26/07/2022 Hodge cycles for cubic hypersurfaces.

$$X = X_t \subseteq \mathbb{P}^{n+1}$$

$$f = x_0^d + x_1^d + \dots + x_{n+1}^d + \sum_{\alpha \in I} t_\alpha x^\alpha = 0 \qquad \deg(x^\alpha) = d.$$

Objective: Understanding Hodge cycles near Fermat. $\mathrm{Hodg}_n(X, \mathbb{Z}) =$

$$H^{\frac{n}{2},\frac{n}{2}} \cap H^n(X, \mathbb{Z}) \cong \left\{ \delta \in H_n(X, \mathbb{Z}) \;\middle|\; \int_\delta F^{\frac{n}{2}+1} H^n_{dR}(X) = 0 \right\}$$

$F^{\frac{n}{2}+1} H^n_{dR}(X) =$ Residues of $\frac{P\Omega}{f^k}$, P homoog. $\Omega = \sum x_i \widehat{dx_i}$.
$k \cdot d = \deg P + n + 2 \quad k \leq \frac{n}{2}$.

For $n=2$ / $\mathbb{C}$ $\mathrm{Pic}(X) = NS(X) \cong \mathrm{Hodge}_n(X, \mathbb{Z})$

Hodge cycles of Fermat

Linear cycles

$$\mathbb{P}^{\frac{n}{2}}: x_0 - \zeta_{2d} x_1 = \dots = x_n - \zeta_{2d} x_{n+1} = 0 \subseteq X_0$$

There are $1 \cdot 3 \cdot \dots \cdot (n+1) \cdot d^{\frac{n}{2}+1}$ of these

Ran 81, Shioda 79, Aoki-Shioda 83: $d =$ prime, $d = 4$, $(d, (n+1)!)$ then Hodge cycles over / $\mathbb{Q}$ are generated by linear cycles.

Schuett, Shioda, van Luijk 2010, Degtyarev 2015. $d \leq 4$, $(d, 6) = 1$, $n = 2$ then Hodge cycles / $\mathbb{Z}$ are generated by lines.

Aljovin, M. Villaflor 2019, Degtyarev-Shimada 2016: Similar statements $n = 4, 6, \dots$

Obs: Still we don't know a complete list of curves generating Picard group for $n=2$, $d=12$ $\qquad x_0^{12} + x_1^{12} + x_2^{12} + x_3^{12} = 0$.

Picard rank $644 \geq$ lattices generate by lines $= 332$

Hodge Locus

$\delta_0 \in H_n(X_0, \mathbb{Z})$ Hodge cycle

$\delta_t \in H_n(X, \mathbb{Z})$ parallel transport = monodromy of δ_0 = flat...

$$V_{\delta_0} = \{ t \in T \mid \delta_t \text{ is Hodge} \} \qquad \text{this is an analytic scheme}$$

$$= \{ f_1(t) = f_2(t) = \dots = f_k(t) = 0 \}$$

$f_i = \int_{\delta_t} \omega_i$, ω a section of $F^{\frac{n}{2}+1} H^n_{dR}(X)$

$$\mathcal{O}_{V_{\delta_0}} = \mathcal{O}_{T,0} / \langle f_1, f_2, \dots f_k \rangle \quad \text{might be non-reduced.}$$

if $\delta_0 = \sum_{i=1}^{r} n_i [Z_i]$, $\dim Z_i = \frac{n}{2}$

when V_{δ_0} is bigger than the expected deformation space of $(X_0, \sum n_i Z_i)$?

Example: $(n,d)=(2,4)$ K3 surface $(n,d)=(4,3)$ cubic fourfold

Example (p. 316): Veronese cycle

$\mathbb{P}^2 \hookrightarrow \mathbb{P}^5 \subseteq \mathbb{P}^7$

veronese with degree two monomials

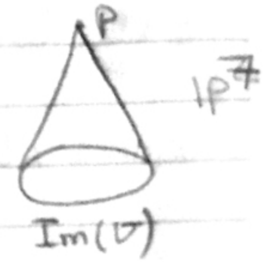
$n=6$, $d=3$

Hodge numbers 0,0,8,71,8,0,0

codimension of Hodge locus ≤ 8

codimension of deformation space of (X, Z)

$= 10$.



