Modular foliations and periods of hypersurfaces

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Chapter 0

Introduction

The study of algebraic numbers leads naturally to the study of transcendental numbers and among them the numbers obtained by integration. Of particular interest is the case in which the integrand is a differential form obtained by algebraic operations and the integration takes place over a topological cycle of an affine variety. The first non trivial class of such integrals are elliptic integrals $\int R(x, \sqrt{f(x)})$, where $f(x)$ is a polynomial of degree 3 or 4 and $R(x, y)$ is a rational function in x, y. Since the 19th century, many distinguished mathematicians have worked on the theory of elliptic integrals, including Gauss, Abel, Bernoulli, Ramanujan and many others, and still it is an active area mainly due to its application on the arithmetic of elliptic curves (see for instance [68] for a historical account on this). Going to higher genus one has the theory of Jacobian and Abelian varieties and in higher dimension one has the Hodge theory. However, with the development of all these elegant areas it has become difficult to relate them to some simple classical integrals. "... students who have sat through courses on differential and algebraic geometry (read by respected mathematicians) turned out to be acquainted neither with the Riemann surface of an elliptic curve $y^2 = x^3 + ax + b$ nor, in fact, with the topological classification of surfaces (not even mentioning elliptic integrals of first kind and the group property of an elliptic curve, that is, the Euler-Abel addition theorem). They were only taught Hodge structures and Jacobi varieties!" [1].

The present text was written during the many years that I was looking for different aspects of Abelian integrals, or periods, from analytic number theory to Hodge theory and differential equations. There are many books written on this subject, each one of them emphasizing the interests of its author. However, a unified approach to periods seems to be lacking in the literature.

The objective of the present text is twofold: First, we collect all the necessary machinery, such as the algebraic de Rham cohomology of affine varieties, Picard-Lefschetz theory, Hodge structures, Gauss-Manin connections and so on, for studying periods of affine hypersurfaces, and then, we give a unified approach which could be useful for different areas of mathematics. From this point of view the algorithms for calculating the Gauss-Manin connection of affine hypersurfaces, Picard-Fuchs equations and their computer implementation are new. Second, we want to study a new class of differential equations, which have a rich arithmetic and dynamical structure. We consider the case in which an integral depends on many parameters and we look for local analytic subvarieties in the parameter space where the integral is constant for any choice of the underlying topological cycle. It

[1] V.I. Arnold, On teaching mathematics, Palais de Découverte in Paris, 7 March 1997.

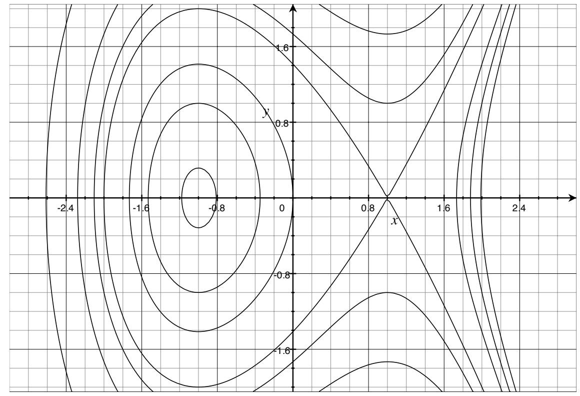

Figure 1: Elliptic curves: $y^2 - x^3 + 3x - t,\ t = -1.9, -1, 0, 2, 3, 5, 10$

turns out that such varieties are part of the leaves of an algebraic foliation in the parameter space. We call them modular foliations. We use the machinery of periods in order to give explicit expressions for such foliations and then we investigate their dynamics and arithmetic.

0.1 Some aspects of Abelian integrals

Let us first clarify what we mean by an Abelian integral or a period. We will use some elementary notations related to algebraic varieties over the field of complex numbers.

Let f be a polynomial in $(n+1)$-variables $x = (x_1, x_2, \ldots, x_{n+1})$. For $n = 0$ (resp. $n = 1$ and $n = 2$) we will use x (resp. (x, y) and (x, y, z)) instead of x_1 (resp. (x_1, x_2) and (x_1, x_2, x_3)). Let also $L_0 := \{f = 0\} \subset \mathbb{C}^{n+1}$ be the corresponding affine variety, ω be a polynomial n-form in $\mathbb{C}^{n+1}$ and $\delta_0 \cong \mathbb{S}^n$ be an n-dimensional sphere C^∞-embedded in L_0 (we call it a cycle). For simplicity we assume that L_0 is smooth. The protagonist of the present text is the number obtained by the integration $\int_\delta \omega$, which we call it an Abelian integral. In fact one can take δ any element in the n-th homology of L_0. Such a number is also called a period of ω (in the literature the name Abelian integral is mainly used for the case $n = 1$). If $f = f_t$ depends on a parameter $t \in T$ with $0 \in T$ then L_0 is a member of the family $L_t := \{f_t = 0\}, t \in T$ and we can talk about the continuous family of cycles $\delta_t \subset L_t$ obtained by the monodromy of δ_0 in the nearby fibers. Therefore, the Abelian integral $\int_{\delta_t} \omega$ is a holomorphic function in a neighborhood of $0 \in T$. To carry an example in mind, take the polynomial $f = y^2 - x^3 + 3x$ in two variables x and y and $f_t := f - t, t \in \mathbb{C}$. Only for $t = -2, 2$ the affine variety L_t is singular and for other values of t, L_t is topologically a torus minus one point (point at infinity). For t a real number between 2 and -2 the level surface of f intersects the real plane $\mathbb{R}^2$ in two connected pieces which one of them is an oval and we can take it as δ_t (with an arbitrary orientation). In this example as t moves from -2 to 2, δ_t is born from the critical point $(-1, 0)$ of f and ends up in the α-shaped piece of the fiber $f^{-1}(2) \cap \mathbb{R}^2$ (see Figure 1).

Planar differential equations and holomorphic foliations: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial mapping and $\delta_t \cong \mathbb{S}^1,\ t \in (\mathbb{R}, 0)$ be a continuous family of ovals in the fibers of f. The level surfaces of f are the images of the solutions of the ordinary differential

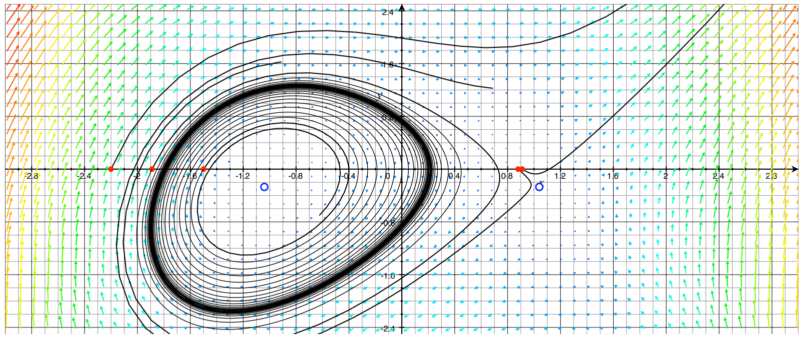

Figure 2: A limit cycle crossing $(x, y) \sim (-1.79, 0)$

equation

$$\mathcal{F}_0 : \begin{cases} \dot{x} = f_y \\ \dot{y} = -f_x \end{cases} . \tag{1}$$

We make a perturbation of $\mathcal{F}_0$

$$\mathcal{F}_\epsilon : \begin{cases} \dot{x} = f_y + \epsilon P(x, y) \\ \dot{y} = -f_x + \epsilon Q(x, y) \end{cases} , \epsilon \in (\mathbb{R}, 0), \tag{2}$$

where P and Q are two polynomials with real coefficients. Usually one expects that in the new ordinary differential equation the cycle δ_0 breaks and accumulates, in positive or negative time, on some part of the real plane or infinity. However, if the Abelian integral $\int_{\delta_t}(Pdy - Qdx)$ is zero for $t = 0$, but not identically zero, then for any small ϵ there will be a limit cycle of $\mathcal{F}_\epsilon$ near enough to δ_0 (see for instance [36, 54, 53]). In other words, δ_0 persists as a limit cycle in the perturbed differential equation. If the Abelian integral is identically zero (for instance if δ_t is homotopic to zero in the complex fiber of f) then the birth of limit cycles is controlled by iterated integrals (see for instance [19, 60]). In our main example take the ordinary differential equation

$$\mathcal{F}_\epsilon : \begin{cases} \dot{x} = 2y + \epsilon \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + \epsilon s y \end{cases} , \epsilon \in (\mathbb{R}, 0). \tag{3}$$

If $\int_{\delta_0}(\frac{x^2}{2}dy - sydx) = 0$ or equivalently

$$s := \frac{-\int_{\Delta_0} x dx \wedge dy}{\int_{\Delta_0} dx \wedge dy} = \frac{5}{7} \frac{\Gamma(\frac{5}{12})\Gamma(\frac{13}{12})}{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})} \sim 0.9025,$$

where Δ_0 is the bounded open set in $\mathbb{R}^2$ with the boundary δ_0, then for ϵ near to 0, $\mathcal{F}_\epsilon$ has a limit cycle near δ_0. In fact for $\epsilon = 1$ and $s = 0.9$ such a limit cycle still exists and it is depicted in Figure (2). The origin of the above discussion comes from the second part of the Hilbert sixteen problem (shortly H16). A weaker version of H16, known as the infinitesimal Arnold-Hilbert problem asks for a reasonable bound for the number of zeros of real Abelian integrals when the degrees of f, P and Q are bounded. There are some partial solutions to this problem but the original problem is still open (see [37, 18]). Even the zero dimensional version of this problem, in which Abelian integrals are algebraic functions, is not completely solved (see [20]).

De Rham cohomologies: A combination of Atiyah-Hodge theorem and Kodaira vanishing theorem implies that the n-th de Rham cohomology (see [65, 56]) of the affine variety L_0 is finite dimensional and it is given by polynomial differential n-forms in $\mathbb{C}^{n+1}$ modulo relatively exact n-forms. This implies that every Abelian integral $\int_{\delta_t}\omega$ can be written as a $\mathbb{C}(t)$-linear combination of $\int_{\delta_t}\omega_i,\ i=1,2,\ldots$, where the ω_i's form a basis of the n-th de Rham cohomology of L_0. In our example, the arithmetic algebraic geometers usually take the differential forms $\frac{dx}{y},\ \frac{xdx}{y}$, which restricted to the regular fibers of f are holomorphic and form a basis of the corresponding de Rham cohomology. The relation of these differential forms and those in the previous paragraph is given by:

$$\int_{\delta_t}(\frac{x^2}{2}dy-sydx)=(-\frac{3}{5}st+\frac{6}{7})\int_{\delta_t}\frac{dx}{y}+(\frac{6}{5}s-\frac{3}{7}t)\int_{\delta_t}\frac{xdx}{y} \tag{4}$$

(see Chapter 3).

Picard-Fuchs equations and Gauss-Manin connections: The Abelian integral $\int_{\delta_t}\frac{dx}{y}$ (resp. $\int_{\delta_t}\frac{xdx}{y}$) satisfies the differential equation

$$\frac{5}{36}I+2tI'+(t^2-4)I''=0 \qquad (\text{ resp. } \frac{-7}{36}I+2tI'+(t^2-4)I''=0) \tag{5}$$

which is called a Picard-Fuchs equation. If we choose another cycle $\delta_t'\in H_1(L_t,\mathbb{Z})$ which together with δ_t form a basis of $H_1(L_t,\mathbb{Z})$ then the matrix $Y=\begin{pmatrix}\int_{\delta_t}\frac{dx}{y} & \int_{\delta_t'}\frac{dx}{y}\\ \int_{\delta_t}\frac{xdx}{y} & \int_{\delta_t'}\frac{xdx}{y}\end{pmatrix}$ forms a fundamental system of the linear differential equation:

$$Y'=\frac{1}{t^2-4}\begin{pmatrix}\frac{-1}{6}t & \frac{1}{3}\\ \frac{-1}{3} & \frac{1}{6}t\end{pmatrix}Y \tag{6}$$

which we call it the Gauss-Manin connection of the family $L_t,\ t\in\mathbb{C}$. The main point behind the calculation of Picard-Fuchs equations and Gauss-Manin connections is the technique of derivation of an integral with respect to a parameter and the simplification of the result in a similar way as in (4). For more details see Chapter 3.

Special functions: The reader may transfer the singularities $-2,2$ of (5) to 0 and 1 and obtain a recursive formula for the coefficients of the Taylor series around 0 of its solutions. Since the integrals $\int_{\delta_t}\frac{dx}{y}$ and $\int_{\delta_t}\frac{xdx}{y}$ are holomorphic around $t=-2$ (this follows from (4)), doing in this way we get:

$$\int_{\delta_t}\frac{dx}{y}=\frac{-2\pi}{\sqrt{3}}F(\frac{1}{6},\frac{5}{6},1|\frac{t+2}{4}),\int_{\delta_t}\frac{xdx}{y}=\frac{2\pi}{\sqrt{3}}F(\frac{-1}{6},\frac{7}{6},1|\frac{t+2}{4})$$

[2] where

$$F(a,b,c|z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,\ c\notin\{0,-1,-2,-3,\ldots\},$$

is the Gauss hypergeometric function and $(a)_n:=a(a+1)(a+2)\cdots(a+n-1)$. Therefore, the Abelian integrals give us a rich class of special functions which can be written as explicit

[2]In order to calculate $a_i=\int_{\delta_{-2}}\frac{x^idx}{y},\ i=0,1$, we use (6) and obtain $a_0+a_1=0$. We also use $\int_2^{\infty}\frac{dx}{(x+1)\sqrt{x-2}}=-\frac{\pi}{\sqrt{3}}$.

convergent series by calculating their Picard-Fuchs equation (Chapter 3) and their values at just one point (Chapter 6).

For more arithmetic oriented aspects of Abelian integrals the reader is referred to [61, 57].

0.2 Modular foliations

A classical approach to the study of a mathematical object is to put it inside a good family and then study it as a member of the family. This is also the case of Abelian integrals. If the parameter space T is 'good' enough then the locus of parameters t, for which $\int_{\delta_t}\omega, \forall \delta_t \in H_n(L_t,\mathbb{Z})$ is constant, is a local holomorphic foliation and one can show that it is a part of a global algebraic foliation in T which we call it a (geometric) modular foliation (see Chapter 3 and 9).

Ramanujan relations: For the family of elliptic curves

$$y^2-4(x-t_1)^3+t_2(x-t_1)+t_3,\ t\in T:=\mathbb{C}^3\backslash\{27t_3^2-t_2^3=0\} \tag{7}$$

and the differential form $\frac{xdx}{y}$ the corresponding modular foliation is given by the ordinary differential equation

$$\begin{cases} \dot{t}_1 = t_1^2-\frac{1}{12}t_2 \\ \dot{t}_2 = 4t_1t_2-6t_3 \\ \dot{t}_3 = 6t_1t_3-\frac{1}{3}t_2^2 \end{cases}. \tag{8}$$

In other words, along the solutions of (8) the differential form $\frac{xdx}{y}$ is a flat section of the Gauss-Manin connection of the family of elliptic curves (7). The above differential equation is obtained using the explicit calculations of the Gauss-Manin connection of the family (7). Such a differential equation in analytic number theory is known as the Ramanujan relations, because Ramanujan observed that the Eisenstein series form a solution of (8) (for more details on this example see Chapter 2 and [57, 58]).

Darboux-Halphen equations: Differential equations of type (8) are even older and many mathematicians such as Darboux, Halphen and Brioschi had already studied the ordinary differential equation

$$\begin{cases} \dot{t}_1+\dot{t}_2 = 2t_1t_2 \\ \dot{t}_2+\dot{t}_3 = 2t_2t_3 \\ \dot{t}_1+\dot{t}_3 = 2t_1t_3 \end{cases} \tag{9}$$

from the analytic point of view (see Chapter 2). The differential equation (9) is related to the family of elliptic curves $y^2=(x-t_1)(x-t_2)(x-t_3)$ in a similar way as the Ramanujan relations and it has also a special solution given by theta series.

In this text we calculate more differential equations similar to (8) and (9). For instance, the system of ordinary differential equations

$$\dot{t}_i = t_1t_2t_3t_4t_5\sum_{i=1}^{5}\left(\frac{-2}{t_i}+\sum_{j=1}^{5}\frac{1}{t_j}\right),\ i=1,2,\ldots,5$$

is related to the family of hyperelliptic curves $y^2 = (x-t_1)(x-t_2)(x-t_3)(x-t_4)(x-t_5)$ in a similar way as the Ramanujan relations (see Chapter 9). Whether the above differential equations have a special solution by some theta series or not, is an open question. An answer to this question needs the development of the theory of differential Siegel modular forms (see for instance [57]).

0.3 Motivations and new results

The main motivation for writing up the present text is the study of a new class of holomorphic foliations which live in the parameter spaces of algebraic varieties. Examples of such differential equations such as (8) and (9) are as old as the word differential equation itself and this might show their importance historically. The text is mainly written for those who work on differential equations and holomorphic foliations and those who wish to explore, apart from the dynamical properties, the arithmetic properties of differential equations. However, the hope is that the text will be also useful for those working on analytic number theory and mathematical physics.

There are many new results in this text. Some of them are: the calculation of de Rham cohomologies, Gauss-Manin connections, Picard-Fuchs equations (Chapter 3), constructing a complex manifold over the Griffiths period domain (Chapter 10), the relation between modular foliations and the Lefschetz-Hodge loci (Chapter 7), geometric interpretation of Ramanujan relations and Darboux-Halphen equations (Chapter 2), the classification of certain holomorphic foliations on abelian varieties with a first integral (§1.6) explicit examples of new modular foliations (Chapter 9). The arithmetic properties of modular foliations is a vast and difficult arena yet to be discovered. For instance, in [58] it is proved that each transcendent leaf of (8) crosses a point with algebraic coordinates at most once. Another important aspect of modular foliations is the algebraic varieties invariant by them. In all the cases which we know, such varieties have geometric interpretations for the fibration.

In the present text we have tried to develop all the machineries for studying integrals associated to a general tame polynomial. Our impression is that further developments of the present text must be done by just considering examples of tame polynomials. The development of differential modular forms for the case of elliptic curves (see [57] and the references there) and the Mirror symmetry associated to a three dimensional family of Calabi-Yau manifolds (see [51]) are testimonies to the fact that each example has its own theory.

The algorithms of the present text are implemented in the library `foliation.lib` of SINGULAR(see [21]) which can be downloaded from the author's web page. However, I have tried to write the text in such a way that the reader can do the calculations by any software in Commutative Algebra. A very important observation is that the calculations in the coefficient space of tame polynomials is not a matter of working with polynomials with small size which fit into a mathematical text. Even for a simple example like a hyperelliptic polynomial of degree 5 each entry of the Gauss-Manin connection matrix occupies half a page. For the mentioned example the modular foliations have simple expressions (see Chapter 9). Therefore, some of the proofs in this text make use of computer and it is almost impossible to follow the proof by hand calculations.

