17 / Jul / 2022 Modular and Automorphic forms & beyond.
Heidelberg.

modular forms, Hilbert, Siegel, Jacobi
Yukawa coupling, Topol. String partition
function, theta series

unified language $\rightsquigarrow$

Differential equation of these objects

A new moduli space T and Gauss-Manin connection in Disguise

$\mathcal{M}$ a classical moduli of a projective variety $X \subseteq \mathbb{P}^N$.
Hilbert scheme $\mathcal{M} = \mathrm{Hilb}(X)/G, \dots$ Physics: Just parameter space?
$X_1, X_2 \in \mathcal{M}$ $X_1 \overset{C^\infty}{\simeq} X_2$ but not analytically/alg. any deformation of $X \in \mathcal{M}$ with the prop. is included in $\mathcal{M}$.

X/k, $\mathrm{char}(k)=0$, $H^*_{dR}(X/k)$, $H^{m_1}_{dR}(X/k) \times H^{m_2}_{dR}(X/k) \to H^{m_1+m_2}_{dR}(X/k)$

$$(\alpha, \omega) \longmapsto \alpha \cup \omega.$$

Hodge filtration $0 = F^{m+1} \subseteq F^m \subseteq \dots \subseteq F^0 = H^m_{dR}(X/k)$

$$F^i H^{m_1} \cup F^j H^{m_2} \subseteq F^{i+j} H^{m_1+m_2}$$

$$\theta = \text{generator of } H^2_{dR}(\mathbb{P}^N)\Big|_X \in F^1 H^2_{dR}(X)$$

Hodge decomposition cannot be defined over k.

Prop 2.4: let $X_t, t \in T$ be a family of smooth projective varieties and X, X_0 be two members of this family. Then

$$(H^*_{dR}(X), F^*, \cup, \theta) \overset{\alpha}{\simeq} (H^*_{dR}(X_0), F_0^*, \cup, \theta_0)$$

Proof: I wrote the whole book.... Deligne SGA....

THERE IS NO CANONICAL α!!

$k = \mathbb{C}$,

$$(H^*(X, \mathbb{Z}), \cup, \theta) \overset{h}{\simeq} (H^*(X_0, \mathbb{Z}), \cup, \theta_0)$$

h is unique attached to a path which connects 0 to t $X = X_t$.

credeal

$T :=$ the moduli of (X, α), $X \in \mathcal{M}$ (Ibipóranga)
X_0 fixed and α as before

This is my south america before its discovery by Europeans. pretty land full of mystery, exotic plants and animals.

We can also use mixed Hodge structures, additional enhancements, like divisors, line bundles, etc.

Conj: T is an a (quasi-) affine variety (at least for CY3).

$$\underset{\text{open subset}}{T \subseteq} \operatorname{Spec}\left(\mathbb{Q}[t_1, t_2, \dots]/\text{ideal}\right)$$

the algebra of modular and Automorphic forms closed under canonical derivations (differential Automorphic)

Elliptic curves (K. Saito, N. Katz, ..., M. 2012)

$$T = \operatorname{Spec}\left(\mathbb{Q}\left[t_1, t_2, t_3, \frac{1}{27t_3^2 - t_2^3}\right]\right)$$ theory of q.m.f for $SL(2,\mathbb{Z})$

lattice polarized K3 surfaces of high rank. (Alim et al) q.m.f for $\Gamma \subseteq SL(2,\mathbb{Z})$
+ Open CY3

CY3 (Alim, M., Sheidegger 16) A mixture of rigor and ...
Yukawa couplings, Topological string partition function $\in \mathcal{O}_T$
BCOV anomaly equation

Mirror quintic CY3 (Book: GMCD: CY modular forms)

$$T := \operatorname{Spec}\left(\mathbb{Q}\left[t_0, t_1, \dots, t_6, \frac{1}{t_5(t_4 - t_0^5)}\right]\right)$$

$$Y = 5^8 (t_4 - t_0^5)^2 / t_5^3$$

$$F_1^{alg} = \ln\left(t_4^{25/12} (t_4 - t_0^5)^{-5/12} t_5^{\frac{1}{2}}\right)$$

$g \geq 2$

$$F_g^{alg} = \frac{P_g}{(t_4 - t_0^5)^{2g-2} t_5^{3g-3}}$$

$P_g \longrightarrow$ hom. of degree $69(g-1)$ with

$\deg(t_i) := 3(i+1)$, $i = 0, 1, 2, \dots, 4$
$\deg t_5 = 11$, $\deg t_6 = 8$

J. Walcher: One says that there is gold in California and then one colonize it.

genus two curves (Cao-M.-Yau) $\longrightarrow$ differential Siegel modular forms

elliptic curves with two marked points and MHS $\longrightarrow$ Jacobi-forms of index zero (Cao-M.-Villaflor).

