Selfish

Hodge conjecture

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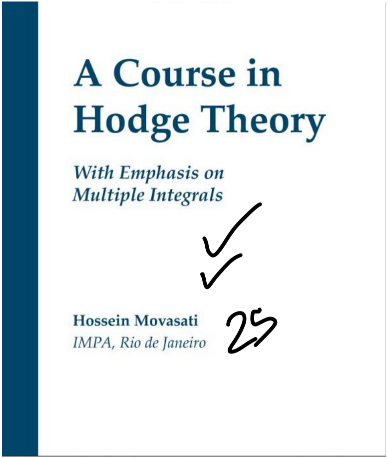
A Course in Hodge Theory
With Emphasis on Multiple Integrals
Hossein Movasati
IMPA, Rio de Janeiro

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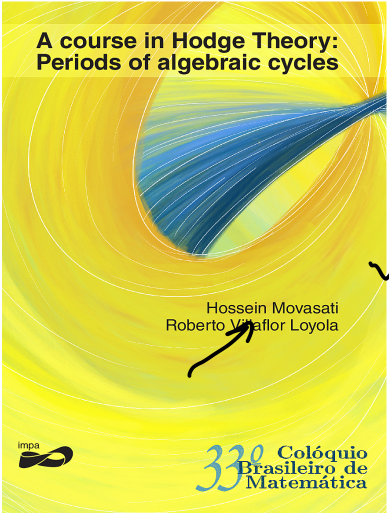
A course in Hodge Theory:
Periods of algebraic cycles
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در آتش خویش چون دمی جوش کنم

خواهم که دمی ترا فراموش کنم

گیرم جانی که عقل بیهوش کند

در جام درآئی و ترا نوش کنم

A poem by Jalal al-Din Muhammad Balkhi. Calligraphy: `nastaliqonline.ir`.
If I boil in the fire of my existence for a while,
that is because I want to forget you for a while,
to get a new soul and put away my wisdom,
and then you become the wine of my glass.

Chapters 2: Prehistory: Elliptic and abelian integrals

< 1900

One dim. curves

$\mathbb{C}$

two dim

$$\int_{a_1}^{a_2} \frac{dx}{\sqrt{P(x)}}$$

y

δ

a_1, a_2 roots of $P(x)$ or $\pm\infty$

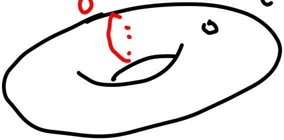
$\deg P = 3.$

$$\left\{ \begin{array}{c} y^2 = P(x) \\ (x,y) \in \mathbb{C}^2 \end{array} \right\}$$

$\|$

elliptic curve

δ



Chapters 2: Lefschetz's puzzle and Picard's ρ_0-formula

$X \subseteq \mathbb{P}^3$ $\quad X: f(x_1,x_2,x_3,x_4)=0 \quad /\mathbb{C}$

homog.

É Picard 1900 + Simart

Lefschetz.

$X \setminus U = \{x_4=0\} \cap X$ $\qquad X: x_1^d+x_2^d+x_3^d+x_4^d=0$

$\qquad\qquad\qquad\qquad\qquad\quad \cup$

curve inside X $\qquad U: x_1^d+x_2^d+x_3^d+1=0$ $\quad x_4=1$

X $\quad 2_{\mathbb{C}}$ $\quad 4_{\mathbb{R}}$

Homology class of algebraic cycles.

C^∞ $\quad \mathbb{Z}$ $\qquad$ polynomial.

Chapters 5,6,7: Topology and Lefschetz' theorems

smooth

$X \subseteq \mathbb{P}^N$

Singular homology

$H_m(X, \mathbb{Z})$ $\quad m = 0, 1, \ldots 2n$

$H^m(X, \mathbb{Z})$ $\quad n = \dim X$

$[x_0 : x_1 : \dots : x_{n+1}] \in \mathbb{P}^{n+1}$

$X : \mathbb{P}\{x_0^d + x_1^d + x_2^d + \dots + x_{n+1}^d = 0\} \subset \mathbb{P}^{n+1}$

The dimension of the n-th cohomology of a smooth hypersurface X of degree d in $\mathbb{P}^{n+1}$ is given by

$$\dim H_n(X, \mathbb{Q}) =$$

$$\boxed{(d-1)^{n+1} - (d-1)^n + (d-1)^{n-1} - \dots + (-1)^n(d-1) + (0 \text{ or } 1)}$$

where $+0$ if n is odd and $+1$ of n is ~~even.~~

$S^2 \hookrightarrow X$

$= (y^2 = P(x), (x, y) \in \mathbb{C}^2) \subset \mathbb{P}^2$

$x_0^d + x_1^d + x_2^d + x_3^d = 0$

$\cap \{x_0 = 0\}$

$C = \{x_1^d + x_2^d + x_3^d = 0\} \simeq$

$g = \frac{(d-1)(d-2)}{2}$

$[C] \in H_2(X, \mathbb{Z})$

De Rham cohomology for complex C^∞ manifolds

$$H^m_{dR}(X) := \frac{\text{closed } C^\infty\ m\text{-forms} / \mathbb{C}}{\text{exact } C^\infty\ m\text{-forms.} / \mathbb{C}}$$

$$H^m_{dR}(X) \simeq \underbrace{H^m(X, \mathbb{C})}_{\to \text{ singular cohomo}}$$

