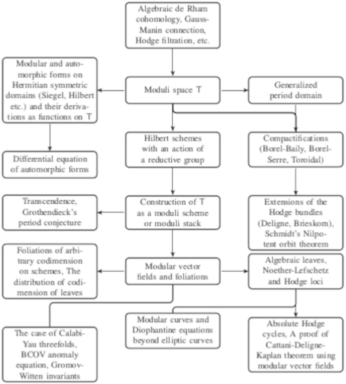
Gauss-Manin connection in disguise: Quasi Jacobi forms of index zero

Hossein Movasati

IMPA,
www.impa.br/∼hossein/

Labyrinth of This Book



Modular and Automorphic Forms & Beyond, Monographs in Number Theory, World Scientific, to be published November 2021.

Why do we have to generalize modular/automorphic forms?

A consequence of Ehressman's fibaration theorem (1947)

Let $\pi : X \to T$ be a family of smooth projective varieties, that is, $X \subset \mathbb{P}^N \times T$, π is the projection on T and π has no singular points and values.

Theorem

Algebraic de Rham cohomology and Hodge filtraton after Grothendieck (1966) and Deligne (1970)

A proposition which seemed to me trivial!

Proposition

Let $X_t, t \in \mathrm{T}$ be a family of smooth projective varieties and let X, X_0 be two regular fibers of this family. We have an isomorphism

$$(H^*_{\mathrm{dR}}(X), F^*, \cup, \theta) \overset{\alpha}{\simeq} (H^*_{\mathrm{dR}}(X_0), F^*_0, \cup, \theta_0) \tag{1}$$

Proof.

...... SGA3-II: "IX 3 uses cohomology to obtain infinitesimal statements. XI 4 proves representability of the functor M of subgroupschemes of multiplicative type. XI 5 puts it all together....." (P. Deligne, personal communication, August 14, 2019) □

Itacuatiara(written or painted stone): Varieties enhanced with ...

Ibiporanga: The moduli T of projective varieties enhanced with ...

Wishful thinking

The moduli space T is a quasi-affine variety over $\mathbb{Q}$ and we have a universal family over it!

Elliptic curves after [Weierstrass, Klein, Fricke, Deligne, Katz .....(M. 2008, 2012)]

$$\mathsf{X}: y^2-4(x-t_1)^3+t_2(x-t_1)+t_3=0,\quad \alpha=\left[\frac{dx}{y}\right],\quad \omega=\left[\frac{xdx}{y}\right] \tag{2}$$

$$\mathsf{T}:=\mathrm{Spec}\left(\mathbb{C}\left[t_1,t_2,t_3,\frac{1}{27t_3^2-t_2^3}\right]\right).$$

How about mixed Hodge structures?

We consider $H^1(X, Y)$, X an elliptic curve and $Y = \{O, P\}$:

$$W_0 H^1_{\mathrm{dR}}(X, Y) = \ker(H^1_{\mathrm{dR}}(X, Y) \to H^1_{\mathrm{dR}}(X))$$

$$W_1 H^1_{\mathrm{dR}}(X, Y) = H^1_{\mathrm{dR}}(X, Y)$$

$$F^1 H^1_{\mathrm{dR}}(X, Y) \cong F^1 H^1_{\mathrm{dR}}(X).$$

Ibiporanga with mixed Hodge structures

Theorem

$$\mathsf{T} = \mathrm{Spec}\,\mathbb{C}[a, b, c, t_1, t_2, \frac{1}{\Delta}]$$

where $\Delta = 27t_3^2 - t_2^3$ *and* $t_3 = 4a^3 - t_2 a - b^2$*. Moreover,* T *admits the universal family given by*

$$X = \{y^2 = 4x^3 - t_2 x - t_3\}, \quad Y = \{O, P\}, \; O = (0:1:0), \; P = (a:b:1$$

and the frame of differential forms

$$d\left(\frac{x-a}{x}\right), \; \frac{dx}{y}, \; \left(c - \frac{b}{2a}\right) d\left(\frac{x-a}{x}\right) + t_1 \frac{dx}{y} + \frac{xdx}{y} - d\left(\frac{y}{2x}\right).$$

Gauss-Manin connection

Theorem

There are unique global vector fields R_τ *and* R_z *on* T *such that*

$$\nabla_{\mathsf{R}_\tau}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}=\begin{pmatrix}0&0&0\\ 0&0&-1\\ 0&0&0\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} \tag{3}$$

and

$$\nabla_{\mathsf{R}_z}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}=\begin{pmatrix}0&0&0\\ -1&0&0\\ 0&0&0\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \tag{4}$$

where ∇ *is the Gauss-Manin connection of* T*.*

More precisely, we let $t_3 = 4a^3 - t_2 a - b^2$ and then

$$\mathrm{R}_\tau = (-2a^2 + 2at_1 + bc + \frac{t_2}{3})\frac{\partial}{\partial a} + (6a^2 c - \frac{ct_2}{2} - 3ab + 3bt_1)\frac{\partial}{\partial b} + (ac + ct_1 - \frac{b}{2})\frac{\partial}{\partial c} + (t_1^2 - \frac{t_2}{12})\frac{\partial}{\partial t_1} + (4t_1 t_2 - 6t_3)\frac{\partial}{\partial t_2} \tag{5}$$

$$\mathrm{R}_z = b\frac{\partial}{\partial a} + (6a^2 - \frac{t_2}{2})\frac{\partial}{\partial b} + (a + t_1)\frac{\partial}{\partial c}. \tag{6}$$

The t-map

$$t : \mathbb{H} \times \mathbb{C} \to \mathrm{T}$$

$$\begin{pmatrix} \int_{\delta_1} \alpha_1 & \int_{\delta_1} \alpha_2 & \int_{\delta_1} \alpha_3 \\ \int_{\delta_2} \alpha_1 & \int_{\delta_2} \alpha_2 & \int_{\delta_2} \alpha_3 \\ \int_{\delta_3} \alpha_1 & \int_{\delta_3} \alpha_2 & \int_{\delta_3} \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 & z & 0 \\ 0 & \tau & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Theorem

The pullback of a, b, c, t_1, t_2, t_3 under the t-map are are

$$(2\pi i)\wp(\tau, z)\ ,\ \ (2\pi i)^{\frac{3}{2}}\wp'(\tau, z)\ ,\ \ -(2\pi i)^{\frac{1}{2}}J_1(\tau, z),$$

$$-\frac{2\pi i}{12}E_2(\tau)\ ,\ \ 12\left(\frac{2\pi i}{12}\right)^2 E_4(\tau)\ ,\ \ -8\left(\frac{2\pi i}{12}\right)^3 E_6(\tau),$$

respectively, where

$$J_1(\tau, z) = y \frac{d}{dy} \log F(y, q) \tag{7}$$

$$y = -e^{2\pi iz}, \quad q = e^{2\pi i\tau}$$

$$F(\tau, z) = \frac{\theta_1(\tau, z)}{\eta^3(\tau)} = (y^{1/2} + y^{-1/2}) \prod_{m \geq 1} \frac{(1 + yq^m)(1 + y^{-1}q^m)}{(1 - q^m)^2} \tag{8}$$

$$\theta(z, \tau) = \sum_{n=-\infty}^{+\infty} e^{2\pi inz + \pi in^2\tau}, \; z \in \mathbb{C}, \quad \tau \in \mathbb{H}$$