Differential Equations and Algebraic Geometry

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The talk is based on my lecture notes [Mov12] on quasi-modular forms. The most general context has been worked out in [Mov21].

Elliptic integrals

In many calculus books we find tables of integrals and there we never find a formula for elliptic integrals

$$\int_a^b \frac{Q(x)dx}{\sqrt{P(x)}}, \tag{1}$$

where $P(x)$ is a degree three polynomial in one variable x and with real coefficients, for simplicity we assume that it has real roots, and a, b are two consecutive roots of P or $\pm\infty$. Since Abel and Gauss it was known that if we choose P randomly (in other words for generic P) such integrals cannot be calculated in terms of until then well-known functions.

Elliptic integrals

Problem

Compute the indefinite integral

$$\int \frac{dx}{\sqrt{p(x)}}, \tag{2}$$

where p is a polynomial of degree 1 and 2. Compute it also when p is of degree 3 but it has double roots. These integrals are computable because $y^2 = p(x)$ is a rational curve!
Example:

$$\int_2^\infty \frac{dx}{2(x+1)\sqrt{x-2}} = \frac{\mathrm{tang}^{-1}(\frac{\sqrt{x-2}}{\sqrt{3}})}{\sqrt{3}}\Bigg|_2^\infty = \frac{\pi}{2\sqrt{3}}$$

Problem

For particular examples of polynomials p of degree 3, there are some formulas for elliptic integrals 2 in terms of the values of the Gamma function on rational numbers. For instance, verify the equality

$$\int_7^{+\infty} \frac{dx}{\sqrt{x^3 - 35x - 98}} = \frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{2\pi i\sqrt{-7}}. \tag{3}$$

$$\int_0^1 \frac{dx}{\sqrt{1 - x^3}} = \frac{\Gamma(\frac{1}{3})^2}{2^{\frac{4}{3}} 3^{\frac{1}{2}} \pi},$$

$$\int_0^1 \frac{dx}{\sqrt{x - x^3}} = \frac{\Gamma(\frac{1}{4})^2}{2^{\frac{3}{2}} \pi^{\frac{1}{2}}}.$$

The Chowla-Selberg theorem, see for instance Gross's articles 1978 and 1979, describes this phenomenon in a complete way.

Geometrization

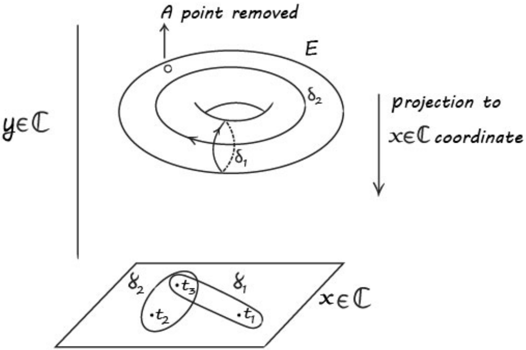

Figure: The elliptic curve $y^2 = p(x)$ in the four dimensional space $\mathbb{C}^2$.

A one parameter family of elliptic curves

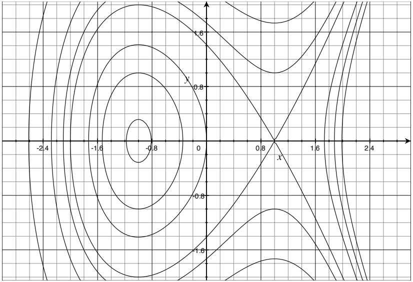

Figure: Elliptic curves: $y^2 - x^3 + 12x - 4\psi,\ \psi = -1.9, -1, 0, 2, 3, 5, 10$

Picard-Fuchs equation

The elliptic integrals $\int_\delta \frac{dx}{y}$ (resp. $\int_\delta \frac{xdx}{y}$) satisfies the differential equation

$$\frac{5}{36}I + 2\psi I' + (\psi^2 - 4)I'' = 0 \qquad (\text{ resp. } \frac{-7}{36}I + 2\psi I' + (\psi^2 - 4)I'' = 0) \tag{4}$$

which is called a Picard-Fuchs equation.

Gauss-Manin connection I

The matrix

$$Y = \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_2} \frac{dx}{y} \\ \int_{\delta_1} \frac{xdx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix}$$

forms a fundamental system of the linear differential equation:

$$Y' = \frac{1}{\psi^2 - 4} \begin{pmatrix} \frac{-1}{6}\psi & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{6}\psi \end{pmatrix} Y, \tag{5}$$

that is, any solution of (5) is a linear combination of the columns of Y.

