Relative Cohomology with Respect to a Lefschetz Pencil

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Abstract

Let M be a complex projective manifold of dimension $n+1$ and f a meromorphic function on M obtained by a generic pencil of hyperplane sections of M. The n-th cohomology vector bundle of $f_0 = f|_{M-\mathcal{R}}$, where $\mathcal{R}$ is the set of indeterminacy points of f, is defined on the set of regular values of f_0 and we have the usual Gauss-Manin connection on it. Following Brieskorn's methods in [**Br**], we extend the n-th cohomology vector bundle of f_0 and the associated Gauss-Manin connection to $\mathbb{P}^1$ by means of differential forms. The new connection turns out to be meromorphic on the critical values of f_0. We prove that the meromorphic global sections of the vector bundle with poles of arbitrary order at $\infty \in \mathbb{P}^1$ is isomorphic to the Brieskorn module of f in a natural way, and so the Brieskorn module in this case is a free $\mathbb{C}[t]$-module of rank β_n, where $\mathbb{C}[t]$ is the ring of polynomials in t and β_n is the dimension of n-th cohomology group of a regular fiber of f_0.

0 Introduction

The algebraic description of the monodromy of a germ of an isolated singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ was done by E. Brieskorn in [**Br**]. In this article he considers the Milnor fibration associated to f and then the n-th cohomology vector bundle $\mathcal{H}$ of f over a punctured neighborhood of $0 \in \mathbb{C}$ and the associated Gauss-Manin connection. Then he constructs three extension of this vector bundle (the sheaf of its holomorphic sections) $\mathcal{H}, '\mathcal{H}, ''\mathcal{H}$ by means of holomorphic forms in $(\mathbb{C}^{n+1}, 0)$. He constructs them up to torsions which may appear in the stalk over zero but later M. Sebastiani in [**Se**] proves that there is no torsion and so Brieskorn's extension is complete. By a slight modification of his argument we can obtain a meromorphic connection $\nabla : V \to \Omega^1_{\mathbb{C},0}(k0) \otimes_{\mathcal{O}_{\mathbb{C},0}} V$ which is the Gauss-Manin connection of the Milnor fibration in $(\mathbb{C}, 0) - \{0\}$. Here V stands for one of $\mathcal{H}, '\mathcal{H}$ and $''\mathcal{H}$, k is the smallest number with this property that the multiplication by f^k induces the zero map in the Jacobi algebra $\frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{< f_{x_i} | i=1,\ldots,n+1 >}$ of f and $\Omega^1_{\mathbb{C},0}(k0)$ is the sheaf of meromorphic 1-forms in $(\mathbb{C}, 0)$ with poles of order at most k at 0. The stalk of $\mathcal{H}, '\mathcal{H}, ''\mathcal{H}$ over $0 \in (\mathbb{C}, 0)$, namely $H, 'H, ''H$, are called Brieskorn modules and they are very useful objects in singularity theory. They are freely generated $\mathcal{O}_{\mathbb{C},0}$-modules of rank μ, where μ is the Milnor number of f.

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We consider a projective manifold M of dimension $n+1$ and we intersect it by a pencil of hyperplanes (see for instance Lamotke's article [**La**] for the definitions concerning a pencil). We can define a holomorphic map f_0 in $M-\mathcal{R}$, where $\mathcal{R}$ is the intersection of the axis A of the pencil with M, such that $\overline{f_0^{-1}(t)}$'s are hyperplane sections. We assume that A intersects M transversally and f_0 has only isolated singularities. Now f_0 is a C^∞ fiber bundle over $\mathbb{P}^1 - C$, where C is the set of critical values of f_0. Therefore we have the cohomology fiber bundle over $\mathbb{P}^1 - C$ and the associated Gauss-Manin connection.

In the above context we will generalize Brieskorn methods as follows: We make a blow up $\pi : V \to M$ along $\mathcal{R}$ and we obtain our extensions by means of meromorphic forms in V with poles of arbitrary order along $\pi^{-1}(\mathcal{R})$. In this way we obtain three analytic sheaves $\mathcal{H}^n, '\mathcal{H}^n, ''\mathcal{H}^n$ on $\mathbb{P}^1$ and connections $\nabla : W \to \Omega^1_{\mathbb{P}^1}(\tilde{C}) \otimes_{\mathcal{O}_{\mathbb{P}^1}} W$. Here W stands for one of $\mathcal{H}^n, '\mathcal{H}^n$ and $''\mathcal{H}^n$, $\tilde{C}$ is a divisor in $\mathbb{P}^1$ with support at the critical values of f_0 and $\Omega^1_{\mathbb{P}^1}(\tilde{C})$ is the sheaf of meromorphic 1-forms in $\mathbb{P}^1$ with pole divisor less than or equal $\tilde{C}$.

Our main result in this article is that W is a locally free sheaf of rank β_n over $\mathbb{P}^1$, where β_n is the dimension of the n-th cohomology of a regular fiber of f_0, and there is a canonical isomorphism between $W\mid_{\mathbb{P}^1-C}$ and the n-th cohomology vector bundle of f_0 over $\mathbb{P}^1-C$ and ∇ is the Gauss-Manin connection by this isomorphism. Then we introduce global Brieskorn modules $H^n, 'H^n, ''H^n$ in our context. They are $\mathbb{C}[t]$-modules, where $\mathbb{C}[t]$ is the ring of polynomials in t, and we prove that there is a $\mathbb{C}[t]$-module isomorphism between H^n (resp. $'H^n$ and $''H^n$) and the module of global meromorphic sections of $\mathcal{H}^n$ (resp. $'\mathcal{H}^n$ and $''\mathcal{H}^n$) with poles of arbitrary order at $\infty \in \mathbb{P}^1$.

In the lower dimension $i < n$ there is no vanishing cycle and the monodromy around a critical value is identity. It is not difficult to see that in this dimension the cohomology of the critical fiber gives us the desired extension and the Gauss-Manin connection on the i-th cohomology vector bundle is holomorphic even in the critical value. This implies that the obtained vector bundle is trivial. However, we construct this extension by means of meromorphic forms.

The notion of global Brieskorn modules has been considered recently by many people, see for instance C. Sabbah, A. Dimca, M. Saito and P. Bonnet's works [**Sa1**], [**DS**], [**BD**]. As an immediate consequence of our last result we prove that H^n and $'H^n$ and $''H^n$ are freely generated $\mathbb{C}[t]$-modules of rank β_n. This result was already known by C. Sabbah in [**Sa1**]. In the context of differential equations (n=1) $'H^1$ appears in the works of G.S. Petrov for polynomials of the type $y^2 + P(x)$ in $\mathbb{C}^2$ and is called Petrov module by L. Gavrilov. (see [**Ga2**] p. 572). Recently some applications of this module in differential equations have been introduced by the author of these lines in [**Mo2**].

Brieskorn module $'H$ in differential equations: Consider the case $M = \mathbb{P}^2$ and $f = \frac{F}{G}$, where F and G are two polynomials of the same degree in an affine coordinate $\mathbb{C}^2$ of $\mathbb{P}^2$. Assume that $F = 0$ intersects $G = 0$ transversally and the critical points of $\frac{F}{G}$ are non-degenerate with distinct images. Consider the foliation

$$\mathcal{F}_\epsilon : df + \epsilon.\omega = 0 \tag{1}$$

where ω is a meromorphic 1-form in $\mathbb{P}^2$ with poles of arbitrary order along $G = 0$. Let $\{\delta_t\}_{t\in(\mathbb{C},0)}$ be a continuous family of vanishing cycles. We call $h(t) := \int_{\delta_t} \omega$ an Abelian integral. If $h(t) \not\equiv 0$ then the cycle δ_{t_0} persists in being cycle after this deformation if and only if $h(t_0) = 0$. Therefore the study of the number of limit cycles appearing from δ_t's after the deformation $\mathcal{F}_t$ leads to the study of the zeros of Abelian integrals. In [**Mo**] (see also [**Mo1**]) it is shown that if $h(t) \equiv 0$ then $\omega = Pdf + dQ$, where P and Q are two meromorphic functions in $\mathbb{P}^2$ with poles of arbitrary order along $G = 0$. Therefore $\omega = 0$ in $'H^1$ and $'H^1$ represents the space of deformations (1) for which the birth of limit cycles can be studied by Abelian integrals.

Now let us explain the structure of this article. In § 1 we have explained in details the extension of the cohomology vector bundles of f_0 to the critical values of f_0 and the associated Gauss-Manin connection to a meromorphic connection. Theorem 1.1 which is the central result in this article is stated there together with Theorem 1.2. The reader who is interested only on the construction and the main results is invited to read only this section. Theorem 1.2 is proved in § 2. The terminology and propositions in Appendix A are used in this section. § 3 is devoted to the proof of Theorem 1.1. The proof can be considered as a kind of variational Atiyah-Hodge theorem, therefore it is recommended to the reader to know the proof of this theorem stated in [**Nr**].

Perhaps three appendices for this article is too many, but in each of them we have obtained some partial results which we need them in this article and I did not find them in the literature. In Appendix A we have listed some necessary concepts and theorems in complex geometry. The first result is A.10 which is a kind of Kodaira vanishing theorem for direct limit of coherent sheaves. A.16 is the main result in this appendix. It is a kind of variational Kodaira vanishing theorem and is frequently used in this article. After doing a blow-up in the indeterminacy locus of our pencil we obtain a holomorphic map $g : V \to \mathbb{P}^1$ and a divisor A in V, in such a way that the intersection of A with a regular fiber is a positive divisor in that fiber. A.16 claims that $R^i g_* \mathcal{S}(*A) = 0, i > 0$, where $\mathcal{S}$ is a coherent sheaf in V and $\mathcal{S}(*A)$ is the sheaf of meromorphic sections of $\mathcal{S}$ with poles of arbitrary order along A. Note that g has critical fibers. In Appendix B we list some information about the topology of the fibers of f_0. Any kind of singularities can appear in our pencil therefore we had to combine some technics of [**La**] and [**AGV**] to obtain B.1, the main result of this appendix. In particular we prove that a distinguished basis of vanishing cycles in the singularities with a same value must be linearly independent. Let us consider the restriction map from a global Brieskorn module of a pencil to a local Brieskorn module of a singularity of the pencil. It is believed that this map is surjective but I was not able to prove this fact. In Appendix C we prove C.3 which says that the local Brieskorn module divided by the image of the mentioned map is a vector space of finite dimension. This easily implies that the image of the mentioned map is a freely generated $\mathcal{O}_{\mathbb{C},0}$-module, the statement of C.2.

