$\nearrow$ 21, 29, 5, 11

14–16 Jum
21–23

14 / 06 / 2022 Few lectures in Hodge theory.

Why are you intersted in Hodge Theory?

Discussion with the students: Some concepts

X, C^∞-variety

$H_m(X,\mathbb{Z})$, $H^m(X,\mathbb{Z})$, $H^m_{dR}(X)$,

$\int_\delta \omega$ $\delta \in H_m$, $\omega \in H^m_{dR}$, Line bundle,

Vanishing cycle, Gauss-Manin connection

How many people have heared of these words?

Lefschetz thms:
$X \subseteq \mathbb{P}^N$
smooth proj

$$\begin{cases} H^q(X,Y,\mathbb{Z}) = 0 \quad \leq q \leq n-1 \\ H^{n-q}(X,\mathbb{Q}) \xrightarrow{\sim} H^{n+q}(X,\mathbb{Q}) \quad \text{H.L.T} \\ x \longrightarrow x \cup h^q \end{cases}$$

$h \in H^2(X,\mathbb{Z})$ polarization $\rightarrow$ cup product

$f_1, f_2, \ldots f_n \in \mathbb{C}[x,y]$ $\quad f_0 \underset{?}{\in} \mathbb{C}[f_1 \ldots f_n]$, Groebner basis

Hodge Theory ≤ 1930, Abel, Poincaré, Picard, Lefschetz

Chapters 2,3. Book I.

Complex manifold: $\mathbb{C}^N / \mathbb{Z}e_1 + \mathbb{Z}e_2 + \ldots \mathbb{Z}e_{2N}$

algebraic subvarieties of $\mathbb{C}^N$: ex: $\{(x,y) \in \mathbb{C}^2 \mid y^2 + x^3 = 1\}$

Chapter 2: elliptic integrals:

Chapter 3: Certain Toric $\mathbb{C}^g/\Lambda$

Λ, a $\mathbb{Z}$-module gen. by columns of $\tau \in \mathbb{H}_2$ $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

$\tau \in \mathbb{H}_g$ :: $\mathrm{Im}(\tau)$ is positive definite.

$\hookrightarrow$ Siegel domain

Riemann theta series: $\theta : \mathbb{C}^g \times \mathbb{H}_g \xrightarrow{hol} \mathbb{C}$

$$\theta(z + \tau . m + n, \tau) = \times \; \theta(z, \tau)$$

independent of z

Fix τ, and let $\theta_1, \theta_2, \theta_3, \theta_4$ be four R. theta series.

$$\mathbb{C}^2/\Lambda \longrightarrow \mathbb{P}^3 \qquad z \longmapsto [\theta_1(z) : \theta_2(z) : \theta_3(z) : \theta_4(z)]$$

Its image is an algebraic surface

Picard, Humbert, ... called it hyperelliptic surface

After A. Weil, (a singular model) of an abelian surf

Picard & many others $\mathbb{C}^3 \hookrightarrow \mathbb{P}^3$

$(x, y, z) \longrightarrow [x : y : z : \omega]$

The plane at ∞ $= \mathbb{P}^2_\infty = \mathbb{P}^3 \setminus \mathbb{C}^3 = \{ z\omega = 0 \} \subseteq \mathbb{P}^3$

Picard was not able to see the hyp. section $\mathbb{P}^2_\infty \cap X = Y$

Long exact seq. of $U : X \setminus Y \subseteq X$

$$H_3(U) \to H_3(X) \to H_3(X, U) \to H_2(U) \to H_2(X) \to H_2(X, U) \to H_1(U) \to H_1(X) \to \; H_1(X,U)$$

4 4 5 6 1 4 4

Picard $\nearrow$ Topology, $\searrow$ Algebraic Geometry. ~~28~~ : 30 June

~~30~~ : 5

7

Chapter 4,5,6 Book I.

16 / 06 2022 Lecture two: Topology of algebraic varieties. 13:30

$X \subseteq \mathbb{P}^N$ smooth projective variety $\rightarrow$ complex manifold of dim n

$X: \ f_1 = f_2 = \dots = f_k = 0$ f_i homog. polynomials. what are the conditions of smoothness?

