Differential Equations and Algebraic Geometry

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Lecture 2.

The talk is based on my lecture notes [Mov12] on quasi-modular forms. The most general context has been worked out in [Mov21].

Elliptic integrals

In many calculus books we find tables of integrals and there we never find a formula for elliptic integrals

$$\int_a^b \frac{Q(x)dx}{\sqrt{P(x)}}, \tag{1}$$

where $P(x)$ is a degree three polynomial in one variable x and with real coefficients, for simplicity we assume that it has real roots, and a, b are two consecutive roots of P or $\pm\infty$. Since Abel and Gauss it was known that if we choose P randomly (in other words for generic P) such integrals cannot be calculated in terms of until then well-known functions.

Elliptic integrals

Problem
Compute the indefinite integral

$$\int \frac{dx}{\sqrt{p(x)}}, \tag{2}$$

where p is a polynomial of degree 1 and 2. Compute it also when p is of degree 3 but it has double roots. These integrals are computable because $y^2 = p(x)$ is a rational curve! Example:

$$\int_2^\infty \frac{dx}{2(x+1)\sqrt{x-2}} = \left.\frac{\tang^{-1}(\frac{\sqrt{x-2}}{\sqrt{3}})}{\sqrt{3}}\right|_2^\infty = \frac{\pi}{2\sqrt{3}}$$

Problem

For particular examples of polynomials p of degree 3, *there are some formulas for elliptic integrals 2 in terms of the values of the Gamma function on rational numbers. For instance, verify the equality*

$$\int_7^{+\infty} \frac{dx}{\sqrt{x^3 - 35x - 98}} = \frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{2\pi i \sqrt{-7}}. \tag{3}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{\Gamma(\frac{1}{3})^2}{2^{\frac{4}{3}} 3^{\frac{1}{2}} \pi},$$

$$\int_0^1 \frac{dx}{\sqrt{x-x^3}} = \frac{\Gamma(\frac{1}{4})^2}{2^{\frac{3}{2}} \pi^{\frac{1}{2}}}.$$

The Chowla-Selberg theorem, see for instance Gross's articles 1978 and 1979, describes this phenomenon in a complete way.
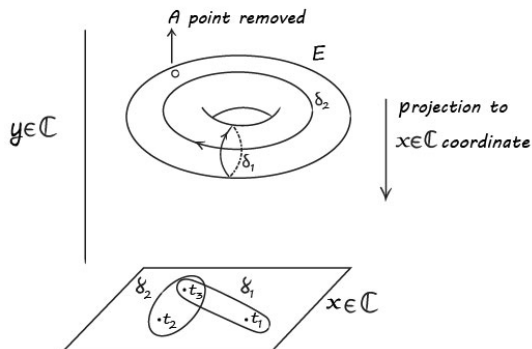
Geometrization

Figure: The elliptic curve $y^2 = p(x)$ in the four dimensional space $\mathbb{C}^2$.
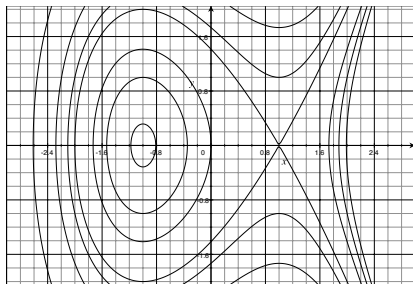
A one parameter family of elliptic curves

Figure: Elliptic curves: $y^2 - x^3 + 12x - 4\psi, \ \psi = -1.9, -1, 0, 2, 3, 5, 10$

Picard-Fuchs equation

The elliptic integrals $\int_\delta \frac{dx}{y}$ (resp. $\int_\delta \frac{x\,dx}{y}$) satisfies the differential equation

$$\frac{5}{36}I + 2\psi I' + (\psi^2 - 4)I'' = 0 \qquad (\text{ resp. } \frac{-7}{36}I + 2\psi I' + (\psi^2 - 4)I'' = 0)$$

(4)

which is called a Picard-Fuchs equation.