0.4 Synopsis of the contents of this text

Let us now explain the content of each chapter. The reader may skip parts of the text according to the descriptions that we provide below.

In Chapter 1 we introduce a general notion of a modular foliation associated to a connection. Then, we give a list of examples of modular foliations which various authors have worked out. The reader can skip this chapter if he is only interested on the modular foliations coming from geometry, i.e. associated to the Gauss-Manin connection of a fibration. This chapter is mainly written for those who like to work with modular foliations in the complex geometrical context.

Chapter 2 is dedicated to the study of the two examples (8) and (9). Despite of the simplicity of the relation between the Ramanujan relations and the Darboux-Halphen equation (Proposition 2.3), it seems that it has been neglected in the literature until recently. In this chapter we give also the geometric interpretation of the Eisenstein series and theta series in terms of Abelian integrals (Proposition 2.6). This chapter is recommended to those who want to get a flavor of the material of this text.

Chapters 3 and 4 are technical. They contain algebraic and computational pieces of the present text. We introduce our main protagonist, namely a tame polynomial in $n+1$ variables with coefficients in a ring and the corresponding affine variety. We find a canonical basis of the de Rham cohomology of the affine variety, explain the algorithms for calculating discriminants, Gauss-Manin connections and modular foliations. In Chapter 4 we introduce the Gauss-Manin system associated to a tame polynomial. It plays the same role as the de Rham cohomology. The difference between differential forms, which can be formulated in precise words using the notion of the mixed Hodge structure, is more transparent using the Gauss-Manin system. We state the Griffiths transversality theorem which has a direct effect on the codimension of modular foliations. The notion of a mixed Hodge structure associated to a tame polynomial is introduced in this chapter. The reader may skip these Chapters, specially Chapter 3, and return to them occasionally when he needs to know some definitions or notations.

In Chapter 5 we study the topology of an affine variety associated to a tame polynomial. A good source for the topics of this chapter is the book [2]. Since this book is mainly concerned with the local theory of tame polynomials, we have collected and proved some theorems on the topology of tame polynomials. In particular, our approach to the calculations of the intersection matrices of tame polynomials and joint cycles has a slightly new feature. The reader who is interested to know which kind of monodromy groups in the context of the present text appear, must read this chapter and also its abstract version presented in §8.5.

Chapter 6 is dedicated to integrals. In this chapter we combine the algebraic methods of Chapters 3 and 4 and the topological methods of Chapter 5. In this chapter we also introduce some techniques for reducing higher dimensional integrals to lower dimensions.

In Chapter 7 we introduce the Lefschetz-Hodge locus which is invariant under certain modular foliations. We will also state some conjectures which are consequences of the Hodge conjecture. This chapter may be skipped by the reader who is not interested on Hodge theory.

Chapter 8 is dedicated to the discussion of various topological and algebraic concepts related to Fermat type varieties. This chapter may be considered as a source for many examples of tame polynomials.

In Chapter 9 we calculate many modular foliations for tame polynomials in two variables and discuss their properties.

In Chapter 10 we introduce the abstract notion of Abelian integrals, namely a polarized Hodge structure. We construct a moduli of polarized Hodge structures $\mathcal{P}$ which is a complex manifold living on the Griffiths domain D. The modular foliations appear in this new space and not in D. This chapter might be of special interest to readers who are familiar with the Griffiths problem on automorphic type functions on the period domain. The similar problem on $\mathcal{P}$ may be much more reasonable than the Griffith's problem. There, they will find a kind of uniformization of geometric modular foliations.

One may follow the following chart for reading the present text:

$$
\begin{array}{cccccc}
10 & \leftarrow & 1 & 3 & \Rightarrow & 4 \\
 & & \downarrow & \downarrow & & \\
 & & 2 & 6 & \Leftarrow & 5 \\
 & & & \downarrow & \searrow & \\
 & & & 8 & & 7 \\
 & & & \downarrow & & \\
 & & & 9 & &
\end{array}
$$

$A \Rightarrow B$ means that to read Chapter B one has to know the subject of Chapter A and $A \rightarrow B$ means that Chapter B can be read independently of A but frequently one has to consult the material of Chapter A. The preliminaries and notations used at each chapter is explained at the beginning of the chapter.

The reader who wants to get an idea of the contents and results of this text can completely skip the chapters 1, 3, 4,5 6,7, 8 and 10. He may consult only Chapter 2 for a historical account and Chapter 9 for new examples of modular foliations.

During the preparation of the present text I benefited from useful discussions with César Camacho and Jorge Vitório Pereira. I would like to thank them for their help. I wrote some chapters of the present text when I was in Japan and I would like to thank Japan Society for the Promotion of Sciences for the financial support. I would also like to thank IMPA for the lovely research ambient. I would like also to thank Max-Planck Institute for Mathematics in Bonn for a short visit.

Chapter 1

Modular foliations

In this chapter we define the notion of a modular foliation in a smooth complex variety M. It is associated to a global meromorphic section of a vector bundle over M equipped with an integrable connection. Well-known examples of such foliations are due to Darboux, Halphen, Brioschi, Ramanujan and recently Lins-Neto which are presented in this chapter. We assume that the reader is acquainted with a basic knowledge on complex manifolds, vector bundles, sheaves of differential forms and etc.. For a complete account on connections the reader is referred to [12, 40].

1.1 Connections on vector bundles

Let M be a complex manifold, V be a locally free sheaf of rank μ on M and $D = \sum_{i=1}^{s} n_i D_i,\ n_i \in \mathbb{N}$ be a divisor in M. If there is no confusion we will also use V for the corresponding vector bundle. By $\omega \in V$ we mean either a section of V in some open subset of M or a germ of a section. Let

$$\nabla : V \to \Omega^1_M(D) \otimes_{\mathcal{O}_M} V$$

be a connection on V, where $\Omega^1_M(D)$ is the sheaf of meromorphic differential 1-forms η in M such that the pole order of η along $D_i,\ i = 1, 2, \ldots, s$ is less than or equal to n_i and $\mathcal{O}_M$ is the sheaf of holomorphic functions on M. By definition ∇ is $\mathbb{C}$-linear and satisfies the Leibniz rule:

$$\nabla(f\omega) = df \otimes \omega + f\nabla\omega,\ f \in \mathcal{O}_M,\ \omega \in V.$$

The connection ∇ induces:

$$\nabla_p : \Omega^p_M(*D) \otimes_{\mathcal{O}_M} V \to \Omega^{p+1}_M(*D) \otimes_{\mathcal{O}_M} V,$$

$$\nabla_p(\alpha \otimes \omega) = d\alpha \otimes \omega + (-1)^p \alpha \wedge \nabla\omega,\ \alpha \in \Omega^p_M(*D),\ \omega \in V,$$

where $\Omega^p_M(*D)$ is the sheaf of meromorphic differential p-forms η in M with poles of arbitrary order along the support $|D| := \cup_{i=1}^{s} D_i$ of D. If there is no risk of confusion we will drop the subscript p of ∇_p. We say that ∇ is integrable if $\nabla_1 \circ \nabla_0 = 0$. Throughout the text we assume that ∇ is integrable. The set $|D|$ is also called the singular set of ∇.

An element $\omega \in V$ with $\nabla\omega = 0$ is called a flat section. The integrability condition implies that the space of flat sections in a small neighborhood of $b \in M\backslash|D|$ is a $\mathbb{C}$-vector space of dimension μ. The analytic continuation of flat sections gives us the monodromy representation of ∇:

$$h : \pi_1(M\backslash|D|, b) \to \mathrm{GL}(V_b),$$

where V_b is the fiber of the vector bundle V over b (equivalently the μ-dimensional $\mathbb{C}$-vector space of the germs of flat sections around b) and $\pi_1(M\backslash|D|, b)$ is the homotopy group of $M\backslash|D|$ with the base point b.

It is sometimes useful to consider the case in which V is a trivial vector bundle and so it has μ global sections $\omega_i,\ i = 1, 2, \cdots, \mu$ such that in each fiber $V_x,\ x \in M$ they form a basis. We write ∇ in the basis $\omega := (\omega_1, \omega_2, \cdots, \omega_\mu)^{\mathtt{t}}$:

$$\nabla(\omega) = A \otimes \omega,\ A = [\omega_{ij}]_{1\le i,j\le\mu} = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{\mu} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2\mu} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{\mu 1} & \omega_{\mu 2} & \cdots & \omega_{\mu\mu} \end{pmatrix},\ \omega_{ij} \in H^0(M, \Omega^1_M(D)),$$

where $\mathtt{t}$ means the transpose of matrices. The matrix A is called the connection matrix of ∇. We have

$$\nabla(\nabla(\omega)) = \nabla(A \cdot \omega) = d(A) \otimes \omega - A \wedge \nabla(\omega) = (dA - A \wedge A) \otimes \omega$$

and so the integrability condition is given by:

$$dA = A \wedge A$$

or equivalently

$$d\omega_{ij} = \sum_{k=1}^{\mu} \omega_{ik} \wedge \omega_{kj},\ i, j = 1, 2, \ldots, \mu. \tag{1.1}$$

Similar formula as in(10.3) appears in the discussion of frames in Hermitian Geometry (See P.A. Griffiths article [28]). The Leibniz rule implies that for a flat section $\tilde{Y}(t) = Y(t) \cdot \omega, t \in U$ written in the basis ω, $X(t) := Y(t)^{\mathtt{t}}$ satisfies the linear multivariable differential equation:

$$dX = -A^{\mathtt{t}} \cdot X. \tag{1.2}$$

We may also take μ global meromorphic sections $\omega_1, \omega_2, \cdots, \omega_\mu$ of V such that they form a basis of V_x for an x in a dense open subset of M. This will produce unnecessary poles for the differential equation (1.2) which we call them apparent singularities. The monodromy around such singularities is the identity map.

Remark 1.1. We usually use the connection matrix A and the differential equation (1.2) instead of vector bundles and connections. One may associate A to the connection matrix of the trivial vector bundle $V = M \times \mathbb{C}^\mu$ with the connection on V given by

$$\nabla(f) = df + fA,\ f \in \mathcal{O}_M^\mu.$$

Example 1.1. Let V be the trivial bundle and let $\omega = \{\omega_i \mid i = 1, 2, \cdots, \mu\}$ be a set of trivializing sections of V. For a section $v = (f_1, f_2, \ldots, f_\mu) = \sum_{i=1}^{\mu} f_i\omega_i,\ f_i \in \mathcal{O}_M$ of V written in the basis ω, the trivial connection on V is given by:

$$\nabla(v) = (df_1, df_2, \cdots, df_\mu) = \sum_{i=1}^{\mu} df_i \otimes \omega_i$$

Flat sections of ∇ are constant vectors.

Remark 1.2. Let $\tilde{\omega} = S\omega$ be another ordered set of global meromorphic sections of V with the same property as ω. Then

$$\nabla(\tilde{\omega}) = S(S^{-1}dS + A)S^{-1} \otimes \tilde{\omega},$$

where $\nabla\omega = A \otimes \omega$. This can be verified using the Leibniz rule as follows:

$$\begin{aligned}\nabla(\tilde{\omega}) &= \nabla(S\omega) = dS \otimes \omega + S\nabla\omega = dS.S^{-1} \otimes \tilde{\omega} + SAS^{-1} \otimes \tilde{\omega} \\ &= (dS.S^{-1} + SAS^{-1}) \otimes \tilde{\omega}.\end{aligned}$$

Therefore, the connection matrix in the basis $\tilde{\omega}$ is given by:

$$\tilde{A} = dS.S^{-1} + SAS^{-1}.$$

1.2 Linear differential equations

We consider an integrable connection ∇ on a vector bundle V on M and a global meromorphic vector field v in M. Let ∇_v denote the composition

$$V(*) \xrightarrow{\nabla} \Omega^1_M(D) \otimes_{\mathcal{O}_M} V(*) \xrightarrow{v \otimes \mathrm{id}} V(*),$$

where $V(*)$ is the sheaf of meromorphic sections of V, and write

$$\nabla_v^i := \underbrace{\nabla_v \circ \nabla_v \circ \cdots \circ \nabla_v}_{i\text{-times}},\ \nabla_v^0 = \mathrm{id},\ i = 0, 1, 2, \ldots.$$

We can iterate a global meromorphic section η of V under ∇_v and get global meromorphic sections $\nabla_v^i \eta, i = 0, 1, 2, \ldots$ of V. Since V is a vector bundle of rank μ, there exist $m \leq \mu$, μ the rank of V, and global meromorphic functions $p_0, p_1, \ldots, p_m$ on M such that

$$p_0\eta + p_2\nabla_v\eta + p_2\nabla_v^2\eta + \cdots + \nabla_v^m\eta = 0.$$

This is called the linear differential equation of η along the vector field v and associated to the connection V.

1.3 Modular foliations

Consider an integrable connection ∇ on a vector bundle V on M. To each global meromorphic section η of V we associate the following distribution:

$$\mathcal{F}_\eta = \{F_p \mid p \in M \backslash (|D| \cup \mathrm{pol}(\eta))\},$$

where

$$F_p := \{v_0 \in T_pM \mid \nabla_v(\eta)(p) = 0,$$
$$\text{for some vector field } v \text{ in a neighborhood of } p \text{ with } v(p) = v_0\}.$$

There is a dense Zariski open subset U of M such that $\dim_{\mathbb{C}} F_p,\ p \in U$ is a fixed number. We call it the dimension of the distribution $\mathcal{F}_\eta$. The distribution $\mathcal{F}_\eta$ is integrable, i.e. for two holomorphic vector fields v_1, v_2 in some open set U' of U with $v_i(p) \in F_p,\ p \in U'$ we have $[v_1, v_2](p) \in F_p,\ p \in U'$, where $[\cdot, \cdot]$ is the Lie bracket. This follows from

$$\nabla_{[v_1,v_2]} = \nabla_{v_1} \circ \nabla_{v_2} - \nabla_{v_2} \circ \nabla_{v_1}.$$

See [12], p.11 (the reader who knows Persian is also referred to [74] p. 261). The integrability of the distribution $\mathcal{F}_\eta$ implies that there is a foliation, which we denote it again by $\mathcal{F}_\eta$, in U such that for $p \in U$ the tangent space of the foliation $\mathcal{F}_\eta$ at p is given by F_p. Geometrically the leaves of $\mathcal{F}_\eta$ are the loci L of points of M such that η is a flat section of $\nabla\mid_L$, i.e.

$$(\nabla\mid_L)(\eta\mid_L) = 0.$$

A foliation $\mathcal{F}_\eta$ obtained in this way is called a modular foliation. For a meromorphic vector field v on M which is tangent to the foliation $\mathcal{F}_\eta$, by definition we have $\nabla_v(\eta) = 0$ and so the linear differential equation of η along the vector field has the simple form $0 \cdot \eta + 1 \cdot \nabla_v(\eta) = 0$.

The modular foliation $\mathcal{F}_\eta$ can be regarded as a singular holomorphic foliation in M in the following way: We write

$$\nabla\eta = [\eta_1, \eta_2, \cdots, \eta_\mu]\omega, \ \eta_i \in \Omega^1_M(*), \ i = 1, 2, \ldots, \mu,$$

where $\Omega^1_M(*)$ is the sheaf of meromorphic 1-forms in M and $\omega = (\omega_1, \omega_2, \ldots, \omega_\mu)^{\mathsf{t}}$ is a set of global meromorphic sections of V such that for x in a Zariski open subset of M, the entries of $\omega(x)$ form a basis of the fiber of V over x. It is left to the reader to verify that:

$$\mathcal{F}_\eta : \eta_1 = 0, \eta_2 = 0, \cdots, \eta_\mu = 0.$$

Therefore, a modular foliation extends to a singular foliation in M. We also write $\mathcal{F}(\eta_1, \eta_2, \cdots, \eta_\mu)$ to denote the foliation induced by $\eta_1 = 0, \eta_2 = 0, \cdots, \eta_\mu = 0$.

The description of $\mathcal{F}_\eta$ in terms of the connection matrix can be done in the following way: Let us write $\eta = p\omega$, where $p = (p_1, p_2, \ldots, p_\mu)$ and p_i's are global meromorphic functions on M. If $\nabla\omega = A\omega$, where A is the connection matrix of ∇ in the basis ω, then

$$\nabla(\eta) = \nabla(p\omega) = (dp + pA)\omega$$

and so

$$\mathcal{F}_\eta : dp_j + \sum_{i=1}^{\mu} p_i\omega_{ij} = 0, \ j = 1, 2, \ldots, \mu, \tag{1.3}$$

where $A = [\omega_{ij}]_{1\le i,j\le\mu}$. In particular, the foliation $\mathcal{F}_{\omega_i}$ is given by the differential forms of the i-th row of A.

Now, we discuss some examples due the particular form of the connection matrix.

Example 1.2. The integrability condition for the diagonal connection matrix formed by $\omega_{ii}, \ i = 1, 2, \ldots, \mu$ is $d\omega_{ii} = 0$ and so $\mathcal{F}_i : \omega_{ii} = 0$ is modular. The modular foliation $\mathcal{F}_i$ can be also obtained by the 1×1 connection matrix $[\omega_{ii}]$. We will discuss this example in §1.5.

Example 1.3. Assume that all the columns of a connection matrix A are zero except the i-th one. The integrability condition is:

$$d\omega_{1i} = \omega_{1i} \wedge \omega_{ii}, \ d\omega_{2i} = \omega_{2i} \wedge \omega_{ii}, \ \ldots, d\omega_{ii} = 0, \ldots, d\omega_{\mu i} = \omega_{\mu i} \wedge \omega_{ii}$$

The modular foliation $\mathcal{F}_{ji} : \omega_{ji} = 0, \ j \neq i$ can be also obtained from a rank two connection matrix $\begin{pmatrix} \omega_{ii} & 0 \\ \omega_{ji} & 0 \end{pmatrix}$. Lins Neto's examples (§1.7) fit into this class of modular foliations.

Example 1.4. (Triangular connections) For a lower triangular matrix A the integrability condition is:

$$d\omega_{ii} = 0,\ d\omega_{i+1,i} = (\omega_{i+1,i+1} - \omega_{i,i}) \wedge \omega_{i+1,i},\ \cdots,$$

For the connection matrix

$$\begin{pmatrix} \omega_1 & 0 & 0 & \cdots & 0 \\ \omega_2 & 2\omega_1 & 0 & \cdots & 0 \\ \omega_3 & \omega_2 & 3\omega_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega_\mu & \omega_{\mu-1} & \omega_{\mu-2} & \cdots & \mu\omega_1 \end{pmatrix}$$

the integrability condition reads:

$$d\omega_1 = 0,\ d\omega_i = (i-1)\omega_1 \wedge \omega_i,\ i = 2, \cdots, \mu.$$

This shows that modular foliations associated to a connection may have all possible codimensions. This can be also seen by taking the diagonal connection matrix.