$X \subseteq \mathbb{P}^{n+1}$ smooth hypersurface $X: \ f(x_0, x_1, \dots, x_{n+1}) = 0$

Fermat hypersurface $X: \ x_0^d + x_1^d + \dots + x_{n+1}^d = 0.$

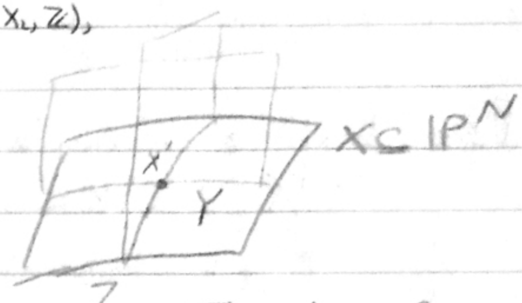
$n=1$, How many people $X \overset{C^\infty}{\simeq}$ (∘∘∘) $\# \circ = (d-1)(d-2)/2.$ / Ehresmann's theorem $X_1 \overset{C^\infty}{\simeq} X_2$ for hypersurface

Chapter 4: Homolog & cohomology & cup product & cap product

$$X \longrightarrow H^m(X, \mathbb{Z}),\ H_m(X, \mathbb{Z}), \dots H^m(X, Y, \mathbb{Z})$$

$$X_1 \overset{C^\infty}{\rightarrow} X_2 \Rightarrow H_m(X_1, \mathbb{Z}) \cong H_m(X_2, \mathbb{Z}),$$

Fig 5.2



Thm 5.1:

$$H_q(X \setminus Z, Y \setminus X') = \begin{cases} 0 & q \neq n \\ \mathbb{Z}^\mu & q = n \end{cases}$$

Thom-Leroy-Gysin

Thm 5.2 (Lefschetz hyp. plane section thm)

$$H_q(X, Y; \mathbb{Z}) = 0 \quad 0 \leqslant q \leq n-1$$

which implies that inclusion induces

$$H_q(Y, \mathbb{Z}) \xrightarrow{\sim} H_q(X, \mathbb{Z}) \quad q < n-1, \quad H_{n-1}(Y, \mathbb{Z}) \rightarrow H_{n-1}(X, \mathbb{Z}) \rightarrow 0$$

$\boxed{\text{Print page 53-54}}$

Thm 5.3 $\boxed{\text{page 56}}$ $X \subseteq \mathbb{P}^{n+1}$ smooth hypersurface

$$H_q(X, \mathbb{P}^{n+1}, \mathbb{Z}) = 0 \quad q \leqslant n$$

In particular

$$H_q(X) \xrightarrow{\sim} H_q(\mathbb{P}^{n+1}) = \begin{cases} \mathbb{Z} & q \text{ even} \\ 0 & q \text{ odd} \end{cases} \qquad q \leqslant n-1$$

The discussion of the generator of $H_q(X)$

Thm. 5.3 $\boxed{\text{page 57}}$

$$H_q(X) \cong \begin{cases} 0 & q \text{ is odd } q \neq n \\ \mathbb{Z} & q \text{ is even } q = n \end{cases}$$

$H_n(X, \mathbb{Z})$ is free.

Hard Lefschetz thm: $X_0 \subseteq X_{n-1} \cdots \subseteq X_2 \subseteq X_1 \subseteq X = X_0$

$\quad 0 \quad 1 \quad n-2 \quad n-1 \quad n \quad$ dim

Thm 5.4:

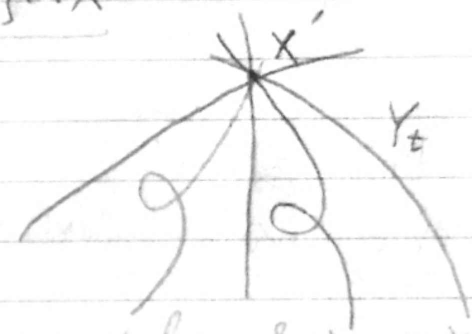
$$H_{n+q}(X,\mathbb{Q}) \xrightarrow{\sim} H_{n-q}(X,\mathbb{Q}) \quad x \longmapsto x \cdot [X_q]$$

- It is enough the case $q=1$
- No topological proof is well known. / using Hodge's harmonic forms / l-adic cohomology

Picard-Lefschetz theory (proof of the main thm):

