B-model of mirror symmetry for compact non-rigid Calabi-Yau manifolds

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Abstract:

In B-model of mirror symmetry, period manipulations play an important role for computing the Gromov-Witten invariants of the A-model. This requires computing power series of periods, finding a maximal unipotent monodromy, mirror map etc. In this talk I will present a purely algebraic version of such computations for Calabi-Yau varieties of arbitrary dimension. It involves a construction of the moduli space of enhanced Calabi-Yau varieties and modular vector fields on it. This will give us an algebraic BCOV anomaly equation and will eventually lead us to the the theory of Calabi-Yau modular forms.

Inspired by and have used the works of:

Candelas et. al 1991, Bershadsky-Cecotti-Ooguri-Vafa 1994, Huang-Klemm-Quackenbush 2009, Yamaguchi-Yau 2004 and many works of D. van Straten, M. Alim, E. Scheidegger, B. Haghighat, Ch. Doran, Sh. Hosono, B. Lian, S.-T. Yau and ....

Period manipulation in B-model:

Most of the time we do not need to know the $CY3$ geometry. We only need to know the Picard-Fuchs equation/system of a holomorphic $(3,0)$-form. For mirror quintic $X_z, z \in \mathbb{P}^1$ this is:

$$\theta^4 - z(\theta + \frac{1}{5})(\theta + \frac{2}{5})(\theta + \frac{3}{5})(\theta + \frac{4}{5}) = 0, \quad \theta = z\frac{\partial}{\partial z}. \qquad (1)$$

A basis of the solution space of (1) is given by:

$$\psi_i(z) = \frac{1}{i!}\frac{\partial^i}{\partial \epsilon^i}(5^{-5\epsilon}F(\epsilon, z)), \quad i = 0, 1, 2, 3,$$

where

$$F(\epsilon, z) := \sum_{n=0}^{\infty} \frac{(\frac{1}{5}+\epsilon)_n(\frac{2}{5}+\epsilon)_n(\frac{3}{5}+\epsilon)_n(\frac{4}{5}+\epsilon)_n}{(1+\epsilon)_n^4} z^{\epsilon+n}$$

and $(a)_n := a(a+1)\cdots(a+n-1)$ for $n > 0$ and $(a)_0 := 1$.

Integration of four differential forms over four cycles

We use the base change

$$\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 5 & \frac{5}{2} & -\frac{25}{12} \\ -5 & 0 & -\frac{25}{12} & 200\frac{\zeta(3)}{(2\pi i)^3} \end{pmatrix} \begin{pmatrix} \frac{1}{5^4}\psi_3 \\ \frac{2\pi i}{5^4}\psi_2 \\ \frac{(2\pi i)^2}{5^4}\psi_1 \\ \frac{(2\pi i)^3}{5^4}\psi_0 \end{pmatrix}.$$

We have $x_{i1} = \int_{\delta_i} \eta, \quad i = 1, 2, 3, 4$, where η is a holomorphic three form on X_z and $\delta_i \in H_3(X_z, \mathbb{Z}), \; i = 1, 2, 3, 4$ is a symplectic basis.

Mirror map:

$$\tau_0 := \frac{x_{11}}{x_{21}}, \quad q := e^{2\pi i \tau_0},$$

16 **periods:**

$$x_{ij} := \theta^{j-1} x_{i1}, \quad i, j = 1, 2, 3, 4.$$

Seven holomorphic quantities at MUM:

The seven functions

$$\begin{aligned}
t_0 &= x_{21}\,, \\
t_1 &= 5^4 x_{21}\left((6z-1)x_{21} + 5(11z-1)x_{22} + 25(6z-1)x_{23} + 125(z - \right. \\
t_2 &= 5^4 x_{21}^2\left((2z-7)x_{21} + 15(z-1)x_{22} + 25(z-1)x_{23}\right), \\
t_3 &= 5^4 x_{21}^3\left((z-6)x_{21} + 5(z-1)x_{22}\right), \\
t_4 &= z x_{21}^5, \\
t_5 &= 5^5(z-1)x_{21}^2\left(x_{12}x_{21} - x_{11}x_{22}\right), \\
t_6 &= 5^5(z-1)x_{21}\left(3(x_{12}x_{21} - x_{11}x_{22}) + 5(x_{13}x_{21} - x_{11}x_{23})\right).
\end{aligned}$$

