Variational Hodge Conjecture and Hodge Loci

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Thus tallk/$\mathbb{C}$, Complex manifold

CY – moduler forms – Arithmetic? → Hecke operators – Isogeny – Algebraic cycles.

$X \longrightarrow T$ a family of smooth projective varieties

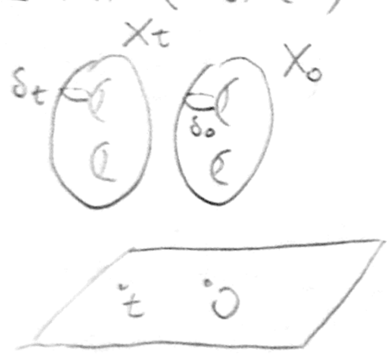
$X_t \subseteq \mathbb{P}^N$, $t\in T$ fiber, Marked point $0 \in T$, $n = \dim_{\mathbb{C}} X_t$

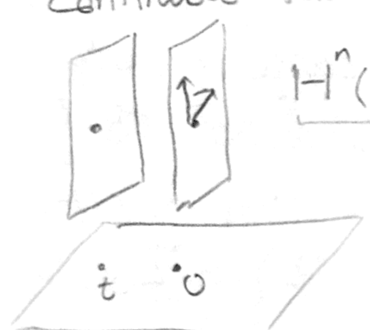
$\delta_0 \in H^{\frac{n}{2},\frac{n}{2}} \cap H^n(X_0, \mathbb{Q})$

is called a Hodge cycle/class

Hodge Conjecture: $\delta_0 = [Z]$, $Z = n_i Z_i$ $n_i \in \mathbb{Q}$, $\dim Z_i = \frac{n}{2}$

$\delta_t \in H^n(X_t, \mathbb{Q})$ flat section of the Gauss–Manin connection continuous family of cycles.





$\underbrace{H^n(X_0, \mathbb{Z})}$ $\subseteq H^n_{dR}(X_0)$

→ replacet with its torision free part

Hodge Locus

$$V_{\delta_0} := \{ t \in (T, 0) \mid \delta_t \text{ is Hodge} \}$$

usuall/analytic topology.

set, analytic variety, analytic space/scheme

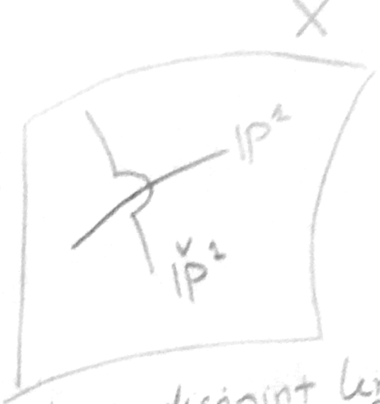
Thm: Cattani–Deligne–Kaplan 1995. V_{δ_0} is a branch of an algebraic variety in T.

VHC*: If the H.C. is true for δ_0 then it is true for all δ_t, $t \in V_{\delta_0}$

CDK thm $\Rightarrow$ VHC* is equivalent to VHC of Grothendieck

AHC (Alternative Hodge): If $\delta_0 = [Z_0]$ then for all $t \in V_{\delta_0}$, $\delta_t = [Z_t]$, wher (X_t, Z_t) is obtained by deformation of (X_0, Z_0).

A.H.C is wrong:



codim deform. space X together with $\mathbb{P}^1 + \check{\mathbb{P}}^1$ $\geqslant 2$

codim 〃 〃 〃 〃 with $\gamma[\mathbb{P}^1] + \check{\gamma}[\check{\mathbb{P}}^1] \in H^2_{dR}(X)$
$\gamma, \check{\gamma} \in \mathbb{Z}$, $\gamma, \check{\gamma} \neq 0$

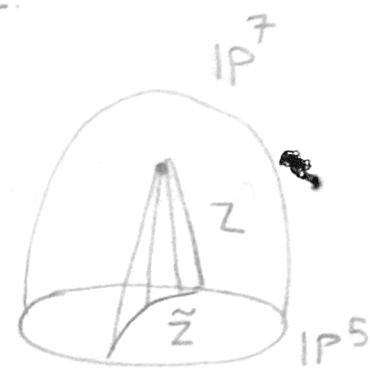
two disjoint lines inside $X \subseteq \mathbb{P}^3$
$\deg X = 4$.

