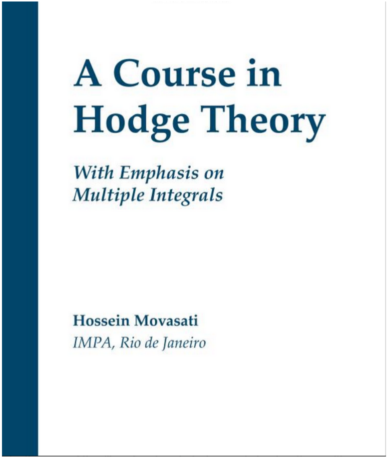
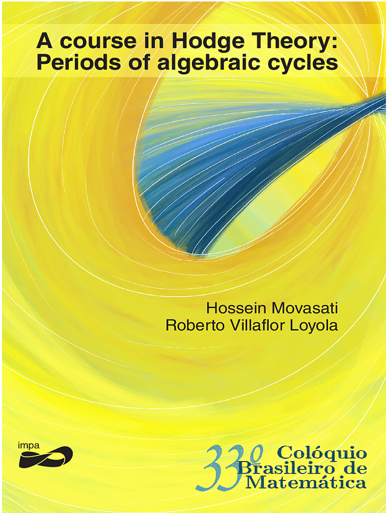
Hodge Cycles for Cubic Hypersurfaces

Hossein Movasati

IMPA,
www.impa.br/∼hossein/


A Course in Hodge Theory
With Emphasis on Multiple Integrals
Hossein Movasati
IMPA, Rio de Janeiro


A course in Hodge Theory: Periods of algebraic cycles
Hossein Movasati
Roberto Villaflor Loyola
impa
33º Colóquio Brasileiro de Matemática

در آتش خویش چون دمی جوش کنم

خواهم که دمی ترا فراموش کنم

گیرم جانی که عقل بیهوش کند

در جام درآئی و ترا نوش کنم

A poem by Jalal al-Din Muhammad Balkhi. Calligraphy: `nastaliqonline.ir`.
If I boil in the fire of my existence for a while,
that is because I want to forget you for a while,
to get a new soul and put away my wisdom,
and then you become the wine of my glass.

27 lines of a smooth cubic surface

$\boxed{\text{Any smooth cubic surface in } \mathbb{P}^3_{\mathbb{C}} \text{ has 27 lines.}}$

$\mathbb{P}\{F(x_0,x_1,x_2,x_3)=0\} \subseteq \mathbb{P}^3$

$\deg F = 3$

$\mathbb{R}^3 \subseteq \mathbb{P}^3_{\mathbb{R}}$

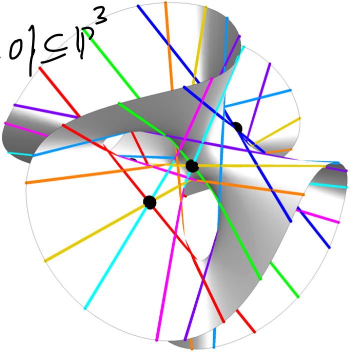


Figure: Clebsch surface[1]

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[1] blogs.ams.org/visualinsight/2016/02/15/27-lines-on-a-cubic-surface/

27 lines of the Fermat cubic surface

$3 \cdot 9 = 27$

$x_0 \; x_1 \qquad x_2 \; x_3$

$$\boxed{X_0 : x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.}$$

$\mathbb{P}^1 \subseteq X_0$

$$\mathbb{P}^1 : \begin{cases} x_0 - \zeta_1 x_1 = 0, \\ x_2 - \zeta_2 x_3 = 0, \end{cases} \qquad \zeta_1^3 = \zeta_2^3 = -1,$$

9

Higher dimensional Fermat

n = even

algebraic cycle

$\zeta_{2d}^{2d} = 1$ $\frac{n}{2}$

$$X_0 \subset \mathbb{P}^{n+1} : \{x_0^d + x_1^d + x_2^d + \cdots + x_{n+1}^d = 0\}.$$

$$\mathbb{P}^{\frac{n}{2}} : x_0 - \zeta_{2d} x_1 = x_2 - \zeta_{2d} x_3 = x_4 - \zeta_{2d} x_5 = \cdots = x_n - \zeta_{2d} x_{n+1} = 0.$$

We have

$$(n+1) \cdot (n-2) \cdots 3 \cdot 1 d^{\frac{n}{2}+1}$$

such algebraic cycles (for $(n, d) = (2, 3)$ this is 27).

