Differential Equations of Modular forms, 29/01/2020
Loughborough, UK. Ref: Joint work with S.T Yau
Jin Cao
weierstrass CY-modular forms.

Theorem (~~N. Katz 1979?~~ M. 2008, 2012). Let T be the moduli space of triples (E, α_1, α_2), E elliptic curve$/k$, $\alpha_1, \alpha_2 \in H^1_{dR}(E/k)$, α_1 hol. 1-form, $\mathrm{Tr}(\alpha_1 \cup \alpha_2) = 1$. There is a unique vector field R in T such that

$$\begin{pmatrix} \nabla_R \alpha_1 \\ \nabla_R \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

where $\nabla : H^1_{dR}(\mathcal{X}/T) \to \Omega^1_T \otimes H^1_{dR}(\mathcal{X}/T)$ is the Gauss-Manin connection of the universal family over T with global sections of α_1, α_2 of $H^1_{dR}(\mathcal{X}/T)$. We have

$$T := \mathrm{Spec}\left(k\left[t_1, t_2, t_3, \frac{1}{27t_3^2 - t_2^3}\right]\right)$$

$$\mathcal{X}/T \quad y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3$$

$$\alpha_1 = \left[\frac{dx}{y}\right], \quad \alpha_2 = \left[\frac{x\,dx}{y}\right]$$

and

$$R = \left(t_1^2 - \frac{1}{12}t_2\right)\frac{\partial}{\partial t_1} +$$
$$\left(4t_1t_2 - 6t_3\right)\frac{\partial}{\partial t_2}$$
$$\left(6t_1t_3 - \frac{1}{3}t_2^2\right)\frac{\partial}{\partial t_3}$$

A solution of $R = \left(\left(\frac{2\pi i}{12}\right)E_2(\tau),\ 12\left(\frac{2\pi i}{12}\right)^2 E_4(\tau),\ 8\left(\frac{2\pi i}{12}\right.\right.$

Halphen diff. Equation (M. 2012)

$$\dot{t}_1 = \frac{a-1}{a+b+c-2}(t_1t_2+t_1t_3-t_2t_3)+\frac{b+c}{a+b+c-2}t_1^2$$

$\dot{t}_2 = \dots, \dot{t}_3 = \dots$ $\quad a,b,c\in\mathbb{Q}$.

Families of curves $\quad y = * (x-t_1)^a(x-t_2)^b(x-t_3)^c$.

Siegel Modular Forms of genus two:

$$\begin{pmatrix} 1 & & \\ 0 & & \\ 1 & & \\ 0 & & \end{pmatrix}, \begin{pmatrix} & -1 & 1 \\ & 1 & -1 \end{pmatrix}, \begin{pmatrix} & & & \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} & 0 & 0 \\ & 0 & -1 \end{pmatrix}$$

$\Gamma\subseteq Sp(4,\mathbb{Z})$ generated by above matrices.

$[Sp(4,\mathbb{Z}):\Gamma]=6$

$\mathbb{H}_2 = \left\{\tau = \begin{pmatrix}\tau_1 & \tau_3\\ \tau_3 & \tau_1\end{pmatrix} \,\middle|\, \mathrm{Im}(\tau)>0\right\}$. Siegel domain

$f((a\tau+b)(c\tau+d)^{-1})\cdot \det(c\tau+d)^{-k} = f(\tau)$

$\forall \tau\in\mathbb{H}_2 \quad \forall \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in Sp(4,\mathbb{Z})$

Theorem (J. Cao, S.T. Yau). There are 153 meromorphic functions

$X_i: \mathbb{H}_2 \dashrightarrow \mathbb{C}$, $i=1,2,\dots,153$ with possible poles along

$\tau_2=0$. s.t.

1. $X_1, X_2, \dots, X_5$ are Siegel m.f. for Γ $X_4^2 = X_5X_3$, X_5 hol.
2. The ideal I of polynomial relations between X_i is defined over $\mathbb{Q}$, $X_4^2 - X_5X_3\in I$
3. $\qquad \mathrm{Spec}(\mathbb{Q}[X]/I) \xrightarrow[\text{open}]{} \mathbb{P}^{6,8,10,3,3,3,3,1,1,1,1,\dots}$ homog.

complement is given by a degree 4 polynomial

4. $X_5 \dfrac{\partial X_i}{\partial \tau_k}\in \mathbb{Q}[X]$

$$E_k(\tau) := 1 + \frac{2k}{B_k} \sum \sigma_{2k-1}(n)\, e^{2\pi i n \tau} \qquad k=2,4,6$$

$$(t_1, t_2, t_3) := \left(\left(\frac{2\pi i}{12}\right), 12\left(\frac{2\pi i}{12}\right)^2 E_4(\tau), 8\cdot\left(\frac{2\pi i}{12}\right)^3 E_6 \right.$$

satisfies
$$\begin{cases} \dot{t}_1 = \left(t_1^2 - \frac{1}{12} t_2\right) \\ \dot{t}_2 = \left(4t_1 t_2 - 6t_3\right) \\ \dot{t}_3 = 6t_1 t_3 - \frac{1}{3} t_2^2 \end{cases}$$

Geometrization:

$$T := \mathrm{Proj}\left(\frac{\mathbb{C}[t_2, t_3, t_4, t_5, s_{11}, s_{21}, s_{31}, s_{41}, s_{12}, s_{22}, s_{32}, s_{42}, \frac{1}{t_5 \Delta (s_{11}s_{22} - s_{12}s_{21})}}{\langle s_{42}s_{21} - s_{41}s_{22} + s_{32}s_{11} - s_{31}s_{12} - \frac{t_2}{4} \rangle} \right.$$

$$\subseteq \mathbb{P}^{4,6,8,10,3,3,3,3,1,1,1,1}$$

we construct ~~v~~vector fields R_1, R_2, R_3 corresponding to $\frac{\partial}{\partial \tau_i}$ $i=1,2,3.$

Classical Igusa's invariants.

$$E_4 := B,\ E_6 := 4AB - 3C,\ \chi_{10} := D,\ \chi_{35} = D^2 E$$

$$A = -3t_2^2 - 20t_4$$
$$B := -3t_2 t_3^2 + 9t_2^2 t_4 - 20t_4^2 + 75 t_3 t_5, \ldots$$

Resnikov's diff. equations

$$\partial = \frac{\partial}{\partial \tau_1} \cdot \frac{\partial}{\partial \tau_2} - \frac{1}{4} \frac{\partial^2}{\partial \tau_3^2}$$

$$D(f) = \frac{4\omega - 1}{4\omega^2} f \partial f + \frac{1 - 2\omega}{8\omega^2} \partial f^2$$

ω weight of f.

$$D E_4 = \frac{984375}{1024} \chi_{10}, \ldots$$