On reconstructing subvarieties from their periods

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The talk is based on my two books [Mov21, MV21] on Hodge theory and a joint article with [MS21].

Fermat surface of degree 12

[Mov21, Chapter 15] and Aoki-Shioda 1983, 1987

$$X \subset \mathbb{P}^3 : \quad x_0^{12} + x_1^{12} + x_2^{12} + x_3^{12} = 0$$

$$\rho_{\mathbb{P}^1} := 332 < \rho(X) = 644 < h^{11} := 892,$$

1. Lines

$$x_0 - \zeta_{24} x_1 = x_2 - \zeta_{24} x_3 = 0$$

2. Aoki-Shioda curves: you can produce more curves using

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\zeta_3 y+\zeta_3^2 z)(x+\zeta_3^2 y+\zeta_3 z).$$

Confession of a master

But the whole program [Grothendieck's program on how to prove the Weil conjectures] relied on finding enough algebraic cycles on algebraic varieties, and on this question one has made essentially no progress since the 1970s.... For the proposed definition [of Grothendieck on a category of pure motives] to be viable, one needs the existence of “enough” algebraic cycles. On this question almost no progress has been made, (P. Deligne in an inteview 2014)....la construction de cycles algébriques intéressants, les progrès ont été maigres, (P. Deligne in 1994).

Computational algebraic geometry

1. Membership in an ideal
2. Constructing line bundles vs. constructing sections of line bundles.

Singular homology and de Rham cohomology

Let X be a C^∞ compact oriented manifold.

$$\int_\delta \omega, \quad \delta \in H_n(X, \mathbb{Z}),\ \omega \in H^n_{\mathrm{dR}}(X)$$

Homology class of algebraic cycles and algebraic de Rham cohomology

Let X be a smooth projective variety over $\mathrm{k} \subset \mathbb{C}$ and $Z \subset X$ a subvariety of dimension $\frac{n}{2}$ and defined over $\bar{\mathrm{k}}$:

1. Homology class of Z: $[Z] \in H_n(X, \mathbb{Z})$.
2. Algebraic de Rham cohomology $H^n(X/\mathrm{k})$.

Hodge cycles and Hodge conjecture

Definition

A homological cycle $\delta \in H_n(X, \mathbb{Z})$ is called a Hodge cycle if

$$\int_\delta F^{\frac{n}{2}+1} H^n_{\mathrm{dR}}(X) = 0.$$

A simple but fundamental observation

Theorem (P. Deligne 1982)

For X/k and $Z/\bar{\mathrm{k}}$ as before, we have

$$\frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}}\int_{\delta}\omega \in \bar{\mathrm{k}}, \quad \forall\omega \in H^n(X/\mathrm{k}). \tag{1}$$

where $\delta = [Z]$. Let $\mathrm{k}_{\delta} :=$ is the smallest field containing k and its periods.

Proposition (M-, Sertöz)

For any algebraic cycle $\delta \in H_n(X, \mathbb{Z})$, there are subvarieties $Z_1, Z_2, \ldots, Z_s \subset X$ of pure dimension $\frac{n}{2}$ defined over k_δ and integers $a_0, \ldots, a_s,\ a_0 > 0$ such that

$$a_0\delta = a_1[Z_1] + a_2[Z_2] + \cdots + a_s[Z_s]. \tag{2}$$

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Artinian-Gorenstein ring attached to a Hodge cycle

Let $X \subset \mathbb{P}^{n+1}$ be a snooth hypersurface.

Definition

For every Hodge cycle $\delta \in H_n(X, \mathbb{Z})$ we define its associated Artinian Gorenstein ideal as the homogeneous ideal

$$I(\delta)_a := \left\{ Q \in \mathbb{C}[x]_a \middle| \int_\delta \mathrm{Resi}\left(\frac{QP\Omega}{F^{\frac{n}{2}+1}}\right) = 0, \ \ \forall P \in \mathbb{C}[x]_{(n+2)(d-2)-a} \right\}.$$

By definition $I(\delta)_m = \mathbb{C}[x]_m$ for all $m \geq (n+2)(d-2)+1$.

Proposition

If $\delta = [Z]$ and Z is given by the ideal I_Z then

$$I_Z \subset I(\delta). \tag{3}$$

Computing algebraic cycles is difficult, but for computing δ and $I(\delta)$ we have algorithms.

Proposition (M-, Sertöz)

The quartic surface

$$X = Z(5x^4 - 4x^2zw + 8y^4 - 5z^4 + 4zw^3) \subset \mathbb{P}^3$$

has Picard number 14*. In fact, X contains* 102 *conics and* 4 *lines whose classes generate the Picard group of X.*

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A course in Hodge theory. With emphasis on multiple integrals.
Somerville, MA: International Press, 2021.

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