Example 1.5. For rank two connection matrix A from the integrability condition one reads:

$$d\omega_{11} = \omega_{12} \wedge \omega_{21}, d\omega_{12} = \omega_{12} \wedge (\omega_{22} - \omega_{11})$$

This implies that the $d\omega_{12} \wedge \omega_{12} = 0$ and so we have the foliation $\mathcal{F}(\omega_{12})$ which contains the modular foliation $\mathcal{F}(\omega_{11}, \omega_{12})$. A similar discussion holds for ω_{21}. Examples for this situation are the Darboux-Halphen equation (§2.1) and the modular foliation given by Ramanujan relations (§2.2).

1.4 Operations on connections and modular foliations

Given an operation on vector bundles and their connections such as direct sum, tensor product and etc., we may construct the corresponding operation on modular foliations. In this section we explain two such operations, namely dual and wedge product connection, and leave others to the reader.

Let $M, V, \nabla, \cdots$ be as in §1.1. Let also $\check{V}$ be the dual vector bundle of V. There is defined a natural dual connection:

$$\check{\nabla} : \check{V} \to \Omega^1_M(D) \otimes_{\mathcal{O}_M} \check{V},$$

which satisfies

$$\langle \check{\nabla}\delta, \omega\rangle = d\langle \delta, \omega\rangle - \langle \delta, \nabla\omega\rangle,\ \delta \in \check{V},\ \omega \in V,$$

where

$$\langle \delta, \omega\rangle := \delta(\omega),\ \langle \alpha \otimes \delta, \omega\rangle = \langle \delta, \alpha \otimes \omega\rangle = \alpha \cdot \langle \delta, \omega\rangle,\ \delta \in \check{V},\ \omega \in V,\ \alpha \in \Omega^1_M(D).$$

The integrability of ∇ implies that $\check{\nabla}$ is also integrable. If $e = \{e_1, e_2, \ldots, e_\mu\}$ is a basis of flat sections in a neighborhood of $b \in M\backslash|D|$ then we can define its dual as follows: $\langle \delta_i, e_j\rangle = 0$ if $i \neq j$ and $= 1$ if $i = j$. We can easily check that δ_i's are flat sections. The associated monodromy for $\check{\nabla}$ with respect to this basis is just the transpose of the monodromy of ∇ in the basis e. It is easy to verify that if A is the connection matrix of ∇ in a basis ω then $-A^{\mathsf{t}}$ is the connection matrix of $\check{\nabla}$ in the dual basis $\check{\omega}$. We knew that for a connection matrix A, the differential forms in a row of A define a modular foliation. Now, we know that the differential forms in a column of A also define a modular foliation.

We can define a natural connection on $\wedge^k V = \{\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k \mid \omega_i \in V\}$ with the pole divisor D as follows:

$$\nabla(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} \omega_1 \wedge \omega_2 \wedge \cdots \widehat{\omega_i, \nabla\omega_i} \cdots \wedge \omega_k,$$

where $\widehat{\omega_i, \nabla\omega_i}$ means that we replace ω_i by $\nabla\omega_i$. The integrability of ∇ implies the integrability of $\wedge^k V$. Using wedge product operation, one can derive more modular foliations from a connection. For instance, if $A = [\omega_{ij}]_{1\leq i,j\leq\mu}$ is the connection matrix of ∇ then $[\omega]_{1\times 1}$, where $\omega := \sum_{i=1}^{\mu} \omega_{ii}$, is the connection matrix of $\wedge^\mu \nabla$ and so we have the modular foliation associated to the closed 1-form ω (the fact that ω is closed can be checked directly from the equalities (1.1)).

In the next sections we discuss some examples of modular foliations which are already studied by many authors.

1.5 Foliations induced by closed forms

Let us consider a connection on a line bundle V. For a global meromorphic section ω of V we can write $\nabla\omega = \omega_{11} \otimes \omega$. The integrability condition implies that $d\omega_{11} = 0$. The choice of another $\omega' = p\omega$, p being a global meromorphic function on M, will replace ω_{11} with $\omega_{11} + \frac{dp}{p}$. Now, the modular foliation $\mathcal{F}_{\omega_1}$ is given by $\omega_{11} = 0$.

Conversely, if a foliation is given by a meromorphic closed differential form ω_{11} then it is modular in the following way: we consider the connection on the trivial line bundle whose connection matrix is the 1×1 matrix $[\omega_{11}]$.

The classification of meromorphic closed 1-forms in the projective spaces is as follows: Let $M = \mathbb{P}^n$ be the projective space of dimension n and $D = -\sum_{i=1}^{s} n_i D_i,\ n_i \in \mathbb{N}$ be the pole divisor of a differential 1-form ω_{11} in $\mathbb{P}^n$. Denote the homogeneous coordinates of $\mathbb{P}^n$ by $(x_0, x_1, \ldots, x_n)$ and let D_i be given by the homogeneous polynomial $f_i(x_0, x_1, \ldots, x_n)$. Let Ω be the pull-back of the 1-form ω_{11} by the canonical projection $\mathbb{C}^{n+1} \to \mathbb{P}^n$. If $d(\omega_{11}) = 0$ then there are complex numbers $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$ and a homogeneous polynomial $g(x_0, x_1, \ldots, x_n)$ such that:

1. If $n_i = 1$ then $\lambda_i \neq 0$ and if $n_i > 1$ then f_i does not divide g,

2. Ω can be written

$$\Omega = (\sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}) + d(\frac{g}{f_1^{n_1-1} \cdots f_s^{n_s-1}}), \tag{1.4}$$

3.
$$\sum_{i=1}^{s} \deg(f_i)\lambda_i = 0,\ \deg(g) = \sum_{i=1}^{s} (n_i - 1)\deg(f_i),$$

(See [9]).

Remark 1.3. The foliation $\mathcal{F}(\omega_{11})$ has the first integral

$$f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_s^{\lambda_s} \exp\left(\frac{g}{f_1^{n_1-1} \cdots f_s^{n_s-1}}\right).$$

If λ_i's are rational numbers then by taking a power of the above function we can assume that λ_i's are integers and so the first integral is of the form Ae^B, where A, B are two rational functions with poles along $|D|$. Moreover, A has zeros only along $|D|$.

1.6 Foliations on abelian varieties

In this section we are going to consider modular foliations in toruses/abelian varieties. For an introduction to such varieties, the reader is referred to [44] in the analytic context and to [49] in the algebraic context. We state and prove Proposition 1.2 which, as far as I know, has not been appeared in the literature, although it is a simple corollary of many theorems on abelian varieties.

Let $G = (\mathbb{Z}^{2n}, +)$ and $e_1, e_2, \cdots, e_{2n}$ be a basis of the $\mathbb{R}$-vector space $\mathbb{C}^n$. We have the action of G on $\mathbb{C}^n$ given by:

$$(a_1, a_2, \cdots, a_{2n}) \cdot z := z + \sum_{i=1}^{2n} a_i e_i, \ (a_1, a_2, \cdots, a_{2n}) \in G, \ z \in \mathbb{C}^n.$$

The torus $A := G \backslash \mathbb{C}^n = \mathbb{C}^n / \Gamma$, where $\Gamma = G \cdot 0$ and 0 is the origin of $\mathbb{C}^n$, is a complex compact manifold with the trivial tangent bundle. It is called an abelian variety if in addition it is a projective manifold. For each linear map $f : \mathbb{C}^n \to \mathbb{C}$, the differential form df is invariant under the action of G and so it gives us a holomorphic differential form ω in A with $d\omega = 0$. Vice versa any holomorphic 1-forms in A is induced by a linear map f. We denote by Ω^1_A the space of holomorphic differential forms in A. We conclude that for $\omega \in \Omega^1_A$ we have the modular foliation $\mathcal{F}(\omega)$ induced by $\omega = 0$.

A torus A has the canonical holomorphic maps

$$g_a : A \to A, \ g_a(x) = x + a, \ n_A : A \to A, \ n_A(x) = nx, \ n \in \mathbb{N}, \ a \in A.$$

We have $g_a^*(\omega) = \omega$ and $n_A^* \omega = n\omega$ for $\omega \in \Omega^1_A$. Therefore, we have a biholomorphism $g_{b-a} : L_a \to L_b$ and a holomorphic map $L_a \to L_{na}$, where L_a denotes the leaf of $\mathcal{F}(\omega)$ through $a \in A$. In this way, L_{0_A} turns out to be a complex manifold with a group structure and every leaf of $\mathcal{F}(\omega)$ is biholomorphic to L_{0_A}. Here 0_A is the zero of the group $(A, +)$.

We are interested to know when a leaf of $\mathcal{F}(\omega), \omega \in \Omega^1_A$ is an analytic subvariety of A. If A is an abelian variety then by GAGA principle an analytic closed subvariety of A is an algebraic subvariety of A. From now on we work with abelian varieties. According to the above discussion if $\mathcal{F}(\omega)$ has an algebraic leaf then all the leaves of $\mathcal{F}(\omega)$ are algebraic subvarieties of A. Let us first recall some terminology related to abelian varieties.

Let A_1, A_2 be two abelian varieties of the same dimension. An isogeny between A_1 and A_2 is a surjective morphism $f : A_1 \to A_2$ of algebraic varieties with $f(0_{A_1}) = 0_{A_2}$. It is well-known that every isogeny is a group homomorphism and there is another isogeny $g : A_2 \to A_1$ such that $g \circ f = n_{A_1}, \ f \circ g = n_{A_2}$. The isogeny f induces an isomorphism $f^* : \Omega^1_{A_2} \to \Omega^1_{A_1}$ of $\mathbb{C}$-vector spaces.

An abelian variety is called simple if it does not contain a non trivial abelian subvariety. For $A = A_1 = A_2$ simple, it turns out that $\mathrm{End}_0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a division algebra, i.e. it is a ring, possibly non-commutative, in which every non-zero element has an inverse. If A is an elliptic curve, i.e. $\dim A = 1$, $\mathrm{End}_0(A)$ is $\mathbb{Q}$ or a quadratic imaginary field ($\mathbb{Q}(\sqrt{-d})$, d square free positive integer). In the second case it is called a CM elliptic curve. Every abelian variety is isogenous to the direct product $A_1^{k_1} \times A_2^{k_2} \times \cdots \times A_s^{k_s}$ of simple, pairwise non-isogenous abelian varieties A_i and this decomposition is unique up to isogeny and permutation of the components.

Proposition 1.1. *For a holomorphic differential form on an abelian variety A, a leaf of $\mathcal{F}(\omega)$ is algebraic if and only if there is a morphism of abelian varieties $f : A \to E$ for some elliptic curve E such that ω is the pull-back of some holomorphic differential form in E.*

Proof. We prove the non-trivial part of the proposition. We assume that a leaf of $\mathcal{F}(\omega)$ is algebraic. Using g_a's we have seen that all the leaves of $\mathcal{F}(\omega)$ are algebraic and in particular, the leaf A_1 through $0 \in A$ is algebraic. It has the induced group structure and so it is an abelian subvariety of A. Poincaré reducibility theorem (see [70] p. 86 and [49] Proposition 12.1 p. 122) implies that there is an abelian subvariety E of A such that $f : E \times A_1 \to A,\ (a, b) \mapsto a + b$ is an isogeny (here we need that A to be an abelian variety and not just a torus). Take an isogeny $g : A \to E \times A_1$ such that $f \circ g = n_A$ for some $n \in \mathbb{N}$. Since $f^*\omega$ restricted to each fiber of the projection on the first coordinate map $\pi : E \times A_1 \to E$ is zero, there is a differential form ω_1 in E such that $\pi^*\omega_1 = f^*\omega$. The composition $\pi \circ g$ and the differential form $\frac{1}{n}\omega_1$ are the desired objects. Since $\dim(A_1) = n - 1$, E is an elliptic curve. □

Remark 1.4. Let $\omega_{11}, \omega_{22}, \ldots, \omega_{nn}$ be a basis of the $\mathbb{C}$-vector space Ω_A^1. To associate all the foliations $\mathcal{F}(\omega),\ \omega := \sum_{i=1}^n t_i\omega_{ii},\ t_i \in \mathbb{C}$ to one connection we proceed as follows: In the trivial bundle $V = A \times \mathbb{C}^{n+1}$ we consider the connection matrix:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \omega_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{nn} & 0 & \cdots & 0 \end{pmatrix},$$

given in the canonical sections $\omega_i,\ i = 0, 1, \ldots, n$ of V (ω_0 is a global flat section). In other words, the connection is given by:

$$\nabla(\omega_i) = \omega_{ii} \otimes \omega_1,\ i = 0, 1, \ldots, n + 1, \omega_{00} := 0.$$

This connection is integrable and $\mathcal{F}(\omega) = \mathcal{F}_\eta$, where $\eta = \sum_{i=1}^n t_i\omega_i$. Note that η runs through all holomorphic sections of V.

Let $\mathbb{P}^{n-1}(A) \cong \mathbb{P}(\Omega_A^1)$ be the space of holomorphic foliations $\mathcal{F}(\omega),\ \omega \in \Omega_A^1$.

Proposition 1.2. *There is an isomorphism $\mathbb{P}^{n-1}(A) \cong \mathbb{P}^{n-1}$ such that under this isomorphism the subspace of $\mathbb{P}^{n-1}(A)$ containing foliations with only algebraic leaves corresponds to:*

$$\mathbb{P}_1^{n_1-1}(k_1) \cup \mathbb{P}_2^{n_2-1}(k_2) \cup \cdots \cup \mathbb{P}_r^{n_r-1}(k_r),$$

where $\mathbb{P}_i^{n_i-1},\ i = 1, 2, \ldots, r$ are projective subspaces of $\mathbb{P}^{n-1}$ (defined over $\mathbb{Q}$) which do not intersect each others, $k_i \subset \mathbb{C}$ is $\mathbb{Q}$ or a quadratic imaginary field, and $\mathbb{P}_i^{n_i-1}(k_i)$ is the set of k_i-rational points of $\mathbb{P}_i^{n_i-1}$.

Note that we have

$$\sum_{i=1}^{r} n_i \leq n.$$

Proof. Let $P(A)$ be the subspace of $\mathbb{P}^{n-1}(A)$ containing foliations with only algebraic leaves. An isogeny $A \to B$ between two abelian varieties induces an isomorphism $\mathbb{P}^{n-1}(A) \to \mathbb{P}^{n-1}(B)$ which sends $P(A)$ to $P(B)$. Therefore, it is enough to prove the proposition for $A = A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_s^{n_s}$, where A_i's are pairwise non-isogenous simple abelian varieties. Let us order A_i's in such a way that $A_i,\ i = 1, 2, \cdots r$ are elliptic curves and other components $A_i, r < i \leq s$ are abelian varieties of dimension bigger than 1. The fields mentioned in the proposition are $k_i := \mathrm{End}_0(A_i), i = 1, 2, \ldots, r$. It is well-known that k_i is either $\mathbb{Q}$ or a quadratic imaginary field. In the second case there is two different embedding of k_i in $\mathbb{C}$. Both have the same image in $\mathbb{C}$. We choose differential forms $\omega_i \in \Omega^1_{A_i},\ i = 1, 2, \ldots, r$ and this gives us an embedding of k_i in $\mathbb{C}$ obtained by:

$$a \mapsto \tilde{a}, \quad \text{where } a \in \mathrm{End}(A_i),\ a^*\omega_i = \tilde{a}\omega_i.$$

Define $\omega_{ij} = \pi_{ij}^*(\omega_i),\ j = 1, 2, \ldots, n_i$, where $\pi_{ij} : A_i^{n_i} \to A_i$ is the projection in the j-th coordinate. The differential forms $\omega_{ij},\ i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, n_i$ form a basis of $\Omega^1_{A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_r^{n_r}}$. Let $\delta_{ij} : A_i \to A$ be the map of the form $x \mapsto (\cdots, 0, x, 0, \cdots)$, where x lies in the j-th coordinate of $A_i^{n_i}$. We have

$$P(A) = P(A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_r^{n_r}) = \cup_{i=1}^r P(A_i^{n_i}) = \cup_{i=1}^r \mathbb{P}^{n_i-1}(k_i).$$

Here we have identified each piece of A by its image in A through the maps δ_{ij}. The first and second equalities are obtained from the following: If $f : A \to E$, E simple, is a non-trivial morphism of abelian varieties then E is exactly isogenous to one of A_i's and the composition $A_j \overset{\delta_{js}}{\to} A \to E,\ s = 1, 2, \ldots, n_j$ is zero if $j \neq i$ and is an isogeny or zero for $j = i$. For $\mathcal{F}(\omega),\ \omega \in \Omega^1_A$ with only algebraic leaves, we have used Proposition 1.1 and obtained a first integral $f : A \to E$ and $\omega' \in \Omega^1_E$ with $f^*\omega' = \omega$, where E an elliptic curve.

Let us now prove the last equality. For a morphism $f : A_i^{n_i} \to E$, E an elliptic curve, and $\omega' \in \Omega^1_E$ we remark that $(f \circ \delta_{ij})^*\omega' = a_{ij}\omega_i$ and so $f^*(\omega') = \sum_j a_{ij}\omega_{ij}$. We have $[a_{i1} : a_{i2} : \cdots : a_{in_i}] \in \mathbb{P}^{n_i-1}(k_i)$ because

$$((f \circ \delta_{ij})^{-1} \circ (f \circ \delta_{ij'}))^*(\omega_i) = \frac{a_{ij}}{a_{ij'}}\omega_i$$

up to multiplication by a rational number (here by $(f \circ \delta_{ij})^{-1}$ we mean any isogeny $g : E \to A_i$ such that $g \circ (f \circ \delta_{ij}) = n_{A_i}$ and $(f \circ \delta_{ij}) \circ g = n_E$ for some $n \in \mathbb{N}$).

Conversely, let $\omega = \sum_j a_{ij}\omega_{ij},\ a_{ij} \in k_i$. After multiplication of ω with an integer number, there are isogenies $f_j : A_i \to A_i,\ j = 1, 2, \ldots, n_i$ such that $f_j^*(\omega_i) = a_{ij}\omega_i$. For $g : A_i^{n_i} \to A_i,\ g(x) = f_1(x) + f_2(x) + \cdots + f_{n_i}(x)$ we have $g^*(\omega_i) = \omega$.