$Y := \{F=0\} \cap X$, $Z = \{G=0\} \cap X$ $\quad F, G$ linear polynomials in $[x_0 \dots x_N] \in \mathbb{P}^N$

Pencil of hyper planes sections $\quad \underbrace{\{F - tG = 0\} \cap X}_{Y_t}$



$$f := \frac{F}{G}\Big|_X : X \dashrightarrow \mathbb{P}^1$$
$$X \setminus X' \longrightarrow \mathbb{P}^1$$

$$Y_t := \overline{f^{-1}(t)}$$

Transversality conditions $\Rightarrow Y_t$ is smooth except for a finite number $\{c_1, c_2, \dots c_\mu\} \longleftrightarrow$ of t, which in turn has isolated singularities

more generic conditions $\Rightarrow Y_{c_i}$ has at most one non-degenerated singularity

Morse lemma (complex): $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ hol function with non deg sing at 0, that is, $\frac{\partial f}{\partial x_1}(0) = \cdots = \frac{\partial f}{\partial x_n}(0) = 0$ but $\det\left[\frac{\partial f}{\partial x_i \partial x_j}(0)\right] \neq 0$. Then in a hol. coordinate sys. $(y_1, y_2, \dots y_n) \in (\mathbb{C}^n, 0)$

$$f = y_1^2 + y_2^2 + \cdots + y_n^2$$

$L_{\mathbb{R}} = \{(x) \in \mathbb{R}^n \mid f(x) = t\} \quad t \in \mathbb{R}_+$

$L_{\mathbb{C}} = \{x \in \mathbb{C}^n \mid f(x) = t\} \qquad L_{\mathbb{R}} \simeq \mathbb{S}^{n-1} \subseteq L_{\mathbb{C}}$

Ex 2.1 Book I.

page 42

Thom-Leray-Gysin iso: Compact oriented submanifold Y of real codim. c in an oriented manifold X

$$H_{m-c}(Y, Z) \simeq H_m(X, X \setminus Y, Z)$$

$$x \longmapsto x$$

$$\longleftarrow$$

intersection with Y.

Chapter 9, Book I

thm

01 July 2022 3rd lecture: Toward a computational proof of Lefschetz (1,1)

$X \subseteq \mathbb{P}^N$ smooth, curves in X. Can be done for X of arbitrary dim and divisors

$$H^{20} = H^0(X, \Omega_X^2) = \text{holomorphic 2-forms in } X$$

$$H^{20} \hookrightarrow H^2_{dR}(X) := \frac{\text{closed } C^\infty \text{ 2-forms}}{\text{exact } C^\infty \text{ 2-forms}}$$

$$\omega \longmapsto [\omega]$$

$$H^2_{dR}(X) \times H^2_{dR}(X) \longrightarrow H^4_{dR}(X) \xrightarrow{\sim} \mathbb{C}$$

$$\omega_1 \quad \omega_2 \longrightarrow \omega_1 \wedge \omega_2$$

$$\omega \longmapsto \int_X \omega$$

This pairing is non-degenerate.

$$H^2_{dR}(X) \xrightarrow{\sim} H^2_{dR}(X) \quad \omega \mapsto \overline{\omega}$$

$$H^{02} := \overline{H^{20}} \qquad H^{20} \cap H^{02} = \{0\}$$

$$H^{1,1} := (H^{20} \oplus H^{02})^{\perp}$$

Hodge decomposition:

$$H^2_{dR}(X) = H^{20} \oplus H^{1,1} \oplus H^{02}$$

$$\cup\ H^2(X, \mathbb{Z}) / \text{up to torsions.}$$

$Z \subseteq X$ (algebraic) curve or $\sum n_i Z_i$ $n_i \in \mathbb{Z}$ algebraic cycle.

Goal: To classify $\delta = \sum n_i [Z_i] \in H_2(X, \mathbb{Z})$.

Simple observation: $\int_\delta H^{20} = 0$ why? There is no hol. 2-form on a curve.

Def: $\delta \in H_2(X, \mathbb{Z})$ is called a Hodge cycle $\int_\delta H^{20} = 0$.

In particular, all torsions are Hodge.

We have proved

algebraic cycle of dim 1 is a Hodge cycle.