are holomorphic at $z = 0$ and so there are holomorphic functions h_i defined in some neighborhood of $0 \in \mathbb{C}$ such that

$$t_i = \left(\frac{2\pi i}{5}\right)^{d_i} h_i(e^{2\pi i \tau_0}), \qquad (3)$$

where

$$d_i := 3(i+1),\ i = 0,1,2,3,4,\quad d_5 := 11,\quad d_6 := 8.$$

q-expansion

	q^0	q^1	q^2	q^3	q^4	q^5	q^6
$\frac{1}{24} t_0$	$\frac{1}{120}$	1	175	117625	111784375	126958105626	160715581780591
$\frac{-1}{750} t_1$	$\frac{1}{30}$	3	930	566375	526770000	592132503858	745012928951258
$\frac{-1}{50} t_2$	$\frac{7}{10}$	107	50390	29007975	26014527500	28743493632402	35790559257796542
$\frac{-1}{5} t_3$	$\frac{6}{5}$	71	188330	100324275	86097977000	93009679497426	114266677893238146
$-t_4$	0	-1	170	41475	32183000	32678171250	38612049889554
$\frac{1}{125} t_5$	$-\frac{1}{125}$	15	938	587805	525369650	577718296190	716515428667010
$\frac{1}{25} t_6$	-$\frac{3}{5}$	187	28760	16677425	15028305250	16597280453022	20644227272244012
$\frac{1}{125} t_7$	$-\frac{1}{5}$	13	2860	1855775	1750773750	1981335668498	2502724752660128
$\frac{1}{10} t_8$	$-\frac{1}{50}$	13	6425	6744325	8719953625	12525150549888	19171976431076873
$\frac{1}{10} t_9$	$-\frac{1}{10}$	17	11185	12261425	16166719625	23478405649152	36191848368238417

Yukawa coupling

$$Y = \frac{5^8(t_4 - t_0^5)^2}{t_5^3} \tag{4}$$

$$= (\frac{2\pi i}{5})^{-3} \left(5 + 2875\frac{q}{1-q} + 609250 \cdot 2^3 \frac{q^2}{1-q^2} + \cdots + n_d d^3 \frac{q^d}{1-q^d} \tag{5}\right.$$

Here, n_d is the virtual number of rational curves in a generic quintic threefold. The numbers n_d are also called instanton numbers or BPS degeneracies (Computed for the fist time in 1991 by Candelas et al.)

Modular vector field or GMCD

t_i's satisfy the ordinary differential equation R_0, with $\dot{*} := \frac{\partial *}{\partial \tau_0}$.

$$\begin{cases} \dot{t}_0 = \frac{1}{t_5}(6 \cdot 5^4 t_0^5 + t_0 t_3 - 5^4 t_4) \\ \dot{t}_1 = \frac{1}{t_5}(-5^8 t_0^6 + 5^5 t_0^4 t_1 + 5^8 t_0 t_4 + t_1 t_3) \\ \dot{t}_2 = \frac{1}{t_5}(-3 \cdot 5^9 t_0^7 - 5^4 t_0^5 t_1 + 2 \cdot 5^5 t_0^4 t_2 + 3 \cdot 5^9 t_0^2 t_4 + 5^4 t_1 t_4 + 2 t_2 t_3) \\ \dot{t}_3 = \frac{1}{t_5}(-5^{10} t_0^8 - 5^4 t_0^5 t_2 + 3 \cdot 5^5 t_0^4 t_3 + 5^{10} t_0^3 t_4 + 5^4 t_2 t_4 + 3 t_3^2) \\ \dot{t}_4 = \frac{1}{t_5}(5^6 t_0^4 t_4 + 5 t_3 t_4) \\ \dot{t}_5 = \frac{1}{t_5}(-5^4 t_0^5 t_6 + 3 \cdot 5^5 t_0^4 t_5 + 2 t_3 t_5 + 5^4 t_4 t_6) \\ \dot{t}_6 = \frac{1}{t_5}(3 \cdot 5^5 t_0^4 t_6 - 5^5 t_0^3 t_5 - 2 t_2 t_5 + 3 t_3 t_6) \end{cases} \tag{6}$$

