Differential Equations and Algebraic Geometry

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q. m.f attached to elliptic curve I.

2012

The talk is based on my lecture notes [Mov12] on quasi-modular forms. The most general context has been worked out in [Mov21].

2021

Modular and Auto Forms & beyond

2021

Elliptic integrals

In many calculus books we find tables of integrals and there we never find a formula for elliptic integrals

$$\int_a^b \frac{Q(x)dx}{\sqrt{P(x)}}, \tag{1}$$

$P(x) = (x-a)\cdot(x-b)(x-$

where $P(x)$ is a degree three polynomial in one variable x and with real coefficients, for simplicity we assume that it has real roots, and a, b are two consecutive roots of P or $\pm\infty$. Since Abel and Gauss it was known that if we choose P randomly (in other words for generic P) such integrals cannot be calculated in terms of until then well-known functions.

Elliptic integrals

Problem

Compute the indefinite integral

$$\int^a \frac{dx}{\sqrt{p(x)}}, \qquad (2)$$

$$\frac{1}{g(x)} \cdot dx \qquad f'(x) \qquad y^2 = P(x)$$

$$y = \sqrt{P(x)}$$

where p is a polynomial of degree 1 and 2. Compute it also when p is of degree 3 but it has double roots. These integrals are computable because $y^2 = p(x)$ is a rational curve!

Example:

$$P(x) = (x+1)^2(x-2)$$

$$\boxed{\int_2^\infty \frac{dx}{2(x+1)\sqrt{x-2}} = \frac{\operatorname{tang}^{-1}\left(\frac{\sqrt{x-2}}{\sqrt{3}}\right)}{\sqrt{3}}\Bigg|_2^\infty = \frac{\pi}{2\sqrt{3}}}$$

$$y^2 = P(x)$$

Problem

For particular examples of polynomials p of degree 3*, there are some formulas for elliptic integrals 2 in terms of the values of the Gamma function on rational numbers. For instance, verify the equality*

$$\boxed{\int_7^{+\infty} \frac{dx}{\sqrt{x^3 - 35x - 98}} = \frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{2\pi i\sqrt{-7}}.} \tag{3}$$

$$\int_0^1 \frac{dx}{\sqrt{1 - x^3}} = \frac{\Gamma(\frac{1}{3})^2}{2^{\frac{4}{3}}3^{\frac{1}{2}}\pi},$$

$$\int_0^1 \frac{dx}{\sqrt{x - x^3}} = \frac{\Gamma(\frac{1}{4})^2}{2^{\frac{3}{2}}\pi^{\frac{1}{2}}}.$$

The Chowla-Selberg theorem, see for instance Gross's articles 1978 and 1979, describes this phenomenon in a complete way.

Geometrization

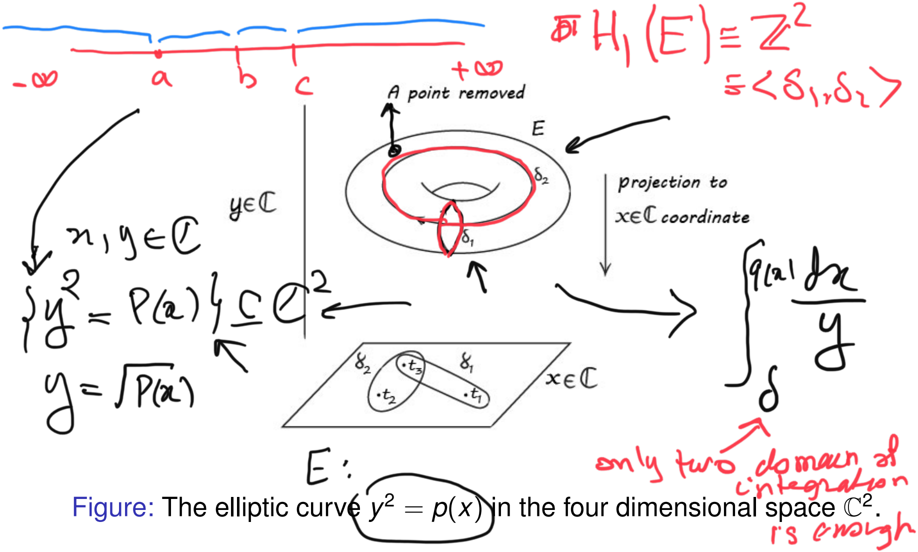


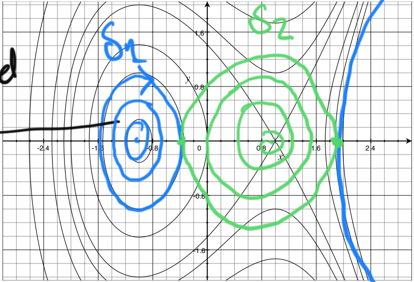
Figure: The elliptic curve $y^2 = p(x)$ in the four dimensional space $\mathbb{C}^2$.