Lefschetz (1,1) thm: Hodge cycles are algebraic.

Sketch of the proof:

$Z = \sum n_i Z_i \rightsquigarrow$ line bundle L_Z, L_Z has a meromorphic section s

$\operatorname{div}(s) = Z$.

Long exact seq. of $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$

$$\to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{d} H^2(X, \mathbb{Z}) \to$$

- $cl(L_Y)$ = Poincaré dual of $[Y]$.

Recall Poincaré duality: $P: H^q(X,\mathbb{Z}) \xrightarrow{\sim} H_{n-q}(X,\mathbb{Z})$

$\alpha \longmapsto \alpha \cap [X]$

X compact, oriented C^∞ dim $= n$

- $\int_\delta \omega = 0 \Rightarrow \begin{matrix} \omega \cdot \delta^{pd} = 0 \\ \bar{\omega} \cdot \delta^{pd} = 0 \end{matrix} \Longrightarrow \begin{matrix} \delta^{pd} \in H^{11} \\ \delta^{pd} \in H^{1,1} \cap H^2(X,\mathbb{Z}) \end{matrix}$

- The map $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \simeq$

$H^2_{dR}(X) \otimes \mathbb{C}$ projection on $H^2_{dR}(X)/H^{20} \oplus H^{11}$

- Now $\delta = cl(L)$, L has a meromorphic section s, $div(L) = Z$

$[Z] = \delta$. Chapter 17 Book I.

Computational Lefschetz (1,1): I give you δ then construct Z for me.

Open problem: We don't know a complete list of curves generating the Picard group of

$X_2^d \quad \mathbb{P}(x_0^d + x_1^d + x_2^d + x_3^d = 0) \subseteq \mathbb{P}^3 \qquad d \geq 2.$

$x_0 - \xi x_1 = x_2 - \xi x_3 = 0$, do all $\xi^d = -1$, permute all x_i's

we get. $(n+1)(n-1)\ 3 \cdot 1 d^{2+}$ rational curves in X_2^d $\quad n=2$

$d=3$, this is 27. the number of lines inside a cubic surface

Ran-Shiod-Aoki 198~: d prime, or $d=4$ or $(d,(n+1)!)$ then the Picard group of Fermat $\otimes \mathbb{Q}$ is generated by lines

Schütt-Shioda, van Luijk, Degtyarev 2010, 2015 $d \leq 4$, $(d,6)=1$ then the Picard group over $\mathbb{Z}$ is generated by lines

Aoki-shioda, new algebraic curves inside Fermat surface.

But they are not enough

Book II

05 July 2022 Lecture 4: Cohomology of hypersurfaces

given a smooth curve $X \subseteq \mathbb{P}^2$ who knows to write down hol. 1-forms on X?

$$X : F(x,y,z) = 0, \qquad \Omega = \frac{P(x,y,z)(x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy)}{F}$$

$$\deg P + 3 = \deg F = d.$$

write down this in the affine coordinate $z = 1$ $\quad f = F(x,y,1)$

$$\frac{P(x,y)\,dx \wedge dy}{f} \xrightarrow{\text{Resi}} \frac{P\,dy}{f_x}$$

$$0 \leqslant \deg P \leqslant d-3$$

Griffiths thm: $X \subseteq \mathbb{P}^{n+1} \ni [x_0 : x_1 : \cdots : x_{n+1}]$.

Thom-Leray-Gysin sequence of $\mathbb{P}^{n+1} \setminus X \subseteq \mathbb{P}^{n+1}$

$$H^{n+1}(\mathbb{P}^{n+1}) \to H^{n+1}(\mathbb{P}^{n+1} \setminus X) \xrightarrow{\text{Resi}} H^n(X) \to H^{n+2}(\mathbb{P}^{n+2}) \to \cdots$$