X locally $\mathbb{C}^n \ni [z_1, z_2 .. z_n]$

$dz_1, dz_2 .. dz_n, d\bar{z}_1, .. d\bar{z}_n.$

ω (p,q)-forms in X: locally it is given by $* \, dz_{i_1} \wedge .. dz_{i_p} \wedge d\bar{z}_{j_1} \wedge .. d\bar{z}_{j_q}$

$\underbrace{H^{p,q} \subseteq H^{p+q}_{dR}(X)}$ $\to$ represented by (p,q)-form

Hodge: 1943. $H^m_{dR}(X) \simeq H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{0,m}.$

Chapter 8: Hodge conjecture without Hodge decomposition

$X \subseteq \mathbb{P}^N$

$\dim_{\mathbb{C}} X = n$

$\dim_{\mathbb{R}} X = 2n$

$0 \leq m \leq 2n$ even

$Z \subseteq X$ subvariety of $\boxed{\dim_{\mathbb{C}} \frac{m}{2}}$, $\dim_{\mathbb{R}} = m$.

singular.

$[Z] \in H_m(X, \mathbb{Z})$.

$\frac{m}{2}+1$

$H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{\frac{m}{2}+1, \frac{m}{2}-1}$ $F^{\frac{m}{2}+1}$

$H^m_{dR}(X)$

$H_m(X, \mathbb{Z})$

$\int_{[Z]} \equiv 0$

$(p,q), \quad p \geq \frac{m}{2}+1.$

Definition: $\delta \in H_m(X, \mathbb{Z})$ is called a Hodge cycle if

$\int_\delta F^{\frac{m}{2}+1} \equiv 0.$

H.C: Hodge cycles are algebraic cycles. $/\mathbb{Q}$

$Z_1, Z_2, \ldots Z_k$

$\sum n_i [Z_i]$

$n_i \in \mathbb{Z}$

Computational Hodge conjecture

$$\delta \in H_m(X,\mathbb{Q}) \overset{H.C}{\Longrightarrow} \delta = \sum n_i [Z_i] \quad n_i \in \mathbb{Q}.$$

$$\exists Z_i \qquad Z_i \subseteq X, \dim Z_i = \frac{m}{2}$$

But the whole program [Grothendieck's program on how to prove the Weil conjectures] relied on finding enough algebraic cycles on algebraic varieties, and on this question one has made essentially no progress since the 1970s.... For the proposed definition [of Grothendieck on a category of pure motives] to be viable, one needs the existence of "enough" algebraic cycles. On this question almost no progress has been made, P. Deligne 2014....la construction de cycles algébriques intéressants, les progrès ont été maigres, P. Deligne 1994.

Do you believe in Hodge conjecture?

Yes → Theory.

No → explicit examples computations.

Lefschetz $(1,1)$ theorem

H.C for $\frac{m}{2} = \dim_{\mathbb{C}} X$

$m = 2.$

divisors and curves in X

$\parallel$ $\quad$ $|$

$m = 2n-2$ $\quad$ $m=2$

Chapter 15,16,17: Fermat varieties

$$x_0^d + x_1^d + \cdots + x_{n+1}^d = 0$$

H.C is true, Moreover

space of Hodge cycles

Theorem (Ran 1980, Shioda 1983, Aoki-Shioda 1983)

Suppose that either d is a prime number or $d = 4$ or d is relatively prime with $(n+1)!$. Then $\mathrm{Hodge}_n(X_n^d, \mathbb{Q})$ is generated by the homology classes of the linear cycles $\mathbb{P}^{\frac{n}{2}}$, and in particular, the Hodge conjecture for X_n^d is true.

$\mathbb{P}^{\frac{n}{2}} \subseteq X.$

$$x_0^d + x_1^d + \cdots = (x_0 + \zeta x_1)(\cdots) + (x_2 + \zeta x_3)(\,)\cdots$$

$\zeta^d + 1 = 0$ $\quad$ $x_0 + \zeta x_1 = 0$ $\quad$ $x_2 + \zeta x_3 = 0$

$$\mathbb{P}^1 \subseteq \{ x_0^d + x_1^d + x_2^d + x_3^d = 0 \}$$

Theorem (Schuett-Shioda-van Luijk, 2010, Degtyarev, 2015)

$d \leq 100$ space of 2 Hodge cycles $/\mathbb{Z}$

If $d < 4$ or $\gcd(d, 6) = 1$ then the ~~Neron-Severi group~~ of the Fermat surface of degree d is generated by lines.