$\square$

Corollary 1.1. *For an abelian variety A of dimension two and the space of holomorphic foliations $\mathcal{F}(\omega),\ \omega \in \mathbb{P}(\Omega^1_A)$ exactly one of the following statements is true:*

1. *There is no holomorphic foliation with only algebraic leaves;*

2. *There are exactly two holomorphic foliations with only algebraic leaves;*

3. *There are two foliations $\mathcal{F}(\omega_i)$, $i = 1, 2$ with only algebraic leaves and all other foliations $\mathcal{F}(\omega)$ with this property are given by $\omega = \omega_1 + t\omega_2, t \in k$, where k is either $\mathbb{Q}$ or an imaginary quadratic field.*

Proof. Up to isogeny every abelian variety of dimension two is either simple or $A_1 \times A_2$ or A_1^2, where A_1 and A_2 are two non-isogenous elliptic curves. These three cases correspond to the three cases of the corollary. □

A typical example of an abelian variety with many elliptic factors is the Jacobian of the Fermat curve $x^n + y^n = 1$ (see [42]). A situation similar to the third item of Corollary 1.1 will occur in Lins Neto's examples (§1.7).

1.7 Lins-Neto's examples

The pencils $P_i : \mathcal{F}(\omega_i + t\eta_i),\ i = 1, 2,\ t \in \mathbb{P}^1$, where

$$\omega_1 = (4x - 9x^2 + y^2)dy - 6y(1 - 2x)dx, \eta_1 = 2y(1 - 2x)dy - 3(x^2 - y^2)dx,$$

$$\omega_2 = y(x^2 - y^2)dy - 2x(y^2 - 1)dx, \eta_2 = (4x - x^3 - x^2y - 3xy^2 + y^3)dy + 2(x + y)(y^2 - 1)dx$$

are studied by A. Lins Neto in [45]. They satisfy

$$d\omega_i = \alpha_i \wedge \omega_i,\ i = 1, 2,$$

where

$$\alpha_i := \lambda_i \frac{dQ_i}{Q_i},\ \lambda_1 = \frac{5}{6},\ \lambda_2 = \frac{3}{4},$$

$$Q_1 = -4y^2 + 4x^3 + 12xy^2 - 9x^4 - 6x^2y^2 - y^4,\ Q_2 = (y^2 - 1)(x + 2 + y^2 - 2x)(x^2 + y^2 + 2x).$$

Consider the connection in the trivial rank 3 bundle V over $\mathbb{C}^2$ given by the connection matrix:

$$A_i := \begin{pmatrix} \alpha_i & 0 & 0 \\ \omega_i & 0 & 0 \\ \eta_i & 0 & 0 \end{pmatrix},\ i = 1, 2.$$

We have $\mathcal{F}(\omega_i + t\eta_i) = \mathcal{F}_{v_{i,t}}$, where $v_{i,t}$ is the section of V given by $x \mapsto x \times (0, 1, t),\ x \in \mathbb{C}^2$. Therefore, all the elements of the pencil P_i are associated to a linear family of sections of V. Lins Neto has proved that the set

$$E_i = \{t \in \mathbb{P}^1 \mid \mathcal{F}_{v_{i,t}} \text{ has a meromorphic first integral}\}$$

is $\mathbb{Q} + \mathbb{Q}e^{2\pi i/3}$ for $i = 1$ and is $\mathbb{Q} + i\mathbb{Q}$ for $i = 2$. This is similar to Corollary 1.1 for abelian varieties.

1.8 Halphen equations

In a series of article ([32, 33, 31]) Halphen studied the following system of ODE's:

$$\mathrm{H}(\alpha) = \mathrm{H}(\alpha_1, \alpha_2, \alpha_3) : \begin{cases} \dot{t}_1 = (1 - \alpha_1)(t_1t_2 + t_1t_3 - t_2t_3) + \alpha_1 t_1^2 \\ \dot{t}_2 = (1 - \alpha_2)(t_2t_1 + t_2t_3 - t_1t_3) + \alpha_2 t_2^2 \\ \dot{t}_3 = (1 - \alpha_3)(t_3t_1 + t_3t_2 - t_1t_2) + \alpha_3 t_3^2 \end{cases} \tag{1.5}$$

with $\alpha_i \in \mathbb{C} \cup \{\infty\}$ (if for instance $\alpha_1 = \infty$ then the first row is replaced with $-t_1t_2 - t_1t_3 + t_2t_3 + t_1^2$). He showed that if $\phi_i,\ i = 1, 2, 3$ are the coordinates of a solution of $\mathrm{H}(\alpha)$ then

$$\frac{1}{(cz+d)^2}\phi_i(\frac{az+b}{cz+d}) - \frac{c}{cz+d},\ i = 1, 2, 3,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

are also coordinates of a solution of $\mathrm{H}(\alpha)$. He conclude that it is enough to find one solution of $\mathrm{H}(\alpha)$ and then use the above formula to obtain the general solution. He then construct a particular solution of $\mathrm{H}(\alpha)$ using the hypergeometric functions.

Let a, b, c be defined by the equations:

$$1 - \alpha_1 = \frac{a-1}{a+b+c-2}, 1 - \alpha_2 = \frac{b-1}{a+b+c-2},\ 1 - \alpha_3 = \frac{c-1}{a+b+c-2}.$$

Proposition 1.3. *The foliation $\mathcal{F}(\mathrm{H}(\alpha))$ is modular associated to the second row of the connection matrix $A = \sum_{i=1}^{3} A_i dt_i$, where*

(1.6)
$$\mathrm{A}_1 = \frac{1}{(\mathrm{t}_1 - \mathrm{t}_2)(\mathrm{t}_1 - \mathrm{t}_3)}\cdot$$

$$\begin{pmatrix} \frac{1}{2}((a+c-1)t_2 + (a+b-1)t_3 + (b+c-2)t_1) & -a-b-c+2 \\ at_2t_3 + (b-1)t_1t_3 + (c-1)t_1t_2 & -\frac{1}{2}((a+c-1)t_2 + (a+b-1)t_3 + (b+c-2)t_1) \end{pmatrix}$$

and A_2 (resp. A_3) is obtained from A_1 by changing the role of t_1 and t_2 (resp. t_1 and t_3).

Proof. We write $A = \begin{pmatrix} * & * \\ \omega_{21} & \omega_{22} \end{pmatrix}$ and look at $\mathrm{H}(\alpha)$ as a vector field. An explicit calculation shows that $\omega_{21} \wedge \omega_{22} \neq 0$ and $\omega_{21}(\mathrm{H}(\alpha)) = \omega_{22}(\mathrm{H}(\alpha)) = 0$ for any $a, b, c \in \mathbb{C} \cup \{\infty\}$. The first one implies that the modular foliation $\mathcal{F} : \omega_{21} = 0, \omega_{22} = 0$ is of codimension two and the second one implies that it is given by the trajectories of $\mathrm{H}(\alpha)$. □

The reader is referred to [59] for the geometric interpretation of the connection matrix A. In particular, one can find the proof of the following proposition:

Proposition 1.4. *The integrals*

$$(t_1 - t_3)^{-\frac{1}{2}(1-a-c)}(t_1 - t_2)^{-\frac{1}{2}(1-a-b)}(t_2 - t_3)^{-\frac{1}{2}(1-b-c)} \int_\delta \frac{xdx}{(x-t_1)^a(x-t_2)^b(x-t_3)^c},$$

where δ is path in $\mathbb{C}\backslash\{t_1, t_2, t_3\}$ connecting two points in t_1, t_2, t_3, ∞, as local multi valued functions in t_1, t_2, t_3 are constant along the solutions of the Halphen equation $\mathrm{H}(\alpha)$.

Remark 1.5. Consider the connection matrix $A = \sum_{i=1}^{3} A_i dt_i$, where

(1.7)
$$\mathrm{A}_1 := \frac{1}{(\mathrm{t}_1 - \mathrm{t}_2)(\mathrm{t}_1 - \mathrm{t}_3)} \begin{pmatrix} -at_1 + (a+c-1)t_2 + (a+b-1)t_3 & -a-b-c+2 \\ at_2t_3 + (b-1)t_1t_3 + (c-1)t_1t_2 & (-a-b-c+2)t_1 \end{pmatrix}$$

and A_2 (resp. A_3) is obtained from A_1 by changing the role of t_1 and t_2 (resp. t_1 and t_3). It is obtained from the connection in Proposition 1.3 by subtracting it from $\frac{dp}{p} I_{2\times 2}$, where

$$p = (t_1 - t_3)^{-\frac{1}{2}(1-a-c)}(t_1 - t_2)^{-\frac{1}{2}(1-a-b)}(t_2 - t_3)^{-\frac{1}{2}(1-b-c)}$$

and I_2 is the identity 2×2 matrix. The modular foliation associated to the second row of A is given by

$$\left\{\begin{array}{l}\dot t_1 = (a-1)t_2t_3 + bt_1t_3 + ct_1t_2\\ \dot t_2 = at_2t_3 + (b-1)t_1t_3 + ct_1t_2\\ \dot t_3 = at_2t_3 + bt_1t_3 + (c-1)t_1t_2\end{array}\right. \tag{1.8}$$

if $a+b+c\neq 2$ and is is given by the $(2,1)$-entry of A if $a+b+c=2$ (in the first case the modular foliation is of codimension two and in the second case it is of codimension one). The integral

$$\int_\delta \frac{xdx}{(x-t_1)^a(x-t_2)^b(x-t_3)^c},$$

where δ is as in Proposition 1.4, is constant along the trajectories of (1.8).

Chapter 2

Darboux, Halphen, Brioschi and Ramanujan equations

In this chapter, we discuss some examples of differential equations which are historically old and their common feature is that they have special solutions given by explicit series. Historically, the first example of differential equations which has a particular solution given by theta constants was studied by Jacobi in 1848. Later, Halphen 1881, Brioschi 1881, Chazy 1909 and Ramanujan 1916 found differential equations for various convergent series. For a recent account on this topic the reader is referred to the works [30, 1, 67, 66] and the references within there. We are mainly interested in those differential equations whose associated foliations are modular in our context.

2.1 Darboux-Halphen equations

In 1881, G. Halphen considered the non-linear differential system

$$\begin{cases} \dot{t}_1+\dot{t}_2=2t_1t_2 \\ \dot{t}_2+\dot{t}_3=2t_2t_3 \\ \dot{t}_1+\dot{t}_3=2t_1t_3 \end{cases},$$

(see [33]) which originally appeared in G. Darboux's work in 1878 on triply orthogonal surfaces in $\mathbb{R}^3$(see [11]). We write the above equations in the ODE's form:

$$\mathrm{H}:\begin{cases} \dot{t}_1=t_1(t_2+t_3)-t_2t_3 \\ \dot{t}_2=t_2(t_1+t_3)-t_1t_3 \\ \dot{t}_3=t_3(t_1+t_2)-t_1t_2 \end{cases} \tag{2.1}$$

Halphen expressed a solution of the system (2.1) in terms of the logarithmic derivatives of the null theta functions; namely,

$$u_1=\frac{1}{2}(\ln\theta_4(0|z))', u_2=\frac{1}{2}(\ln\theta_2(0|z))', u_3=\frac{1}{2}(\ln\theta_3(0|z))'$$

where

$$\begin{cases} \theta_2(0|z):=\sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \\ \theta_3(0|z):=\sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \\ \theta_4(0|z):=\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \end{cases}, \ q=e^{2\pi iz},\ z\in\mathbb{H}.$$

In fact H is the special case $\mathrm{H}(0,0,0)$ of (1.5) considered in §1.8. Therefore, we have

Proposition 2.1. The foliation $\mathcal{F}(\mathrm{H})$ is modular associated to the second row of the connection matrix

$$A_{\mathrm{H}} = \frac{dt_1}{2(t_1-t_2)(t_1-t_3)}\begin{pmatrix} -t_1 & 1 \\ t_2t_3 - t_1(t_2+t_3) & t_1 \end{pmatrix} +$$

$$\frac{dt_2}{2(t_2-t_1)(t_2-t_3)}\begin{pmatrix} -t_2 & 1 \\ t_1t_3 - t_2(t_1+t_3) & t_2 \end{pmatrix} + \frac{dt_3}{2(t_3-t_1)(t_3-t_2)}\begin{pmatrix} -t_3 & 1 \\ t_1t_2 - t_3(t_1+t_2) & t_3 \end{pmatrix}.$$

2.2 Ramanujan relations

S. Ramanujan in 1916 proved that $g = (g_1, g_2, g_3)$, where g_k's are the Eisenstein series

$$g_k(z) = a_k(1 + (-1)^k \frac{4k}{B_k}\sum_{n\geq 1}\sigma_{2k-1}(n)q^n), \quad k = 1,2,3, \ z \in \mathbb{H},$$

$$B_1 = \frac{1}{6},\ B_2 = \frac{1}{30},\ B_3 = \frac{1}{42},\ \ldots,\ \sigma_i(n) := \sum_{d|n} d^i,$$

$$(a_1, a_2, a_3) = (\frac{2\pi i}{12}, 12(\frac{2\pi i}{12})^2, 8(\frac{2\pi i}{12})^3),$$

satisfies the ODE's

$$\mathrm{R}: \begin{cases} \dot{t}_1 = t_1^2 - \frac{1}{12}t_2 \\ \dot{t}_2 = 4t_1t_2 - 6t_3 \\ \dot{t}_3 = 6t_1t_3 - \frac{1}{3}t_2^2 \end{cases} \tag{2.2}$$

(see [69]). One can easily verify that:

Proposition 2.2. *The foliation $\mathcal{F}(\mathrm{R})$ is modular associated to the second row of the connection matrix:*

$$A_{\mathrm{R}} := \frac{1}{\Delta}\begin{pmatrix} -\frac{3}{2}t_1\alpha - \frac{1}{12}d\Delta & \frac{3}{2}\alpha \\ \Delta dt_1 - \frac{1}{6}t_1 d\Delta - (\frac{3}{2}t_1^2 + \frac{1}{8}t_2)\alpha & \frac{3}{2}t_1\alpha + \frac{1}{12}d\Delta \end{pmatrix}, \tag{2.3}$$

where $\Delta = 27t_3^2 - t_2^3$, $\alpha = 3t_3dt_2 - 2t_2dt_3$.

The proof is an explicit calculation similar to the one in Proposition 1.3.

2.3 Ramanujan vs. Darboux-Halphen

The reader may have noticed that there must be a relation between the foliation induced by the Ramanujan relations and the Halphen equation. It seems to me that historically such a relation is neglected by mathematicians until recently.

Proposition 2.3. *1. The algebraic morphism*

$$\alpha : (t_1, t_2, t_3) \mapsto (T, 4\sum_{1\leq i<j\leq 3}(T-t_i)(T-t_j), 4(T-t_1)(T-t_2)(T-t_3)),$$

$$\textit{where } \ T := \frac{1}{3}(t_1+t_2+t_3),$$

satisfies

$$\alpha^* A_{\mathrm{R}} = A_{\mathrm{H}},$$

where $\alpha^*\omega$ is the pull-back of the differential form ω. For ω a matrix it is the pull-back of entries.

2. *Looking* H *and* R *as vector fields in* $\mathbb{C}^3$, α *maps* H *to* R *and*

3. *the solution of* H *given by theta series is mapped to the solution of* R *given by Eisenstein series.*

Proof. The first and second part of the above proposition are mere calculation. The reader who does not like to do calculations will find another proof in the next section. The third part follows from the equality $\frac{1}{3}(u_1+u_2+u_3) = g_1$ (see [1]) and the fact that two solutions of R whose first coordinates coincides are the same because:

$$t_2 = 12(t_1^2 - \dot{t}_1),\ t_3 = \frac{1}{6}(4t_1t_2 - \dot{t}_2)$$

□

Both Halphen equation and Ramanujan relations are intimately related to the theory of elliptic curves and integrals. This will be explained in §2.4.

2.4 The geometric interpretation of Darboux-Halphen and Ramanujan equations

Let f be one of the polynomials

$$f_{\mathrm{R}} = y^2 - 4(x-t_1)^3 + t_2(x-t_1) + t_3, \tag{2.4}$$

$$f_{\mathrm{H}} = y^2 - 4(x-t_1)(x-t_2)(x-t_3),\ t = (t_1,t_2,t_3) \in \mathbb{C}^3$$

which depends on the parameter $t = (t_1, t_2, t_3)$. We will use subscript R (resp H) to denote the corresponding object associated to f_{R} (resp. f_{H}). We have the following family of elliptic curves

$$E_t : f = 0,\ t \in \mathbb{C}^3 \backslash \{\Delta = 0\},$$

where Δ is the discriminant of f:

$$\Delta_{\mathrm{R}} = 27t_3^2 - t_2^3,\quad \Delta_{\mathrm{H}} = -(t_1-t_2)^2(t_2-t_3)^2(t_3-t_1)^2. \tag{2.5}$$

The map α in Proposition (2.3) maps the parameter space of f_{H} to the parameter space of f_{R}. Note that

$$\Delta_{\mathrm{R}} \circ \alpha = \Delta_{\mathrm{H}}.$$

The period domain is defined to be

$$\mathcal{P} := \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}) \mid \mathrm{Im}(x_1\overline{x_3}) > 0, x_1x_4 - x_2x_3 = 1 \right\}. \tag{2.6}$$

It lives over the Poincaré upper half plane $\mathbb{H} := \{x+iy \mid Im(y) > 0\}$ (the mapping $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto \frac{x_1}{x_2}$). We have the period map

$$\mathsf{pm} : T \to \mathcal{P},\ t \mapsto \frac{1}{\sqrt{2\pi i}} \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix}$$

where $\sqrt{i} = e^{\frac{2\pi i}{4}}$,

$$T := \mathbb{C}^3 \backslash \{t \in \mathbb{C}^3 \mid \Delta(t) = 0\}$$

and (δ_1, δ_2) is a basis of the $\mathbb{Z}$-module $H_1(\{f=0\}, \mathbb{Z})$ such that the intersection matrix in this basis is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In fact pm is multivalued and its different values correspond to the different choices of (δ_1, δ_2). The fact that the image of pm lies in determinant one matrices follows from the Legendre relation between elliptic integrals.

Proposition 2.4. *For $f = f_{\mathrm{R}}$ (resp. $f = f_{\mathrm{H}}$) we have*

$$d\mathsf{pm} = \mathsf{pm} \cdot A^{\mathsf{t}}$$

where A is the matrix A_{R} in Proposition 2.2 (resp. A_{H} in Proposition 2.1). In other words, a fundamental system of solutions for the linear differential equation $dx = A \cdot x$ in $\mathbb{C}^3$ is given by pm^{t}.