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1 Extension of the cohomology vector bundle and the Gauss-Manin connection

Let M be a complex projective manifold of dimension $n+1$, $\{M_t\}_{t\in\mathbb{P}^1}$ a pencil of hyperplane sections of M and f the meromorphic function on M whose level sets are M_t's. Suppose that the axis of the pencil intersects M transversally (see [**La**]). This implies that the set of indeterminacy points $\mathcal{R}$ of f is a smooth submanifold of M of codimension two and every two M_t's intersect each other transversally in $\mathcal{R}$. Define $f_0 := f\mid_{M-\mathcal{R}}$ and $L_t := M_t - \mathcal{R} = f_0^{-1}(t)$. We assume that the critical points of f_0 are isolated and we denote by $C = \{c_1, c_2, c_3, \dots, c_r\}$ the set of critical values of f_0. Note that a critical fiber M_{c_j} may have more than one critical point. Put

$$\beta_i = dim(H^i(L_t, \mathbb{C})),\ t \in \mathbb{P}^1 - C,\ 0 \leq i \leq n$$

f_0 is a C^∞ fibration over $\mathbb{P}^1 - C$ (see for instance [**La**]) and so β_i is independent of t.

The set $\tilde{H}^i = \cup_{t\in\mathbb{P}^1-C} H^i(L_t, \mathbb{C})$ has a natural structure of a complex manifold and the natural projection $\tilde{H}^i \to \mathbb{P}^1 - C$ is a holomorphic vector bundle map which is called the i-th cohomology vector bundle. Let $\mathbb{C}_{M-\mathcal{R}}$ be the sheaf of constant functions in $M-\mathcal{R}$ and $R^i f_{0*}\mathbb{C}_{M-\mathcal{R}}$ be the i-th direct image of the sheaf $\mathbb{C}_{M-\mathcal{R}}$ (see [**GrRe**] and Appendix A). Any element of $R^i f_{0*}\mathbb{C}_{M-\mathcal{R}}(U)$, U being an open set in $\mathbb{P}^1 - C$, is a holomorphic section of the cohomology vector bundle map and is called a constant section. It is easy to verify that

$$\mathcal{O}(\tilde{H}^i) \cong R^i f_{0*}\mathbb{C}_{M-\mathcal{R}} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1-C}$$

where $\mathcal{O}(\tilde{H}^i)$ denotes the sheaf of holomorphic sections of $\tilde{H}^i$. We define the sheaf

$$\tilde{\mathcal{H}}^i = R^i f_{0*}\mathbb{C}_{M-\mathcal{R}} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1-C},\ 0 \leq\ i \leq n$$

The Gauss-Manin connection on $\tilde{\mathcal{H}}^i$ is given by

$$\nabla : \tilde{\mathcal{H}}^i \to \Omega^1_{\mathbb{P}^1-C} \otimes_{\mathcal{O}_{\mathbb{P}^1-C}} \tilde{\mathcal{H}}^i \tag{2}$$

$$\nabla(g \otimes c) = dg \otimes c,\ c \in R^i f_{0*}\mathbb{C}_{M-\mathcal{R}}(U),\ g \in \mathcal{O}_{\mathbb{P}^1-C}(U)$$

where U is an open set in $\mathbb{P}^1 - C$. Now we have the problem of extension of $\tilde{\mathcal{H}}^i$ to a locally free sheaf on $\mathbb{P}^1$ and ∇ to a (meromorphic) connection (with possible poles in C) defined in the extended sheaf. We could define $\tilde{\mathcal{H}}^i$ and ∇ in the whole $\mathbb{P}^1$. In B.1 Appendix B we have proved that $\tilde{\mathcal{H}}^i$, $0 \leq i < n$ is a locally free sheaf of rank β_i in $\mathbb{P}^1$ and so in this case there is no serious problem. But in the case $i = n$ we have the notion of vanishing cycle in a critical point of f_0 and so the definition (2) does not give us the desired extension. Using Brieskorn's ideas we are going to construct some extensions and reconstruct $\tilde{\mathcal{H}}^i$'s by means of meromorphic forms.

Let $\pi : V \to M$ be the blow-up along $\mathcal{R}$ (see [**La**]). V is a smooth manifold and $g = \pi \circ f$ is a well-defined holomorphic function in V. We have

$$(3) \qquad A := \pi^{-1}(\mathcal{R}) \cong \mathbb{P}^1 \times \mathcal{R},\ V_t := g^{-1}(t) = \pi^{-1}(M_t),\quad L_t \cong V_t - A$$

Each fiber V_t intersects A transversally in $\{t\} \times \mathcal{R}$. For $U \subset \mathbb{P}^1$ we define

$$(4) \qquad V_U := g^{-1}(U),\ L_U := V_U - A$$

Let $\tilde{\Omega}^i$ be the sheaf of holomorphic i-forms in V and $\tilde{\Omega}^i(*A)$ be the sheaf of its meromorphic sections with poles of arbitrary order along A. Let also Ω^i be the direct image by g of $\tilde{\Omega}^i(*A)$, i.e. $\Omega^i = g_* \tilde{\Omega}^i(*A)$. The following sheaf is well-defined

$$\Omega^i_{V/\mathbb{P}^1} = \frac{\Omega^i}{\Omega^1_{\mathbb{P}^1} \wedge \Omega^{i-1}}$$

We have the following long sequence:

$$(5) \qquad 0 \stackrel{d^{-1}}{\to} \Omega^0_{V/\mathbb{P}^1} \stackrel{d^0}{\to} \Omega^1_{V/\mathbb{P}^1} \stackrel{d^1}{\to} \cdots \stackrel{d^{n-1}}{\to} \Omega^n_{V/\mathbb{P}^1} \stackrel{d^n}{\to} \Omega^{n+1}_{V/\mathbb{P}^1}$$

We define

$$\mathcal{H}^i = \frac{ker d^i}{Im d^{i-1}}$$

Note that $\Omega^i_{V/\mathbb{P}^1}$ are $\mathcal{O}_{\mathbb{P}^1}$-module and the differential operators d^i's are $\mathcal{O}_{\mathbb{P}^1}$ linear and so the cohomology sheaves $\mathcal{H}^i$'s are $\mathcal{O}_{\mathbb{P}^1}$-modules. We also define the sheaves:

$$'\mathcal{H}^n = \frac{\Omega^n}{\Omega^1_{\mathbb{P}^1} \wedge \Omega^{n-1} + d\Omega^{n-1}},\ ''\mathcal{H}^n = \frac{\Omega^{n+1}}{\Omega^1_{\mathbb{P}^1} \wedge d\Omega^n}$$

From now on the index i stands for $0, 1, 2, \ldots, n,' n,'' n$. For instance if $i =' n$ then $\mathcal{H}^i = '\mathcal{H}^n$ and $\beta_i = \beta_n$. To construct connections on $\mathcal{H}^i$'s we need the following lemmas. The key of the proof in both lemmas is A.16 in Appendix A. Let $c \in \mathbb{P}^1$, U a small open disk around c and t a regular holomorphic function in U.

Lemma 1.1. *(generalized de Rham lemma) An element $\omega \in \Omega^i(U), i \leq n$ is of the form $dt \wedge \eta,\ \eta \in \Omega^{i-1}(U)$ if and only if $dt \wedge \omega = 0$.*

Proof. It is enough to prove that if $dt \wedge \omega = 0$ then ω is of the form $dt \wedge \eta,\ \eta \in \Omega^{i-1}(U)$. By de Rham lemma (see [**Br**], p. 110) we can write

$$\omega = dt \wedge \eta_\alpha, \eta_\alpha \in \tilde{\Omega}^{i-1}(*A)(U_\alpha)$$

where $\{U_\alpha\}_{\alpha \in I}$ is an open covering of V_U. Now $\{\eta_\alpha - \eta_\beta\}_{\alpha,\beta \in I}$ is an element of $H^1(V_U, \mathcal{S}(*A))$, where

$$\mathcal{S} = Ker(\tilde{\Omega}^{i-1} \stackrel{dt\wedge}{\to} \tilde{\Omega}^i)$$

By A.16 $H^1(V_U, \mathcal{S}(*A)) = 0$ and so we can find $\eta'_\alpha \in \mathcal{S}(*A)(U_\alpha)$ such that $\eta_\alpha - \eta_\beta = \eta'_\alpha - \eta'_\beta$. The $(i-1)$-form $\eta \mid_{U_\alpha} = \eta_\alpha - \eta'_\alpha$ satisfies $\omega = dt \wedge \eta$ and is the desired $(i-1)$-form. □

Lemma 1.2. *$\Omega^{n+1}_{V/\mathbb{P}^1}$ is a discrete sheaf with support at C. The stalk of $\Omega^{n+1}_{V/\mathbb{P}^1}$ over $c_j \in C$ is a vector space of dimension μ_{c_j}, where μ_{c_j} is the sum of Milnor numbers of the singularities of f within L_{c_j}. In particular there is a natural number k_j such that $(f-c_j)^{k_j}\Omega^{n+1}_{V/\mathbb{P}^1}$ is zero in c_j.*

Proof. Put $\mu_c = 0$ if c is not a critical value of f. We know that

$$(\tilde{\Omega}^{n+1}_{V/\mathbb{P}^1})_x := (\frac{\tilde{\Omega}^{n+1}}{g^*\Omega^1_{\mathbb{P}^1} \wedge \tilde{\Omega}^n})_x =$$

$$\begin{cases} \mathbb{C}^{\mu_x} & x \text{ is a critical point of } g \text{ with the Milnor number } \mu_x \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

By A.16 $H^1(V_U, g^*(\Omega^1_{\mathbb{P}^1}) \wedge \tilde{\Omega}^n(*A)) = 0$ and so

$$\Omega^{n+1}_{V/\mathbb{P}^1}(U) = \frac{H^0(V_U, \tilde{\Omega}^{n+1}(*A))}{H^0(V_U, g^*(\Omega^1_{\mathbb{P}^1}) \wedge \tilde{\Omega}^n(*A))} = H^0(V_U, \frac{\tilde{\Omega}^{n+1}(*A)}{g^*(\Omega^1_{\mathbb{P}^1}) \wedge \tilde{\Omega}^n(*A)}) =$$

$$H^0(V_U, \frac{\tilde{\Omega}^{n+1}}{g^*(\Omega^1_{\mathbb{P}^1}) \wedge \tilde{\Omega}^n}(*A))$$

Since (6) is a discrete sheaf with support at $V - A$, we conclude that

$$\Omega^{n+1}_{V/\mathbb{P}^1}(*A)(U) = \oplus_{x \in V_c}(\tilde{\Omega}^{n+1}_{V/\mathbb{P}^1})_x = \mathbb{C}^{\mu_c}$$

where x runs through all critical points of f within V_c. □

We put k_j the minimum number with the property in Lemma 1.2. For every critical point p in the fiber V_{c_j} there exists a natural number k_p depending only on the type of the critical point p such that $(g-c)^{k_p}(\tilde{\Omega}^{n+1}_{V/\mathbb{P}^1})$ is zero in p (see [**Br**], p. 110 and 125). Choose always the minimum k_p. We have $k_j = \max_p\{k_p\}$, where p runs through all critical points of f within L_{c_j}.