A new moduli space:

Let T be the moduli of pairs $(X, [\alpha_1, \alpha_2, \alpha_3, \alpha_4])$, where X is a mirror quintic Calabi-Yau threefold and

$$\alpha_i \in F^{4-i} \backslash F^{5-i}, \quad i = 1, 2, 3, 4,$$

$$[\langle \alpha_i, \alpha_j \rangle] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

Here, $H^3_{\mathrm{dR}}(X)$ is the third algebraic de Rham cohomology of X, F^i is the i-th piece of the Hodge filtration of $H^3_{\mathrm{dR}}(X)$, $\langle \cdot, \cdot \rangle$ is the intersection form in $H^3_{\mathrm{dR}}(X)$.

Gauss-Manin connection

We construct the universal family $\mathsf{X} \to \mathsf{T}$ together with global sections $\alpha_i, \;\; i = 1, 2, 3, 4$ of the relative algebraic de Rham cohomology $H^3(\mathsf{X}/\mathsf{T})$. Let

$$\nabla : H^3_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \Omega^1_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} H^3_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}),$$

be the algebraic Gauss-Manin connection on $H^3(\mathsf{X}/\mathsf{T})$.

[Mov15, Mov17]

There is a unique vector field R_0 in T such the Gauss-Manin connection of the universal family of mirror quintic Calabi-Yau threefolds over T composed with the vector field R_0, namely ∇_{R_0}, satisfies:

$$\nabla_{\mathsf{R}_0} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \tag{7}$$

for some regular function Y in T. In fact,

$$\mathsf{T} := \mathrm{Spec}(\mathbb{Q}[t_0, t_1, \ldots, t_6, \frac{1}{t_4 t_5 (t_4 - t_0^5)}]), \tag{8}$$

and the vector field R_0 and Y are given as before.

Main goal: A new theory of modular forms

These are not classical modular forms! Modular type functions?
Zagier called classical modular forms for $\mathrm{SL}(2,\mathbb{Z})$:

Elliptic modular forms.

I decided to call these new theories:

CY modular forms

A general theorem: [Mov20]

Let X/T be a universal family of enhanced smooth projective Calabi-Yau varieties of dimension n. There exist unique global vector fields $v_j,\ j = 1, 2, \ldots, \mathrm{h}^{n-1,1}_{\text{prim}}$ in T and unique $\mathrm{h}^{n-i+1,i-1}_{\text{prim}} \times \mathrm{h}^{n-i,i}_{\text{prim}}$ matrices $Y_j^{i-1,i},\quad i = 1, 2, \ldots, n$ with entries as regular functions in T such that

$$\nabla_{v_j}\alpha = \begin{pmatrix} 0 & Y_j^{01} & 0 & \cdots & 0 \\ 0 & 0 & Y_j^{12} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Y_j^{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \alpha$$

with

$$
\begin{aligned}
Y_j^{01} &= [0, 0, \dots, 0, 1, 0, \dots, 0], \quad \text{1 is in the } j\text{-th place} \\
Y_j^{i-1,i} &= (-1)^{n-1} \left(Y_j^{n-i,n-i+1} \right)^{\mathrm{tr}}, \\
v_j(Y_k^{i-1,i}) &= v_k(Y_j^{i-1,i}), \\
Y_j^{i-1,i} Y_k^{i,i+1} &= Y_k^{i-1,i} Y_j^{i,i+1}.
\end{aligned}
$$

Quasi affinness conjecture:

We know that the moduli of smooth Calabi-Yau varieties exists as a quasi-projective variety (Viehweg 1995). This implies the same statement for T. However,

Conjecture

The moduli space T *is quasi-affine and moreover, the universal family* $X \to T$ *exists.*