Gauss-Manin connection I

The matrix

$$Y = \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_2} \frac{dx}{y} \\ \int_{\delta_1} \frac{xdx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix}$$

forms a fundamental system of the linear differential equation:

$$Y' = \frac{1}{\psi^2 - 4} \begin{pmatrix} \frac{-1}{6}\psi & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{6}\psi \end{pmatrix} Y, \tag{5}$$

that is, any solution of (5) is a linear combination of the columns of Y.

Hypergeometric functions

$$Y = \begin{pmatrix} \frac{\pi i}{\sqrt{3}} F(\frac{1}{6}, \frac{5}{6}, 1 | \frac{-\psi+2}{4}) & \frac{\pi}{\sqrt{3}} F(\frac{1}{6}, \frac{5}{6}, 1 | \frac{\psi+2}{4}) \\ \frac{\pi i}{\sqrt{3}} F(\frac{-1}{6}, \frac{7}{6}, 1 | \frac{-\psi+2}{4}) & -\frac{\pi}{\sqrt{3}} F(\frac{-1}{6}, \frac{7}{6}, 1 | \frac{\psi+2}{4}) \end{pmatrix}$$

where

$$F(a, b, c | z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \ c \notin \{0, -1, -2, -3, \ldots\}, \quad (6)$$

Three parameter family of elliptic curve

$$E_t : y^2 = 4(x - t_1)^3 - t_2(x - t_1) - t_3$$

$$t \in \operatorname{Spec}(\mathbb{C}[t_1, t_2, t_3, \frac{1}{\Delta}]), \ \Delta := 27t_3^2 - t_2^3.$$

Gauss-Manin connection II

Problem
Let $P(x) := 4(x - t_1)^3 + t_2(x - t_1) + t_3$. We have

$$\begin{pmatrix} d\left(\int \frac{dx}{\sqrt{P(x)}}\right) \\ d\left(\int \frac{xdx}{\sqrt{P(x)}}\right) \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}t_1\frac{\alpha}{\Delta} - \frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ dt_1 - \frac{1}{6}t_1\frac{d\Delta}{\Delta} - (\frac{3}{2}t_1^2 + \frac{1}{8}t_2)\frac{\alpha}{\Delta}, & \frac{3}{2}t_1\frac{\alpha}{\Delta} + \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} \begin{pmatrix} \int \frac{dx}{\sqrt{P(x)}} \\ \int \frac{xdx}{\sqrt{P(x)}} \end{pmatrix}$$

where

$$\Delta := 27t_3^2 - t_2^3, \ \alpha := 3t_3 dt_2 - 2t_2 dt_3.$$

Classical moduli space

1. Any elliptic curve can be written in the Weierstrass format
   $E_t : y^2 = 4x^3 - t_2 x - t_3$

2. We have

$$E_{k^{-6}t_2, k^{-4}} \cong E_{t_2, t_3}, \quad (x, y) \mapsto (k^2 x, k^3 y).$$

3. The j-invariant $j := \frac{t_2^3}{t_2^3 - 27 t_3^2}$ classifies elliptic curves.

Ibiporanga: A new moduli space

Ibiporanga: A new moduli space

End of Lecture 1

Ramanujan's relations between Eisenstein series

$$E_2 = 1 - 24\left(1\,q^1 + 3\,q^2 + \cdots\right)$$

In 1916 Ramanujan verified the identities

$$\begin{cases} q\frac{\partial E_2}{\partial q} = \frac{1}{12}(E_2^2 - E_4) \\ q\frac{\partial E_4}{\partial q} = \frac{1}{3}(E_2 E_4 - E_6) \\ q\frac{\partial E_6}{\partial q} = \frac{1}{2}(E_2 E_6 - E_4^2) \end{cases} \qquad (7)$$

where E_i's are the Eisenstein series:

$$\sigma_{2i-1}(n)$$

$$E_{2i}(q) := 1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1} \right) q^n, \quad i = 1, 2, 3 \qquad (8)$$

and $(b_1, b_2, b_3) = (-24, 240, -504)$.