A one parameter family of elliptic curves

δ_1, δ_2

$\langle \delta_1, \delta_2 \rangle = 1$

δ_1, δ_2 are called vanishing cycles



$\mathbb{R}^2$

Figure: Elliptic curves: $y^2 - x^3 + 12x - 4\psi$, $\psi = -1.9, -1, 0, 2, 3, 5, 10$

$(x, y) \in \mathbb{R}^2$.

Picard-Fuchs equation

$$\frac{\partial}{\partial \psi}\int_a^b \frac{dx}{\sqrt{P(x)}}$$

a, b roots of $P(x)$

$\delta_\psi \in H_1(E_\psi, \mathbb{Z})$

The elliptic integrals $\int_\delta \frac{dx}{y}$ (resp. $\int_\delta \frac{xdx}{y}$) satisfies the differential equation

$$\frac{5}{36} I + 2\psi I' + (\psi^2 - 4) I'' = 0 \qquad \left(\text{ resp. } \frac{-7}{36} I + 2\psi I' + (\psi^2 - 4) I'' = 0\right) \tag{4}$$

which is called a Picard-Fuchs equation.

linear dif equ

Gauss-Manin connection I

$H_1(E_\psi;\mathbb{Z}) = \mathbb{Z}^2$

δ_1, δ_2

Period matrix

$\left[\frac{dx}{y}\right], \left[\frac{x\,dx}{y}\right] \in H^1_{dR}(E_\psi)$

basis.

2 dim.

The matrix

$$Y = \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_2} \frac{dx}{y} \\ \int_{\delta_1} \frac{xdx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix}$$

function in ψ.

forms a fundamental system of the linear differential equation:

A

$' = \frac{\partial}{\partial\psi}$

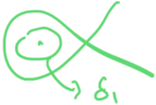
$$\boxed{Y' = \frac{1}{\psi^2 - 4} \begin{pmatrix} \frac{-1}{6}\psi & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{6}\psi \end{pmatrix} Y,} \tag{5}$$

that is, any solution of (5) is a linear combination of the columns of Y.

$\tilde{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad \tilde{Y} = AY$

Hypergeometric functions

$\int \frac{dx}{\sqrt{P(x)}}$ $= \int_{\delta_1} \frac{dx}{y}$ = functions in ψ.



$P(x)$ with double root

$$Y = \begin{pmatrix} \frac{\pi i}{\sqrt{3}} F(\frac{1}{6}, \frac{5}{6}, 1 | \frac{-\psi+2}{4}) & \frac{\pi}{\sqrt{3}} F(\frac{1}{6}, \frac{5}{6}, 1 | \frac{\psi+2}{4}) \\ \frac{\pi i}{\sqrt{3}} F(\frac{-1}{6}, \frac{7}{6}, 1 | \frac{-\psi+2}{4}) & -\frac{\pi}{\sqrt{3}} F(\frac{-1}{6}, \frac{7}{6}, 1 | \frac{\psi+2}{4}) \end{pmatrix}$$

where

Gauss hyp.

$$\boxed{F(a, b, c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad c \notin \{0, -1, -2, -3, \dots\},} \tag{6}$$

$a, b, c.$ $\uparrow$ ψ $\quad z$

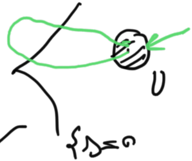
$(a)_n = a(a+1)\cdots(a+n-1)$. Pochhammer Sym

~~$a+1 \quad a+n-1$~~

Three parameter family of elliptic curve

$U \subseteq \mathbb{C}^3 \backslash \{\Delta = 0\}$

multivalued hol. function in t_1, t_2, t_3



$$E_t : y^2 = \overbrace{4(x-t_1)^3 - t_2(x-t_1) - t_3}^{P(x)} \qquad y^2 = P(x).$$

$$t \in \mathrm{Spec}(\mathbb{C}[t_1, t_2, t_3, \frac{1}{\Delta}]), \quad \Delta := 27t_3^2 - t_2^3.$$

U

$$\int_{\delta_i} \frac{x^j dx}{y} \qquad i = 1, 2 \quad j = 0, 1$$

$(t_1, t_2, t_3) \in \mathbb{C}^3 \backslash \{\Delta = 0\}.$

hol. functions in t_1, t_2, t_3

Gauss-Manin connection II

$$\begin{pmatrix}\frac{\partial}{\partial t_3}\int\frac{dx}{y}\\ \frac{\partial}{\partial t_3}\int\frac{x\,dx}{y}\end{pmatrix} = \overset{A_3}{\begin{pmatrix} * & * \\ * & *\end{pmatrix}}\begin{pmatrix}\int \frac{dx}{y}\\ \int\frac{x\,dx}{y}\end{pmatrix}$$

