equiv 0$

$$M_k = -\int_{\delta_t} \sum_{i=1}^{k-1} P_i \omega_{k-i} \equiv 0$$

$$\Rightarrow \quad -\sum_{i=1}^{k-1} P_i \omega_{k-i} = dg_k + P_k df$$

$$\left(1 - \sum_{i=1}^{k} P_i \varepsilon^i\right)\omega_\varepsilon = d\left(f - \sum_{i=1}^{k} g_i \varepsilon^i\right) + \left(\sum_{i=0}^{k} P_i \omega_{k+1-i}\right)\varepsilon^{k+1} + O(\varepsilon^{k+2})$$

As before, restrict this equality to $\delta_{\varepsilon, h_\varepsilon(t)}$

$$h_\varepsilon(t) - t - \left(\sum_{i=1}^{k} g_i \varepsilon^i\right)\Big|_t^{h_\varepsilon(t)} + \varepsilon^{k+1} \int_{\delta_{t, h_\varepsilon(t)}} \left(\sum_{i=0}^{k} P_i \omega_{k+1-i}\right) + O(\varepsilon^{k+2})$$

coef of ε^{k+1}