Hypergeometric functions

$$Y = \begin{pmatrix} \frac{\pi i}{\sqrt{3}} F(\frac{1}{6}, \frac{5}{6}, 1 | \frac{-\psi+2}{4}) & \frac{\pi}{\sqrt{3}} F(\frac{1}{6}, \frac{5}{6}, 1 | \frac{\psi+2}{4}) \\ \frac{\pi i}{\sqrt{3}} F(\frac{-1}{6}, \frac{7}{6}, 1 | \frac{-\psi+2}{4}) & -\frac{\pi}{\sqrt{3}} F(\frac{-1}{6}, \frac{7}{6}, 1 | \frac{\psi+2}{4}) \end{pmatrix}$$

where

$$F(a, b, c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \; c \notin \{0, -1, -2, -3, \ldots\}, \qquad (6)$$

Three parameter family of elliptic curve

$$E_t : y^2 = 4(x - t_1)^3 - t_2(x - t_1) - t_3$$

$$t \in \mathrm{Spec}(\mathbb{C}[t_1, t_2, t_3, \frac{1}{\Delta}]),\ \Delta := 27t_3^2 - t_2^3.$$

Gauss-Manin connection II

Problem

Let $P(x) := 4(x-t_1)^3 + t_2(x-t_1) + t_3$. *We have*

$$\begin{pmatrix} d\left(\int \frac{dx}{\sqrt{P(x)}}\right) \\ d\left(\int \frac{xdx}{\sqrt{P(x)}}\right) \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}t_1\frac{\alpha}{\Delta} - \frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ dt_1 - \frac{1}{6}t_1\frac{d\Delta}{\Delta} - (\frac{3}{2}t_1^2 + \frac{1}{8}t_2)\frac{\alpha}{\Delta}, & \frac{3}{2}t_1\frac{\alpha}{\Delta} + \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} \begin{pmatrix} \int \frac{dx}{\sqrt{P(x)}} \\ \int \frac{xdx}{\sqrt{P(x)}} \end{pmatrix}$$

where

$$\Delta := 27t_3^2 - t_2^3, \ \alpha := 3t_3dt_2 - 2t_2dt_3.$$

Classical moduli space

1. Any elliptic curve can be written in the Weierstrass format $E_t : y^2 = 4x^3 - t_2 x - t_3$
2. We have

$$E_{k^{-6}t_2, k^{-4}} \cong E_{t_2,t_3}, \quad (x, y) \mapsto (k^2 x, k^3 y).$$

3. The j-invariant $j := \frac{t_2^3}{t_2^3 - 27t_3^2}$ classifies elliptic curves.

Ibiporanga: A new moduli space

Ibiporanga: A new moduli space

End of Lecture 1

Ramanujan's relations between Eisenstein series

In 1916 Ramanujan verified the identities

$$\begin{cases} q\frac{\partial E_2}{\partial q} = \frac{1}{12}(E_2^2 - E_4) \\ q\frac{\partial E_4}{\partial q} = \frac{1}{3}(E_2 E_4 - E_6) \\ q\frac{\partial E_6}{\partial q} = \frac{1}{2}(E_2 E_6 - E_4^2) \end{cases}, \qquad (7)$$

where E_i's are the Eisenstein series:

$$E_{2i}(q) := 1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1} \right) q^n, \quad i = 1, 2, 3 \qquad (8)$$

and $(b_1, b_2, b_3) = (-24, 240, -504)$.

Verifying few coefficients

Darboux-Halphen differential equation

Darboux in 1878 considered

$$\mathrm{H}: \begin{cases} \dot{t}_1 = t_1(t_2+t_3) - t_2 t_3 \\ \dot{t}_2 = t_2(t_1+t_3) - t_1 t_3 \\ \dot{t}_3 = t_3(t_1+t_2) - t_1 t_2 \end{cases} \tag{9}$$

Halphen in 1881 found a solution in terms of the logarithmic derivatives of the null theta functions:

$$t_1 = 2(\ln\theta_4(0|\tau))', t_2 = 2(\ln\theta_2(0|\tau))', t_3 = 2(\ln\theta_3(0|\tau))'$$

where

$$\begin{cases} \theta_2(0|\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \\ \theta_3(0|\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \\ \theta_4(0|\tau) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \end{cases}, \; q = e^{2\pi i\tau}, \; \tau \in \mathbb{H}, \; ' = \frac{\partial}{\partial\tau}$$