Middle cohomology of a smooth hypersurface

Lefschetz thms: $H_m(X,\mathbb{Z}) = \begin{cases} 0 & m \text{ odd} \\ \mathbb{Z} & m \text{ even} \end{cases}$ $m \neq n$

The dimension of the n-th cohomology of a smooth hypersurface X of degree d in $\mathbb{P}^{n+1}$ is given by

$$\dim H_n(X,\mathbb{Q}) =$$

$$(d-1)^{n+1} - (d-1)^n + (d-1)^{n-1} - \cdots + (-1)^n(d-1) + (0 \text{ or } 1)$$

where $+0$ if n is odd and $+1$ of n is even (for $(n,d) = (2,3)$ this is 7).

$$\boxed{[\mathbb{P}^{\frac{n}{2}}] \in H_n(X_0,\mathbb{Z})}$$

$\mathbb{P}^{\frac{n}{2}}$ complex.

n

In cohomology

$\mathbb{Z}$

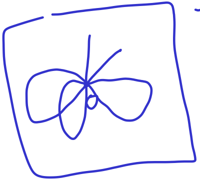
$$[\mathbb{P}^{\frac{n}{2}}] \in H^n(X_0,\mathbb{Z}) \cap H^{\frac{n}{2},\frac{n}{2}},$$

$\ni \delta$ Hodge cycle.

These are our first examples of a Hodge class/cycle.

$H^n(X,\mathbb{C}) = H^{n,0} \oplus H^{n-1,1} \oplus \cdots \oplus H^{\frac{n}{2},\frac{n}{2}} \oplus \cdots \oplus H^{0,n}$

Hypersurface containing $\mathbb{P}^{\frac{n}{2}}$



$\mathbb{P}^{\frac{n}{2}} \subseteq X$

$X_0 \quad X_t \quad t \in T \subseteq \mathbb{C}[x_0, \dots, x_{n+1}]_d$

The space of smooth hypersurfaces containing a $\mathbb{P}^{\frac{n}{2}}$ is of codimension

$$\boxed{\binom{\frac{n}{2}+d}{d} - \left(\frac{n}{2}+1\right)^2}$$

$\mathbb{P}^1$

$\mathbb{P}^2$

0

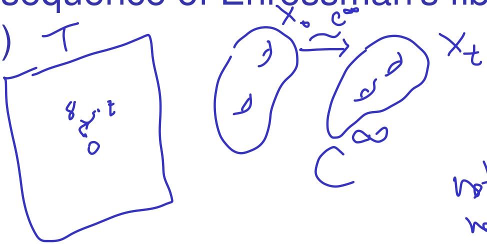
For $(n, d) = (2, 3)$ this is zero.

Cubic surfaces.

$\mathbb{P}^{\frac{n}{2}}$

$1 \cdot 3 \cdot 5 \cdots (n-1)(n+1) \, d^{\frac{n}{2}+1}$

A consequence of Ehressman's fibaration theorem (1947)



not alg.
not hol.

Theorem

For any two smooth hypersurfaces X_1, X_2 of degree d and dimension n we have

$$X_1 \overset{C^\infty}{\cong} X_2$$

$f : X_1 \xrightarrow[\sim]{C^\infty} X_2$

Let us consider the hypersurface X_t in the projective space $\mathbb{P}^{n+1}$ given by the homogeneous polynomial:

$$\boxed{f_t := x_0^d + x_1^d + \cdots + x_{n+1}^d - \sum_{\alpha} t_\alpha x^\alpha = 0,} \qquad (1)$$

$X_t : \{f_t = 0\}$

$$t = (t_\alpha)_{\alpha \in I} \in (\mathrm{T}, 0),$$

usual

where x^α runs through all monomials of degree d in $x_0, x_1, \cdots, x_{n+1}$. For $\delta_0 \in H_n(X_0, \mathbb{Z})$ we have also a unique

$$\delta_t \in H_n(X_t, \mathbb{Z}).$$

$t \in (\mathrm{T}, 0)$

(monodromy, parallel transport, flat section,....)

Finding Hodge cycles by deformation

'In searching for possible counterexamples to the "Hodge conjecture", one has to look for varieties whose Hodge ring is not generated by its elements of degree 2' (A. Weil). The description of such elements for CM abelian varieties is fairly understood and it is due to D. Mumford (1966) and A. Weil (1977) himself. Other examples are Y. André's (1996) motivated Hodge cycles. None of these methods can be applied to hypersurfaces. By Lefschetz hyperplane section theorem, for hypersurfaces of even dimension $n \geq 4$ we have only a one dimensional subspace of $H^{\frac{n}{2},\frac{n}{2}}$ generated by elements of degree 2 (the class of a hyperplane section), and producing interesting Hodge cycles in this case, for which the Hodge conjecture is not known, is extremely difficult, and hence, they might be a better candidate for a counterexample to the Hodge conjecture.

Hodge loci

$= \sum r_i [\mathbb{P}^{\frac{n}{2}}_i]$ Shioda 1980

Aoki

$\delta_0 \in H_n(X_0, \mathbb{Z})$. $\delta_t \in H_n(X_t, \mathbb{Z})$

$(d, (n+1)!) = 1$ δ_t

Take δ_0 a linear combination of $\mathbb{P}^{\frac{n}{2}}$ with $\mathbb{Z}$ coefficients.

$$V_{\delta_0} := \{t \in (\mathrm{T}, 0) \mid \delta_t \in H^n(X_0, \mathbb{Z}) \cap H^{\frac{n}{2}, \frac{n}{2}}\}$$

→ analytic var.