Elliptic curves: [Mov12]

$$\mathsf{T} := \mathrm{Spec}(\mathbb{Q}[t_1, t_2, t_3, \frac{1}{27t_3^2 - t_2^3}])$$

Universal family

$$y^2 = 4(x - t_1)^3 - t_2(x - t_1) - t_3, \ \ \alpha_1 = [\frac{dx}{y}], \ \alpha_2 = [\frac{xdx}{y}]$$

Ramanujan vector field:

$$\mathsf{R} = (t_1^2 - \frac{1}{12}t_2)\frac{\partial}{\partial t_1} + (4t_1t_2 - 6t_3)\frac{\partial}{\partial t_2} + (6t_1t_3 - \frac{1}{3}t_2^2)\frac{\partial}{\partial t_3}.$$

(partially inspired by K. Saito's work on primitive forms!)

Dwork family: M.+Nikdelan [MN16]

$$x_0^{n+2} + x_1^{n+2} + \ldots + x_{n+1}^{n+2} - (n+2)\psi x_0 x_1 \ldots x_n = 0,$$

For $n = 1, 2$ one gets vector fields which can be solved with modular forms.
It seems that that the theory of CY modular forms for $n = 2k+1$ is related to the same theory for $n = 2k+2$!!

Lattice polarized K3 surfaces: Alim 2014, [Mov20]

Let X/T be a universal family of enhanced K3 surfaces. There are unique vector fields $v_k,\ k = 1, 2, \ldots, h^{1,1}_{\mathrm{prim}}$ in T such that

$$A_{v_k} = \begin{pmatrix} 0 & \delta^j_k & 0 \\ 0 & 0 & -\delta^i_k \\ 0 & 0 & 0 \end{pmatrix} \qquad (9)$$

$H^0(\mathrm{T}, \mathcal{O}_{\mathrm{T}})$ is going to be the algebra of automorphic forms and their derivations for the classical mouli of lattice polarized K3 surfaces $\Gamma_{\mathbb{Z}} \backslash M$.

Elliptically fibered CY3 and CY4: Haghighat+M.+Yau, [HMY17]

The computations in this article suggest that
There is a partial compactification $\bar{\mathsf{T}} = S \cup \mathsf{T}$ of T such that the modular vector fields are tangent to S and, restricted to S they have solutions by modular forms.

CY3: Alim+M.+Scheidegger+Yau, [AMSY16]

There are unique vector fields $\mathsf{R}_k,\ k = 1, 2, \ldots, \mathsf{h} := \mathsf{h}^{21}$ in T and unique $Y_{ijk} \in \mathcal{O}_{\mathsf{T}},\ \ i, j, k = 1, 2, \ldots, \mathsf{h}$ symmetric in i, j, k such that

$$\mathsf{A}_{\mathsf{R}_k} = \begin{pmatrix} 0 & \delta_k^j & 0 & 0 \\ 0 & 0 & Y_{kij} & 0 \\ 0 & 0 & 0 & \delta_k^i \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

Further

$$\mathsf{R}_{i_1} Y_{i_2 i_3 i_4} = \mathsf{R}_{i_2} Y_{i_1 i_3 i_4}. \tag{11}$$

$$\dim(\mathsf{T}) = \mathsf{h} + \frac{3\mathsf{h}^2 + 5\mathsf{h} + 4}{2}.$$

Abelian varieties: [Mov20],T. Fonseca 2017

There are unique vector fields $v_{ij}, i, j = 1, 2, \ldots, n, i \leq j$ defined over $\mathbb{Q}$ in the moduli space T of enhanced principally polarized abelian varieties such that the Gauss-Manin connection $\mathsf{A}_{v_{ij}}$ is the constant matrix C_{ij}, where all the entries of C_{ij} are zero except $(i, \mathsf{n}+j)$ and $(j, \mathsf{n}+i)$ entries which are -1. In other words, the Gauss-Manin connection ∇ satisfies

$$\nabla_{v_{ij}}\alpha_i = -\alpha_{\mathsf{n}+j}, \ \ \nabla\alpha_j = -\alpha_{\mathsf{n}+i}, \quad i, j = 1, 2 \ldots, \mathsf{n}$$

and $\nabla_{v_{ij}}\alpha_k = 0$ otherwise.