Consider the sheaf $\mathcal{H}^i, 1 \leq i \leq n-1$. Let $[\omega] \in \mathcal{H}^i(U)$. We can write $d\omega = dt \wedge \eta, \eta \in \Omega^i(U)$. We have $dt \wedge d\eta = 0$ and so by Lemma 1.1 we have $d\eta = dt \wedge \eta'$ for some $\eta' \in \Omega^i(U)$. Therefore we can define the following connection:

$$\nabla : \mathcal{H}^i \to \Omega^1_{\mathbb{P}^1} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{H}^i$$

$$\nabla[\omega] = dt \otimes [\eta],\ d\omega = dt \wedge \eta$$

Now we are going to define the connections

$$\nabla : \mathcal{H}^i \to \Omega^1_{\mathbb{P}^1}(\tilde{C}) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{H}^i,\ i = n,' n,'' n$$

where $\tilde{C} = \sum k_j c_j$ and $\Omega^1_{\mathbb{P}^1}(\tilde{C})$ is the sheaf of meromorphic sections of $\mathcal{H}^n$ with a pole of order at most k_j at c_j. The reader is referred to [**EV1**] for more informations about meromorphic connections on sheaves on $\mathbb{P}^1$.

Let us define $p(t)$ in U as follows: $p(t) = 1$ if $U \subset \mathbb{P}^1 - C$ and $p(t) = (t-c_j)^{k_j}$ if U is an open disk around c_j. Let $[\omega] \in \mathcal{H}^n(U)$. We can write $d\omega = dt \wedge \eta, \eta \in \Omega^n(U)$.

Since $d\eta$ may not be of the form $dt \wedge \eta',\ \eta' \in \Omega^n(U)$, we had to multiply $d\eta$ by $p(t)$ and therefore by Lemma 1.2 we have:

$$d(p(t)\eta) = p(t)d\eta + p'(t)dt \wedge \eta = dt \wedge \eta' + p'(t)dt \wedge \eta,\ \eta' \in \Omega^n(U)$$

Therefore we can define

$$\nabla[\omega] = \frac{dt}{p(t)} \otimes [\eta'],\ \eta' = p(t)\eta,\ d\omega = dt \wedge \eta$$

In a similar way for $[\omega] \in {}'\mathcal{H}^n(U)$

$$\nabla[\omega] = \frac{dt}{p(t)} \otimes [\eta],\ p(t)d\omega = dt \wedge \eta$$

and for $[\omega] \in {}''\mathcal{H}^n(U)$ we have

$$\nabla[\omega] = \frac{dt}{p(t)} \otimes [d\eta],\ p(t)\omega = dt \wedge \eta$$

It is not difficult to see that these definitions are well-defined and do not depend on the choice of the coordinate t and the choice of ω in the class $[\omega]$. From now on we use the notation ω instead of $[\omega]$. The main theorem of this article is:

Theorem 1.1. *$\mathcal{H}^i, i = 0, 1, \ldots, n, ' n, '' n$ is a locally free sheaf of rank β_i on $\mathbb{P}^1$. The natural map $\mathcal{H}^i \to \tilde{\mathcal{H}}^i$ in $\mathbb{P}^1 - C$ which is obtained by the restriction of differential forms to the fibers of g induces an isomorphism between $(\mathcal{H}^i, \nabla)$ and $(\tilde{\mathcal{H}}^i, \nabla)$.*

We have used the convention $\tilde{\mathcal{H}}^i = \tilde{\mathcal{H}}^n$ for $i = ' n, '' n$. If we consider only one fiber V_t then by Atiyah-Hodge theorem (see A.17) we know that meromorphic differential forms in V_t with poles of arbitrary order along $A \cap V_t$ give us the cohomology groups of $V_t - A$. This shows that the above theorem in $\mathbb{P}^1 - C$ is a natural statement. Main difficulty in the proof of the above theorem lies in the critical values of f. To prove it we will have to look more precisely to the proof of Atiyah-Hodge theorem stated in [**Nr**].

Now we can look at $\mathcal{H}^i$ as a vector bundle. In the case $i < n$ the obtained connection is holomorphic in $\mathbb{P}^1$. This implies that the vector bundle $\mathcal{H}^i, i < n$ is a trivial bundle. We have already expected this fact.

Note that the above extensions are not necessarily logarithmic. All logarithmic extension to C are described in [**AB**], p. 89 and also in [**EV**], [**He**] for arbitrary dimension of the base space. The extensions introduced above have a peculiar property which we are going to explain below:

Choose $p = \infty \in \mathbb{P}^1 - C$. This implies that $D := M_\infty$ is smooth. Let t be an affine coordinate of $\mathbb{C} = \mathbb{P}^1 - \{p\}$ and $\overline{\Omega}^i(*D)$ be the set of meromorphic i-forms in M with poles of arbitrary order along D. It is a $\mathbb{C}[t]$-module in a trivial way

$$p(t).\omega = p(f)\omega,\ \omega \in \overline{\Omega}^i(*D),\ p(t) \in \mathbb{C}[t]$$

where $\mathbb{C}[t]$ is the ring of polynomials in t. The set of relative meromorphic i-forms with poles of arbitrary order along D is defined as follows:

$$\overline{\Omega}^i_{V/\mathbb{P}^1}(*D) = \frac{\overline{\Omega}^i(*D)}{df \wedge \overline{\Omega}^{i-1}(*D)}$$

The differential operator

$$d^i : \overline{\Omega}^i_{V/\mathbb{P}^1}(*D) \to \overline{\Omega}^{i+1}_{V/\mathbb{P}^1}(*D)$$

$$\omega_1 \to d\omega_1$$

is well-defined and $\mathbb{C}[t]$-linear. Now we have the complex of relative meromorphic forms $(\overline{\Omega}^*_{V/\mathbb{P}^1}(*D), d^*)$ and so we can form the cohomology groups

$$H^i = H^i(\overline{\Omega}^*_{V/\mathbb{P}^1}(*D), d^*) = \frac{Ker(d^i)}{Im(d^{i-1})},\ d^{-1} = 0$$

H^i is called the i-th relative cohomology of M with respect to f. It is easy to see that $H^0 = \mathbb{C}[t]$. In dimension n there are two other useful $\mathbb{C}[t]$-modules

$$'H^n = \frac{\overline{\Omega}^n(*D)}{df \wedge \overline{\Omega}^{n-1}(*D) + d\overline{\Omega}^{n-1}(*D)},\ ''H^n = \frac{\overline{\Omega}^{n+1}(*D)}{df \wedge d\overline{\Omega}^n(*D)}$$

The $\mathbb{C}[t]$-modules H^n, H'^n, H''^n were introduced by Brieskorn in [**Br**] for a germ of a holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and are also called Global Brieskorn modules (see Appendix C).

We consider the $\mathbb{C}[t]$-morphism

$$H^i \to H^0(\mathbb{P}^1, \mathcal{H}^i(*p)),\ i = 0, 1, 2, \ldots, n, 'n, ''n \tag{7}$$

and the morphism of vector spaces

$$\overline{\Omega}^{n+1}_{V/\mathbb{P}^1}(*D) \to H^0(\mathbb{P}^1, \Omega^{n+1}_{V/\mathbb{P}^1}(*p)) \tag{8}$$

obtained by restrictions to the fibers of g, where $\mathcal{H}^i(*p)$ denotes the sheaf of meromorphic sections of $\mathcal{H}^i$ with poles of arbitrary order at p. The next main result in this article is:

Theorem 1.2. *The morphisms (7) and (8) are isomorphisms.*

Corollary 1.1. *Keeping the notations used above, we have*

1. $\overline{\Omega}^{n+1}_{V/\mathbb{P}^1}(*D)$ *is a vector space of dimension μ, where μ is the sum of local Milnor numbers of f;*

2. $H^i,\ i = 0, 1, \ldots, n, 'n, ''n$ *is a free $\mathbb{C}[t]$ module of rank β_i;*

Proof. By Theorem 1.2 the first statement is trivial. For the second one it is enough to prove that $H^0(\mathbb{P}^1, \mathcal{H}^i(*p))$ is a free $\mathbb{C}[t]$-module of rank β_i. The sheaf $\mathcal{H}^i$ is a locally free sheaf of rank β_i over $\mathbb{P}^1$. By Birkhoff-Grothendieck decomposition theorem (see [**GrRe1**]) there exist integers $n_1, n_2, \ldots, n_{\beta_i}$ (uniquely determined up to a permutation) such that

$$\mathcal{H}^i \cong \mathcal{O}(n_1 p) \oplus \mathcal{O}(n_2 p) \oplus \cdots \oplus \mathcal{O}(n_{\beta_i} p)$$

Meromorphic sections of $\mathcal{O}(n_i p)$ with poles of arbitrary order at p is a $\mathbb{C}[t]$-module of rank one and therefore $H^0(\mathbb{P}^1, \mathcal{H}^i(*p))$ is a free $\mathbb{C}[t]$-module of rank β_i. □

If $i < n$ then the vector bundle $\mathcal{H}^i$ is trivial and so the numbers $n_1, n_2, \ldots, n_{\beta_i}$ are zero. It would be interesting, if one tries to understand the nature of the numbers $n_1, n_2, \ldots, n_{\beta_i}$ by some numerical invariants of the manifold and the singularities of f_0 in the case $i = n,' n,'' n$.

The above corollary generalizes Brieskorn and Sebastiani's results in [**Br**] and [**Se**] in the local case $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. When I finished this article I was informed that similar results to Corollary 1.1 are obtained by C. Sabbah [**Sa1**] in the context of algebraic geometry and therefore in view of Serre's GAGA principal it is true in the context of analytic geometry.