Discuss n = even $\qquad H^n(X)_0 = \mathrm{Im}(\mathrm{Resi})$

Take a basis $P_1, P_2, \ldots, P_\mu$ of $\mathbb{C}[x]/\mathrm{Jacob}(F)$

$H^{n+1}(\mathbb{P}^{n+1} \setminus X)$ is generated by $\dfrac{P\Omega}{F^q}$ $\quad n+2+\deg P = d(q+1)$

$$\Omega = \sum (-1)^i x_i \widehat{dx_i}$$

$\dfrac{P_i \Omega}{F^{q+1}}$ $i = 1, 2, \ldots, \mu$ form a basis of $H^{n+1}(\mathbb{P}^{n+1} \setminus X)$

pole filtration $\qquad // \; F^q H^{n+1}_{dR}(\mathbb{P}^{n+1} \setminus X)$

pole order $q=1$, $\subseteq$ pole order $q=2$ $\qquad \subseteq$ pole order $q = n+1$

$$\| \; H^{n+1}_{dR}(\mathbb{P}^{n+1} \setminus X)$$

By the residue map this goes to the Hodge filtration of $H^n_{dR}(X)$.

$$\mathrm{Resi}\left(F^q H^{n+1}_{dR}(\mathbb{P}^{n+1} \setminus X)\right) = F^{n+1-q} H^n_{dR}(X)_0$$

X complex compact Kähler (for example projective)

$$H^m_{dR}(X) = H^{m0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{1,m-1} \oplus H^{0,m}$$

F^{m-1} F^m

given by (p,q)-forms

F^0 F^1

Deligne + Grothendieck: de Rham cohomology and Hodge filtration can be defined by polynomial for $X \subseteq \mathbb{P}^N$.

.HYPERCOHOMOLOGY.

Atiyah - Hodge thm: X smooth affine algebraic variety $/\mathbb{C}$, The natural map is an isomorphism

$$H^q(\Gamma(\Omega^{\bullet}_X), d) \xrightarrow{\sim} H^q_{dR}(X).$$

$$\downarrow$$

$$X \hookrightarrow \mathbb{P}^{n+1} \setminus X$$

Algebraic de Rham cohomology: Chapter 5, book II,

Hodge cycles and Hodge conjecture for hypersurfaces:

$Z \subseteq X$ $\dim Z = \frac{n}{2}$ algebraic cycle.

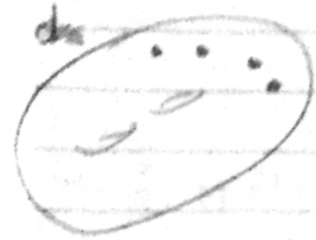
$$\int_{[Z]} F^{\frac{n}{2}+1} H^n_{dR}(X) = 0.$$

Hodge cycle $\delta \in H_n(X, \mathbb{Z})$ has this property.

07 / 06 / 2022 **Lecture 5: Hodge theory of Fermat varieties.**

$X = X^d_n : \mathbb{P}(x_0^d + x_1^d + \cdots + x_{n+1}^d = 0) \subseteq \mathbb{P}^{n+1}$

$d=$



$n=1$, points at ∞.

$$L \; : \; x_1^d + x_2^d + \cdots + x_{n+1}^d = -1 \subseteq \mathbb{C}^{n+1}$$

$$x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} = 1 \subseteq \mathbb{C}^{n+1} \qquad m_i \geq 2$$

$$\underset{n}{X} = \underset{n}{L} \cup \underset{n-1}{X^d_{n-1}}$$

$$\mathbb{P}^{n+1} = \mathbb{C}^{n+1} \cup \mathbb{P}^n$$

A basis of $H_n(L, \mathbb{Z})$: $I_i := \{0, 1, \dots, m_i - 2\}$ $\quad I = I_1 \times I_2 \times \cdots I_{n+1}$

$$\Delta_n := \{(t_1, t_2, \dots t_{n+1}) \in \mathbb{R}^{n+1} \mid t_i \geq 0 \; \sum t_i = 1\}$$

$\beta \in I$, $a \in \{0,1\}^{n+1}$ $\quad \Delta_{\beta+a}(t) = (t_1^{\frac{1}{m_1}} \zeta_{m_1}^{\beta_1 + a_1}, \dots)$

$$\Delta_{\beta+a} : \Delta_n \to L$$

$$\delta_\beta = \sum_a (-1)^{\sum(1-a_i)} \Delta_{\beta+a} \in H_n(L, \mathbb{Z})$$

Then $\delta_\beta, \beta \in I$ form a basis of $H_n(L, \mathbb{Z})$

Thm: $\omega_\beta = x^\beta \left(\sum_{i=1}^{n+1} \frac{(-1)^{i-1}}{m_i} x_i \widehat{dx_i} \right)$, $\beta \in I$ form a basis of $H^n_{dR}(L)$.