Vector fields and differential forms

Gauss-Manin connection

The Gauss-Manin connection of the family

$$y^2 = 4(x - t_1)^3 + t_2(x - t_1) + t_3$$

is reduced to the following linear differential equation (system)

$$dY = \begin{pmatrix} -\frac{3}{2}t_1\frac{\alpha}{\Delta} - \frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ dt_1 - \frac{1}{6}t_1\frac{d\Delta}{\Delta} - (\frac{3}{2}t_1^2 + \frac{1}{8}t_2)\frac{\alpha}{\Delta}, & \frac{3}{2}t_1\frac{\alpha}{\Delta} + \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} Y$$

where

$$\Delta := 27t_3^2 - t_2^3, \ \alpha := 3t_3dt_2 - 2t_2dt_3.$$

Gauss-Manin connection

The Gauss-Manin connection of the family

$$y^2 = 4(x - t_1)(x - t_2)(x - t_3)$$

is reduced to the following linear differential equation (system) $dY = AY$, where

$$A = \frac{dt_1}{2(t_1 - t_2)(t_1 - t_3)} \begin{pmatrix} -t_1 & 1 \\ t_2 t_3 - t_1(t_2 + t_3) & t_1 \end{pmatrix} +$$

$$\frac{dt_2}{2(t_2 - t_1)(t_2 - t_3)} \begin{pmatrix} -t_2 & 1 \\ t_1 t_3 - t_2(t_1 + t_3) & t_2 \end{pmatrix} +$$

$$\frac{dt_3}{2(t_3 - t_1)(t_3 - t_2)} \begin{pmatrix} -t_3 & 1 \\ t_1 t_2 - t_3(t_1 + t_2) & t_3 \end{pmatrix}.$$

Explaining the computation of Gauss-Manin connection

Gauss-Manin connection along vector fields

There are unique vector fields e, h, f in the parameter space of $y^2 = 4(x - t_1)^3 + t_2(x - t_1) + t_3$ such that

$$A_h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_e = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We can compute explicit expressions for e, f, h.

$$f = -(t_1^2 - \frac{1}{12}t_2)\frac{\partial}{\partial t_1} - (4t_1t_2 - 6t_3)\frac{\partial}{\partial t_2} - (6t_1t_3 - \frac{1}{3}t_2^2)\frac{\partial}{\partial t_3}, \quad (10)$$

$$h = -6t_3\frac{\partial}{\partial t_3} - 4t_2\frac{\partial}{\partial t_2} - 2t_1\frac{\partial}{\partial t_1}, \; e = \frac{\partial}{\partial t_1}.$$

The Lie algebra $\mathfrak{sl}_2$

The $\mathbb{C}$-vector space generated by h, e, f equipped with the classical bracket of vector fields is isomorphic to the Lie Algebra $\mathfrak{sl}_2$, and hence, it gives a representation of $\mathfrak{sl}_2$ in the polynomial ring $\mathbb{Q}[t_1, t_2, t_3]$ which is infinite dimensional.

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{11}$$

There is a unique vector field D in the parametr space of $y^2 = (x - t_1)(x - t_2)(x - t_3)$ such that

$$A_D = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

This is

$$D = (t_1(t_2+t_3)-t_2 t_3)\frac{\partial}{\partial t_1}+(t_2(t_1+t_3)-t_1 t_3)\frac{\partial}{\partial t_2}+(t_3(t_1+t_2)-t_1 t_2)\frac{\partial}{\partial t_3}.$$

and it is called the Darboux-Halphen vector field.

Vector field as an ODE

Ramanujan's ODE

The function $t = (t_1, t_2, t_3) = (a_1 E_2(\tau), a_2 E_4(\tau), a_3 E_6(\tau))$ with $(a_1, a_2, a_3) = \left(\frac{2\pi i}{12}, 12(\frac{2\pi i}{12})^2, 8(\frac{2\pi i}{12})^3\right)$ satisfies the ODE's

$$\begin{cases} \dot{t}_1 = t_1^2 - \frac{1}{12} t_2 \\ \dot{t}_2 = 4 t_1 t_2 - 6 t_3 \\ \dot{t}_3 = 6 t_1 t_3 - \frac{1}{3} t_2^2 \end{cases},$$

where the derivation is with respect to τ.

Recovering Eisenstein from Ramanujan

Halphen property

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Quasi-modular forms attached to elliptic curves, I.
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