2 Proof of Theorem 1.2

Recall the notations introduced in (4) and (3). We will also use notations introduced in Appendix A. Let $\tilde{\Omega}^i(kA)$ be the sheaf of meromorphic i-forms in V with poles of order $\leq k$ along A and $\Omega^i(k) = g_*\tilde{\Omega}^i(kA)$. By Grauert direct image theorem $\Omega^i(k)$ is a coherent sheaf and so we have a direct system of coherent sheaves $\{\Omega^i(k)\}_k$ (the map $\Omega^i(k) \to \Omega^i(k+1)$ is the inclusion). The sheaf $\Omega^i = \lim_{k\to\infty} \Omega^i(k)$ is the direct image of $\tilde{\Omega}^i(*A)$. The quotient direct system of sheaves

$$\{\Omega^i_{V/\mathbb{P}^1}(k)\}_k = \frac{\{\Omega^i(k)\}_k}{\{\Omega^1_{\mathbb{P}^1} \wedge \Omega^{i-1}(k)\}_k}$$

is called the direct system of sheaves of relative meromorphic i-forms. We can easily check that the differential operator

$$d^i : \{\Omega^i_{V/\mathbb{P}^1}(k)\}_k \to \{\Omega^{i+1}_{V/\mathbb{P}^1}(k+1)\}_k,\ d^i(\omega) = d\omega$$

is well-defined and $\mathcal{O}_{\mathbb{P}^1}$-linear. Now we have the following, not necessarily exact, sequence

$$\{\Omega^0_{V/\mathbb{P}^1}(k)\}_k \xrightarrow{d^0} \{\Omega^1_{V/\mathbb{P}^1}(k+1)\}_k \xrightarrow{d^1} \cdots \xrightarrow{d^{i-1}} \{\Omega^i_{V/\mathbb{P}^1}(k+i)\}_k \xrightarrow{d^i} \cdots$$

We have the complex of relative meromorphic i-forms $(\{\Omega^*_{V/\mathbb{P}^1}(k+*)\}_k, d^*)$, and so we can form the direct system of cohomology sheaves

$$\{\mathcal{H}^i(k)\}_k = H^i(\{\Omega^*_{V/\mathbb{P}^1}(k+*)\}_k, d^*) = \frac{\{Ker(d^i)\}_k}{\{Im(d^{i-1})\}_k},\ i \geq 0,\ d^{-1} = 0$$

The differential operators d^i's are $\mathcal{O}_{\mathbb{P}^1}$ linear and so $\{\mathcal{H}^i(k)\}_k$ is a direct system of $\mathcal{O}_{\mathbb{P}^1}$-module sheaves. We also define the sheaves:

$$\mathcal{H}^i = \lim_{k\to\infty} \mathcal{H}^i(k)$$

$$\{'\mathcal{H}^n(k)\}_k = \frac{\{\Omega^n(k)\}_k}{\{\Omega^1_{\mathbb{P}^1} \wedge \Omega^{n-1}(k) + d\Omega^{n-1}(k-1)\}_k},\ '\mathcal{H}^n = \lim_{k\to\infty} {}'\mathcal{H}^n(k)$$

$$''\mathcal{H}^n(k) = \frac{\{\Omega^{n+1}(k)\}_k}{\{dt \wedge d\Omega^n(k-1)\}_k},\ ''\mathcal{H}^n = \lim_{k\to\infty} {}''\mathcal{H}^n(k)$$

It is not clear at all that the sheaf $\mathcal{H}^i$ is coherent. By properties A.2 and A.3 in Appendix A it is not difficult to see that this way of definition of $\mathcal{H}^i$'s coincide with the one introduced in the first section.

Theorem 2.1. *The sheaves* $\mathcal{H}^i(k),\ i = 0, 1, \ldots, n,' n,'' n, k = 0, 1, 2, \ldots$ *are coherent.*

Proof. According to Grauert direct image theorem (Theorem A.13 in Appendix A) $\Omega^i(k)$ is coherent for all finite integer number k. Now the coherence of $\mathcal{H}^i(k), k \in \mathbb{N}$ can be obtained from the following facts: Let $\mathcal{S}$ and $\mathcal{S}'$ be two coherent sheaves on a complex manifold P. If $\mathcal{S} \subset \mathcal{S}'$ then $\frac{\mathcal{S}}{\mathcal{S}'}$ is coherent. If $i : \mathcal{S} \to \mathcal{S}'$ is a $\mathcal{O}_P$-linear map then kernel and image of i are coherent sheaves (see [**GrRe**] p. 236-237). □

Now let us prove Theorem 1.2. We only prove the isomorphism (7). The proof of the second is similar. First we observe that

$$\mathcal{H}^i(*p) = \frac{Ker(d^i)}{Im(d^{i-1})}(*p) \overset{A.5}{=} \frac{Ker(d^i)(*p)}{Im(d^{i-1})(*p)}$$

The number over an equality means the corresponding property in Appendix A. $Im(d^i)$ is a direct limit of coherent sheaves and so by A.10 we have $H^1(\mathbb{P}^1, Im(d^{i-1})(*p)) = 0$ and

$$H^0(\mathbb{P}^1, \mathcal{H}^i(*p)) = H^0(\mathbb{P}^1, \frac{Ker(d^i)(*p)}{Im(d^{i-1})(*p)}) \overset{A.10}{=} \frac{H^0(\mathbb{P}^1, Ker(d^i)(*p))}{H^0(\mathbb{P}^1, Im(d^{i-1})(*p))}$$

Let

$$\cdots \to H^0(\mathbb{P}^1, \Omega^{i-1}_{V/\mathbb{P}^1}(*p)) \overset{d_p^{i-1}}{\to} H^0(\mathbb{P}^1, \Omega^i_{V/\mathbb{P}^1}(*p)) \overset{d_p^i}{\to} H^0(\mathbb{P}^1, \Omega^{i+1}_{V/\mathbb{P}^1}(*p)) \to \cdots$$

be obtained from (5). By A.6 and A.11 we have

$$Ker(d_p^i) = H^0(\mathbb{P}^1, Ker(d^i)(*p)),\ Im(d_p^{i-1}) = H^0(\mathbb{P}^1, Im(d^{i-1})(*p))$$

$\Omega^1_{\mathbb{P}^1} \wedge \Omega^{i-1}$ is a direct limit of coherent sheaves, by A.10 $H^1(\mathbb{P}^1, (\Omega^1_{\mathbb{P}^1} \wedge \Omega^{i-1})(*p)) = 0$ and

$$H^0(\mathbb{P}^1, \Omega^i_{V/\mathbb{P}^1}(*p)) = H^0(\mathbb{P}^1, \frac{\Omega^i(*p)}{(\Omega^1_{\mathbb{P}^1} \wedge \Omega^{i-1})(*p)}) \overset{A.10}{=} \frac{H^0(\mathbb{P}^1, \Omega^i(*p))}{H^0(\mathbb{P}^1, (\Omega^1_{\mathbb{P}^1} \wedge \Omega^{i-1})(*p))} \overset{A.8}{=}$$
$$\frac{H^0(\mathbb{P}^1, \Omega^i(*p))}{dt \wedge H^0(\mathbb{P}^1, \Omega^{i-1}(*p)))}$$

where t is the chart map for $\mathbb{P}^1 - \{p\}$ (it has a pole of order one in p). We have

$$\Omega^i(*p) = (g_*\tilde{\Omega}^i(*A))(*p) \overset{A.14}{=} g_*\tilde{\Omega}^i(*A + *V_p),\ \pi^{-1}(D) = A + V_p$$

By blow down along A we can see easily that $H^0(\mathbb{P}^1, \Omega^i(*p)) \cong \overline{\Omega}^i(*D)$. □

3 Proof of Theorem 1.1

The arguments of this section can be considered as a variational Atiyah-Hodge theorem (see A.17 in Appendix A). It is highly recommended to the reader to know the proof of Atiyah-Hodge theorem stated in [**Nr**]. First we will prove the assertion of Theorem 1.1 for $i = 0, 1, \cdots, n-1, ' n$. The same statements for $i = n, '' n$ follows directly.

We have constructed $\mathcal{H}^i$ by means of meromorphic forms in V with poles of arbitrary order along A. The following lemma enables us to reconstruct it by means of holomorphic forms in $V - A$.

Lemma 3.1. *Let $g : X \to Y$ be a continuous map between paracompact Hausdorff spaces and suppose that two complexes $\mathcal{A}$ and $\mathcal{A}'$ of Abelian sheaves over X are given together with mappings h such that the diagram*

$$\begin{array}{ccccccccc} 0 & \overset{d'^{-1}}{\to} & \mathcal{A}'_0 & \overset{d'^0}{\to} & \mathcal{A}'_1 & \overset{d'^1}{\to} & \mathcal{A}'_2 & \overset{d'^2}{\to} & \cdots \\ & & h_0 \downarrow & & h_1 \downarrow & & h_2 \downarrow & & \\ 0 & \overset{d^{-1}}{\to} & \mathcal{A}_0 & \overset{d^0}{\to} & \mathcal{A}_1 & \overset{d^1}{\to} & \mathcal{A}_2 & \overset{d^2}{\to} & \cdots \end{array} \tag{9}$$

is commutative. (The rows are not supposed to be exact, but we have $d \circ d = 0$ and $d' \circ d' = 0$). Suppose further that

$$R^i g_* \mathcal{A}_k = 0, R^i g_* \mathcal{A}'_k = 0, \ \forall i \geq 1, k \geq 0$$

and for $k \geq 0$ h induces isomorphisms of cohomology sheaves

$$\frac{Ker(d'^k)}{Im(d'^{k-1})} \to \frac{Ker(d^k)}{Im(d^{k-1})} \tag{10}$$

Then h induces isomorphisms

$$\frac{Ker(d'^k_*)}{Im(d'^{k-1}_*)} \to \frac{Ker(d^k_*)}{Im(d^{k-1}_*)} \tag{11}$$

for all $k \geq 0$, where d_* and d'_* define the sequences

$$\begin{array}{ccccccccc} 0 & \overset{d'^{-1}_*}{\to} & g_*\mathcal{A}'_0 & \overset{d'^0_*}{\to} & g_*\mathcal{A}'_1 & \overset{d'^1_*}{\to} & g_*\mathcal{A}'_2 & \overset{d'^2_*}{\to} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \overset{d^{-1}_*}{\to} & g_*\mathcal{A}_0 & \overset{d^1_*}{\to} & g_*\mathcal{A}_1 & \overset{d^2_*}{\to} & g_*\mathcal{A}_2 & \overset{d^2_*}{\to} & \cdots \end{array} \tag{12}$$

Proof. We have just rewritten Theorem 6.5 of [**Nr**] in another form. □

Let $\mathcal{E}^i$ be the sheaf on V, which is defined by the presheaf that to every open subset U of V associated the modules of holomorphic i-forms in $U - A$. Let also

$$\mathcal{E}^i_{V/\mathbb{P}^1} = \frac{\mathcal{E}^i}{g^*\Omega^1_{\mathbb{P}^1} \wedge \mathcal{E}^{i-1}}$$

Let U be a small open disk in $\mathbb{P}^1$. Since L_U is a Stein manifold (see B.1) and the restriction of any Stein covering (see [**GrRe**]) of V_U to L_U is again a Stein covering, by Cartan's B theorem we have

$$R^j g_* \mathcal{E}^i_{V/\mathbb{P}^1} = 0, \ j > 0$$

We have the following long sequence:

$$\mathcal{A} := \ 0 \to \mathcal{E}^0_{V/\mathbb{P}^1} \xrightarrow{d^0} \mathcal{E}^1_{V/\mathbb{P}^1} \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathcal{E}^n_{V/\mathbb{P}^1} \to 0 \tag{13}$$