The proof of the above Proposition is a mere calculation. The case $f = f_{\mathrm{R}}$ is established by Griffiths [23] and Sasai [72]. A proof which works for bigger class of polynomials f, namely tame polynomials, is presented in Chapter 3. An immediate corollary of Propositions 2.1, 2.2, 2.4 is the following:

Proposition 2.5. *Let $\mathcal{F}$ be one of the foliations $\mathcal{F}(\mathrm{R})$ or $\mathcal{F}(\mathrm{H})$.*

1. *The integrals $\int_\delta \frac{xdx}{y}, \delta \in H_1(E_t, \mathbb{Z}),\ \Delta(t) \neq 0$ as a function in t are constants along the leaves of $\mathcal{F}$.*
2. *In particular, we have the real one-valued first integral*

$$B(t) := \mathrm{Im}\left(\int_{\delta_1} \frac{xdx}{y} \overline{\int_{\delta_2} \frac{xdx}{y}}\right)$$

for the foliation $\mathcal{F}$ restricted to $\mathbb{C}^3 - \{\Delta = 0\}$, where $\{\delta_1,\ \delta_2\}$ is a basis of $H_1(E_t, \mathbb{Z})$ with $\langle \delta_1, \delta_2 \rangle = 1$.

Recall that a function B is called a first integral of a foliation $\mathcal{F}$ if its level surfaces are invariant under the foliation $\mathcal{F}$.

Proof. Let $f = f_{\mathrm{R}}$. The first part follows form:

$$d(\mathsf{pm}(t))(\mathrm{R}(t)) = \mathsf{pm}(t) A^{\mathsf{t}}(\mathrm{R}(t)) = \mathsf{pm}(t) \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}.$$

To prove the second part we first note that B is one valued: Let $\delta' = (\delta'_1, \delta'_2)^{\mathsf{t}}$ be the monodromy of $\delta = (\delta_1, \delta_2)^{\mathsf{t}}$ along a closed path in $\mathbb{C}^3 \backslash \{\Delta = 0\}$. Since the monodromy is an isomorphism in the corresponding homologies and it preserves the intersection of cycles we have

$$\delta' = B\delta,\ B \in \mathrm{SL}(2,\mathbb{Z}).$$

We write B using the cycles in δ' and then substitute $B\delta$ for δ' and we get the definition of B using δ. The fact that B is constant along the leaves of $\mathcal{F}$ follows from the first part and the definition of B. $\square$

Proof of Proposition 2.3, 1,2 . Let pm_1 (resp. pm_2) be the period map associated to f_{R} (resp. f_{H}). By definition we have $\mathsf{pm}_1 \circ \alpha = \mathsf{pm}_2$ and so by Proposition 2.4 we have

$$\begin{aligned} \mathsf{pm}_2 \cdot \alpha^* A_{\mathrm{R}}^{\mathsf{t}} &= (\mathsf{pm}_1 \circ \alpha)\cdot \alpha^* A_{\mathrm{R}}^{\mathsf{t}} = \alpha^*(\mathsf{pm}_1 \circ A_{\mathrm{R}}^{\mathsf{t}}) = \alpha^*(d\mathsf{pm}_1) \\ &= d(\mathsf{pm}_1 \circ \alpha) = d\mathsf{pm}_2 = \mathsf{pm}_2 \cdot A_{\mathrm{H}}^{\mathsf{t}} \end{aligned}$$

which implies $\alpha^* A_{\mathrm{R}} = A_{\mathrm{H}}$.

To prove the second part note that Propositions 2.1,2.2 and the first part imply that α maps H to cR for some rational number c. In order to prove that $c = 1$ we proceed as follows: the analytic function $a(z) := \alpha(u_1(z), u_2(z), u_3(z))$ is a solution of cR and hence $a(c^{-1}z)$ is a solution of R. The first coordinate of any solution of R satisfies the so called Chazy equation[1]

$$t_1''' + 18(t_1')^2 - 12t_1t_1'' = 0. \tag{2.7}$$

Since we know that $\frac{1}{3}(u_1 + u_2 + u_3) = g_1$ (see [1]), we conclude that $g_1(z)$ and $g_1(c^{-1}z)$ are solutions of the Chazy equation. Replacing both in (2.7) we conclude that $c = 1$. $\square$

2.5 Special solutions

For the Darboux-Halphen vector field H (resp. Ramanujan vector field R) we have an special solution given by theta series (resp. Eisenstein series) and the reader may ask from where they come from. In Ramanujan's case the answer is easy because he knew first the Eisenstein series and then by some derivation manipulations he obtained the differential equation (2.2). In Darboux-Halphen case, I did not found the answer from Halphen's original works. In any way, it is natural to expect that such special solutions can be written in terms of elliptic integrals:

Let $\{\delta_1,\ \delta_2\}$ be a basis of $H_1(E_t,\mathbb{Z}),\ \Delta(t) \neq 0$ with $\langle \delta_1, \delta_2\rangle = 1$.

Proposition 2.6. *Consider the vector field* R *and the corresponding family of elliptic curves.*

1. *The functions*

$$I_1 := t_1(\int_{\delta_2} \frac{dx}{y})^2 - (\int_{\delta_2} \frac{xdx}{y})(\int_{\delta_2} \frac{dx}{y}),\ I_2 := t_2(\int_{\delta_2} \frac{dx}{y})^4,\ I_3 := t_3(\int_{\delta_2} \frac{dx}{y})^6$$

   can be written in terms of the variable

$$z := \frac{\int_{\delta_1} \frac{dx}{y}}{\int_{\delta_2} \frac{dx}{y}}$$

[1] The classical Chazy equation is written in the form $t_1''' + 3(t_1')^2 - 2t_1t_1'' = 0$. We have to multiply a solution of (2.7) with 6 in order to obtain this one.

2. *The vector $I(z) = (I_1, I_2, I_3)$ viewed as a function of z is a solution of the vector field* R.

3. *More precisely, we have*
$$I_1 = g_1,\ I_2 = g_2,\ I_3 = g_3,$$
where g_i's are the Eisenstein series in (2.2).

In a similar way for H we have

$$t_i(\int_{\delta_2} \frac{dx}{y})^2 - (\int_{\delta_2} \frac{xdx}{y})(\int_{\delta_2} \frac{dx}{y}) = u_i(z),\ i = 1, 2, 3,$$

where (u_1, u_2, u_3) is the Halphen's solution for H (see §2.1). We could state only the third part of Proposition 2.6 because the first and second part follows from the third one. However, note that in general finding explicit convergent series as solutions to differential equations is hard or impossible.

Proof. For simplicity define the functions x_i according to the equality:

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} := \frac{1}{\sqrt{2\pi i}} \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix}. \tag{2.8}$$

By Proposition 2.4 we have
$$dx = x \cdot A_{\mathrm{R}}^{\mathrm{t}}.$$

To prove the first part of Proposition we must verify that

$$dz \wedge dI_i = 0,\ i = 1, 2, 3, \tag{2.9}$$

and z is a regular function. The first one implies that I_i's are constant along the fibers of z and the second implies that I_i's can be written in terms of z. We note that

$$\begin{aligned} dz &= d(\frac{x_1}{x_3}) \\ &= \frac{x_3 dx_1 - x_1 dx_3}{x_3^2} \\ &= \frac{x_3(x_1(-\frac{3}{2}t_1\alpha - \frac{1}{12}d\Delta) + x_2(\frac{3}{2}\alpha)) - x_1(x_3(-\frac{3}{2}t_1\alpha - \frac{1}{12}d\Delta) + x_4(\frac{3}{2}\alpha))}{\Delta x_3^2} \\ &= \frac{\frac{3}{2}\alpha(x_2x_3 - x_1x_4)}{\Delta x_3^2} = -\frac{3\alpha}{2\Delta x_3^2}. \end{aligned}$$

The equality $x_1x_4 - x_2x_3 = 1$ is the Legendre relation between elliptic integrals. The above calculation shows that z is regular function in $\mathbb{C}^3 \backslash \{\Delta = 0\}$.

We just prove the equality (2.9) for $i = 2$ and left the others to the reader.

$$\begin{aligned} dI_2 &= d(t_2 x_3^4) \\ &= x_3^3(x_3 dt_2 + 4t_2 dx_3) \\ &= x_3^3(x_3 dt_2 + 4t_2 \frac{1}{\Delta}(x_3(-\frac{3}{2}t_1\alpha - \frac{1}{12}d\Delta) + x_4(\frac{3}{2}\alpha)) \\ &= \frac{1}{\Delta}x_3^3((6x_4t_2 - 6t_2t_1x_3)\alpha + x_3(\Delta dt_2 - \frac{1}{3}t_2 d\Delta)) \\ &= \frac{1}{\Delta}x_3^3((6x_4t_2 - 6t_2t_1x_3)\alpha + x_3(9t_3\alpha)) \\ &= \frac{1}{\Delta}x_3^3(6x_4t_2 - 6t_2t_1x_3 + 9x_3t_3)\alpha. \end{aligned}$$

Now, let us prove the second part. We just prove the second line of 2.2 and leave the others to the reader:

$$\frac{dI_2}{dz} = -\frac{2}{3}x_3^5(6x_4t_2 - 6t_2t_1x_3 + 9x_3t_3) = 4I_1I_2 - 6I_3$$

The proof of the third part follows from Weierstrass uniformization theorem and can be found in [58]. □

2.6 Automorphic properties of the special solutions

Let us define

$$\mathrm{SL}(2,\mathbb{Z}) = \Gamma(1) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad - bc = 1\}$$

and

$$\Gamma(d) := \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{mod}\ d\},\ d \in \mathbb{N}.$$

For a holomorphic function defined in $\mathbb{H}$ let also

$$(f \mid_m^0 A)(z) := (cz+d)^{-m} f(Az),\ (f \mid_m^1 A)(z) := (cz+d)^{-m} f(Az) - c(cz+d)^{-1},$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}),\ m \in \mathbb{N}.$$

Proposition 2.7. *If $\phi_i,\ i = 1,2,3$ are the coordinates of a solution of* R*(resp.* H*) then*

$$\phi_1 \mid_2^1 A,\ \phi_2 \mid_4^0 A,\ \phi \mid_6^0 A$$

(resp.

$$\phi_i \mid_2^1 A,\ i = 1,2,3,$$

) are also coordinates of a solution of R *(resp.* H*) for all $A \in \mathrm{SL}(2,\mathbb{C})$. The subgroup of* $\mathrm{SL}(2,\mathbb{C})$ *which fixes the solution given by Eisenstein series (resp. theta series) is* $\mathrm{SL}(2,\mathbb{Z})$ *(resp.* $\Gamma(2)$*).*

Proof. The first part of the proposition is a mere calculation and it is in fact true for a general Halphen equations (see 1.8). The second part is easy and it is left to the reader. □

2.7 Moduli space interpretation

The Ramanujan foliation or Darboux-Halphen foliation lives on a moduli space and this may be a motivation for the reader to define the general notion of modular foliations on moduli spaces. Let us state the precise statements. For simplicity, we consider elliptic curves over complex numbers.

Proposition 2.8. *The quasi affine variety*

$$T_{\mathrm{R}} := \mathbb{C}^3 \backslash \{t \in \mathbb{C}^3 \mid \Delta_{\mathrm{R}}(t) = 0\}$$

is the moduli of (E,ω_1,ω_2)*'s, where* E *is an elliptic curve defined over* $\mathbb{C}$, $\{\omega_1,\omega_2\}$ *is a basis of* $H^1_{\rm dR}(E)$ *such that* ω_1 *is represented by a holomorphic differential form of the first kind on* E *and* $\frac{1}{2\pi i}\int_E \omega_1\wedge\omega_2=1$*. In a similar way*

$$T_{\rm H}:=\mathbb{C}^3\backslash\{t\in\mathbb{C}^3\mid \Delta_{\rm H}(t)=0\}$$

is the moduli of $(E,\omega_1,\omega_2,a_1,a_2,a_3)$*, where* E,ω_1,ω_2 *are as before and* (a_1,a_2,a_3) *is an ordered triple of non-zero 2-torsion points of* E*, i.e.* $2a_i=0,\ i=1,2,3$.

Proof. Let us prove the first part. We denote by E_t the elliptic curve $\{f_{{\rm R},t}=0\}$. First, note that each point of $T_{\rm R}$ denotes the triple $(E_t,[\frac{dx}{y}],[\frac{xdx}{y}])$. The equality $\int_E[\frac{dx}{y}]\wedge[\frac{xdx}{y}]=2\pi i$ follows from the Legendre relation between elliptic integrals. We have the action of

$$G_0=\left\{\begin{pmatrix} k_1 & k_2\\ 0 & k_1^{-1}\end{pmatrix}\mid k_2\in\mathbb{C}, k_1\in\mathbb{C}^*\right\} \tag{2.10}$$

on $T_{\rm R}$ given by

$$t\bullet g:=(t_1k_1^{-2}+k_2k_1^{-1},t_2k_1^{-4},t_3k_1^{-6}),\ t\in T_{\rm R}, g=\begin{pmatrix} k_1 & k_2\\ 0 & k_1^{-1}\end{pmatrix}\in G_0. \tag{2.11}$$

It turns out that $(E_{t\bullet g},[\frac{dx}{y}],[\frac{xdx}{y}])$ and $(E_t,k_1[\frac{dx}{y}],[k_1^{-1}\frac{xdx}{y}+k_2\frac{dx}{y}])$ are isomorphic triples. The j invariant

$$j:T_{\rm R}\to\mathbb{C},\ j(t):=\frac{t_2^3}{27t_3^2-t_2^3}$$

classifies the elliptic curves over $\mathbb{C}$ (see [34] Theorem 4.1). Therefore, for a triple (E,ω_1,ω_2) as in the proposition we can find a parameter $t\in T_{\rm R}$ such that $E\cong E_t$ over $\mathbb{C}$. Under this isomorphism we write

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}=g^{\mathsf{t}}\begin{pmatrix}[\frac{dx}{y}]\\ [\frac{xdx}{y}]\end{pmatrix},\quad \text{in}\quad H^1_{\rm dR}(E_t)$$

for some $g\in G_0$. Now, the triple (E,ω_1,ω_2) is isomorphic to $(E_{t\bullet g},\frac{dx}{y},\frac{xdx}{y})$. Since $j:\mathbb{C}^3/G_0\to\mathbb{C}$ is an isomorphism, every triple (E,ω_1,ω_2) is represented exactly by one parameter $t\in T$.

The proof of the second part of the proposition is similar. Note that the non-zero 2-torsion points of $E_t:=\{f_{{\rm H},t}=0\}$ are $(x,y)=(t_1,0),(t_2,0),(t_3,0)$. The zero element of the group E_t is the point at infinity. □

2.8 Transcendency of leaves vs. transcendency of numbers

An interesting property of the the holomorphic foliation $\mathcal{F}({\rm R})$ is the following:

Theorem 2.1. *([58]) We have:*

1. *For any point* $t\in T_{\rm R}$*, the set* $\bar{\mathbb{Q}}^3\cap L_t$*, where* L_t *is the leaf of* $\mathcal{F}({\rm R})$ *through* t*, is empty or has only one element. In other words, every transcendent leaf contains at most one point with algebraic coordinates.*

2. *The image of the solution* g *given by Eisenstein series never intersects* $\bar{\mathbb{Q}}^3$.

Proof. Concerning the second part of the theorem we need the following: We look at $I = (I_1, I_2, I_3)$ in Proposition 2.6 as a function from $T_{\rm R}$ to itself. Using the notation (2.8) we have

$$\mathsf{pm}(I) = \mathsf{pm}(t \bullet \begin{pmatrix} x_3^{-1} & -x_4 \\ 0 & x_3 \end{pmatrix}) = x \begin{pmatrix} x_3^{-1} & -x_4 \\ 0 & x_3 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{x_3} & -1 \\ 1 & 0 \end{pmatrix}$$

Therefore, for the cycle $\delta_2 \in H_1(E_I, \mathbb{Z})$ we have $\int_{\delta_2} \frac{xdx}{y} = 0$.

Using Proposition 2.5, part 1, the proposition follows from:

1. For an elliptic curve E defined over $\bar{\mathbb{Q}}$ and $0 \neq \delta \in H_1(E(\mathbb{C}), \mathbb{Z})$, $0 \neq \omega \in H^1_{\rm dR}(E)$ the period $\int_\delta \omega$ is never zero. Here $E(\mathbb{C})$ is the underlying complex manifold of E and $H^1_{\rm dR}(E)$ is the algebraic de Rham cohomology of E (see [29]).

2. Let E_i, $i = 1, 2$ be two elliptic curves defined over $\bar{\mathbb{Q}}$ and $0 \neq \omega_i \in H^1_{\rm dR}(E_i)$ such that the $\mathbb{Z}$-modules $\{\int_\delta \omega_i \mid \delta \in H_1(E_i, \mathbb{Z})\}$ coincide. Then there is an isomorphism $a : E_1 \to E_2$ defined over $\bar{\mathbb{Q}}$ such that $a^*\omega_2 = \omega_1$, where $a^* : H^1_{\rm dR}(E_2) \to H^1_{\rm dR}(E_1)$ is the induced map in the de Rham cohomologies.

The above statements follow from the abelian subvariety theorem (see the appendix of [75]) on the periods of abelian varieties. For more details see [58]. □

2.9 Differential equations for special functions

It is natural to expect that the material of the present chapter to be generalized in direction of finding simple differential equations for many special functions that one has in the literature. In fact, this seems to have many applications in mathematical physics (see for instance [1] and the reference there). In this section we present one of such differential equations.

Let

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \ q = e^{2\pi i z}$$

be the Dedekind's η-function. In [67] Y. Ohyama has found that

$$W = (3 \log \eta(\frac{z}{3}) - \log \eta(z))' \tag{2.12}$$

$$X = (3 \log \eta(3z) - \log \eta(z))' \tag{2.13}$$

$$Y = (3 \log \eta(\frac{z+2}{3}) - \log \eta(z))' \tag{2.14}$$

$$Z = (3 \log \eta(\frac{z+1}{3}) - \log \eta(z))' \tag{2.15}$$

satisfy the equations:

$$\begin{cases} t_1' + t_2' + t_3' = t_1t_2 + t_2t_3 + t_3t_1 \\ t_1' + t_3' + t_4' = t_1t_3 + t_3t_4 + t_4t_1 \\ t_1' + t_2' + t_4' = t_1t_2 + t_2t_4 + t_4t_1 \\ t_2' + t_3' + t_4' = t_2t_3 + t_3t_4 + t_4t_2 \\ \zeta_3^2(t_2t_4 + t_3t_1) + \zeta_3(t_2t_1 + t_3t_4) + (t_2t_3 + t_4t_1) = 0. \end{cases}$$

where $\zeta_3 = e^{\frac{2\pi i}{3}}$. We write the first four lines of the above equation as a solution to a vector field V in $\mathbb{C}^4$ and let $F(t_1, t_2, t_3, t_4)$ be the polynomial in the fifth line. Using a computer, or by hand if one has a good patience for calculations, one can verify the equality $dF(V) = 0$ and so F is constant along the solutions of V, in other words V has a first integral.