$$\sigma_1(n) = \sum_{d|n} d$$

Verifying few coefficients

$$q \frac{\partial}{\partial q} E_2 = \frac{1}{12} \left(E_2^2 - E_4 \right)$$

E_2, E_4, E_6

$\cdot q^1, \cdot q^2, \cdots q^9 \qquad q^{10}$

Darboux-Halphen differential equation

Darboux in 1878 considered

$$\mathrm{H} : \begin{cases} \dot{t}_1 = t_1(t_2 + t_3) - t_2 t_3 \\ \dot{t}_2 = t_2(t_1 + t_3) - t_1 t_3 \\ \dot{t}_3 = t_3(t_1 + t_2) - t_1 t_2 \end{cases} \qquad (9)$$

Halphen in 1881 found a solution in terms of the logarithmic derivatives of the null theta functions:

$$t_1 = 2(\ln \theta_4(0|\tau))', \ t_2 = 2(\ln \theta_2(0|\tau))', \ t_3 = 2(\ln \theta_3(0|\tau))'$$

where

$$\begin{cases} \theta_2(0|\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \\ \theta_3(0|\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \\ \theta_4(0|\tau) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \end{cases}, \ q = e^{2\pi i \tau}, \ \tau \in \mathbb{H}, \ ' = \frac{\partial}{\partial \tau}$$

Vector fields and differential forms

$$\not{X} \rightarrow \mathbb{R}^n \ni (x_1, x_2, \ldots x_n) = x$$

vector field $\quad P_1(x) \dfrac{\partial}{\partial x_1} + P_2(x) \dfrac{\partial}{\partial x_2} + \cdots P_n(x) \dfrac{\partial}{\partial x_n}$

dif. 1-forms $\quad q_1(x)\, dx_1 + q_2(x)\, dx_2 + \cdots$

$$P_i, Q_i \in \mathbb{R}[x_1 \cdots x_n]$$

$$dx_i \left(\dfrac{\partial}{\partial x_j} \right) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\underline{\mathbb{R}\text{-algebra} \quad R[x]\text{-linear.}}$$

$$Q_i\, dx_i \left[P_i \dfrac{\partial}{\partial x_j} \right] \sim \quad q_i, P_i$$

$$q_i, P_i\, dx_i \left[\dfrac{\partial}{\partial x_j} \right]$$

Gauss-Manin connection

The Gauss-Manin connection of the family

$$y^2 = 4(x - t_1)^3 + t_2(x - t_1) + t_3$$

$t = (t_1, t_2, t_3) \in$

$\mathbb{C}^3 \setminus \{\Delta = 0\}$

is reduced to the following linear differential equation (system)

$$dY = \begin{pmatrix} -\frac{3}{2} t_1 \frac{\alpha}{\Delta} - \frac{1}{12} \frac{d\Delta}{\Delta}, & \frac{3}{2} \frac{\alpha}{\Delta} \\ dt_1 - \frac{1}{6} t_1 \frac{d\Delta}{\Delta} - (\frac{3}{2} t_1^2 + \frac{1}{8} t_2) \frac{\alpha}{\Delta}, & \frac{3}{2} t_1 \frac{\alpha}{\Delta} + \frac{1}{12} \frac{d\Delta}{\Delta} \end{pmatrix} Y$$

$\| \quad T$

where

$$\Delta := 27 t_3^2 - t_2^3, \quad \alpha := 3 t_3 dt_2 - 2 t_2 dt_3.$$

$U \subseteq T$

$$d \quad Y = c_1 \begin{pmatrix} \int_{\delta_1} y \\ \int_{\delta_1} \frac{x}{y} \end{pmatrix} + c_2 \begin{pmatrix} \int_{\delta_2} y \\ \int_{\delta_2} \frac{x}{y} \end{pmatrix}.$$

$$Y = \begin{pmatrix} y_1^{(t)} \\ y_2^{(t)} \end{pmatrix} \quad t \in U.$$

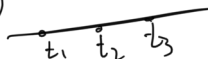# Gauss-Manin connection

The Gauss-Manin connection of the family

$$y^2 = 4(x - t_1)(x - t_2)(x - t_3)$$

is reduced to the following linear differential equation (system) $dY = AY$, where

$$A = \frac{dt_1}{2(t_1 - t_2)(t_1 - t_3)} \begin{pmatrix} -t_1 & 1 \\ t_2 t_3 - t_1(t_2 + t_3) & t_1 \end{pmatrix} +$$

$$\frac{dt_2}{2(t_2 - t_1)(t_2 - t_3)} \begin{pmatrix} -t_2 & 1 \\ t_1 t_3 - t_2(t_1 + t_3) & t_2 \end{pmatrix} +$$

$$\frac{dt_3}{2(t_3 - t_1)(t_3 - t_2)} \begin{pmatrix} -t_3 & 1 \\ t_1 t_2 - t_3(t_1 + t_2) & t_3 \end{pmatrix}.$$