Recall that $\tilde{\Omega}^i(*A)$ is the sheaf of meromorphic i-forms in V with poles of arbitrary order along A and

$$\tilde{\Omega}^i_{V/\mathbb{P}^1}(*A) = \frac{\tilde{\Omega}^i(*A)}{g^*\Omega^1_{\mathbb{P}^1} \wedge \tilde{\Omega}^{i-1}(*A)}$$

By A.16 we have $R^j g_* \tilde{\Omega}^i_{V/\mathbb{P}^1}(*A) = 0, j > 0$. We have the following long sequence

$$\mathcal{A}' := \ 0 \to \tilde{\Omega}^0_{V/\mathbb{P}^1}(*A) \xrightarrow{d'^0} \tilde{\Omega}^1_{V/\mathbb{P}^1}(*A) \xrightarrow{d'^1} \cdots \xrightarrow{d'^{n-1}} \tilde{\Omega}^n_{V/\mathbb{P}^1}(*A) \to 0 \tag{14}$$

Now we would like to verify the hypothesis of Lemma 3.1 for $\mathcal{A}'$ and $\mathcal{A}$. The maps h are inclusions. The only non-trivial hypothesis is the isomorphism (10) in a point $p \in A$. Choose a Stein neighborhood U and a coordinate system $(z_1, z_2, \ldots, z_n, t)$ around $p \in A$ such that in this system $p = 0$, A is given by $z_1 = 0$ and L_{t_0} by $t = t_0$. We have proved in C.5 in Appendix C that in U

$$\frac{Ker(d'^i)}{Im(d'^{i-1})} = \frac{Ker(d^i)}{Im(d^{i-1})} = 0, i \geq 2$$

$$\frac{Ker(d'^1)}{Im(d'^0)} = \frac{Ker(d^1)}{Im(d^0)} = \{p(t)\frac{dz_1}{z_1} \mid p(t) \in \mathcal{O}_{\mathbb{C},0}\}$$

which proves the desired isomorphism in $p \in A$. The conclusion is that:

$$\mathcal{H}^i \cong \frac{Ker(d^i_*)}{Im(d^{i-1}_*)}, 0 \leq i \leq n-1, \ {'\mathcal{H}^n} \cong \frac{g_*\mathcal{E}^n_{V/\mathbb{P}^1}}{Im(d^{n-1}_*)} \tag{15}$$

where

$$0 \xrightarrow{d^{-1}_*} g_*\mathcal{E}^0_{V/\mathbb{P}^1} \xrightarrow{d^0_*} g_*\mathcal{E}^1_{V/\mathbb{P}^1} \xrightarrow{d^1_*} \cdots \xrightarrow{d^{n-1}_*} g_*\mathcal{E}^n_{V/\mathbb{P}^1} \to 0 \tag{16}$$

Lemma 3.2. *Let X be a paracompact Hausdorff space and*

$$0 \to F \xrightarrow{i} F_0 \xrightarrow{d^0} F_1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} F_n \tag{17}$$

an exact sequence of sheaves of Abelian groups. Let also Y be another paracompact Hausdorff space and $g : X \to Y$ a continuous map. Suppose that

$$R^q g_* F_p = 0, \ \forall \ q \geq 1, n > p \geq 0 \tag{18}$$

Then

$$R^p g_* F \cong \frac{Ker(d_*^p)}{Im(d_*^{p-1})},\ 0 \leq p < n$$

and there exists a natural inclusion $R^n g_* F \to \frac{g_* F_n}{Im(d_*^{n-1})}$ *such that we have*

$$0 \to R^n g_* F \to \frac{g_* F_n}{Im(d_*^{n-1})} \to \frac{g_* F_n}{g_* Im(d^{n-1})} \to 0$$

where d_*^p*'s define the sequence*

$$0 \xrightarrow{d_*^{-1}} g_* F_0 \xrightarrow{d_*^0} g_* F_1 \xrightarrow{d_*^1} \cdots \xrightarrow{d_*^{n-1}} g_* F_n$$

Proof. The proof is a slight modification of Lemma 6.3 of [**Nr**]. Put $Z_p = Ker(d^p), 0 \leq p < n$. The first statement is trivial for $p = 0$. Therefore let us prove the first statement for $p \geq 1$. The exactness of (17) at F_p gives us

(19) $$0 \to Z_{p-1} \to F_{p-1} \to Z_p \to 0,\ 1 \leq p < n$$

and we get the long exact sequence

$$\cdots \to R^q g_* F_{p-1} \to R^q g_* Z_p \to R^{q+1} g_* Z_{p-1} \to R^{q+1} g_* F_{p-1} \to \cdots$$

By (18) we conclude that

$$R^q g_* Z_p \cong R^{q+1} g_* Z_{p-1}, 1 \leq p < n, q \geq 1$$

Since $F \cong Z_0$, we have

(20) $$R^p g_* F \cong R^{p-1} g_* Z_1 \cong \cdots \cong R^1 g_* Z_{p-1}, 1 \leq p \leq n$$

(19) gives us also

$$g_* F_{p-1} \xrightarrow{d_*^{p-1}} g_* Z_p \to R^1 g_* Z_{p-1} \to 0,\ 1 \leq p < n$$

and thus

$$R^1 g_* Z_{p-1} \cong \frac{g_* Z_{p-1}}{Im(d_*^{p-1})} = \frac{Ker(d_*^p)}{Im(d_*^{p-1})}, 0 \leq p < n$$

We have proved the first part of the lemma. Now let us prove the second part. We have the short exact sequence

$$0 \to g_* Im(d^{n-1}) \to g_* F_n \to \frac{g_* F_n}{g_* Im(d^{n-1})} \to 0$$

$Im(d_*^{n-1})$ is a subsheaf of both $g_* Im(d^{n-1})$ and $g_* F_n$ so we can rewrite the above exact sequence as:

(21) $$0 \to \frac{g_* Im(d^{n-1})}{Im d_*^{n-1}} \to \frac{g_* F_n}{Im(d_*^{n-1})} \to \frac{g_* F_n}{g_* Im(d^{n-1})} \to 0$$

The short exact sequence

$$0 \to Z_{n-1} \to F_{n-1} \to Imd^{n-1} \to 0$$

gives us

$$g_* F_{n-1} \overset{d_*^{n-1}}{\to} g_* Im(d^{n-1}) \to R^1 g_* Z_{n-1} \to 0$$

Therefore by (20) we have

$$R^n g_* F = \frac{g_* Im(d^{n-1})}{Im(d_*^{n-1})} \tag{22}$$

Note that for this we do not need to have $R^q g_* Im(d^{n-1}) = 0, \forall q \geq 1$. Now (21) and (22) finish the proof. □

Since $V - A \overset{\pi}{\cong} M - \mathcal{R}$ and $f_0 \circ \pi = g \mid_{V-A}$, we can use the symbol f_0 instead of $g \mid_{V-A}$. The following sequence

$$0 \to f_0^* \mathcal{O}_{\mathbb{P}^1} \overset{i}{\to} \mathcal{E}^0_{V/\mathbb{P}^1} \overset{d^0}{\to} \mathcal{E}^1_{V/\mathbb{P}^1} \overset{d^1}{\to} \cdots \overset{d^{n-1}}{\to} \mathcal{E}^n_{V/\mathbb{P}^1} \text{ in } V - A \tag{23}$$

is exact even in the critical points of f_0 (see [**Br**], Proposition 1.7, iii). We can apply Lemma 3.2 to the above sequence and obtain

$$R^i f_{0*}(f_0^* \mathcal{O}_{\mathbb{P}^1}) \cong \frac{Ker(d_*^i)}{Im(d_*^{i-1})} \cong \mathcal{H}^i, \ i < n$$

$$0 \to R^n f_{0*}(f_0^* \mathcal{O}_{\mathbb{P}^1}) \to \frac{g_* \mathcal{E}^n_{V/\mathbb{P}^1}}{Im(d_*^{n-1})} \to \frac{g_* \mathcal{E}^n_{V/\mathbb{P}^1}}{g_* Im(d^{n-1})} \to 0$$

where d_*^i is defined in (16).

Let $'H(p_i)$ be the Brieskorn module of a singularity p_i of g (see the first paragraph of Appendix C). Define

$$\mathcal{C}_c = \begin{cases} 0 & c \text{ is a regular value} \\ \oplus_i {'H}(p_i) & p_i\text{'s are the critical points within } L_c \end{cases} \tag{24}$$

Each stalk $\mathcal{C}_c$ is a free $\mathcal{O}_{\mathbb{P}^1,c}$-module of rank μ_c. There is defined a natural restriction map

$$\pi : {'\mathcal{H}}^n \to \mathcal{C} \tag{25}$$

We denote by $\mathcal{C}'$ its image. Now fix a critical value $c \in C$. The stalk $\mathcal{C}'_c$ is a $\mathcal{O}_{\mathbb{P}^1,c}$-submodule of $\mathcal{C}_c$.

Lemma 3.3. *We have*

$$\mathcal{C}' = \frac{g_* \mathcal{E}^n_{V/\mathbb{P}^1}}{g_* Im(d^{n-1})}$$

Proof. By (15) for $i = n$, $\mathcal{C}'$ is the image of $g_* \tilde{\Omega}^n_{V/\mathbb{P}^1}$ under the projection $g_* \tilde{\Omega}^n_{V/\mathbb{P}^1} \to \mathcal{C}$. The kernel of this map is exactly $g_* Im(d^{n-1})$ and so the proof is finished. □

We know that

$$(R^i f_{0*}(f_0^* \mathcal{O}_{\mathbb{P}^1})) \cong (R^i g_* \mathbb{C}_{V-A} \otimes_{\mathbb{C}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1})$$

(see [**Br**] p. 120). Let $\bar{\mathcal{H}}^i$ be the above sheaf. By definition we have $\bar{\mathcal{H}}^i \mid_{\mathbb{P}^1 - C} = \tilde{\mathcal{H}}^i$. Lemma 3.3 implies that

$$\mathcal{H}^i \cong \bar{\mathcal{H}}^i \ i < n, \ '\mathcal{H}^n \mid_{\mathbb{P}^1 - C} \cong \bar{\mathcal{H}}^n \mid_{\mathbb{P}^1 - C} \tag{26}$$

$$0 \to \bar{\mathcal{H}}^n \to {}'\mathcal{H}^n \to \mathcal{C}' \to 0 \tag{27}$$

(26) and B.1 imply that $\mathcal{H}^i, \ i < n$ (resp. $'\mathcal{H}^n \mid_{\mathbb{P}^1 - C}$) is a freely generated sheaf of rank β_i (resp. β_n). Now for $c \in C$, since the division of $'\mathcal{H}^n_c$ over the freely generated of rank $\beta_n - \mu_c$ $\mathcal{O}_{\mathbb{P}^1, c}$-module $\bar{\mathcal{H}}^n_c$ (see B.1) is freely generated of rank μ_c (see C.2), we conclude that $'\mathcal{H}^n_c$ is freely generated of rank β_n.