$\Rightarrow$ AHC is wrong for $(X, \mathbb{P}^1 + \check{\mathbb{P}}^1)$.

More examples due to P. Deligne. Book "A course in Hodge Theory" my webpage. Chapter 18

Thm* (S. Bloch 1972): Z_0 local complete intersection and semi-reguler and V_{δ_0} is smooth $\Rightarrow$ A.H.C.

(M. 2019)



$\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ vernoese embedding by deg 2 monomials

Z = the cone over the image $\tilde{Z}$ of the veronese embedding

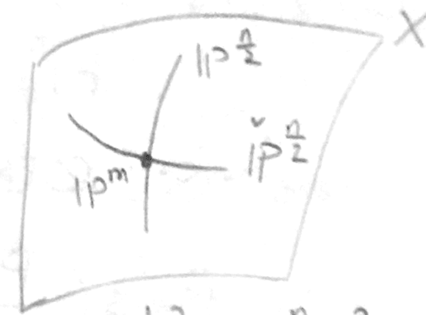
cubic 6-fold $X \subseteq \mathbb{P}^7$ with $Z \subseteq X$.

codim deformation space of X together with Z $= 10$

〃 〃 〃 〃 〃 $[Z]$ $\leqslant 8 = h^{31}(X)$.

$\Rightarrow$ AHC is wrong for (X, Z).

X smooth hyp of deg. d and dim n.
$T := \mathbb{C}[x_0, x_1, \dots, x_{n+1}]_d \setminus$ singular hypersurfaces.



$m = -1, 2, \dots, \frac{n}{2} - 2$

Thm 1 (M. + Villaflor, 2018, Villaflor 2019).
if $m < \frac{n}{2} - \frac{d}{d-2}$ then

AHC is true for generic

$$(X, \ r\mathbb{P}^{\frac{n}{2}} + \check{r}\,\check{\mathbb{P}}^{\frac{n}{2}})$$

$$r, \check{r} \in \mathbb{Z}, \ r, \check{r} \neq 0$$

$m = \frac{n}{2} - 1$: $\mathbb{P}^{\frac{n}{2}} + \check{\mathbb{P}}^{\frac{n}{2}}$ can be deformed into a smooth complete intersection of type $(1, 1, \dots, 1, 2)$.

$d \geqslant 5$: No counterexample to A.H.C
$d = 4$: ...
$d = 3$: $m = \frac{n}{2} - 2, \ \frac{n}{2} - 3$

After many months of writing computer codes + running computer for months

Conj*: AHC is wrong for $(X, \ r\mathbb{P}^{\frac{n}{2}} + \check{r}\,\check{\mathbb{P}}^{\frac{n}{2}})$ as above with

$$d = 3, \quad m = \frac{n}{2} - 3, \quad n = 4, 6, 8, 10, 12, \dots$$

trivial.

Why?
1. It makes Hodge Theory more effective/computational
2. I want to produce more and more Hodge cycles
3. Counter example to the Hodge conj.?

Residue map: long exact sequence of $(\mathbb{P}^{n+1}, \mathbb{P}^{n+1} \setminus X)$:

$$H^{n+1}_{dR}(\mathbb{P}^{n+1}) \to H^{n+1}_{dR}(\mathbb{P}^{n+1} \setminus X) \xrightarrow{Res} H^n_{dR}(X) \xrightarrow{\alpha} H^n_{dR}(\mathbb{P}^{n+1}) \to$$

$$H^{n+1}_{dR}(\mathbb{P}^{n+1}) = 0, \qquad H^n_{dR}(\mathbb{P}^{n+1}) \cong \mathbb{C}$$

$$H^n_{dR}(X)_0 := \mathrm{Im}(Res) = \ker(\alpha)$$

Griffiths 1969:

$$H^n_{dR}(X)_0 \xrightarrow{\sim} \bigoplus_{k=1}^{n+1} S_{kd-n-2}$$

where $S = \mathbb{C}[x]/\mathrm{Jacob}(f)$

$$\mathrm{Res}\left(\frac{P\Omega}{f^k}\right) \longleftarrow P \in S_{kd-n-2}$$

where $\Omega: \sum_{i=0}^{n+1} (-1)^i x_i \widehat{dx_i}$

$$F^{\frac{n}{2}+1} H^n_{dR}(X)_0 \xrightarrow[\sim]{*} \bigoplus_{k=1}^{\frac{n}{2}} S_{kd-n-2}$$