Prop. 15.1: $$\int_{\delta_\beta} \omega_{\beta'} = \frac{(-1)^n}{\prod m_i} B\left(\frac{\beta_1' + 1}{m_1}, \dots \right) \prod_{i=1}^{n+1} \left(\zeta_{m_i}^{(\beta_i+1)(\beta_i'+1)} - \zeta_{m_i}^{\beta_i(\beta_i'+1)} \right)$$

$$A_\beta := \sum \frac{\beta_i + 1}{m_i}$$

Obs: $H^n_{dR}(X) \to H^n_{dR}(L)$ its image is generated by ω_β, $A_\beta \notin \mathbb{N}$.

Cycles at ∞: $\delta \in H_n(L, \mathbb{Z})$ $\langle \delta, \delta_\beta \rangle = 0 \quad \forall \beta$.

$$\Updownarrow$$

$$\int_\delta \omega_\beta = 0 \qquad \forall \beta, \; A_\beta \notin \mathbb{N}.$$

Hodge filtration

Hodge cycles: $\delta \in H_n(L, \mathbb{Z})$ $\quad \int_\delta \omega_\beta = 0 \quad \forall \beta, \; A_\beta \notin \mathbb{N}$, $A_\beta < \frac{n}{2}$

Hodge cycles modulo cycles at ∞: Table 15.7

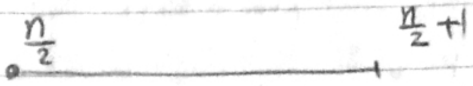
$d=$	1	2	3	4	5	6	7
dim Hodge	1	2	21	192	401	1752	1861
h^{22}	1	2	21	192	581	1752	4333

Deligne's fundamental observation: conjecture $X/\overline{\mathbb{Q}}$, $\delta \in H_n(X,\mathbb{Q})$ Hodge

$$\int_\delta H^n_{dR}(X/\overline{\mathbb{Q}}) \in (2\pi i)^{\frac{n}{2}}\,\overline{\mathbb{Q}}.$$

he observed this for algebraic cycles.

For Fermat:



B_β is algebraic if $\int_\delta \omega_\beta$ is not zero!

Algebraic cycles

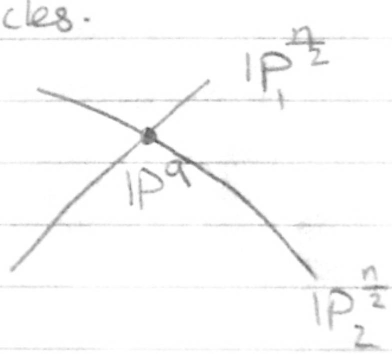
$$x_0^d + x_1^d + \cdots = (x_0 - \xi x_1)(\cdots) + (x_2 - \xi' x_3)(\cdots) + \cdots$$

$$\xi^d + 1 = 0$$

$$\mathbb{P}^{\frac{n}{2}} : x_0 - \xi x_1 = x_2 - \xi' x_2 = \cdots = 0 \subseteq X^d_n.$$

we want to verify the H.C for X^d_n using linear cycles.

$$\mathbb{P}_1^{\frac{n}{2}} \cdot \mathbb{P}_2^{\frac{n}{2}} = \frac{1-(-d+1)^{a+1}}{d}$$



we know also the intersection between δ_β's

$$\langle \delta_\beta, \delta_{\beta'} \rangle = (-1)^{\frac{n(n-1)}{2}} (1 + (-1)^n) \quad \beta = \beta'$$

$$= (-1)^{\frac{n(n+1)}{2}} (-1)^{[\beta'_k - \beta_k}$$

for $\beta_k < \beta'_k \leq \beta_k + 1$ otherwise $= 0$

we have to work with primitive algebraic cycles

$$Z = \sum n_i Z_i \qquad Z \cdot [Z_\infty] = 0$$

Hodge index thm: If you find primitive Hodge cycles $Z_1, Z_2, \dots Z_k$ such that $\det[Z_i \cdot Z_j] \neq 0$ then $[Z_i] \in H_n(X,\mathbb{Z})$ are linearly indep.

Compute the rank etc.