03/06/2022 Clemens' conjecture

$\underbrace{[x_1:x_2:x_3:x_4:x_5]}_{x}\in\mathbb{P}^4$ $\quad f(x)\in\mathbb{C}[x]_5$ = homog. poly of degree 5

$X=\mathbb{P}\{f(x)=0\}\subseteq\mathbb{P}^4$ assume that it is smooth

X is a Calabi-Yau threefold.

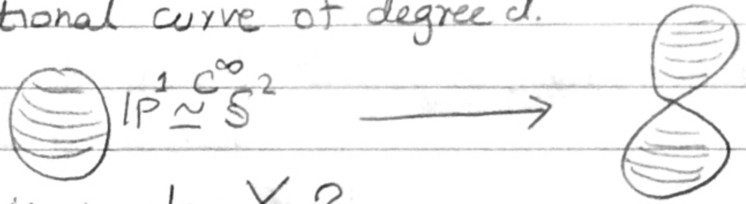
Rational curves in X:

$$p:\mathbb{P}^1\longrightarrow\mathbb{P}^4$$

$$p([s:t])=[f_1(s,t):f_2(s,t):\cdots:f_5(s,t)]$$

$$f_i\in\mathbb{C}[t,s]_d$$

This is called a rational curve of degree d.



$\mathbb{P}^1\overset{C^\infty}{\simeq}S^2$ might be singular.

when the image of p is inside X?

$$f(f_1(s,t),f_2(s,t),\dots f_5(s,t))=0$$

Fixing the coefs of f, we get many equations in the coef. of f_i's

\# equations = # of monomials of degree $5d$ in t,s = $5d+1$

\# unknowns = # coef of f_i's = $5(d+1)$

but we are only interested on the image of P: $[s,t]\to\left[\begin{smallmatrix}as+bt,\\ cs+dt\end{smallmatrix}\right]$ doesn't change the image

$$5d+1$$

\# equations = number of variables

Clemens' conjecture: The number of rational curves of degree d in a generic quintic is finite.

Stronger conjecture:

Conjecture: The number 〃 〃 〃 in $x_1x_2^4+x_2x_3^4+x_3x_4^4+x_4x_1^4+x_5^5=0$

credeal

Noether-Lefschetz thm: For a generic surface $X \subseteq \mathbb{P}^3$
$\mathrm{Pic}(X) \simeq \mathbb{Z}$
The only curves in X are $X \cap Y$, Y another surface.

Max Noether $\lesssim 1900$, Lefschetz ~ 1924 Griffiths 1980 ...

T. Shioda 1980: $x_1 x_2^{p-1} + x_2 x_3^{p-1} + x_3 x_1^{p-1} + x_4^p = 0$ for p prime
has Picard rank one.

Fermat: Picard rank $(x_1^d + x_2^d + x_3^d + x_4^d)$ = 7, 20, 37, 86, 147
d = 3, 4, 5, 6, 7

The idea of the proof of Noether-Lefschetz: (IVHS Griffiths', M. Book chapter 14

1. Hilbert scheme argument, families of curves in full families of surfaces

$$C_t \subseteq X_t \quad t \in \mathbb{C}[x]_d \setminus \text{discriminant}$$

2. Picard puzzle

$$\int_{[C_t] \in H_2(X_t, \mathbb{Z})} H^0(X_t, \Omega^2) = 0$$

3. Gauss-Manin connection: apply derivation

$$\frac{\partial}{\partial t_i} \int = 0 \Rightarrow \int_{[C_t]} H^2_{dR}(X_t)_{\text{primitive}} = 0$$

String theory & mirror symmetry.

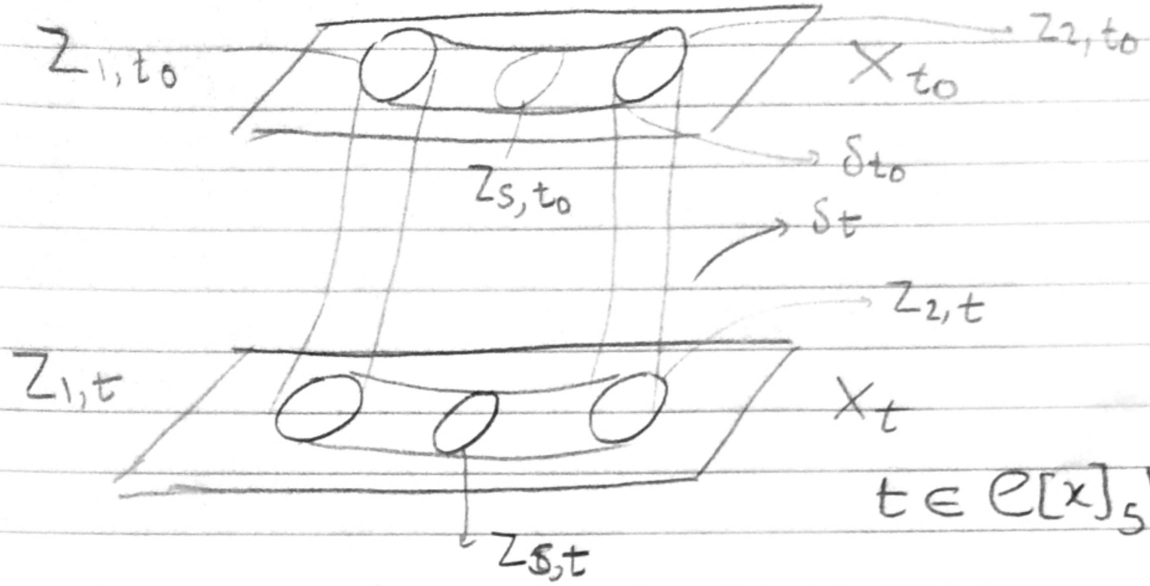
Candelas, de la Ossa. 1991

$$\text{Yukawa coupling} = 5 + 2875 \frac{q}{1-q} + 609250 \cdot 2^3 \frac{q^2}{1-q^2} + \cdots + n_d \, d^3 \frac{q^d}{1-q^d} + \cdots$$

n_d is the # of rational curves in a generic quintic

After Givental, Kontsevich.. $\rightsquigarrow$ virtual number of rational curves ...

1. Hilber schemes $\rightsquigarrow$ families of rational curves inside families of quintics



$t \in \mathbb{C}[x]_5 \setminus \text{disc.}$

2. $$\dim H^0(X, \Omega^3_X) = 1 \qquad \omega^{3,0} \in \text{''}$$

$$\int_{\delta_t} \omega \equiv 0$$

because for fixed t $Z_{t,s}$ moves inside a surface.

3. Gauss-Manin connection: apply $\frac{\partial}{\partial t_i}$

$$\int_{\delta t} \omega = 0 \quad \forall \omega \in F^2 H^3_{dR}(X_t) = H^{30} \oplus H^{21}$$

$\int_{\delta_t} H^3_{dR}(X_t)$ doesn't make sense as δ_t is not a closed cycle.

$$\int_{Z_{t,s}} \frac{\omega}{ds} = 0$$

If time permits, talk about computing such periods