Consider a continuous family $\{\delta_t\}_{t \in U}$ of i-dimensional cycles in $g^{-1}(U) - A$ in such a way that δ_t lies in L_t. For any $\omega \in \Omega^i(U)$ the integral $\int_{\delta_t} \omega$ is well-defined. Let γ be a path in U going around t anti-clockwise and Γ be the surface in V formed by the union of curves $\Gamma = \cup_{s \in \gamma} \delta_s$. With the above notation we have

$$\int_{\delta_t} \omega = \frac{1}{2\pi i} \int_{\Gamma} \frac{df \wedge \omega}{f - t}, \quad \frac{d}{dt} \int_{\delta_t} \omega = \int_{\delta_t} \nabla_{\frac{\partial}{\partial t}} \omega \tag{28}$$

For the proof of above equalities see [**AGV**]. By the second formula in (28) we can see that the flat sections of ∇ in $\mathcal{H}^i$ go to the flat sections of ∇ in $\bar{\mathcal{H}}^i$ by the isomorphism in (26) and we know that this isomorphism is obtained by restriction of $\omega \in \mathcal{H}^i(U)$ to the fibers $L_t, t \in U$. This implies that this isomorphism sends $(\mathcal{H}^i, \nabla)$ to $(\bar{\mathcal{H}}^i, \nabla)$. The proof of Theorem 1.1 for $i = 0, 1, \ldots, n-1, 'n$ is finished.

Now let us prove Theorem 1.1 for $i = n, ''n$. There is a natural inclusion $\mathcal{H}^n \subset {}'\mathcal{H}^n$. Let U be a small open disk in $\mathbb{P}^1$ and t a regular holomorphic function in U. By Lemma 1.1 we have also the inclusion

$$'\mathcal{H}^n \mid_U \stackrel{dt\wedge}{\to} {}''\mathcal{H}^n \mid_U$$

We can see that

$$\frac{''\mathcal{H}^n}{'\mathcal{H}^n} \mid_U \cong \Omega^{n+1}_{V/\mathbb{P}^1} \mid_U$$

and $\frac{'\mathcal{H}^n}{\mathcal{H}^n} \stackrel{d(.)}{\to} \frac{''\mathcal{H}^n}{'\mathcal{H}^n}$ is an inclusion and so by C.4 we conclude that $\mathcal{H}^n, ''\mathcal{H}^n$ are locally free sheaves of rank β_n. If $U \subset \mathbb{P}^1 - C$ then the above inclusions are isomorphism of sheaves with connections. □

By the first part of Corollary 1.1 we know that $\mathcal{C}_c/\mathcal{C}_c, \ c \in C$ is a vector space of dimension less than μ_c. I believe that it is zero.

A Complex Geometry

In this appendix we will give all preliminaries in complex analysis and complex geometry used throughout the article. I did not find a book in the literature of complex analysis containing all of these preliminaries and so I have collected them in this appendix.

In what follows by an analytic sheaf over an analytic variety V we mean a $\mathcal{O}_V$-module sheaf. For a given analytic sheaf $\mathcal{S}$ over V, when we write $x \in \mathcal{S}$ we mean that x is a holomorphic section of $\mathcal{S}$ over some open neighborhood in V or it is an element of some stalk of $\mathcal{S}$; being clear from the text which we mean.

Direct Limit Sheaves: Let $\{\mathcal{S}_i\}_i$ be a direct system of sheaves i.e.,

$$\mathcal{S}_0 \to \mathcal{S}_1 \to \cdots \mathcal{S}_i \to \cdots$$

If there is no confusion we write simply $\{\mathcal{S}_i\}$. We define the direct limit of the system, say $\lim_{i\to\infty} \mathcal{S}_i$, to be the sheaf associated to the presheaf $U \to \lim_{i\to\infty} \mathcal{S}_i(U)$. There are defined natural maps $\mathcal{S}_i \to \lim_{i\to\infty} \mathcal{S}_i$.

Let $\mathcal{S}$ be another analytic sheaf and $\{\mathcal{S}_i \to \mathcal{S}\}$ a collection of compatible analytic homomorphisms. Then there is a unique map $\lim_{i\to\infty} \mathcal{S}_i \to \mathcal{S}$ such that for each i, the original map $\mathcal{S}_i \to \mathcal{S}$ is obtained by composing the maps $\mathcal{S}_i \to \lim_{i\to\infty} \mathcal{S}_i \to \mathcal{S}$.

A.1. *Let $\{\mathcal{S}_i\}$ be a direct system of sheaves and $\{\mathcal{S}_i \to \mathcal{S}\}$ a collection of compatible maps. Then $\lim_{i\to\infty} \mathcal{S}_i \to \mathcal{S}$ is an isomorphism if and only if*

1. *For any $x \in \mathcal{S}$ there exist $i \in \mathbb{N}$ and $x_i \in \mathcal{S}_i$ such that $x_i \to x$;*
2. *If there exist $i_0 \in \mathbb{N}$ and a sequence $x_{i_0} \to x_{i_0+1} \to \cdots$, $x_i \in \mathcal{S}_i$ such that $x_i \to 0 \in \mathcal{S}$ then there exists $i_1 \geq i_0$ such that for all $i \geq i_1$ we have $x_i = 0$.*

Proof. The first statement implies the surjectivity and the second one implies the injectivity of $\lim_{i\to\infty} \mathcal{S}_i \to \mathcal{S}$. □

Using the above proposition we can check the following simple facts:

A.2. The short exact sequence

$$0 \to \{\mathcal{L}_i\} \to \{\mathcal{S}_i\} \to \{\mathcal{T}_i\} \to 0$$

gives

$$0 \to \lim_{i\to\infty} \mathcal{L}_i \to \lim_{i\to\infty} \mathcal{S}_i \to \lim_{i\to\infty} \mathcal{T}_i \to 0$$

A.3. For a collection of compatible maps $\{\mathcal{S}_i\} \to \{\mathcal{T}_i\}$ we have

$$\lim_{i\to\infty} Ker(\{\mathcal{S}_i\} \to \{\mathcal{T}_i\}) = Ker(\lim_{i\to\infty} \mathcal{S}_i \to \lim_{i\to\infty} \mathcal{T}_i)$$

$$\lim_{i\to\infty} Im(\{\mathcal{S}_i\} \to \{\mathcal{T}_i\}) = Im(\lim_{i\to\infty} \mathcal{S}_i \to \lim_{i\to\infty} \mathcal{T}_i)$$

One of the important properties of the direct limit sheaf is:

A.4. Let $\{\mathcal{S}_i\}$ be a direct system of sheaves over V. If V is compact then

$$H^\mu(V, \lim_{i\to\infty} \mathcal{S}_i) = \lim_{i\to\infty} H^\mu(V, \mathcal{S}_i), \mu = 0, 1, 2, \ldots$$

Proof. The trick of the proof is that for a finite covering $\mathcal{U}$ of V with Stein open sets every $\alpha \in H^\mu(\mathcal{U}, \lim_{i\to\infty} \mathcal{S}_i)$ ($Z^\mu(\mathcal{U}, \lim_{i\to\infty} \mathcal{S}_i)$ or $B^\mu(\mathcal{U}, \lim_{i\to\infty} \mathcal{S}_i)$) is represented by a finite number of sections. This enables us to check the properties 1 and 2 of Proposition A.1. □

Sheaves with Pole Divisors: Let $\mathcal{S}$ be an analytic sheaf over an analytic compact variety V and D a divisor in V which does not intersect the singular locus of V. By $\mathcal{S}(kD)$ we denote the sheaf of meromorphic sections of $\mathcal{S}$ with poles of multiplicity at most k along D. Also, $\mathcal{S}(*D) = \lim_{k\to\infty} \mathcal{S}(kD)$ denotes the sheaf of meromorphic sections of $\mathcal{S}$ with poles of arbitrary order along D. We list some natural properties of sheaves with poles.

A.5. The short exact sequence of analytic sheaves $0 \to \mathcal{L} \to \mathcal{S} \to \mathcal{T} \to 0$ gives us the short exact sequence $0 \to \mathcal{L}(*D) \to \mathcal{S}(*D) \to \mathcal{T}(*D) \to 0$. In particular if $\mathcal{L}$ is a subsheaf of $\mathcal{S}$ then $\frac{\mathcal{S}}{\mathcal{T}}(*D) = \frac{\mathcal{S}(*D)}{\mathcal{L}(*D)}$.

A.6. The analytic homomorphism of sheaves $d : \mathcal{S} \to \mathcal{T}$ induces a natural analytic homomorphism $d_D : \mathcal{S}(*D) \to \mathcal{T}(*D)$ and

$$ker(d)(*D) = ker(d_D),\ Im(d)(*D) = Im(d_D)$$

A.7. Let D be a divisor in V. We have

$$(\lim_{i\to\infty} \mathcal{S}_i)(*D) = \lim_{i\to\infty} \mathcal{S}_i(*D)$$

A.8. $(\mathcal{S} \otimes_{\mathcal{O}_V} \mathcal{T})(*D) = \mathcal{S} \otimes_{\mathcal{O}_V} \mathcal{T}(*D) = \mathcal{S}(*D) \otimes_{\mathcal{O}_V} \mathcal{T}$

A.9. If $\mathcal{S}$ is coherent then $\mathcal{S}(kD)$ is also coherent. Moreover if V is a compact manifold and D is a positive divisor then there exists an integer k_0 such that

$$H^\mu(V, \mathcal{S}(kD)) = 0,\ k \geq k_0,\ \mu \geq 1$$

Using A.4 and $\mathcal{S}(*D) = \lim_{k\to\infty} \mathcal{S}(kD)$ we have

$$H^\mu(V, \mathcal{S}(*D)) = 0,\ \mu \geq 1$$

A.10. (Vanishing theorem for limit sheaves) Let $\{\mathcal{S}_i\}$ be a direct system of coherent sheaves and D a positive divisor in V. If V is a compact manifold then

$$H^\mu(V, \lim_{i\to\infty} \mathcal{S}_i(*D)) = 0,\ \mu \geq 1$$

Proof. This is a direct consequence of A.4 and A.9. □

Let $d : S \to \mathcal{T}$ be an analytic map between two coherent sheaves on a complex manifold V, D a positive divisor in V and $H^0(d_D) : H^0(V, S(*D)) \to H^0(V, \mathcal{T}(*D))$.

A.11. *We have* $H^0(V, ker(d_D)) = ker(H^0(d_D)),\ H^0(V, Im(d_D)) = Im(H^0(d_D))$.

The coherence of the sheaves and the positivity of the divisor is strongly used in the second equality.

