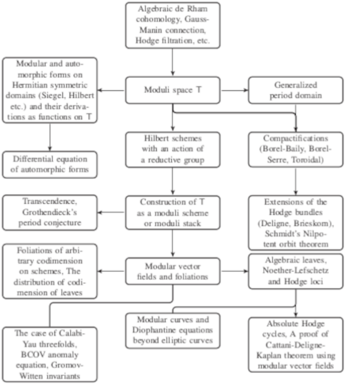
Ibiporanga: A moduli space for differential equations of automorphic forms

Hossein Movasati

IMPA,
www.impa.br/∼hossein/

“There should be almost a book, or at least a nice article, on the history of moduli and moduli spaces; and for that, one should go back to the theory of elliptic functions. A. Weil, Collected Papers. Volume II. page 545.

Labyrinth of This Book



Modular and Automorphic Forms & Beyond, Monographs in Number Theory, World Scientific, to be published November 2021.

Why do we have to generalize modular/automorphic forms?

A consequence of Ehressman's fibaration theorem (1947)

Let $\pi :\to$ be a family of smooth projective varieties, that is, $\subset^N \times$, π is the projection on and π has no singular points and values.

Theorem

Algebraic de Rham cohomology and Hodge filtraton after Grothendieck (1966) and Deligne (1970)

A proposition which seemed to me trivial!

Proposition

Let $X_t, t \in$ be a family of smooth projective varieties and let X, X_0 be two regular fibers of this family. We have an isomorphism

$$(H_(^* X), F^*, \cup, \theta) \overset{\alpha}{\simeq} (H_(^* X_0), F_0^*, \cup, \theta_0) \tag{1}$$

A proposition which seemed to me trivial!

Proposition

Let $X_t, t \in$ be a family of smooth projective varieties and let X, X_0 be two regular fibers of this family. We have an isomorphism

$$(H^*_(X), F^*, \cup, \theta) \overset{\alpha}{\simeq} (H^*_(X_0), F^*_0, \cup, \theta_0) \quad (1)$$

Proof.

...... SGA3-II: “IX 3 uses cohomology to obtain infinitesimal statements. XI 4 proves representability of the functor *M* of subgroupschemes of multiplicative type. XI 5 puts it all together....." (P. Deligne, personal communication, August 14, 2019) □

Itacuatiara(written or painted stone): Varieties enhanced with ...

Ibiporanga: The moduli of projective varieties enhanced with ...

Wishful thinking

The moduli space is a quasi-affine variety over and we have a universal family over it!

Elliptic curves after [Weierstrass, Klein, Fricke, Deligne, Katz .....(M. 2008, 2012)]

$$: y^2 - 4(x - t_1)^3 + t_2(x - t_1) + t_3 = 0, \;\; \alpha = \left[\frac{dx}{y}\right], \;\; \omega = \left[\frac{xdx}{y}\right] \quad (2)$$

$$:= \left(\left[t_1, t_2, t_3, \frac{1}{27t_3^2 - t_2^3}\right]\right).$$

Principally polarized abelian varieties

Theorem (....., Fonseca 2019)

The moduli space for polarized abelian varieties is quasi-affine over .

Principally polarized abelian varieties

Theorem (....., Fonseca 2019)

The moduli space for polarized abelian varieties is quasi-affine over .

Theorem (Igusa 1962)

The algebra of genus 2 *Siegel modular forms is*

$$\mathbb{C}\left[\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}, \mathcal{C}_{35}\right] / \left(\mathcal{C}_{35}^2 = \mathrm{P}(\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12})\right),$$

where $\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}, \mathcal{C}_{35}$ *are generalization of Eisenstein sereis and* P *is an explicit polynomial.*

Principally polarized abelian varieties

Theorem (....., Fonseca 2019)

The moduli space for polarized abelian varieties is quasi-affine over .

Theorem (Igusa 1962)

The algebra of genus 2 Siegel modular forms is

$$\mathbb{C}\left[\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}, \mathcal{C}_{35}\right] / \left(\mathcal{C}_{35}^2 = \mathrm{P}(\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12})\right),$$

where $\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}, \mathcal{C}_{35}$ are generalization of Eisenstein sereis and P is an explicit polynomial.