Chapter 3

Weighted tame polynomials over a ring

The objective of the present chapter is to develop all algebraic aspects related to integrals in dimension $n \in \mathbb{N}$, i.e. the integration takes place on n-dimensional homological cycles living in the affine variety induced by a polynomial $f(x)$ in $(n+1)$-variables $x := (x_1, x_2, \ldots, x_{n+1})$ which may depend on some parameters. In particular, our focus is on calculating of the Gauss-Manin connection of the fibration induced by f. In the literature one can find such a claculation for the polynomial f_{R} in Chapter 2 which is due to Griffiths and Sasai. For the two varible polynomial $f(x_1, x_2) - s$ with the parameter s such a calculation or parts of it is done by many people such as Gavrilov, Novikov, Yakovenko, Zoladeck and etc. in the context of planar differential equations. The many variable case of such a calculation can be interesting from the Hodge theory point of view and it is completely discussed in [63] and [62] for a tame polynomial in the sense of §3.4. Our arguments in the present chapter work for a polynomial f defined on a general ring (instead of $\mathbb{C}$ above) described in the next section. We have tried to keep as much as possible the algebraic language and meantime to explain the theorems and examples by their topological interpretations. The complete geometric interpretation is postponed to Chapter 6. When one works with affine varieties in an algebraic context, one does not need the whole algebraic geometry of schemes and one needs only a basic theory of commutative algebra. This is also the case in this chapter and so from algebraic geometry of schemes we only use some standard notations.

3.1 The base ring

We consider a commutative ring R with multiplicative identity element 1. We assume that R is without zero divisors, i.e. if for some $a, b \in \mathsf{R}$, $ab = 0$ then $a = 0$ or $b = 0$. We also assume that R is Noetherian, i.e. it does not contain an infinite ascending chain of ideals (equivalently every ideal of R is finitely generated/every set of ideals contains a maximal element). From §3.3 on we will further assume that R is a Cohen-Macauly ring. Some algebraic notations that we use frequently are the following:

A multiplicative system in a ring R is a subset S of R containing 1 and closed under multiplication. The localization M_S of an R-module M is defined to be the R-module formed by the quotients $\frac{a}{s}$, $a \in M$, $s \in S$. If $S = \{1, a, a^2, \ldots\}$ for some $a \in \mathsf{R}$, $a \neq 0$ then the corresponding localization is denoted by M_a. Note that by this notation $\mathbb{Z}_a$, $a \in$

$\mathbb{Z}, a \neq 0$ is no more the set of integers modulo $a \in \mathbb{N}$. By $\check{M}$ we mean the dual of the R-module

$$\check{M} := \{a : M \to \mathsf{R},\ a \text{ is } \mathsf{R} \text{ linear }\}.$$

Usually we denote a basis/set of generators of M as a column matrix with entries in M.

We denote by k the field obtained by the localization of R over $\mathsf{R}\setminus\{0\}$ and by $\bar{\mathsf{k}}$ the algebraic closure of k. In many arguments we need that the characteristic of k to be zero. If this is the case then we mention it explicitly.

For the purpose of the present text we will only use a localization of a polynomial ring $\mathbb{Q}[t_1, t_2, \ldots, t_n]$ instead of the general ring R described above. Therefore, the reader may follow the content of this and the next Chapter only for this ring.

3.2 Homogeneous tame polynomials

Let $n \in \mathbb{N}_0$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \in \mathbb{N}^{n+1}$. For a ring R we denote by $\mathsf{R}[x]$ the polynomial ring with coefficients in R and the variable $x := (x_1, x_2, \ldots, x_{n+1})$. We consider

$$\mathsf{R}[x] := \mathsf{R}[x_1, x_2, \ldots, x_{n+1}]$$

as a graded algebra with $\deg(x_i) = \alpha_i$. For $n = 0$ (resp. $n = 2$ and $n = 3$) we use the notations x (resp. x, y and x, y, z).

A polynomial $f \in \mathsf{R}[x]$ is called a homogeneous polynomial of degree d with respect to the grading α if f is a linear combination of monomials of the type

$$x^\beta := x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}},\ \deg(x^\beta) = \alpha \cdot \beta := \sum_{i=1}^{n+1} \alpha_i \beta_i = d.$$

For an arbitrary polynomial $f \in \mathsf{R}[x]$ one can write in a unique way $f = \sum_{i=0}^{d} f_i,\ f_d \neq 0$, where f_i is a homogeneous polynomial of degree i. The number d is called the degree of f. The Jacobian ideal of f is defined to be:

$$\mathrm{jacob}(f) := \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_{n+1}} \rangle \subset \mathsf{R}[x].$$

The Tjurina ideal is

$$\mathrm{tjurina}(f) := \mathrm{jacob}(f) + \langle f \rangle \subset \mathsf{R}[x].$$

We define also the R-modules

$$\mathsf{V}_f := \frac{\mathsf{R}[x]}{\mathrm{jacob}(f)},\ \mathsf{W}_f := \frac{\mathsf{R}[x]}{\mathrm{tjurina}(f)}.$$

These modules may be called the Milnor module and Tjurina module of f, analog to the objects with the same name in singularity theroy (see [6]).

Remark 3.1. In practice one considers V_f as an $\mathsf{R}[f]$-module. If we introduce the new parameter s and define

$$\tilde{f} := f - s \in \tilde{\mathsf{R}}[x],\ \tilde{\mathsf{R}} := \mathsf{R}[s]$$

then $\mathsf{W}_{\tilde{f}}$ as $\tilde{\mathsf{R}}$-module is isomorphic to V_f as $\mathsf{R}[f]$-module. We have introduced V_f because the main machinaries are first developed for $f - s,\ f \in \mathbb{C}[x]$ in the literature of singularities (see [63]).

Definition 3.1. A homogeneous polynomial $g \in \mathsf{R}[x]$ in the weighted ring $\mathsf{R}[x],\ \deg(x_i) = \alpha_i,\ i = 1, 2, \dots, n+1$ has an isolated singularity at the origin if the R-module V_g is freely generated of finite rank. We also say that g is a (homogenous) tame polynomial in $\mathsf{R}[x]$.

In the case $\mathsf{R} = \mathbb{C}$, a homogeneous polynomial g has an isolated singularity at the origin if $Z(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_{n+1}}) = \{0\}$. This justfies the definition geometrically. If the homogeneous polynomial $g \in \mathbb{C}[x]$ is tame then the projective variety induced by $\{g = 0\}$ in $\mathbb{P}^\alpha$ is a V-manifold/quasi-smooth variety (see Steenbrink [77]). For the case $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 1$ the notions of a V-manifold and smooth manifold are equivalent.

Example 3.1. The two variable polynomial $f(x) = x^2 + y^2$ is not tame when it is considered in the ring $\mathbb{Z}[x, y]$ and it is tame in the ring $\mathbb{Z}[\frac{1}{2}][x, y]$. In a similar way $f(x, y) = t^2x^2 + y^2$ is tame in $\mathbb{Q}[t, \frac{1}{t^2}][x, y]$ but not in $\mathbb{Q}[t][x]$.

Example 3.2. Consider the case $n = 0,\ \deg(x) = 1$. For $g = x^d$ we have

$$\mathsf{V}_g = \oplus_{i=0}^{d-2} \mathsf{R} \cdot x^i \oplus \oplus_{i=d-1}^{\infty} \mathsf{R}/(d \cdot \mathsf{R}) \cdot x^i$$

and so g is tame if and only if d is invertible in R. For instance take $\mathsf{R} = \mathbb{Z}[\frac{1}{d}],\ \mathbb{Q}, \mathbb{C}$. A basis of the R-module V_g is given by $I = \{1, x, x^2, \dots, x^{d-2}\}$.

Example 3.3. In the weighted ring $\mathsf{R}[x],\ \deg(x_i) = \alpha_i \in \mathbb{N}$ for a given degree $d \in \mathbb{N}$, we would like to have at least one tame polynomial of degree d. For instance, if

$$m_i := \frac{d}{\alpha_i} \in \mathbb{N}, \quad i = 1, 2, \dots, n+1$$

and all m_i's are invertible in R then the homogeneous polynomial

$$g := x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$$

is tame. A basis of the R-module V_g is given by

$$I = \{x^\beta \mid 0 \le \beta_i \le m_i - 2,\ i = 1, 2, \dots, n+1\}.$$

For other d's we do not have yet a general method which produces a tame polynomial of degree d.

Example 3.4. For $n = 1$ and $\mathsf{R} = \mathbb{C}$, a homogeneous polynomial has an isolated singularity at the origin if and only if in its irreducible decomposition there is no factor of multiplicity greater than one.

Throughout the present text we will work with a fixed homogeneous tame polynomial g and we assume that the degree d of g is invertible in R. We use the following notations related to a homogeneous tame polynomial $g \in \mathsf{R}[x]$: We fix a basis

$$x^I := \{x^\beta \mid \beta \in I\}$$

of monomials for the R-module V_g. We also define

$$w_i := \frac{\alpha_i}{d},\ 1 \le i \le n+1,\ A_\beta := \sum_{i=1}^{n+1} (\beta_i + 1) w_i,\ \mu := \#I = \mathrm{rank} \mathsf{V}_g \tag{3.1}$$

$$\eta := (\sum_{i=1}^{n+1}(-1)^{i-1}w_i x_i \widehat{dx_i}),\ \eta_\beta := x^\beta \eta,\ \omega_\beta = x^\beta dx\ \beta \in I,$$

where

$$dx := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1},\ \widehat{dx_i} = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}.$$

One may call μ the Milnor number of g.[1] To make our notation simpler, we define

$$\mathbb{U}_0 := \mathrm{Spec}(\mathsf{R}),\ \mathbb{U}_1 := \mathrm{Spec}(\mathsf{R}[x])$$

and denote by $\pi : \mathbb{U}_1 \to \mathbb{U}_0$ the canonical morphism. The set of (relative) differential i-forms in $\mathbb{U}_1$ is:

$$\Omega^i_{\mathbb{U}_1/\mathbb{U}_0} := \{\sum f_{k_1,k_2,\ldots,k_i} dx_{k_1} \wedge dx_{k_2} \wedge \cdots \wedge dx_{k_i} \mid f_{k_1,k_2,\ldots,k_i} \in \mathsf{R}[x]\}.$$

The adjective relative is used with respect to the morphism π. The set $\Omega^i_{\mathbb{U}_j},\ j = 0, 1$ of differential i-forms and the differential maps

$$d : \Omega^i_{\mathbb{U}_j} \to \Omega^{i+1}_{\mathbb{U}_j}, i = 0, 1, \ldots$$

can be defined in an algebraic context (see [34], p.17). The set $\mathcal{D}_{\mathbb{U}_0}$ of vector fields in $\mathbb{U}_0$ is by definition the dual of the R-module $\Omega^1_{\mathbb{U}_0}$. Therefore, we have the R-bilinear map

$$\mathcal{D}_{\mathbb{U}_0} \times \Omega^1_{\mathbb{U}_0} \to \mathsf{R},\ (v, \eta) \mapsto \eta(v) := v(\eta)$$

and the map

$$\mathcal{D}_{\mathbb{U}_0} \times \mathsf{R} \to \mathsf{R},\ (v, p) \mapsto dp(v)$$

We define

$$\deg(dx_j) = \alpha_j,\ \deg(\omega_1 \wedge \omega_2) = \deg(\omega_1) + \deg(\omega_2),\ j = 1, 2, \ldots, n+1,\ \omega_1, \omega_2 \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}.$$

With the above rules, $\Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$ turns into a graded $\mathsf{R}[x]$-module and we can talke about homogeneous differential forms and decomposition of a differential form into homogeneous pieces. A geometric way to look at this is the following: The multiplicative group $\mathsf{R}^* = \mathsf{R} \setminus \{0\}$ acts on $\mathbb{U}_1$ by:

$$(x_1, x_2, \ldots, x_{n+1}) \to (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_{n+1}} x_{n+1}),\ \lambda \in \mathsf{R}^*.$$

We also denote the above map by $\lambda : \mathbb{U}_1 \to \mathbb{U}_1$. The polynomial form $\omega \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$ is weighted homogeneous of degree m if

$$\lambda^*(\omega) = \lambda^m \omega,\ \lambda \in \mathsf{R}^*.$$

For the homogeneous polynomial g of degree d this means that

$$g(\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_{n+1}} x_{n+1}) = \lambda^d g(x_1, x_2, \ldots, x_{n+1}),\ \forall \lambda \in \mathsf{R}^*.$$

Remark 3.2. The reader who wants to follow the present text in a geometric context may assume that $\mathsf{R} = \mathbb{C}[t_1, t_2, \ldots, t_s]$ and hence identify $\mathbb{U}_i,\ i = 0, 1$ with its geometric points, i.e.

$$\mathbb{U}_0 = \mathbb{C}^s,\ \mathbb{U}_1 = \mathbb{C}^{n+1} \times \mathbb{C}^s.$$

The map π is now the projection on the last s coordinates.

[1] J. Milnor in [50] proves that in the case $\mathsf{R} = \mathbb{C}$ there are small neighborhoods $U \subset \mathbb{C}^{n+1}$ and $S \subset \mathbb{C}$ of the origins such that $g : U \to S$ is a C^∞ fiber bundle over $S \backslash \{0\}$ whose fiber is of homotopy type of a bouquet of μ n-spheres.We will see a similar statement for tame polynomials in Chapter 5.

3.3 De Rham Lemma

In this section we state the de Rham lemma for a homogeneous tame polynomial. Originally, a similar Lemma was stated for a germ of holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ in [6], p.110. To make the section self sufficient we recall some facts from commutative algebra. The page numbers in the bellow paragraph refer to the book [16].

Let $\tilde{\mathsf{R}}$ be a commutative Noetherin ring with the multiplicative identity 1. The dimension of $\tilde{\mathsf{R}}$ is the supremum s of the lengths of chains $0 \neq I_0 \subsetneqq I_1 \subsetneqq \cdots \subsetneqq I_s$ of prime ideals in $\tilde{\mathsf{R}}$. For a prime ideal $I \subset \tilde{\mathsf{R}}$ we define $\dim(I) = \dim(\frac{\tilde{\mathsf{R}}}{I})$ and $\operatorname{codim}(I) = \dim(\tilde{\mathsf{R}}_I)$ (p. 225), where $\tilde{\mathsf{R}}_I$ is the localization of $\tilde{\mathsf{R}}$ over the complement of I in $\tilde{\mathsf{R}}$.

A sequence of elements $a_1, a_2, \dots, a_{n+1} \in \tilde{\mathsf{R}}$ is called a regular sequence if

$$\langle a_1, a_2, \dots, a_{n+1} \rangle \neq \tilde{\mathsf{R}}$$

and for $i = 1, 2, \dots, n+1$, a_i is a non-zero divisor on $\frac{\tilde{\mathsf{R}}}{\langle a_1, a_2, \dots, a_{i-1} \rangle}$(p. 17). For $I \neq \tilde{\mathsf{R}}$, the depth of the ideal I is the length of a (indeed any) maximal regular sequence in I.

The ring $\tilde{\mathsf{R}}$ is called Cohen-Macaulay if the codimension and the depth of any proper ideal of $\tilde{\mathsf{R}}$ coincide (p. 452). If $\tilde{\mathsf{R}}$ is a domain, i.e. it is finitely generated over a field, then we have

$$\dim(I) + \operatorname{codim}(I) = \dim(\tilde{\mathsf{R}}) \tag{3.2}$$

(this follows from Theorem A, p. 221) but in general the equality does not hold. If $\tilde{\mathsf{R}}$ is a Cohen-Macaulay ring then $\mathsf{R}[x] = \mathsf{R}[x_1, x_2, \dots, x_{n+1}]$ is also Cohen-Macaulay (p. 452 Proposition. 18.9). In particular, any polynomial ring with coefficients in a field and its localizations are Cohen-Macaulay.

Proposition 3.1. *Let* R *be a Cohen-Macaulay ring and* g *be a homogeneous tame polynomial in* $\mathsf{R}[x]$. *The depth of the Jacobian ideal* $\operatorname{jacob}(g) \subset \mathsf{R}[x]$ *of* g *is* $n+1$.

Proof. Let $I := \operatorname{jacob}(g) \subset \mathsf{R}[x]$ we have:

$$\operatorname{codim}(I) := \dim \mathsf{R}[x]_I = \dim \mathsf{k}[x]_{\bar{I}} = \dim \mathsf{k}[x] - \dim \bar{I} = n+1.$$

Here $\bar{I}$ is the Jacobian ideal of g in $\mathsf{k}[x]$, where k is the quotient field of R. In the second and last equalities we have used the fact that g is tame and hence I does no contain any non-zero element of R and $\dim \bar{I} := \dim(\frac{\mathsf{k}[x]}{\bar{I}}) = 0$. We have also used $\dim(\mathsf{k}[x]) = n+1$ (Theorem A, p.221). We conclude that the depth of $\operatorname{jacob}(g) \subset \mathsf{R}[x]$ is $n+1$. $\square$

The purpose of all what we said above is:

Proposition 3.2. *(de Rham Lemma) Let* R *be a Cohen-Macaulay ring and* g *be a homogeneous tame polynomial in* $\mathsf{R}[x]$. *An element* $\omega \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}, i \leq n$ *is of the form* $dg \wedge \eta,\ \eta \in \Omega^{i-1}_{\mathbb{U}_1/\mathbb{U}_0}$ *if and only if* $dg \wedge \omega = 0$. *This means that the following sequnce is exact*

$$0 \to \Omega^0_{\mathbb{U}_1/\mathbb{U}_0} \overset{dg\wedge\cdot}{\to} \Omega^1_{\mathbb{U}_1/\mathbb{U}_0} \overset{dg\wedge\cdot}{\to} \cdots \overset{dg\wedge\cdot}{\to} \Omega^n_{\mathbb{U}_1/\mathbb{U}_0} \overset{dg\wedge\cdot}{\to} \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}. \tag{3.3}$$

In other words

$$H^i(\Omega^\bullet_{\mathbb{U}_1/\mathbb{U}_0}, dg \wedge \cdot) = 0,\ i = 0, 1, \dots, n.$$

Proof. We have proved the depth of $\mathrm{jacob}(g) \subset \mathsf{R}[x]$ is $n+1$. Knowing this the above proposition follows from the main theorem of [71]. See also [16] Corollary 17.5 p. 424, Crollary 17.7 p. 426 for similar topics. $\square$

The sequence in (3.3) is also called the Koszul complex.

Proposition 3.3. *The following sequence is exact*

$$0 \xrightarrow{d} \Omega^0_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{d} \Omega^1_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{d} \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{d} 0.$$

In other words

$$H^i_{\mathrm{dR}}(\mathbb{U}_1/\mathbb{U}_0) := H^i(\Omega^\bullet_{\mathbb{U}_1/\mathbb{U}_0}, d) = 0, \ i = 1, 2, \ldots, n+1.$$

Proof. This is [16], Exercise 16.15 c, p. 414. $\square$

Note that in the above proposition we do not need R to be Cohen-Macaulay. Later, we will need the following proposition.