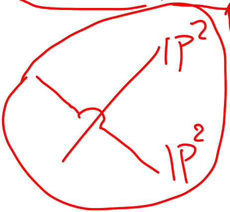
→ analytic space scheme

Main example: $\mathrm{Zar}(I)$, $I \subseteq \mathcal{O}_{T,0}$ $I = \langle f_1, \dots, f_k \rangle$

$$\delta_0 := r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}}, \ \mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m, \ m = \frac{n}{2} - 3.$$

$n = 4$

$\times$



$\subseteq \mathbb{P}^5$

$n \geq 4$

$\mathbb{P}^{-1} = \emptyset$

$d = 3$.

Range of codimensions

codim(hyp $\supseteq \mathbb{P}^{\frac{n}{2}}$).

$$\binom{\frac{n}{2}+d}{d} - \left(\frac{n}{2}+1\right)^2 \leq \operatorname{codim} V_\delta \leq \begin{cases} \binom{\frac{n}{2}d-1}{n+1} & \text{if } d < \frac{2(n+1)}{n-2} \\ \binom{d+n}{n+1} - (n+2) & \text{if } d = \frac{2(n+1)}{n-2} \\ \binom{d+n+1}{n+1} - (n+2)^2 & \text{if } d > \frac{2(n+1)}{n-2} \end{cases}$$

codim V_δ

Higher dimensional cubic scroll

$g_1, g_2 \cdots g_{\frac{n}{2}-1}, f_{ij}$: $\sum_{i=0}^{n+1} a_i x_i$

$$\mathcal{C} : g_1 = g_2 = \cdots = g_{\frac{n}{2}-1} = 0, \quad \text{rank} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix} \leq 1, \qquad (2)$$

where g_i, f_{ij}'s are linear equations.

A table from Chapter 19

dim $H^{\frac{n}{2},\frac{n}{2}}$

$d=3$

$\mathbb{P}^{\frac{n}{2}}$

$\dim(X_0)$	$\dim(\mathrm{T})$	range of codimensions	L	CS	M	QS	V	Hodge numbe
n	$\binom{n+2}{3}$	$\binom{\frac{n}{2}+1}{3}$, $\binom{n+2}{\min\{3,\frac{n}{2}-2\}}$						$h^{n,0}, h^{n-1,1}, \ldots, h^{1}$
4	20	1, 1	1	1	$\boxed{1}$	1	1	0, 1, 21, 1, 0
6	56	4, 8	4	6	$\boxed{7}$	8	$\boxed{10}$	0, 0, 8, 71, 8, 0
8	120	10, 45	10	16	$\boxed{19}$	23	25	0, 0, 0, 45, 253, 45
10	220	20, 220	20	32	$\boxed{38}$	45	47	0, 0, 0, 1, 220, 925, 22
12	364	35, 364	35	55	$\boxed{65}$	75	77	0, 0, 0, 0, 14, 1001, 3432, 10

Table: Codimensions of the components of the Hodge/special loci for cubic hypersurfaces.

Conjecture

codim $\overline{V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}}$ $\quad r, \check{r} \in \mathbb{Z}$, $\neq 0$, $(r,\check{r})=1$.

much bigger than the deformation space of $X, \mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}}$.

I started to compute codimension of Hodge loci with a computer with processor `Intel Core i7-7700`, 16 GB Memory plus 16 GB swap memory and the operating system `Ubuntu 16.04`. It turned out that for many cases we get the 'Memory Full' error. Therefore, we had to increase the swap memory up to 170 GB. Despite the low speed of the swap which slowed down the computation, the computer was able to use the data and give us the desired output. The computation for this example took more than 21 days. We only know that at least 18 GB of the swap were used.

Computational Hodge conjecture

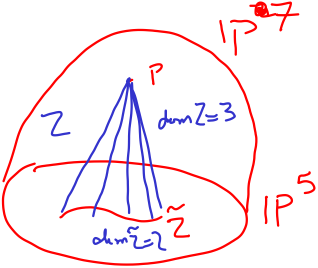
But the whole program [Grothendieck's program on how to prove the Weil conjectures] relied on finding enough algebraic cycles on algebraic varieties, and on this question one has made essentially no progress since the 1970s.... For the proposed definition [of Grothendieck on a category of pure motives] to be viable, one needs the existence of "enough" algebraic cycles. On this question almost no progress has been made, P. Deligne 2014....la construction de cycles algébriques intéressants, les progrès ont été maigres, P. Deligne 1994.

Veronese embedding

$\widetilde{T}$ = space of cubic 6-folds containing Z

$\widetilde{T} \subseteq T$ codim $\widetilde{T} = 10$.

codim(Hodge loci) ≤ 8

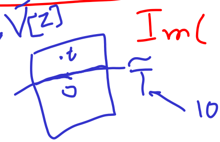


Consider the image of the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ by degree 2 monomials.

Im( ) = $\tilde{Z}$

$Z_0 \subseteq X_0$, $S_0 : [Z_0] \in H_6(X_0, \mathbb{Z})$. 8

codim $V_{[Z]} \leq 8$



$S_t \in H_6(X_t, \mathbb{Z})$

8.

Z_0 is homologous to another algebraic cycle

$C = \sum_{i=1}^{s} r_i Z_i$, such that C has bigger deform. space.