Explaining the computation of Gauss-Manin connection

$$\begin{pmatrix} \dfrac{\partial}{\partial t_1} \displaystyle\int_{t_1}^{t_2} \dfrac{dx}{\sqrt{4(x-t_1)(x-t_2)(x-t_3)}} \\[2em] \dfrac{\partial}{\partial t_1} \displaystyle\int_{t_1}^{t_2} \dfrac{x\,dx}{\sqrt{4(x-t_1)(x-t_2)(x-t_3)}} \end{pmatrix} = A_1 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Gauss-Manin connection along vector fields

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

There are unique vector fields e, h, f in the parameter space of
$y^2 = 4(x - t_1)^3 + t_2(x - t_1) + t_3$ such that

$$v = x(t)\frac{\partial}{\partial t_1} + x_2(t)\frac{\partial}{\partial t_2} + x_3\left(\frac{\partial}{\partial t_3}\right)$$

$$A_h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_e = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We can compute explicit expressions for e, f, h.

$$f = -(t_1^2 - \frac{1}{12}t_2)\frac{\partial}{\partial t_1} - (4t_1 t_2 - 6t_3)\frac{\partial}{\partial t_2} - (6t_1 t_3 - \frac{1}{3}t_2^2)\frac{\partial}{\partial t_3}, \quad (10)$$

$$h = -6t_3\frac{\partial}{\partial t_3} - 4t_2\frac{\partial}{\partial t_2} - 2t_1\frac{\partial}{\partial t_1}, \quad e = \frac{\partial}{\partial t_1}.$$

The Lie algebra $\mathfrak{sl}_2$

$$Sl_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \;\middle|\; a + d = 0 \right\}$$

$$[A, B] = AB - BA$$

The $\mathbb{C}$-vector space generated by h, e, f equipped with the classical bracket of vector fields is isomorphic to the Lie Algebra $\mathfrak{sl}_2$, and hence, it gives a representation of $\mathfrak{sl}_2$ in the polynomial ring $\mathbb{Q}[t_1, t_2, t_3]$ which is infinite dimensional.

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{11}$$

There is a unique vector field D in the parametr space of $y^2 = (x - t_1)(x - t_2)(x - t_3)$ such that

$$A_D = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

$t_1, t_2, t_3.$

Gauss-Manin

This is

$$D = (t_1(t_2+t_3)-t_2 t_3)\frac{\partial}{\partial t_1} + (t_2(t_1+t_3)-t_1 t_3)\frac{\partial}{\partial t_2} + (t_3(t_1+t_2)-t_1 t_2)\frac{\partial}{\partial t_3}.$$

and it is called the Darboux-Halphen vector field.

Vector field as an ODE

$$\mathbb{R}^n \ni (x_1, x_2, \ldots x_n) = x$$

$$\vartheta = P_1(x)\frac{\partial}{\partial x_1} + P_2(x)\frac{\partial}{\partial x_2} \ldots + P_n(x)\frac{\partial}{\partial x_n}$$

$$U \subseteq \mathbb{C}$$

$$\vartheta : \begin{cases} \dot{x}_1 = P_1(x) \\ \dot{x}_2 = P_2(x) \\ \quad \vdots \\ \dot{x}_n = P_n(x) \end{cases}$$

$$x_1(t) \qquad t \in I \subseteq \mathbb{R}$$

$$n = 2 \qquad P \notin Sing(\vartheta)$$

$$x(t)$$

$$\text{no!}$$

$$Sing(\vartheta) = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{l} P_1(x) = \ldots \\ = P_n(x) = 0 \end{array} \right\}$$

$$\mathbb{R}^2$$

$$\dot{x}(t) = \vartheta(x(t))$$

$$x(t) : U \xrightarrow{\subseteq \mathbb{C}} \mathbb{C}^2 \qquad \mathbb{C}^2 \qquad \underset{\mathbb{C}^2}{\cap}$$