Proposition 3.4. *Let R be a Cohen-Macaulay ring. If for $\omega \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$, $1 \leq i \leq n-1$ we have*

$$d\omega = dg \wedge \omega_1, \quad \textit{for some } \omega_1 \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0} \tag{3.4}$$

then there is an $\omega' \in \Omega^{i-1}_{\mathbb{U}_1/\mathbb{U}_0}$ such that

$$d\omega = dg \wedge d\omega'.$$

Proof. Since g is homogeneous, in (3.4) we can assume that

$$\deg_x(\omega_1) = \deg_x(d\omega) - d \text{ and so } \deg_x(\omega_1) < \deg_x(d\omega) \leq \deg_x(\omega).$$

We take differential of (3.4) and use Proposition 3.2. Then we have $d\omega_1 = dg \wedge \omega_2$, and again we can assume that $\deg_x(\omega_2) < \deg_x(\omega_1)$. We obtain a sequence of differential forms ω_k, $k = 0, 1, 2, 3, \ldots$, $\omega_0 = \omega$ with decreasing degrees and $d\omega_{k-1} = dg \wedge \omega_k$. Therefore, for some $k \in \mathbb{N}$ we have $\omega_k = 0$. We claim that for all $0 \leq j \leq k$ we have $d\omega_j = dg \wedge d\omega'_j$ for some $\omega'_j \in \Omega^{i-1}_{\mathbb{U}_1/\mathbb{U}_0}$. We prove our claim by decreasing induction on j. For $j = k$ it is already proved. Assume that it is true for j. Then by Proposition 3.3 we have

$$\omega_j = dg \wedge \omega'_j + d\omega'_{j-1}, \ \omega'_{j-1} \in \Omega^{i-1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

Putting this in $d\omega_{j-1} = dg \wedge \omega_j$, our claim is proved for $j-1$. $\square$

3.4 Tame polynomials

We start this section with the definition of a tame polynomial.

Definition 3.2. A polynomial $f \in \mathsf{R}[x]$ is called a tame polynomial if there exist natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in \mathbb{N}$ such that the R-module V_g is freely generated R-module of finite rank (g has an isolated singularity at the origin), where $g = f_d$ is the last homogeneous piece of f in the graded algebra $\mathsf{R}[x]$, $\deg(x_i) = \alpha_i$.

In practice, we fix up a weighted ring $\mathsf{R}[x]$, $\deg(x_i) = \alpha_i \in \mathbb{N}$ and a homogeneous tame polynomial $g \in \mathsf{R}[x]$. The perturbations $g + g_1$, $\deg(g_1) < \deg(g)$ of g are tame polynomials.

Example 3.5. For $n = 0$, if d is invertible in R then

$$f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0,\ t_i \in \mathsf{R}$$

is a tame polynomial in $\mathsf{R}[x]$ (we have used for simplicity $x = x_1$).

Example 3.6. One of the most important class of tame polynomials are the so called hyperelliptic polynomials

$$f = y^2 + t_d x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0 \in \mathsf{R}[x, y],\ deg(x) = 2,\ \deg(y) = d,$$

with $g = y^2 + t_d x^d$. We assume that t_d and $2d$ are invertible in R. A R-basis of the V_g-module (and hence of V_f) is given by

$$I := \{1, x, x^2, \dots, x^{d-2}\}.$$

In this example we have:

$$A_i = \frac{1}{2} + \frac{i+1}{d},\ \eta := \frac{1}{d}ydy - \frac{1}{2}ydx,$$

$$\frac{x^i dx}{y} = -2\frac{x^i dx \wedge dy}{df} = \frac{-2}{A_i}\nabla_{\frac{\partial}{\partial t_0}}(x^i \eta). \tag{3.5}$$

The last equalities will be explained in §3.11.

Definition 3.3. The polynomial

$$f = \sum_{\deg(x^\alpha) \le d} t_\alpha x^\alpha \in \mathsf{R}[x],\ \mathsf{R} = \mathbb{Q}[\{t_\alpha \mid \deg(x^\alpha) \le d\}]$$

is called a complete polynomial. Let $\tilde{\mathsf{R}} \subset \mathsf{R}$ be the polynomial ring generated by the coefficients of the last homogeneous piece g of f. Let also $\tilde{\mathsf{k}}$ be the field obtained by the localization of $\tilde{\mathsf{R}}$ over $\tilde{\mathsf{R}}\backslash\{0\}$. Assume that the polynomial $g \in \tilde{\mathsf{k}}[x]$ has an isolated singularity at the origin and so it has an isolated singularity at the origin as a polynomial in a localization $\tilde{\mathsf{R}}_a$ of $\tilde{\mathsf{R}}$ for some $a \in \mathsf{R}$. The variety $\{a = 0\}$ contains the locus of parameters for which g has not an isolated zero at the origin. It may contains more points. To find such an a we choose a monomial basis x^β, $\beta \in I$ of $\tilde{\mathsf{k}}[x]/\mathrm{jacob}(g)$ and write all $x_i x^\beta$, $\beta \in I$, $i = 1, 2, \dots, n+1$ as a $\tilde{\mathsf{k}}$-linear combination of x^β's and a residue in $\mathrm{jacob}(g)$. The product of the denominators of all the coefficients (in $\tilde{\mathsf{k}}$) used in the mentioned equalities is a candidate for a. The obtained a depends on the choice of the monomial basis.

Now, a complete polynomial is tame over $\mathsf{R}_{\tilde{\mathsf{R}}\backslash\{0\}}[x]$. An arbitrary tame polynomial $f \in \mathsf{R}[x]$ is a specialization of a unique complete tame polynomial, called the completion of f.

Remark 3.3. In the context of the article [7] the polynomial mapping $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is tame if there is a compact neighborhood U of the critical points of f such that the norm of the Jacobian vector of f is bounded away from zero on $\mathbb{C}^n \backslash U$. It has been proved in the same article (Proposition 3.1) that f is tame if and only if the Milnor number of f is finite and the Milnor numbers of $f^w := f - (w_1x_1 + \cdots + w_{n+1}x_{n+1})$ and f coincide for all sufficiently small $(w_1, \cdots, w_{n+1}) \in \mathbb{C}^{n+1}$. This and Proposition 3.6 imply that every tame polynomial in the sense of this article is also tame in the sense of [7]. However, the inverse may not be true (for instance take $f = x^2 + y^2 + x^2y^2$, see [73] for other examples).

3.5 De Rham Lemma for tame polynomials

Proposition 3.5. *(de Rham lemma for tame polynomials) Proposition 3.2 is valid replacing g with a tame polynomial f.*

Proof. If there is $\omega \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$, $i \leq n$ such that $df \wedge \omega = 0$ then $dg \wedge \omega' = 0$, where ω' is the last homogeneous piece of ω. We apply Proposition 3.2 and conclude that $\omega = df \wedge \omega_1 + \omega_2$ for some $\omega_1 \in \Omega^{i-1}_{\mathbb{U}_1/\mathbb{U}_0}$ and $\omega_2 \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$ with $\deg(\omega_2) < \deg(\omega)$ and $df \wedge \omega_2 = 0$. We repeat this argument for ω_2. Since the degree of ω_2 is decreasing, at some point we will get $\omega_2 = 0$ and then the desired form of ω. □

Recall that in §3.2 we fixed a monomial basis x^I for the R-module V_g.

Proposition 3.6. *For a tame polynomial f, the R-module V_f is freely generated by x^I.*

Proof. Let $f = f_0 + f_1 + f_2 + \cdots + f_{d-1} + f_d$ be the homogeneous decomposition of f in the graded ring $\mathsf{R}[x]$, $\deg(x_i) = \alpha_i$ and $g := f_d$ be the last homogeneous piece of f. Let also $F = f_0x_0^d + f_1x_0^{d-1} + \cdots + f_{d-1}x_0 + g$ be the homogenization of f. We claim that the set x^I generates freely the $\mathsf{R}[x_0]$-module $V := \mathsf{R}[x_0, x]/\langle \frac{\partial F}{\partial x_i} \mid i = 1, 2, \ldots, n+1\rangle$. More precisely, we prove that every element $P \in \mathsf{R}[x_0, x]$ can be written in the form

$$P = \sum_{\beta \in I} C_\beta x^\beta + R, \ R := \sum_{i=1}^{n+1} Q_i \frac{\partial F}{\partial x_i}, \tag{3.6}$$

$$\deg_x(R) \leq \deg_x(P), \ C_\beta \in \mathsf{R}[x_0], \ Q_i \in \mathsf{R}[x_0, x]. \tag{3.7}$$

Since x^I is a basis of V_g, we can write

$$P = \sum_{\beta \in I} c_\beta x^\beta + R', \ R' = \sum_{i=1}^{n+1} q_i \frac{\partial g}{\partial x_i}, \ c_\beta \in \mathsf{R}[x_0], \ q_i \in \mathsf{R}[x_0, x]. \tag{3.8}$$

We can choose q_i's so that

$$\deg_x(R') \leq \deg_x(P). \tag{3.9}$$

If this is not the case then we write the non-trivial homogeneous equation of highest degree obtained from (3.8). Note that $\frac{\partial g}{\partial x_i}$ is homogeneous. If some terms of P occur in this new equation then we have already (3.9). If not we subtract this new equation from (3.8). We repeat this until getting the first case and so the desired inequality. Now we have

$$\frac{\partial g}{\partial x_i} = \frac{\partial F}{\partial x_i} - x_0 \sum_{j=0}^{d-1} \frac{\partial f_j}{\partial x_i} x_0^{d-j-1},$$

and so

$$P=\sum_{\beta\in I}c_\beta x^\beta+R_1-P_1, \tag{3.10}$$

where

$$R_1:=\sum_{i=1}^{n+1}q_i\frac{\partial F}{\partial x_i},\ P_1:=x_0(\sum_{i=1}^{n+1}\sum_{j=0}^{d-1}q_i\frac{\partial f_j}{\partial x_i}x_0^{d-j-1}).$$

From (3.9) we have

$$\deg_x(P_1)\leq\deg_x(P)-1,\ \deg_x(R_1)\leq\deg_x(P).$$

We write again $q_i\frac{\partial f_j}{\partial x_i}$ in the form (3.8) and substitute it in (3.10). By degree conditions this process stops and at the end we get the equation (3.6) with the conditions (3.7).

Now let us prove that x^I generates the $\mathsf{R}[x_0]$-module V freely. If the elements of x^I are not $\mathsf{R}[x_0]$-independent then we have $\sum_{\beta\in I}C_\beta x^\beta=0$ in V for some $C_\beta\in\mathsf{R}[x_0]$ or equivalently

$$\sum_{\beta\in I}C_\beta x^\beta=dF\wedge\omega \tag{3.11}$$

for some $\omega=\sum_{i=1}^{n+1}Q_i[x,x_0]\widehat{dx_i},\ Q_i\in\mathsf{R}[x,x_0]$, where d is the differnetial with respect to $x_i,\ i=1,2,\ldots,n+1$ and hence $dx_0=0$. Since F is homogenous in (x,x_0), we can assume that in the equality (3.11) the $\deg_{(x,x_0)}$ of the left hand side is $d+\deg_{(x,x_0)}(\omega)$. Let $\omega=\omega_0+x_0\omega_1$ and ω_0 does not contain the variable x_0. In the equation obtained from (3.11) by putting $x_0=0$, the right hand side side must be zero otherwise we have a nontrivial relation between the elements of x^I in V_g. Therefore, we have $dg\wedge\omega_0=0$ and so by de Rham lemma (Proposition 3.2)

$$\omega_0=dg\wedge\omega'=dF\wedge\omega'+x_0(\frac{g-F}{x_0})\wedge\omega',$$

with $\deg_x(\omega_0)=d+\deg(\omega')$. Substituting this in ω and then ω in (3.11) we obtain a new ω with the property (3.11) and stricktly less $\deg_x$. □

Proposition 3.6 implies that f and its last homogeneous piece have the same Milnor number.

3.6 The discriminant of a polynomial

Definition 3.4. Let A be the R-linear map in V_f induced by multiplication by f. According to (3.6), V_f is freely generated by x^I and so we can talk about the matrix A_I of A in the basis x^I. For a new parameter s define

$$S(s):=\det(A-s\cdot I_{\mu\times\mu}),$$

where $I_{\mu\times\mu}$ is the identity μ times μ matrix and $\mu=\#I$. It has the property $S(f)\mathsf{V}_f=0$. We define the discriminant of f to be

$$\Delta=\Delta_f:=S(0)\in\mathsf{R}.$$

For the tame polynomials f_{R} and f_{H} in §2.4 we have to multiply Δ_{R} and Δ_{H} with $\frac{1}{27}$ in order to obtain the discriminant in the sense of the above definition. The discriminant has the following property

$$\Delta_f \cdot \mathsf{W}_f = 0. \tag{3.12}$$

In general Δ_f is not the the simplest element in R with the property (3.12).

Remark 3.4. In the zero dimensional case $n = 0$ the discriminant of a monic polynomial $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0 \in \mathsf{R}[x]$ is defined as follows:

$$\Delta'_f := \prod_{1 \le i \ne j \le d} (x_i - x_j) = \prod_{i=1}^{d} f'(x_i) \in \mathsf{R},$$

where $f' = \frac{\partial f}{\partial x}$ is the derivative of f. It is an easy exercise to see that the multiplication of Δ'_f with the number d^d is equal to Δ_f.

Proposition 3.7. *Let R be a closed algebraic field. We have $\Delta_f = 0$ if and only if the affine variety $\{f = 0\} \subset \mathsf{R}^{n+1}$ is singular.*

Proof. $\Leftarrow$: If $\Delta_f \ne 0$ then A is surjective and $1 \in \mathsf{R}[x]$ can be written in the form $1 = \sum_{i=1}^{n+1} \frac{\partial f}{\partial x_i} q_i + qf$. This implies that the variety $Z := \{\frac{\partial f}{\partial x_i} = 0,\ i = 1, 2, \dots, n+1,\ f = 0\}$ is empty.

$\Rightarrow$: If $\{f = 0\}$ is smooth then the variety Z is empty and so by Hilbert Nullstelensatz there exists $\tilde{f} \in \mathsf{R}[x]$ such that $f\tilde{f} = 1$ in V_f. This means that A is invertible and so $\Delta_f \ne 0$. $\square$

The above Proposition implies that in the case of $\mathsf{R} = \mathbb{C}[t_1, t_2, \dots, t_s]$, the affine variety $\{\Delta_f(t) = 0\} \subset \mathbb{C}^s$ is the locus of parameters t such that the affine variety $\{f = 0\} \subset \mathbb{C}^{n+1}$ is singular.

Definition 3.5. For a tame polynomial f we say that the affine variety $\{f = 0\}$ is smooth if the discriminant Δ_f of f is not zero.

Proposition 3.8. *Assume that f is a tame polynomial and $\Delta_f \ne 0$. If*

$$df \wedge \omega_2 = f\omega_1, \quad \textit{for some } \omega_2 \in \Omega^n_{\mathbb{U}_1/\mathbb{U}_0},\ \omega_1 \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$$

then

$$\omega_2 = f\omega_3 + df \wedge \omega_4,\ \omega_1 = df \wedge \omega_3, \quad \textit{for some } \omega_3 \in \Omega^n_{\mathbb{U}_1/\mathbb{U}_0},\ \omega_4 \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

Proof. If ω_1 is not zero in W then the multiplication by f R-linear map in V_f has a non trivial kernel and so $\Delta_f = 0$ which contradicts the hypothesis. Now let $\omega_1 = df \wedge \omega_3$ and so $df \wedge (f\omega_3 - \omega_2) = 0$. The de Rham lemma for f (Proposition 3.5) finishes the proof. $\square$

The example bellow shows that the above proposition is not true for singular affine varieties.

Example 3.7. For a homogeneous polynomial g in the graded ring $\mathsf{R}[x]$, $\deg(x_i) = \alpha_i$ we have

$$g = \sum_{i=1}^{n+1} w_i x_i \frac{\partial g}{\partial x_i} \quad \text{equivalentely } g dx = dg \wedge \eta$$

and so the matrix A in the definition of the discriminat of g is the zero matrix. In particular, the discriminant of $g - s \in \mathsf{R}[s][x]$ is $(-s)^{\mu}$.

Example 3.8. Assume that $2d$ is invertible in R. For the hypergeometric polynomial $f := y^2 - p(x) \in \mathsf{R}[x, y]$, $\deg(p) = d$ we have $\mathsf{V}_f \cong \mathsf{V}_p$ and under this isomorphy the multiplication by f linear map in V_f coincide with the multiplication by p map in V_p. Therefore,

$$\Delta_f = \Delta_p.$$

Bellow there is a table of discriminants.

$$f = x^d + t_{d-1}x^{d-1} + t_{d-2}x^{d-2} + \cdots + t_0$$

d	$d^d \cdot \Delta$
2	$4t_0 - t_1^2$
3	$27t_0^2 - 18t_0t_1t_2 + 4t_0t_2^3 + 4t_1^3 - t_1^2t_2^2$
4	$256t_0^3 - 192t_0^2t_1t_3 - 128t_0^2t_2^2 + 144t_0^2t_2t_3^2 - 27t_0^2t_3^4 + 144t_0t_1^2t_2 - 6t_0t_1^2t_3^2 - 80t_0t_1t_2^2t_3 + 18t_0t_1t_2t_3^3 + 16t_0t_2^4 - 4t_0t_2^3t_3^2 - 27t_1^4 + 18t_1^3t_2t_3 - 4t_1^3t_3^3 - 4t_1^2t_2^3 + t_1^2t_2^2t_3^2$

3.7 The double discriminant of a tame polynomial

Let $f \in \mathsf{R}[x]$ be a tame polynomial. We consider a new parameter s and the tame polynomial $f - s \in \mathsf{R}[s][x]$. The discriminant Δ_{f-s} of $f - s$ as a polynomial in s has degree μ and its coefficients are in R. Its leading coefficient is $(-1)^\mu$ and so if μ is invertible in R then it is tame (as a polynomial in s) in $\mathsf{R}[s]$. Now, we take again the discriminant of Δ_{f-s} with respect to the parameter s and obtain

$$\check{\Delta} = \check{\Delta}_f := \Delta_{\Delta_{f-s}} \in \mathsf{R}$$

which is called the double discriminant of f. We consider a tame polynomial f as a function from $\bar{\mathsf{k}}^{n+1}$ to $\bar{\mathsf{k}}$. The set of critical values of f is defined to be $P = P_f := Z(\text{jacob}(f))$ and the set of critical values of f is $C = C_f := f(P_f)$. It is easy to see that:

Proposition 3.9. *The tame polynomial f has μ distinct critical values (and hence distinct critical points) if and only of its double discriminant is not zero.*

Note that that μ is the maximum possible number for $\#C_f$.