$\mathbb{Z}$

Chapter 18,19: Hunting for Hodge cycles after deformations → Hodge loci

$X \subseteq \mathbb{P}^{n+1}$ $X: f(x)=0$ → hom.

→ depends on many param.

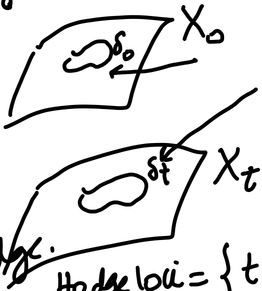
$f = \sum t_\alpha x^\alpha$ $x^\alpha = x_0^{\alpha_0} \cdots x_{n+1}^{\alpha_{n+1}}$

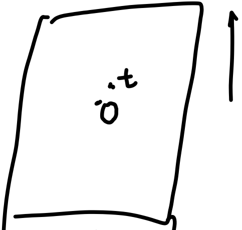
$\sum \alpha_i = d.$

$(\dots t_\alpha \dots) \in T \subseteq \mathbb{C}^{\# N} \setminus -$

X_0: Fermat

$\delta_0 \in H_n(X_0, \mathbb{Z})$ Hodge.





$\delta_t \in H_n(X_t, \mathbb{Z})$

δ_t might not be Hodge.

Hodge loci $= \{ t \in (T, 0) \mid \delta_t \text{ is Hodge} \}$.

A table from Chapter 19

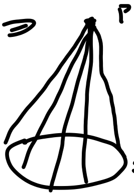
$\dim(X_0)$	$\dim(\mathrm{T})$	range of codimensions	L	CS	M	QS	V	Hodge numbe
n	$\binom{n+2}{3}$	$\binom{\frac{n}{2}+1}{3}, \binom{n+2}{\min\{3, \frac{n}{2}-2\}}$						$h^{n,0}, h^{n-1,1}, \ldots, h^1$
4	20	1, 1	1	1	$\boxed{1}$	1	1	0, 1, 21, 1, 0
6	56	4, 8	4	6	$\boxed{7}$	8	$\boxed{10}$	0, 0, 8, 71, 8, 0
8	120	10, 45	10	16	$\boxed{19}$	23	25	0, 0, 0, 45, 253, 45
10	220	20, 220	20	32	$\boxed{38}$	45	47	0, 0, 0, 1, 220, 925, 22
12	364	35, 364	35	55	$\boxed{65}$	75	77	0, 0, 0, 0, 14, 1001, 3432, 10

Table: Codimensions of the components of the Hodge/special loci for cubic hypersurfaces.

$(d=3.\ n=6)$

Veronese embedding

$$\tilde{Z} \subseteq \mathbb{P}^5 \subseteq \mathbb{P}^7 \supset$$



$\tilde{Z}$

$\dim Z = 3.$ $\quad Z \subseteq \mathbb{P}^7$

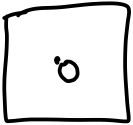
$\tilde{Z}$

Consider the image of the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ by degree 2 monomials.

$[x:y:z] \longrightarrow [x^2, \ldots, z^2]$

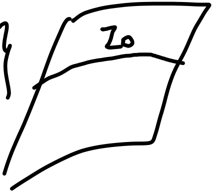
$$Z \subseteq X \subseteq \mathbb{P}^7$$

T

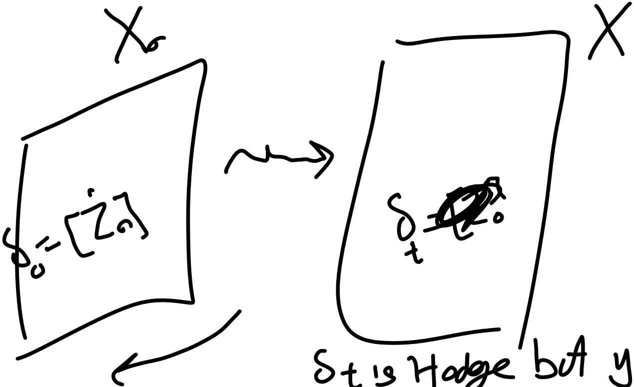


$\operatorname{Codim}\left\{\begin{array}{l}\text{cubic hy} \subseteq \mathbb{P}^7 \\ \text{cont. some } Z\end{array}\right\} = 10$

Codim Hodge loci ≤ 8



X_0



δ_t is Hodge but you cannot verify H.C. by veronese cycles

Chapter 21: Some mathematical olympiad problems