Ramanujan's ODE

The function $t = (t_1, t_2, t_3) = (a_1 E_2(\tau), a_2 E_4(\tau), a_3 E_6(\tau))$ with $(a_1, a_2, a_3) = (\frac{2\pi i}{12}, 12(\frac{2\pi i}{12})^2, 8(\frac{2\pi i}{12})^3)$ satisfies the ODE's

$$\begin{cases} \dot{t}_1 = t_1^2 - \frac{1}{12} t_2 \\ \dot{t}_2 = 4 t_1 t_2 - 6 t_3 \\ \dot{t}_3 = 6 t_1 t_3 - \frac{1}{3} t_2^2 \end{cases},$$
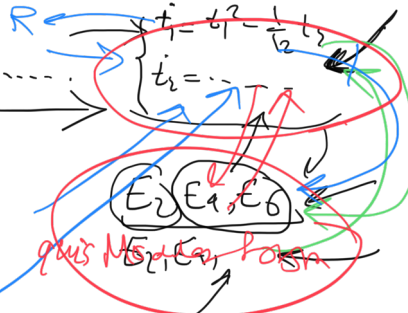
$$\left(t_1^2 - \frac{1}{12}\right) \frac{\partial}{\partial t_1} +$$

$$\left(4 t_1 t_2 \ldots\right) \frac{\partial}{\partial t_2}$$

where the derivation is with respect to τ.

$$\tau \in \mathbb{H}.$$

Recovering Eisenstein from Ramanujan

$$y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3$$

R

$t_1 = t_1^2 - \frac{1}{12}t_2$

$t_2 = \cdots$

$t_2 \quad \boxed{E_4}$

$t_3 \quad \boxed{E_6}$

$t_1 \quad \boxed{E_2}$

E_2, E_4, E_6

qms M_{E_1, C_0}, Poison

$$t_1 = \sum_{n=0}^{\infty} t_{1,n} q^n$$

$$t_2 = \sum_{n=2} t_{2,n} q^n$$

$$t_3 = \sum_{n=3} t_{3,n} q^n$$

$t_{1,n} = ?$

$\cdot = q\dfrac{\partial}{\partial q}$

$\Rightarrow$

R will turn into a recurs in $t_{i,n}$

$\begin{pmatrix} t_{1,n} \\ t_{2,n} \\ t_{3,n} \end{pmatrix} = $ in terms of $t_{i,m}$

$m < n \quad c=1,2,3$

Halphen property

$$H \begin{cases} \dot{t_1} = t_1(t_2 + t_3) - t_2 t_3 \\ \dot{t_2} = t_2(t_1 + t_3) - t_1 t_3 \\ \dot{t_3} = t_3(t_2 + t_1) - t_1 t_2 \end{cases}$$

$$' = \frac{\partial}{\partial t}$$

$$\begin{pmatrix} t_1(\tau) \\ t_2(\tau) \\ t_3(\tau) \end{pmatrix} \text{ a solution of } H$$

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix}, \quad ad - bc = 1$$
$$a, b, c, d \in \mathbb{C}$$

then
$$\begin{cases} \widetilde{t_1}(\tau) = (c\tau + d)^{-2} t_1\left(\frac{a\tau + b}{c\tau + a}\right) - c(c\tau + d)^{-1} \\ \widetilde{t_2}(\tau) = (c\tau + d)^{-2} t_2(\text{———}) - \sim \\ \widetilde{t_3}(\tau) = \sim \qquad t_3(\sim) - \sim \end{cases}$$

is also a solution of H. $\quad \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in SL(2, \mathbb{C})$.

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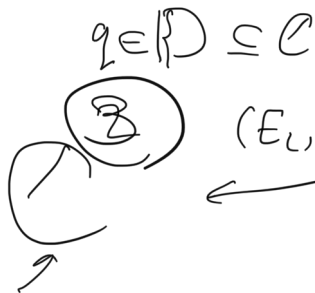Quasi-modular forms attached to elliptic curves, I.
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