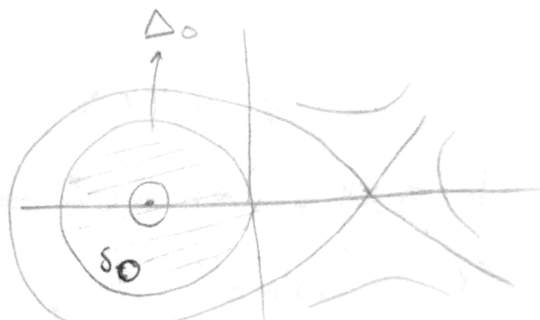
What happens when a period vanishes?
Plymouth, 27/01/2020. Ref. A course in Hodge Theory with emphsis on multiple integrals.
Period ~ elliptic integral - abelian integrals ~ multiple integrals.

$\int_\delta \omega$ $\omega \to$ algebraic dif. form
$\delta \to$ topological object.

Planar differential equations:
Hamiltonian system

$$\begin{cases} \dot{x} = f_y \\ \dot{y} = -f_x \end{cases} \qquad f = y^2 - x^3 + 3x$$

Solutions of level surfaces of f



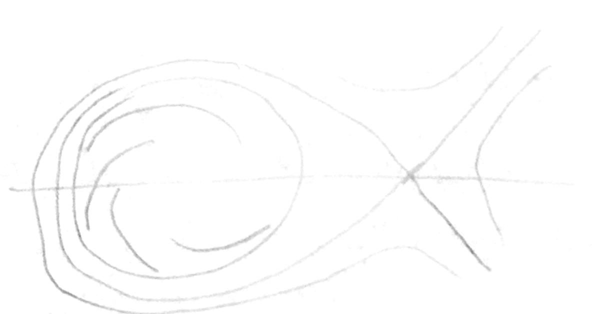
Perturb
$$\begin{cases} \dot{x} = -f_y + \varepsilon P(x,y) \\ \dot{y} = -f_x + \varepsilon Q(x,y) \end{cases}$$

If $\int_{\delta_0} P dy - Q dx = 0$ and $\int_{\delta_t} \neq 0$ then for $0 < \varepsilon \ll 0$, δ_0 persists in the deformed foliation.

$$\begin{cases} \dot{x} = 2y + \varepsilon \frac{x^2}{2} \\ \dot{y} = 3x^2 - 2 + \varepsilon s y \end{cases} \qquad \int_{\delta_0} \left(\frac{x^2}{2} dy - sy\, dx \right) = 0 \quad \Rightarrow$$

$$s = \frac{-\int_{\Delta_0} x\, dx \wedge dy}{\int_{\Delta_0} dx \wedge dy} = \frac{5}{7} \frac{\Gamma(\frac{5}{12})\Gamma(\frac{13}{12})}{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})} \sim 0.9025$$

Hilbert 16-th problem:
There is a number $N(d)$, $d \in \mathbb{N}$ s.t any diff. equation of degree $\leq d$ has $\leq N(d)$ limit cycles

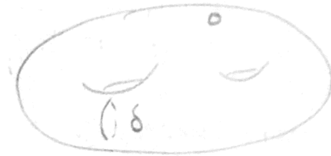


Arnold-Hilbert Infinitesimal.

Let us complexify

$X := \{ (x,y) \in \mathbb{C}^2 \mid f = t \} \simeq$ 

$t \neq \pm 2$

Let us consider a curve of genus 2. 

For instance $X: y^2 = P(x)$, $\deg P = 5$ with

5 different roots.

It has a basis of hol. differential 1-forms: $\frac{dx}{y}, \frac{x\,dx}{y}$

$\delta \in H_1(X, \mathbb{Z})$

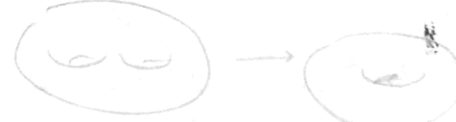
$$\int_\delta a\frac{dx}{y} + \frac{x\,dx}{y} = 0 \implies a = \frac{-\int_\delta \frac{x\,dx}{y}}{\int_\delta \frac{dx}{y}}$$

a is usually a transc. number.

Abelian subvariety theorem due to Wüstholz-Masser $\implies$

If X is a genus two curves, ω a hol. diff. 1-form, both defined over $\overline{\mathbb{Q}}$ and $\delta \in H_1(X, \mathbb{Z})$ s.t that $\int_\delta \omega = 0$

Then there is an elliptic curve E and a non-constant morphism

$$X \longrightarrow E$$ 

all defined over $\overline{\mathbb{Q}}$.

Higher dimensional periods.

E. Picard 1897-1906 two books.

G. Simart

$F(x,y,z,w)$ homog. poly of degree 4

$X: \{ [x:y:z:w] \in \mathbb{P}^3 \mid F(x,y,z,w) = 0 \} \subseteq \mathbb{P}^3$ smooth

$$\left\{ \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = \frac{\partial F}{\partial w} = 0 \right\} = \{0\}$$

Let us restrict to the affine chart $\{w=1\}$

$$f := F(x,y,z,1)$$

$$U := \{ (x,y,z) \in \mathbb{C}^3 \mid f(x,y,z) = 0 \}$$

Two dimensional periods

$$\int_\delta \frac{P\, dx \wedge dy}{f_z} \qquad P \text{ polynomial}, \quad \delta \in H_2(U, \mathbb{Z})$$

- $\omega^{2,0} = \frac{dx \wedge dy}{f_z}$ up to multiplication with a constant is a unique hol. 2-form in X (Recall that $\deg(X) = 4$).

- $\{z=0\} \pitchfork X \Rightarrow \operatorname{rank} H_2(U, \mathbb{Z}) = (d-1)^{n+1} = 3^3 = 27$.

- Long exact sequence of $(X, X/U)$

$$\to H_1(Y) \to H_2(U) \to H_2(X) \to H_0(Y) \to$$
$$\mathbb{Z}^6 \qquad \mathbb{Z}^{27} \qquad \mathbb{Z}^{22} \qquad \mathbb{Z}$$

A glance at Lefschetz (1,1) theorem

$$\int_\delta \omega^{2,0} = 0 \text{ for some } \delta \in H_2(X, \mathbb{Z}) \Longrightarrow$$

$\exists$ curves $C_1, C_2, \dots C_k$ inside X and $n_1, n_2 \dots n_k \in \mathbb{Z}$ s.t

$$\delta = \sum n_i [C_i]$$