Let $P_1, P_2, \dots P_a$ be a basis of the right hand side of $*$

$$f_i := \int_{\delta_t} \mathrm{Res}\left(\frac{P_i\Omega}{f^{k_i}}\right) \in \mathcal{O}_{T,0}$$

The ideal of V_{δ_0} : $\langle f_1, f_2, \dots, f_a\rangle \subseteq \mathcal{O}_{T,0}$.

$$\mathcal{O}_{V_{\delta_0}} := \frac{\mathcal{O}_{T,0}}{\langle f_1, f_2, \dots f_a\rangle} \quad \text{might be non-reduced}$$

Let $\check{T} \subseteq T$ be the parameter space of smooth hyp. containing $\mathbb{P}^{\frac{n}{2}}$, $\check{\mathbb{P}}^{\frac{n}{2}} \subseteq \mathbb{P}^{n+1}$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$

Proof of Thm 1: $\mathrm{codim}\, \check{T} = \mathrm{codim}\, T_0 V_{\delta_0}$

↳ Fermat point

First evidence for conj*:

$$\mathrm{codim}\, \check{T} > \mathrm{codim}\, T_0 V_{\delta_0} \qquad n = 4, 6, 8, 10, 12$$

computing tang. space $T_0 V_{\delta_0}$: IVHS Griffiths & coauthors 1980
+ computing the cohomology class of algebraic cycles (M. + Villaflor)

V_{δ_0} is an analytic scheme : this is not enough

$$X_t: x_0^d + x_1^d + \cdots + x_{n+1}^d - \sum_{\deg(x^\alpha)=d} t_\alpha x^\alpha$$

$$x^\alpha = x_0^{\alpha_0} \cdots x_{n+1}^{\alpha_{n+1}} \qquad t = (\ldots, t_\alpha, \ldots) \in \mathbb{C}^M$$

$$Z_0 \cong \mathbb{P}^{\frac{n}{2}} : x_0 - \xi_0 x_1 = \cdots = x_n - \xi_n x_{n+1} = 0$$

$$\delta_0 = [Z_0] \qquad \xi_i^d = -1, \quad i = 0, 2, 4, \ldots, n.$$

Thm (Section 18.5, Hodge Theory Book) A closed formula for the Taylor series of $\int_{\delta_t} \left(\frac{x^\beta \Omega}{f^k}\right)$ = convergent series in $t \in \mathbb{C}^M$ with coeff. in $\mathbb{Q}(\xi)$, $\xi^{2d} = 1$, primitive.

Evidence to conj*:

$V_{\delta_0}^N$ the scheme obtained by truncating the power series of f_i at degree N

$V_{\delta_0}^1$ = The tangent space of V_{δ_0}.

Thm: For $\delta_0 = [\gamma \mathbb{P}^{\frac{n}{2}} + \check{\gamma}\, \check{\mathbb{P}}^{\frac{n}{2}}]$, $\gamma, \check{\gamma} \in \mathbb{Z}$, $0 < |\gamma|, |\check{\gamma}| \leq 10$

$\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^{\frac{n}{2}-3}$ inside Fermat, $V_{\delta_0}^N$ is smooth.

$n =$	4	6	8	10	$\boxed{12}$
$N =$	∞	14	6	4	$\boxed{3}$
$M =$	2	8	20	39	$\boxed{66}$

1. Increasing the swap memory $\geq$ 34GB
2. 24 days of computer running.

why to invest on conj*.

n	min	CS	my Hodge loci	max
4	1	1	1	1
6	4	6	7	8
8	10	16	19	45
10	20	32	38	220
12	35	55	65	364

components of Hodge Locus with small codim. parameterize (X, Z) with a <u>simple</u> alg. cycle Z