Direct Image Sheaves: The first lines of this paragraph can be found in Chapter 1 Section 4.7 of [**GrRe**]. Let $f : X \to Y$ be a holomorphic map between the analytic varieties X and Y and $\mathcal{S}$ an analytic sheaf on X. For any open Stein subset U of Y we can associate the $\mathcal{O}(U)$-module $H^i(f^{-1}(U), \mathcal{S})$. There are canonical restriction maps and we have an analytic presheaf on Y defined on all open Stein subsets of Y. The associated analytic sheaf on Y is called the i-th direct image of $\mathcal{S}$ and is denoted by $R^i f_* \mathcal{S}$. Every short exact sequence $0 \to \mathcal{L} \to \mathcal{S} \to \mathcal{T} \to 0$ of analytic sheaves over X induces a long exact cohomology sequences

$$0 \to R^0 f_* \mathcal{L} \to R^0 f_* \mathcal{S} \to R^0 f_* \mathcal{T} \to R^1 f_* \mathcal{L} \to R^1 f_* \mathcal{S} \to \cdots$$

The following fact says that the functor $R^i f_*$ and lim commute:

A.12. Let $f : X \to Y$ be a holomorphic map between the analytic varieties X and Y and $\{\mathcal{S}_k\}$ a direct system of analytic sheaves over X. Then

$$\lim_{k \to \infty} R^i f_* \mathcal{S}_k = R^i f_* \lim_{k \to \infty} \mathcal{S}_k$$

The Grauert direct image theorem says when the direct image sheaf $R^i f_* \mathcal{S}$ is coherent:

A.13. (Grauert direct image theorem) Let $f : X \to Y$ be a proper holomorphic map between the analytic varieties X and Y and $\mathcal{S}$ a coherent analytic sheaf over X. Then for any $i \geq 0$ the i-th direct image $R^i f_* \mathcal{S}$ is a coherent analytic sheaf over Y.

Let X' be an analytic subvariety of X and $\mathcal{S}$ an analytic sheaf over X. By structural restriction of $\mathcal{S}$ to X' we mean $\mathcal{S} \mid_{X'} = \frac{\mathcal{S}}{\mathcal{M}.\mathcal{S}}$, where $\mathcal{M}$ is the sheaf of holomorphic functions vanishing on X'. If $\mathcal{S}$ is a coherent $\mathcal{O}_X$-module sheaf then $\mathcal{S} \mid_{X'}$ is a coherent $\mathcal{O}_{X'}$-module sheaf. This restriction is different with the sheaf theorical restriction. In what follows all restrictions we consider are structural except in mentioned cases.

Let $g : V \to \mathbb{P}^1$ be the holomorphic function introduced in in the first section, c a point in $\mathbb{P}^1$ and $\mathcal{S}$ an analytic sheaf on V.

A.14. *Let $\mathcal{S}$ be an analytic sheaf on V. Then*

$$(R^i g_* \mathcal{S})(*D) = R^i g_* (\mathcal{S}(*g^{-1}(D)))$$

where $D = \{p\}$.

The above proposition in general may not be true (for instance when g has multiplicity along $g^{-1}(D)$).

We define $\mathcal{S}_c = \mathcal{S} \mid_{V_c}$ to be the restriction of $\mathcal{S}$ to the fiber $V_c = g^{-1}(c)$. The following natural function is well-defined:

$$g_{c,i} : R^i g_* \mathcal{S} \mid_c \to H^i(V_c, \mathcal{S}_c)$$

A.15. The map $g_{c,i},\ i \geq 0$ is injective.

Proof. Suppose that for an $\alpha \in R^i g_* \mathcal{S} \mid_c$ we have $g_{c,i}(\alpha) = 0$. For a Stein covering $\mathcal{U}$ of V_U, α is represented by an element $\alpha \in H^i(\mathcal{U}, \mathcal{S})$, where U is a small open disk around c. $g_{c,i}(\alpha) = 0$ means that the restriction of α to V_c is zero. In other words there exists a $\beta \in C^{i-1}(\mathcal{U} \cap V_c, \mathcal{S} \mid_{V_c})$ such that $\alpha = \partial\beta$. Since $\mathcal{U}$ is a Stein covering, taking U smaller if it is necessary we can represent β as an element of $C^{i-1}(\mathcal{U}, \mathcal{S})$ (by extending β). Now $\alpha - \partial\beta \mid_{V_c} = 0$ and so $\alpha - \partial\beta = h \circ g.\gamma$ for some $\gamma \in C^i(\mathcal{U}, \mathcal{S})$, where h is a holomorphic regular function on U vanishing on c (here we have used this fact that the multiplicity of g along each irreducible component of V_c is one). Therefore α is zero in $R^i g_* \mathcal{S} \mid_c$. □

The map $g_{c,i}$ need not to be surjective. The obstruction to the surjectivity of $g_{c,i}$ is an element $\alpha \in R^{i+1} g_* \mathcal{S}_c$ with $supp(\alpha) = \{c\}$. Therefore if $R^{i+1} g_* \mathcal{S}$ is freely generated then $g_{c,i}$ is an isomorphism (For more information see [**GrRe**] p. 209).

Our main Theorem in this paragraph which is used frequently in the article is the following:

A.16. (Variational vanishing theorem) Let $g : V \to \mathbb{P}^1$ be as before and $\mathcal{S}$ a coherent sheaf on V. Let also A be the blow-up divisor in V. Then

$$R^i g_* \mathcal{S}(*A) = 0,\ i \geq 1$$

Proof. The main property of A is that it its intersection with each fiber V_c is positive in V_c. Fix a regular value $c \in \mathbb{P}^1$. Since $A_c = A \cap V_c$ is positive in V_c, there exists a natural number k_0 such that

$$H^i(V_c, \mathcal{S}_c(kA_c)) = 0,\ k \geq k_0$$

This and A.15 imply that $R^i g_* \mathcal{S}(kA) \mid_c = 0$. By Grauert direct image theorem $R^i g_* \mathcal{S}(kA)$ is coherent, therefore $R^i g_* \mathcal{S}(kA)$ is the zero sheaf in a neighborhood of c. Now in this neighborhood we have

$$R^i g_* \mathcal{S}(*A) = R^i g_* \lim_{k \to \infty} \mathcal{S}(kA) = \lim_{k \to \infty} R^i g_* \mathcal{S}(kA) = \lim_{k \to \infty} 0 = 0$$

Until now we have proved that $supp(R^i g_* \mathcal{S}(*A)) \subset C$. If we had some type of Kodaira vanishing theorem for a singular variety $V_c, c \in C$ then the proof was complete. But I do not know such a theorem and so I use the following trick: Let b be a regular value in $\mathbb{P}^1$. Since $R^i g_* \mathcal{S}(*A)$ is a discrete sheaf, we have

$$H^0(\mathbb{P}^1, R^i g_* \mathcal{S}(*A)(*b)) = \cup_{c \in C} R^i g_* \mathcal{S}(*A)_c$$

By A.14 we have $R^i g_* \mathcal{S}(*A)(b) = R^i g_* \mathcal{S}(*A + *V_b)$ and so $H^0(\mathbb{P}^1, R^i f_* \mathcal{S}(*A)(*b)) = H^0(\mathbb{P}^1, R^i g_* \mathcal{S}(*A + *V_b)) \subset H^i(V, \mathcal{S}(*A + *V_b))$. $A \cup V_b$ is the pullback of M_b by the blow up map $\pi : V \to M$ and M_b is a hyperplane section of M. Therefore $A \cup V_b$ is a positive divisor and

$$H^i(V, \mathcal{S}(*A + *V_b)) = 0$$

We conclude that $R^i g_* \mathcal{S}(*A)_c = 0$ for $c \in C$ which is the desired. □

Atiyah-Hodge type Theorems: Let V be a projective manifold of dimension n and A a submanifold of V of codimension one. Denote by $\Omega^i(*A)$ the sheaf of meromorphic i-forms in V with poles of arbitrary order along A. We have the following, not necessarily exact, sequence

$$0 \to \mathbb{C} \to H^0(V, \Omega^0(*A)) \xrightarrow{d^0} H^0(V, \Omega^1(*A)) \xrightarrow{d^1} \cdots \xrightarrow{d^{i-1}} H^0(V, \Omega^i(*A)) \xrightarrow{d^i} \cdots$$

We form the cohomology groups

$$\tilde{H}^i = \frac{Ker(d^i)}{Im(d^{i-1})}, \ i \geq 0$$

A.17. *(Atiyah-Hodge Theorem [**Nr**]) Suppose that A is positive in V. Then there are natural isomorphisms*

$$H^i(V - A, \mathbb{C}) \cong \tilde{H}^i$$

Roughly speaking, this theorem says that every cohomology class in $H^i(V - A, \mathbb{C})$ is represented by a closed meromorphic i-form in V with poles along V.

B Some topological facts

All homologies considered in this appendix are with rational coefficients. Recall the notations (3), (4). Let $c \in \mathbb{P}^1$, U a small open disk with center c and b a regular point in the boundary of U. We denote by D the closure of U in $\mathbb{P}^1$. Let also $\{p_i \mid i = 1, 2, \ldots, k\}$ be the singularities within L_c. To each p_i we can associate a set of distinguished vanishing cycles $\{\delta_{ij} \mid j = 1, 2, \ldots, l_k\}$ in $H_n(L_b)$ (see [**AGV**]). Let also μ_c denote the sum of Milnor numbers of singularities within L_c. If c is a regular value of f_0 then $\mu_c = 0$.

B.1. *We have 1. L_c is a deformation retract of L_D 2. $H_{n+1}(L_D) = 0$ 3. $H_i(L_D, L_b) = 0$ for $0 \leq i \leq n$ and $H_{n+1}(L_D, L_b)$ is freely generated of rank μ_c 4. L_U is a Stein manifold 5. There is no linear relation between δ_{ij}'s 6. $\tilde{\mathcal{H}}^i_c, 0 \leq i \leq n-1$ is a freely generated $\mathcal{O}_{\mathbb{P}^1,c}$-module of rank β_i and $\tilde{\mathcal{H}}^n_c$ is a freely generated $\mathcal{O}_{\mathbb{P}^1,c}$-module of rank $\beta_n - \mu_c$.*

Proof. Let us prove the first part. Since out of c the map g is a C^∞ fiber bundle, by homotopy covering theorem (see 14, 11.3,[**St**]) we can take U smaller if it is necessary. Let $B_i, i = 1, 2, \ldots, k$ be an open ball with center p_i whose boundary is transverse to $L_t, t \in D$. $f : (L_D - \cup_i B_i, \partial(L_D - \cup_i B_i)) \to D$ is a C^∞ fibration. Therefore L_D can be retracted to $L_c \cup \cup_i (L_D \cap B_i)$. Now by an argument stated in [**AGV**] p.32 we know that $L_c \cap B_i$ is a deformation retract of $L_D \cap B_i$ and so L_c is a deformation retract of L_D.