Theorem (Cao-M.-Yau, 2021)

Calabi-Yau varieties

Theorem (Viehweg 1995)

The coarse moduli space of polarized smooth Calabi-Yau varieties exists as a quasi-projective variety.

Calabi-Yau varieties

Theorem (Viehweg 1995)

The coarse moduli space of polarized smooth Calabi-Yau varieties exists as a quasi-projective variety.

Theorem (Bogomolov-Tian-Todorov 1987)

The above moduli space is smooth of dimension $h := h^{n-1,1}$.

Calabi-Yau varieties

Theorem (Viehweg 1995)

The coarse moduli space of polarized smooth Calabi-Yau varieties exists as a quasi-projective variety.

Theorem (Bogomolov-Tian-Todorov 1987)

The above moduli space is smooth of dimension $h := h^{n-1,1}$.

Corollary (M. 2021)

is a smooth algebraic variety (of dimension $\frac{3h^2+7h+4}{2}$ *for* $n = 3$*).*

M.-Nikdelan 2018 for Dowrk family
Alim-M.-Yau-Scheidegger 2017, for arbitrary CY3.

Ibiporanga with mixed Hodge structures

Considering $H^*(X, Y)$, X and elliptic curve and $Y = \{O, P\}$...

Theorem (Cao-M.-Villaflor)

$$= [a, b, c, t_1, t_2, \frac{1}{\Delta}]$$

where $\Delta = 27t_3^2 - t_2^3$ *and* $t_3 = 4a^3 - t_2 a - b^2$*. Moreover, admits the universal family given by*

$$X = \{y^2 = 4x^3 - t_2 x - t_3\}, \quad Y = \{O, P\}, \; O = (0:1:0), \; P = (a:b:1$$

and the frame of differential forms

$$d\left(\frac{x-a}{x}\right), \; \frac{dx}{y}, \; \left(c - \frac{b}{2a}\right) d\left(\frac{x-a}{x}\right) + t_1 \frac{dx}{y} + \frac{xdx}{y} - d\left(\frac{y}{2x}\right).$$

Mondromy group

Fix a fiber $X = X_0$.

$$\begin{aligned}\Gamma \quad &\subset \quad (H^*(X_0,), \ \cup, \ \theta_0 \ ) \\ &= \quad \Big\{ \ A : H^*(X_0,) \to H^*(X_0,) \ \Big| \\ &\qquad \text{-linear, respects the homology grading,} \\ &\qquad \forall x, y \in H^*(X_0,), \ Ax \cup Ay = A(x \cup y), \ \ A(\theta_0) = \theta_0 \}.\end{aligned}$$

Monodromy covering

Let˜be the moduli of (X, δ, θ), where X is a projective variety, $\theta \in H^2(X,)$ is the homology class induced by a hyperplane section and

$$\delta : (H^*(X,), \cup, \theta) \cong (H^*(X_0,), \cup, \theta_0)$$

is an isomorphism. We denote by a connected component of˜ which contains the triple $(X_0, \delta_0, \theta_0)$, where δ_0 is the identity map, and call it the monodromy covering.

A bridge between two worlds: $\Gamma - map$

I would go to work either on foot or by tram (called the bonde in Brazilian Portuguese). A servant-girl was employed to clean the department and especially to serve every professor, both before and after classes, as many cafezinhos (small cups of Brazilian-style coffee) as he wished. When I visited Brazil in 1966,1 observed that this excellent custom had been discontinued, and that the mathematics department was now housed with the university's other departments in huge buildings on a sprawling campus devoid of character.

I would go to work either on foot or by tram (called the bonde in Brazilian Portuguese). A servant-girl was employed to clean the department and especially to serve every professor, both before and after classes, as many cafezinhos (small cups of Brazilian-style coffee) as he wished. When I visited Brazil in 1966,1 observed that this excellent custom had been discontinued, and that the mathematics department was now housed with the university's other departments in huge buildings on a sprawling campus devoid of character. (André Weil in The Apprenticeship of a Mathematician (1992), page 188)