CY3 with two homologous rational curves in the case of mirror quintic $\longrightarrow$ $\dim T = 9$, generating function of disc counting.

Obs: algebraic objects $f \in \mathcal{O}_T$ $\longleftrightarrow$ holomorphic functions encoding counting objects $\longleftrightarrow$ C^∞ objects involving hol. and antihol.

choosing a mirror map $\longrightarrow$; $\longleftarrow$ holomorphic limit

Even though Top Journals don't publish this kind of Math $\xrightarrow{\text{until}}$ End of my life I have to colonize this mathematics and say there is Gold in it. will produce

T moduli space is a natural space to study Hodge loci after conversations with J. Walcher + A. Braun $\hookrightarrow$ Tadpole conjecture.

$X = E_1 \times E_2$, E_i elliptic curve

$\mathcal{U} = (\mathbb{P}^1 - \{\infty\}) \times (\mathbb{P}^1 - \{\infty\}) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.

Hodge locus in $\mathbb{P}^1 \times \mathbb{P}^1 = \left\{ (j_1, j_2) \in \mathbb{P}^1 \times \mathbb{P}^1 \;\middle|\; E_{j_1} \longrightarrow E_{j_2} \text{ isogenous of degree } N \right\}$

$$= \left\{ \quad '' \quad \middle|\; E_{j_1} \times E_{j_2} \text{ has a non-trivial curve} \right\}$$

$$= \left\{ \int_\delta \omega_1^{10} \otimes \omega_2^{10} = 0, \quad \delta \in H_2(E_1 \times E_2, \mathbb{Z}) \right\}$$

There exists

$$= \operatorname{rank}\left(H^{1,1}(E_1 \times E_2) \cap H^2(E_1 \times E_2, \mathbb{Z}) \right) \geq 3$$

= Union of highly singular $Y_0(N)$, with complicated equations

$Y_0(N)$ birational $\Gamma_0(N) \backslash \mathbb{H}$, $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \right.$ $c \equiv 0$

credeal

$T = T_{\text{elliptic}} \times T_{\text{elliptic}}$, $\dim T = 6 = 3+3$.

we can introduce

$T =$ moduli of (E_1, E_2, α), $\alpha : H^*_{dR}(E_1) \xrightarrow{\sim} H^*_{dR}(E_2) \cdots$

Thm (M. 2022 MMJ):

1. $$T \simeq \mathrm{Spec}\left(\mathbb{Q}\left[x_2, x_3, y_2, y_3, \frac{1}{27x_3^2 - x_2^3}, \frac{1}{27y_3^2 - y_2^3}\right]\right)$$
$$= \mathbb{C}^4 \setminus \{\Delta_1 = 0\} \cup \{\Delta_2 = 0\}$$

2. The vector field
$$V = \left(2x_2 - 6x_3 + \frac{1}{6}(x_2 - y_2)x_2\right)\frac{\partial}{\partial x_2} + \left(3x_3 - \frac{1}{3}x_2^2 + \frac{1}{4}(x_2 - y_2)x_3\right)\frac{\partial}{\partial x_3}$$
$$- \left(2y_2 - 6y_3 \quad {}^{\prime\prime} \quad {}^{\prime\prime} \quad {}^{\prime\prime}\right)\frac{\partial}{\partial y_2} \quad {}^{\prime\prime}$$

has an enumerable set $S_0(N)$, $N \in \mathbb{N}$ of algebraic solutions. $S_0(N) \xrightarrow{\sim}^{\text{bih.}} \Gamma_0(N)\backslash\mathbb{H}$

singularities are in $\Delta_1 = 0$ or $\Delta_2 = 0$.

3. The solutions of V are to the loci of (E_1, E_2, α) such that α is "topologically constant" $= \nabla_V \alpha = 0$.

4. $S_0(N)$ is given by the image
$$\Gamma_0(N)\backslash\mathbb{H} \longrightarrow \mathbb{P}^{2,3,2,3,1} \supseteq \mathbb{C}^4$$
$$\tau \longmapsto \left(\frac{g_2(\tau)}{(g_1(\tau) - N \cdot g_1(N\tau))^2}, \frac{g_3(\tau)}{(g_1(\tau) - N g_1(N\tau))^3}, \frac{N^2 \cdot g_2(N\tau)}{(\quad)^2}, \frac{N^3 g_3(N\tau)}{(\quad)^3} \right.$$