Action of G on T:

In all these cases there is an algebraic group G acting on T which corresponds to base change in cohomology, and for *CY*3 it plays an essential role in the algebraic BCOV anomaly equation. Its Lie algebra $\mathrm{Lie}(\mathsf{G})$ can be embedded into the set of (global) vector fields in T:

$$\mathfrak{g} \mapsto v_{\mathfrak{g}}.$$

Therefore, we can talk about

$$\nabla_{v_{\mathfrak{g}}}, \quad , \quad \mathfrak{g} \in \mathrm{Lie}(\mathsf{G}).$$

It turn out that

$$\nabla_{v_{\mathfrak{g}}} \alpha = \mathfrak{g}^{\mathsf{tr}} \alpha.$$

G and $\mathrm{Lie}(G)$ for mirror quintic: [Mov17]

There are unique vector fields $\mathsf{R}_i,\ i = 0, 1, 2 \ldots, 6$ in T and a unique regular function Y on T such that $\nabla_{\mathsf{R}_i}\alpha = \mathsf{A}_{\mathsf{R}_i}\alpha$, where

$$\mathsf{A}_{\mathsf{R}_0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{A}_{\mathsf{R}_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{A}_{\mathsf{R}_2} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathsf{A}_{\mathsf{R}_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{A}_{\mathsf{R}_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathsf{A}_{\mathsf{R}_5} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathsf{A}_{\mathsf{R}_6} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

AMSY Lie algebra

Lie brackets of R_i's.

	R_0	R_1	R_2	R_3	R_4	R_5	R_6
R_0	0	R_0	$-R_0$	$-R_2 + R_1$	$Y \cdot R_1$	$2R_4 + Y \cdot R_3$	R_5
R_1	$-R_0$	0	0	R_3	$-2R_4$	$-R_5$	0
R_2	R_0	0	0	$-R_3$	0	$-R_5$	$-2R_6$
R_3	$R_2 - R_1$	$-R_3$	R_3	0	$-R_5$	$-2R_6$	0
R_4	$-Y \cdot R_1$	$2R_4$	0	R_5	0	0	0
R_5	$-2R_4 - Y \cdot R_3$	R_5	R_5	$2R_6$	0	0	0
R_6	$-R_5$	0	$2R_6$	0	0	0	0

BCOV anomaly for mirror quintic: [Mov17], for arbitrary $CY3$ see [AMSY16]

Apart from the Yukawa coupling, we have the generating function of genus g Gromov-Witten invariants of the generic quintic in $\mathbb{P}^4$.

$$\mathsf{F}_g^{\rm hol} := \sum_{d=0}^{\infty} N_{g,d} q^d, \; g \geq 2, \quad \mathsf{F}_1^{\rm hol} := \frac{25}{12} \ln q + \sum_{d=1}^{\infty} N_{1,d} q^d$$

which are called genus g topological string partition function.

$$\mathsf{F}_1^{\rm alg} := \ln(t_4^{\frac{25}{12}} (t_4 - t_0^5)^{\frac{-5}{12}} t_5^{\frac{1}{2}}). \tag{12}$$

$$\begin{aligned} \mathsf{R}_i \mathsf{F}_g^{\rm alg} &= 0, \quad i = 1, 3, \\ \mathsf{R}_2 \mathsf{F}_g^{\rm alg} &= (2g-2)\mathsf{F}_g^{\rm alg}, \\ \mathsf{R}_4 \mathsf{F}_g^{\rm alg} &= \frac{1}{2}(\mathsf{R}_0^2 \mathsf{F}_{g-1}^{\rm alg} + \sum_{r=1}^{g-1} \mathsf{R}_0 \mathsf{F}_r^{\rm alg} \mathsf{R}_0 \mathsf{F}_{g-r}^{\rm alg}). \end{aligned} \tag{13}$$

These collections of equations do not determine $\mathsf{F}_g^{\rm alg}$ uniquely.

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