Problem: Try to verify H.C for these deformed Hodge cycles

My obsession

$Z = r\,\mathbb{P}^{\frac{n}{2}} + \check{r}\,\check{\mathbb{P}}^{\frac{n}{2}}$, $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ $\quad (r,\check{r})=1.$ $\quad -1 \leq m \leq \frac{n}{2}-1$

$\delta_0 = [Z]$ p. 283.

Thm 18.1 (M. + Villaflor) $m < \frac{n}{2} - \frac{d}{d-2}$ then V_{δ_0} is the expected deformation space of $(X, \mathbb{P}^{n/2}, \check{\mathbb{P}}^{n/2})$.

Most of the time or this ↑ or non-reduced / non-smooth V_{δ_0}.

$d=3$, $m = \frac{n}{2}-2, \frac{n}{2}-3, \frac{n}{2}-1$ are not included in this thm.

→ this is not strange
$r=\check{r}=1$ deformation into comp intersection of type 1,1,...,1,2
otherwise, non-reduced

Thm 19.1 page 309: V_{δ_0} is N-smooth according to $(d=3)$

	n	4	6	8	10	12
$m=\frac{n}{2}-2$, $Z_0 = \mathbb{P}^{\frac{n}{2}} \check{\mathbb{P}}^{\frac{n}{2}}$	codim(V_{δ_0})	1	6	16	32	55
				19	38	65
$m=\frac{n}{2}-3$, $Z_0 = r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}}$	codim(V_{δ_0})	1	7			
	N	∞	19	6	4	3

and this is bigger than the deformation space of $(X_0, \mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}})$

After 2 months of emails $\leftrightarrows$ Deligne, the mystry of $m=\frac{n}{2}-2$ is solved

$$\mathbb{P}^{\frac{n}{2}}\ \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}_1^{\frac{n}{2}} + \mathbb{P}_2^{\frac{n}{2}} + \mathbb{P}_3^{\frac{n}{2}} =$$

$\mathbb{P}^{\frac{n}{2}-1}$

A referee told me that I am decorating myself with Deligne!

$n=2$ surfaces

NL loci = Union of all V_{δ_0} = surfaces X s.t $Pic(X)\simeq\mathbb{Z}$

= Union of enumerable algebraic sets.

$d-3 \leqslant$ codim comp. of NL $\leqslant \binom{d-1}{3} = h^{20}$

Voisin, Greeen 1990 classification of comp. with codim $= d-3, \dots$

special component: codim $< \binom{d-1}{3}$, generic comp: codim $= \binom{d-1}{3}$.

J. Herris: number of special components is finite

C. Voisin: For d big Herris' conjecture is false.

(M. 2021): Take a line C_1 and a complete intersection of type $(3,3)$ inside Fermat $d=8, n=2$, $C_1\cap C_2=\emptyset$. For all except a finite number of $r, \check{r}\in\mathbb{Z}$, V_{δ_0}, $\delta_0 = r[C_1] + \check{r}[C_2]$ is 5-smooth of codimension $31 < 35 = \binom{7}{3}$

در حیاط جلوی موسسهٔ ریاضی در هانوور نشسته‌ام و به این فکر می‌کنم چرا مرد همیشه بیشتر و بیشتر می‌خواهد، چرا می‌خواهد بالای کوه اورست برود حتی اگر بمیرد، می‌خواهد بیشتر و بیشتر پول داشته باشد، می‌خواهد جام سایکل پیدا کند، دنبال طلا است، آنقدر حرصش زیاد است که گاهی چشمانش کور می‌شود همچون اسکندر فقط می‌خواهد کشور گشایی کند بدون آنکه فکر کند با این همه کشور چه می‌خواهد بکند، این لحظهٔ لذت مرد از این همه چقدر برای او مهم است. همه چیز برای این لحظه. گویی طبیعت مرد را در همهٔ موارد به این ترتیب آفریده است. عاشق شدن، سال‌ها به دنبال معشوق تنها برای یک لحظه

$$y^2 + x^3 + t^{12} + 1 = 0$$

credeal