Problem

Let $P(x) := 4(x-t_1)^3 + t_2(x-t_1) + t_3$. *We have*

$$A = A_1\,dt_1 + A_2\,dt_2 + A_3\,dt_3$$

$$d\begin{pmatrix}\left(\int\frac{dx}{\sqrt{P(x)}}\right)\\ \left(\int\frac{x\,dx}{\sqrt{P(x)}}\right)\end{pmatrix} = \begin{pmatrix} -\frac{3}{2}t_1\frac{\alpha}{\Delta} - \frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ dt_1 - \frac{1}{6}t_1\frac{d\Delta}{\Delta} - \left(\frac{3}{2}t_1^2 + \frac{1}{8}t_2\right)\frac{\alpha}{\Delta}, & \frac{3}{2}t_1\frac{\alpha}{\Delta} + \frac{1}{12}\frac{d\Delta}{\Delta}\end{pmatrix}\begin{pmatrix}\int\frac{dx}{\sqrt{P(x)}}\\ \int\frac{x\,dx}{\sqrt{P(x)}}\end{pmatrix}$$

3

where

$$\Delta := 27t_3^2 - t_2^3,\ \boxed{\alpha := 3t_3\,dt_2 - 2t_2\,dt_3}.$$

$$d\left(\int_\delta \frac{dx}{y}\right) = \left(\frac{\partial}{\partial t_1}\int\frac{dx}{y}\right)dt_1 + \left(\frac{\partial}{\partial t_2}\int\frac{dx}{y}\right)dt_2 + \cdots$$

1-form.

Classical moduli space

$E_t = E_{t_2, t_3}$

$y^2 = 4x^3 - t_2 x - t_3$

$k^6 y^2 = 4k^6 x^3 - t_2 k^2 x - t_3$

1. Any elliptic curve can be written in the Weierstrass format $E_t : y^2 = 4x^3 - t_2 x - t_3$ $t_2, t_3 \in \mathbb{C}$.
2. We have
$$\boxed{E_{k^{-6} t_2, k^{-4} t_3} \cong E_{t_2, t_3},} \quad \boxed{(x, y) \mapsto (k^2 x, k^3 y).}$$
$k \in \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}.$
3. The j-invariant $j := \frac{t_2^3}{t_2^3 - 27 t_3^2}$ classifies elliptic curves.

$j(t_2, t_3) = j(k^{-4} t_2, k^{-6} t_3).$

Ibiporanga: A new moduli space

Ibiporanga: A new moduli space

$g \in \mathbb{P}^1 \setminus \{\infty\}$ $/\mathbb{C}$

$T :=$ moduli of (E, α, ω),

E elliptic curve, $\alpha, \omega \in \underbrace{H^1_{dR}(E)}$
(E, O) $O \in E$ basis

C^∞ Grothendieck.

α rep. by a hol. 1-form. in E

$H^1_{dR}(E) \times H^1_{dR}(E) \to \mathbb{C}$
$\langle \alpha, \omega \rangle = 1$.

Vector fields and differential forms

$$E_t : \quad y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3$$

$$F^1 \ni \alpha = \left[\frac{dx}{y}\right], \ \omega \left[\frac{x\,dx}{y}\right] \in H^1_{dR}(E_t).$$

$$T = \mathbb{C}^3 \setminus \{\Delta = 0\}$$

$$\rotatebox{90}{$\in$}$$

$$(t_1, t_2, t_3)$$

Gauss-Manin connection along a vector field

Proposition

There are unique vector fields e, h, f in T such that the Gauss-Manin connection matrix matrix A along these vector fields has the form:

$$A_h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_e = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We can compute explicit expressions for e, f, h.

$$f = -(t_1^2 - \frac{1}{12}t_2)\frac{\partial}{\partial t_1} - (4t_1t_2 - 6t_3)\frac{\partial}{\partial t_2} - (6t_1t_3 - \frac{1}{3}t_2^2)\frac{\partial}{\partial t_3}, \tag{7}$$

$$h = -6t_3\frac{\partial}{\partial t_3} - 4t_2\frac{\partial}{\partial t_2} - 2t_1\frac{\partial}{\partial t_1}, \ e = \frac{\partial}{\partial t_1}.$$

The Lie algebra $\mathfrak{sl}_2$

The $\mathbb{C}$-vector space generated by h, e, f equipped with the classical bracket of vector fields is isomorphic to the Lie Algebra $\mathfrak{sl}_2$, and hence, it gives a representation of $\mathfrak{sl}_2$ in the polynomial ring $\mathbb{Q}[t_1, t_2, t_3]$ which is infinite dimensional.

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{8}$$

Vector field as an ODE

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Quasi-modular forms attached to elliptic curves, I.
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