Let us prove the second part. Let δ be an $(n+1)$-cycle in L_D. Taking another cycle in the homological class of δ we can assume that δ does not pass through p_i's. This time we take the ball B_i in such a way that it does not intersect δ. Let D' be another small closed disk inside D with center c such that $L_t, t \in D'$ is transverse to ∂B_i. $L_{D'}$ is a deformation retract of L_D and $L_b \cup \cup_i (L_{D'} \cap B_i)$ is a deformation retract of $L_{D'}$, where b is a regular value in the boundary of D'. Therefore δ is homologous to an $(n+1)$-cycle in L_b. But L_b is a Stein manifold of dimension n and so $H_{n+1}(L_b) = 0$. We conclude that $H_{n+1}(L_U) = 0$.

The proof of the third part is the same as (5.4.1) of [**La**]. Instead of (5.5.9)[**La**] we use a similar statement for an arbitrary isolated singularity (see [**AGV**]).

L_U has no non discrete compact analytic set because $L_U = \cup_{t \in U} L_t$ and each L_t is a Stein analytic space. Now let us prove that L_U is holomorphically convex. To see this fact let $p = \infty \notin U$ and $t \in U$. We can consider L_U as a subset of the Stein manifold $M - M_p = L_{\mathbb{C}}$. Every holomorphic function in L_t extends to $M - M_p$ and hence to L_U. Knowing this and the fact that each $L_t, t \in U$ is Stein, we can easily check that L_U is holomorphically convex.

Now let us prove the fifth part. Writing the long exact sequence of the pair (L_D, L_b) we have:

$$\cdots \to H_{n+1}(L_D) \to H_{n+1}(L_D, L_b) \to H_n(L_b) \to H_n(L_D) \to 0 \tag{29}$$

$H_{n+1}(L_D) = 0$ and the vanishing cycles δ_{ij} are images of a basis of $H_{n+1}(L_D, L_b)$ under the boundary map. Therefore there does not exist any linear relation between δ_{ij}'s.

Now let us prove the last part. Since L_c is a deformation retract of L_D, $\tilde{\mathcal{H}}^i_c, i \le n$ is freely generated of rank $dim H_i(L_c)$. By 1,2,3 and the long exact sequence of the pair (L_D, L_b), we have $dim(H_i(L_c)) = \beta_i, i \le n-1$ and $dim(H_n(L_c)) = \beta_n - \mu_c$. □

Both the inclusions $L_b \subset L_D, b \in D$ and $L_c \subset L_D$ induce isomorphisms in i-th homologies, where $i \le n-1$ and $i = n$ if c is a regular value. This means that we have a natural i-th homology bundle, and hence i-th cohomology bundle over $\mathbb{P}^1$, for $i \le n-1$ (for $i = n$ over $\mathbb{P}^1 - C$).

C Local Brieskorn modules

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated critical point at 0. The Brieskorn module

$$'H(0) = \frac{\Omega^n}{df \wedge \Omega^{n-1} + d\Omega^{n-1}}$$

is a freely generated $\mathcal{O}_{\mathbb{C},0}$-module of rank μ (see [**Br**] and [**Se**]), where Ω^i is the set of i-forms in $(\mathbb{C}^{n+1}, 0)$ and μ is the Milnor number of f. Let t be a coordinate system in $(\mathbb{C}, 0)$.

C.1. *Suppose that the restriction of $\{\omega_i \in \Omega^n \mid 1 \le i \le \mu\}$ to a fiber $f^{-1}(t), t \in (\mathbb{C}, 0) - \{0\}$ generates its cohomology group. Then for all $\omega \in \Omega^n$ there exists a natural number h such that $t^h \omega$ belongs to the $\mathcal{O}_{\mathbb{C},0}$-module generated by ω_i's in $'H(0)$.*

Proof. Let $\{\delta_j(t) \mid 1 \le j \le \mu\}$ be a basis of vanishing cycles in $H_n(f^{-1}(t), \mathbb{Z})$. Define the matrices $A = [\int_{\delta_j(t)} \omega_i]_{\mu\times\mu}$ and $B = [\int_{\delta_j(t)} \omega]_{\mu\times 1}$. Define

$$P := A^{-1}B = \frac{adj(A).B}{det(A)} \tag{30}$$

where $adj(A)$ is the adjoint of A. If we change the basis of $H_n(f^{-1}(t), \mathbb{Z})$ and C is the matrix of this change then A changes to $C.A$ and B to $C.B$, therefore $P = (C.A)^{-1}C.B = A^{-1}B$ does not change, particularly when C is the monodromy operator obtained by turning around 0. We conclude that P is a one valued holomorphic function in $(\mathbb{C}, 0) - \{0\}$. (30) implies that$P = \frac{P'}{t^h}$, where h is a natural number and $P' = [p_j], p_j \in \mathcal{O}_{\mathbb{C},0}$. Now

$$\int_{\delta_j(t)} (t^h\omega - \sum_i p_i\omega_i) = 0 \ \forall \delta_j(t)$$

A basis of the freely generated $\mathcal{O}_{\mathbb{C},0}$-module $'H(0)$ generates $H^n(f^{-1}(t), \mathbb{C}), \forall t \in (\mathbb{C}, 0) - \{0\}$. Therefore $t^h\omega - \sum_i p_i\omega_i$ is zero in $'H(0)$. □

Recall (24) and (25). The main proposition in this appendix is:

C.2. *$\mathcal{C}'_c$ is a free $\mathcal{O}_{\mathbb{P}^1,c}$-module of rank μ_c.*

Its proof consists of various steps.

C.3. *$\frac{\mathcal{C}_c}{\mathcal{C}'_c}$ is a finite dimensional vector space.*

Proof. Let a be a regular point in U. Since $A \cap V_a$ is a hyperplane section of V_a and δ_{ij}, $i = 1, 2, \dots, k$ $j = 1, 2, \dots, l_k$ are linearly independent, by Atiyah-Hodge theorem there are meromorphic n-forms ω_{ij} in V_a with poles along $A \cap V_a$ such that $det[\int_{\delta_{ij}} \omega_{ij}]_{\mu_c\times\mu_c} \neq 0$

Consider the sheaf $\mathcal{S}$ of holomorphic n-forms in V_U which are zero restricted to V_a. $\mathcal{S}$ is a coherent sheaf and so by A.16 $H^1(V_U, \mathcal{S}(*A)) = 0$, where $\mathcal{S}(*A)$ is the sheaf of meromorphic sections of $\mathcal{S}$ with poles of arbitrary order along A. This implies that each ω_{ij} extends to V_U as a meromorphic n-form with poles along A. We use the same notations for the extended ones. We conclude that the restriction of the n-forms ω_{ij} to a regular fiber of a singularity $g : (V, p_i) \to (\mathbb{P}^1, c)$ generate its n-th cohomology group. This and C.1 imply that for every $\omega \in \mathcal{C}_c/\mathcal{C}'_c$ there exists a natural number h such that $(t - c)^h\omega = 0$. Let $\Omega = \{\omega_i \mid i = 1, 2, \dots, \mu_c\}$ freely generate $\mathcal{C}_c$ and h be the minimum number such that $(t - c)^h\Omega = 0$ in $\mathcal{C}_c/\mathcal{C}'_c$. Now $\cup_{0\le i\le h-1}(t - c)^i\Omega$ generates $\mathcal{C}_c/\mathcal{C}'_c$ as a vector space. □

C.4. *If S is a free $\mathcal{O}_{\mathbb{C},0}$ module of finite rank k and if R is a submodule, then R is free of rank $l \le k$; one has $l = k$ if and only if* $\dim_{\mathbb{C}} S/K < \infty$.

$\mathcal{O}_{\mathbb{C},0}$ is a principal ideal domain and the proof follows form the structure theroy of modules of principal ideal domains.

Proof of C.2. We know that $\mathcal{C}_c$ is a free $\mathcal{O}_{\mathbb{P}^1,c}$-module of rank μ_c. Also by C.3, $\mathcal{C}_c/\mathcal{C}'_c$ is a finite dimensional vector space. Therefore we can apply C.4 and conclude the theorem. □

I believe that $\mathcal{C}'_c = \mathcal{C}_c$. But the methods in this article are not sufficient to prove this stronger result.

Let $A = \{(z_1, z_2, \ldots, z_n) \in (\mathbb{C}^n, 0) \mid z_1 = 0\}$ and Ω^i be the set of holomorphic i-forms in $(\mathbb{C}^n, 0) - A$. We have the complex

$$0 \to \Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Omega^n \xrightarrow{d^n} 0$$

and so we define $H^i = \frac{Ker(d^i)}{Im(d^{i-1})}$. Let also Ω^i_* be the subset of Ω^i containing the i-forms with poles of arbitrary order along A. In the same way we can define H^i_*.

C.5. *we have* $H^i = H^i_* = 0, i \geq 2$ *and* $H^1 = H^1_* = L$*, where* $L = \{p\frac{dz_1}{z_1} \mid p \in \mathcal{O}_{\mathbb{C}^k,0}\}$

Proof. The proof is completely formal, for instance see [**Gu**] Theorem 3E. We only prove the proposition for H^i. For the other the argument is similar.

Fix the i-form ω with $d\omega = 0$. We want to prove that $\omega = d\eta$ (up to L if $i = 1$), where $\eta \in \Omega^{i-1}$. Let k be the least integer such that the representation of ω contains only $dz_1, dz_2, \ldots, dz_k$ (we have $i \leq k$). The proof is by induction on k. The case $k = 0$ is trivial. We write $\omega = dz_k \wedge \alpha + \beta$, where α and β are differential forms that involve only $dz_1, dz_2, \ldots, dz_{k-1}$. Since $-dz_k \wedge d\alpha + d\beta = 0$, the coefficients of α and β do not depend on $z_{k+1}, \ldots, z_n$. If $k > 1$ then we can write any coefficient of α, say f, as $f = \frac{dg}{dz_k}$ and if $k = 1$ as $f = \frac{p}{z_1} + \frac{dg}{dz_1}$, where $g \in \Omega^0$ and $p \in \mathcal{O}_{\mathbb{C}^k,0}$. Let γ be the differential form obtained from α by replacing each coefficient f by the corresponding coefficient g. Then $d\gamma = \delta + dz_k \wedge \alpha$ (if $k = 1$ then up to L), where δ is a differential form involving only $dz_1, \ldots, dz_{k-1}$. Next set $\theta = \delta - \beta$. We have $d\theta = 0$ and so by induction $\theta = d\eta$ (if $i = 1$ then up to L). Now $\omega = d(\gamma - \eta)$ (if $i = 1$ then up to L). □

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