3.8 De Rham cohomology

Let $f \in \mathsf{R}[x]$ be a tame polynomial as a in §3.4. The following quotients

$$\mathsf{H}' = \mathsf{H}'_f := \frac{\Omega^n_{\mathbb{U}_1}}{f\Omega^n_{\mathbb{U}_1} + df \wedge \Omega^{n-1}_{\mathbb{U}_1} + \pi^{-1}\Omega^1_{\mathbb{U}_0} \wedge \Omega^{n-1}_{\mathbb{U}_1} + d\Omega^{n-1}_{\mathbb{U}_1}}, \tag{3.13}$$

$$\cong \frac{\Omega^n_{\mathbb{U}_1/\mathbb{U}_0}}{f\Omega^n_{\mathbb{U}_1/\mathbb{U}_0} + df \wedge \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0} + d\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}}$$

$$\mathsf{H}'' = \mathsf{H}''_f := \frac{\Omega^{n+1}_{\mathbb{U}_1}}{f\Omega^{n+1}_{\mathbb{U}_1} + df \wedge d\Omega^{n-1}_{\mathbb{U}_1} + \pi^{-1}\Omega^1_{\mathbb{U}_0} \wedge \Omega^n_{\mathbb{U}_1}} \cong \frac{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}}{f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0} + df \wedge d\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}} \tag{3.14}$$

are R-modules and play the role of de Rham cohomology of the affine variety

$$\{f = 0\} := \text{Spec}(\frac{\mathsf{R}[x]}{f \cdot \mathsf{R}[x]}).$$

Here $\pi : \mathbb{U}_1 \to \mathbb{U}_0$ is the projection corresponding to $\mathsf{R} \subset \mathsf{R}[x]$. We have assumed that $n > 0$. In the case $n = 0$ we define:

$$\mathsf{H}' = \mathsf{H}'_f := \frac{\mathsf{R}[x]}{f \cdot \mathsf{R}[x] + \mathsf{R}}$$

and

$$\mathsf{H}'' = \frac{\Omega^1_{\mathbb{U}_1/\mathbb{U}_0}}{f \cdot \Omega^1_{\mathbb{U}_1/\mathbb{U}_0} + \mathsf{R} \cdot df}.$$

Remark 3.5. We will use H or $H^n_{\mathrm{dR}}(\{f = 0\})$ to denote one of the modules H' or H''. We note that for an arbitrary polynomial f such modules may not coincide with the de Rham cohomology of the affine variety $\{f = 0\}$ defined by Grothendieck, Atiyah and Hodge (see [29]). For instance, for $f = x(1 + xy) - t \in \mathsf{R}[x, y],\ \mathsf{R} = \mathbb{C}[t]$ the differential forms $y^{k+1}dx + xy^k dy,\ k > 0$ are not zero in the corresponding H' but they are relatively exact and so zero in $H^1_{\mathrm{dR}}(\{f = 0\})$ (see [4]).

One may call H' and H'' the Brieskorn modules associated to f in analogy to the local modules introduced by Brieskorn in 1970. In fact, the classical Brieskorn modules for $n > 0$ are

$$H' = H'_f = \frac{\Omega^n_{\mathbb{U}_1/\mathbb{U}_0}}{df \wedge \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0} + d\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}},\ H'' = H''_f := \frac{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}}{df \wedge d\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}}.$$

and for $n = 0$

$$H' := \frac{\mathsf{R}[x]}{\mathsf{R}[f]},\ H'' = \frac{\Omega^1_{\mathbb{U}_1/\mathbb{U}_0}}{\mathsf{R}[f] \cdot df}.$$

We consider them as $\mathsf{R}[f]$-modules. In [63] we have worked with the classical ones.

Remark 3.6. The $\mathsf{R}[f]$-module H'_f is isomorphic to the $\mathsf{R}[s]$-module $\mathsf{H}'_{\tilde{f}}$, where $\tilde{f} = f - s \in \mathsf{R}[s][x]$ and s is a new parameter. A similar statement is true for the other Brieskorn module.

Remark 3.7. We have the following well-defined R-linear map

$$\mathsf{H}' \to \mathsf{H}'',\ \omega \mapsto df \wedge \omega$$

which is an inclusion by Proposition 3.8. When we write $\mathsf{H}' \subset \mathsf{H}''$ then we mean the inclusion obtained by the above map. We have

$$\frac{\mathsf{H}''}{\mathsf{H}'} = \mathsf{W}_f.$$

For $\omega \in \mathsf{H}''$ we define the Gelfand-Leray form

$$\frac{\omega}{df} := \frac{\omega'}{\Delta} \in \mathsf{H}'_\Delta, \quad \text{where } \Delta \cdot \omega = df \wedge \omega'.$$

Recall the definition of ω_β and η_β from §3.2. Let us first state the main results of this section.

Theorem 3.1. *Let* R *be of characteristic zero and* $\mathbb{Q} \subset \mathsf{R}$*. If* f *is a tame polynomial in* $\mathsf{R}[x]$ *then the* $\mathsf{R}[f]$*-modules* H'' *and* H' *are free and* $\omega_\beta, \beta \in I$ *(resp.* $\eta_\beta,\ \beta \in I$*) form a basis of* H'' *(resp.* H'*). More precisely, in the case* $n > 0$ *every* $\omega \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ *(resp.* $\omega \in \Omega^{n}_{\mathbb{U}_1/\mathbb{U}_0}$*) can be written*

$$\omega = \sum_{\beta \in I} p_\beta(f)\omega_\beta + df \wedge d\xi,\ p_\beta \in \mathsf{R}[f],\ \xi \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},\ \deg(p_\beta) \leq \frac{\deg(\omega)}{d} - A_\beta \tag{3.15}$$

(resp.

$$\omega = \sum_{\beta \in I} p_\beta(f)\eta_\beta + df \wedge \xi + d\xi_1,\ p_\beta \in \mathsf{R}[t],\ \xi, \xi_1 \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},\ \deg(p_\beta) \leq \frac{\deg(\omega)}{d} - A_\beta \tag{3.16}$$

).

A similar statemennt holds for the case $n = 0$. We leave its formulation and proof to the reader. We will prove the above theorem in §3.9 and §3.10.

Corollary 3.1. *Let* R *be of characteristic zero and* $\mathbb{Q} \subset \mathsf{R}$*. If* f *is a tame polynomial in* $\mathsf{R}[x]$ *then the* R*-modules* H' *and* H'' *are free and* $\eta_\beta, \beta \in I$ *(resp.* $\omega_\beta,\ \beta \in I$*) form a basis of* H' *(resp.* H''*).*

Note that in the above corollary $\{f = 0\}$ may be singular. We call $\eta_\beta, \beta \in I$ (resp. $\omega_\beta,\ \beta \in I$) the canonical basis of H' (resp. H'').

Proof. We prove the corollary for H'. The proof for H'' is similar. We consider the following canonical exact sequence

$$0 \to fH' \to H' \to \mathsf{H}' \to 0$$

Using this, Theorem 3.1 implies that H' is generated by $\eta_\beta,\ \beta \in I$. It remains to prove that η_β's are R-linear independent. If $a := \Sigma_{\beta \in I} r_\beta \eta_\beta = 0,\ r_\beta \in \mathsf{R}$ in H' then $a = fb,\ b \in H'$. We write b as a $\mathsf{R}[f]$-linear combination of η_β's and we obtain $r_\beta = fc_\beta(f)$ for some $c_\beta(f) \in \mathsf{R}[f]$. This implies that for all $\beta \in I,\ r_\beta = 0$. □

Theorem 3.1 is proved first for the case $\mathsf{R} = \mathbb{C}$ in [63]. In this article we have used a topological argument to prove that the forms $\omega_\beta, \beta \in I$ (resp. $\eta_\beta,\ \beta \in I$) are $\mathsf{R}[f]$-linear independent. It is based on the following facts: 1. η_β's generates the $\mathbb{C}[f]$-module H', 2. $\#I = \mu$ is the dimension of $H^n_{\mathrm{dR}}(\{f = c\})$ for a regular value $c \in \mathbb{C} - C$, 3. H' restricted to $\{f = 0\}$ is isomorphic to $H^n_{\mathrm{dR}}(\{f = c\})$. In the forthcomming sections we present an algebraic proof.

3.9 Proof of Theorem 3.1 for a homogeneous tame polynomial

Let $f = g$ be a homogeneous tame polynomial with an isolated singularity at origin. We explain the algorithm which writes every element of H'' of g as a $\mathsf{R}[g]$-linear combination of ω_β's. Recall that

$$dg \wedge d(P\widehat{dx_i, dx_j}) = (-1)^{i+j+\epsilon_{i,j}}\left(\frac{\partial g}{\partial x_j}\frac{\partial P}{\partial x_i} - \frac{\partial g}{\partial x_i}\frac{\partial P}{\partial x_j}\right)dx,$$

where $\epsilon_{i,j} = 0$ if $i < j$ and $= 1$ if $i > j$ and $\widehat{dx_i, dx_j}$ is dx without dx_i and dx_j (we have not changed the order of $dx_1, dx_2, \ldots$ in dx).

Proposition 3.10. *In the case $n > 0$, for a monomial $P = x^\beta$ we have*

$$\frac{\partial g}{\partial x_i} \cdot Pdx = \frac{d}{d \cdot A_\beta - \alpha_i} \frac{\partial P}{\partial x_i} g dx + dg \wedge d(\sum_{j \neq i} \frac{(-1)^{i+j+1+\epsilon_{i,j}} \alpha_j}{d \cdot A_\beta - \alpha_i} x_j P \widehat{dx_i, dx_j}). \tag{3.17}$$

Proof. The proof is a straightforward calculation:

$$\sum_{j \neq i} \frac{(-1)^{i+j+1+\epsilon_{i,j}} \alpha_j}{d \cdot A_\beta - \alpha_i} dg \wedge d(x_j P \widehat{dx_i, dx_j}) =$$

$$\frac{-1}{d \cdot A_\beta - \alpha_i} \sum_{j \neq i} (\alpha_j \frac{\partial g}{\partial x_j} \frac{\partial (x_j P)}{\partial x_i} - \alpha_j \frac{\partial g}{\partial x_i} \frac{\partial (x_j P)}{\partial x_j}) dx =$$

$$\frac{-1}{d \cdot A_\beta - \alpha_i} ((d \cdot g - \alpha_i x_i \frac{\partial g}{\partial x_i}) \frac{\partial P}{\partial x_i} - P \frac{\partial g}{\partial x_i} \sum_{j \neq i} \alpha_j (\beta_j + 1)) dx =$$

$$\frac{-1}{d \cdot A_\beta - \alpha_i} (d \cdot g \frac{\partial P}{\partial x_i} - \alpha_i \beta_i P \frac{\partial g}{\partial x_i} - P \frac{\partial g}{\partial x_i} \sum_{j \neq i} \alpha_j (\beta_j + 1)) dx.$$

□

The only case in which $dA_\beta - \alpha_i = 0$ is when $n = 0$ and $P = 1$. In the case $n = 0$ for $P \neq 1$ we have

$$\frac{\partial g}{\partial x} \cdot Pdx = \frac{d}{d \cdot A_\beta - \alpha} \frac{\partial P}{\partial x} g dx = \frac{d}{\alpha} x^{\beta - 1} g dx$$

and if $P = 1$ then $\frac{\partial g}{\partial x_i} \cdot Pdx$ is zero in H''. Based on this observation the following works also for $n = 0$.

We use the above Proposition to write every $Pdx \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ in the form

$$Pdx = \sum_{\beta \in I} p_\beta(g) \omega_\beta + dg \wedge d\xi, \tag{3.18}$$

$$p_\beta \in \mathsf{R}[g],\ \xi \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},\ \deg(p_\beta(g)\omega_\beta),\ \deg(dg \wedge d\xi) \leq \deg(Pdx).$$

- Input: The homogeneous tame polynomial g and $P \in \mathsf{R}[x]$ representing $[Pdx] \in H''$.
  Output: $p_\beta, \beta \in I$ and ξ satisfying (3.18)

  We write

  $$Pdx = \sum_{\beta \in I} c_\beta x^\beta \cdot dx + dg \wedge \eta,\ \deg(dg \wedge \eta) \leq \deg(Pdx). \tag{3.19}$$

  Then we apply (3.17) to each monomial component $\tilde{P} \frac{\partial g}{\partial x_i}$ of $dg \wedge \eta$ and then we write each $\frac{\partial \tilde{P}}{\partial x_i} dx$ in the form (3.19). The degree of the components which make Pdx not to be of the form (3.18) always decreases and finally we get the desired form.

To find a similar algorithm for H' we note that if $\eta \in \Omega^n_{\mathbb{U}_1/\mathbb{U}_0}$ is written in the form

$$\eta = \sum_{\beta \in I} p_\beta(g) \eta_\beta + dg \wedge \xi + d\xi_1,\ p_\beta \in \mathsf{R}[g],\ \xi, \xi_1 \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}, \tag{3.20}$$

where each piece in the right hand side of the above equality has degree less than $\deg(\eta)$, then

$$d\eta = \sum_{\beta\in I}(p_\beta(g)A_\beta + p'_\beta(g)g)\omega_\beta - dg\wedge d\xi \tag{3.21}$$

and the inverse of the map $\mathsf{R}[g]\to\mathsf{R}[g],\ p(g)\mapsto A_\beta.p(g)+p'(g)\cdot g$ is given by

$$\sum_{i=0}^{k} a_i g^i \to \sum_{i=1}^{k}\frac{a_i}{A_\beta+i}g^i. \tag{3.22}$$

Now let us prove that there is no $\mathsf{R}[g]$-relation between ω_β's in H''_g. This implies also that there is no $\mathsf{R}[g]$ relation between η_β's in H'_g. If such a relation exists then we take its differential and since $dg\wedge\eta_\beta = g\omega_\beta$ and $d\eta_\beta = A_\beta\omega_\beta$ we obtain a nontrivial relation in H''_g.

Since $g = dg\wedge\eta$ and x^β are R-linear independent in V_g, the existence of a non trivoal $\mathsf{R}[g]$-relation between ω_β's in H''_g implies that there is a $0\neq\omega\in H''_g$ such that $g\omega = 0$ in H''_g. Therefore, we have to prove that H''_g has no torsion. Let $a\in\mathsf{R}[x]$ and

$$g\cdot a\cdot dx = dg\wedge d\omega_1,\quad \text{for some } \omega_1\in\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}. \tag{3.23}$$

Since g is homogeneous, we can assume that a is also homogeneous. Now, the above equality implies that

$$dg\wedge(a\eta - d\omega_1) = 0 \overset{\text{Proposition 3.2}}{\Longrightarrow} a\eta = d\omega_1 + dg\wedge\omega_2,\quad \text{for some } \omega_2\in\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

We take differential of the above equality and we conclude that

$$(\sum_{i=1}^{n+1} w_i + \frac{\deg(a)}{d})a\cdot dx = 0 \ \text{ in } H''_g.$$

Since $\mathbb{Q}\subset\mathsf{R}$, we conclude that $adx = 0$ in H''_g.

Remark 3.8. The reader may have already noticed that Theorem 3.1 is not at all true if R has characteristic different form from zero. In the formulas (3.22) and (3.17) we need to divide over $d\cdot A_\beta - \alpha_i$ and $A_\beta + i$. Also, to prove that H''_g has no torsion we must be able to divide on $\sum_{i=1}^{n+1} w_i + \frac{\deg(a)}{d}$.

3.10 Proof of Theorem 3.1 for an arbitrary tame polynomial

For simplicity we assume that $n>0$. We explain the algorithm which writes every element of H'' of f as a $\mathsf{R}[f]$-linear combination of ω_β's. We write an element $\omega\in\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$, $\deg(\omega) = m$ in the form

$$\omega = \sum_{\beta\in I} p_\beta(g)\omega_\beta + dg\wedge d\psi,\ p_\beta\in\mathsf{R}[g],\ \psi\in\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},\ \deg(p_\beta(g)\omega_\beta)\le m,\ \deg(d\psi)\le m-d.$$

This is possible because g is homogeneous. Now, we write the above equality in the form

$$\omega = \sum_{\beta\in I} p_\beta(f)\omega_\beta + df\wedge d\psi + \omega',\ \text{with } \omega' = \sum_{\beta\in I}(p_\beta(g)-p_\beta(f))\omega_\beta + d(g-f)\wedge d\psi.$$

The degree of ω' is strictly less than m and so we repeat what we have done at the beginning and finally we write ω as a $\mathsf{R}[f]$-linear combination of ω_β's.

The algorithm for H' is similar and uses the fact that for $\eta \in \Omega^n_{\mathbb{U}_1/\mathbb{U}_0}$ one can write

$$\eta = \sum_{\beta \in I} p_\beta(g)\eta_\beta + dg \wedge \psi_1 + d\psi_2 \tag{3.24}$$

and each piece in the right hand side of the above equality has degree less than $\deg(\eta)$.

Let us now prove that the forms $\omega_\beta, \beta \in I$ (resp. $\eta_\beta,\ \beta \in I$) are $\mathsf{R}[f]$-linear independent. If there is a $\mathsf{R}[f]$-relation between ω_β's in H''_f, namely

$$\sum_{\beta \in I} p_\beta(f)\omega_\beta = df \wedge d\omega,\ \ \omega \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}, \tag{3.25}$$

then by taking the last homogeneous piece of the relation, we obtain a nontrivial $\mathsf{R}[g]$-relations between ω_β's in H''_g or

$$dg \wedge d\omega_1 = 0,\ \ \omega_1 \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},$$

where $\omega = \omega_1 + \omega_1'$ with $\deg(\omega_1') < \deg(\omega_1) = \deg(\omega)$. The first case does not happen by the proof of our theorem in the $f = g$ case (see §3.9). In the second case we use Proposition 3.11 and its Proposition 3.4 and obtain

$$d\omega_1 = dg \wedge d\omega_2,\ \ \omega_2 \in \Omega^{n-2}_{\mathbb{U}_1/\mathbb{U}_0},\ \ \deg(d\omega_1) = d + \deg(d\omega_2).$$

Now

$$df \wedge d\omega = df \wedge d(\omega_1 + \omega_1') = df \wedge (d(g - f) \wedge d\omega_1 + d\omega_1').$$

This means that we can substitute ω with another one and with less $\deg_x$. Taking ω the one with the smallest degree and with the property (3.25), we get a contradiction. In the case of H'_f the proof is similar and is left to the reader.

3.11 Gauss-Manin connection

In this section we define the so called Gauss-Manin connection of the R-module H. Its geometric interpretation in terms of a connection on a vector bundle in the sense of Chapter 1 will be explained in Chapter 6.

The Tjurina module of f can be rewritten in the form

$$\mathsf{W}_f := \frac{\Omega^{n+1}_{\mathbb{U}_1}}{df \wedge \Omega^n_{\mathbb{U}_1} + f\Omega^{n+1}_{\mathbb{U}_1} + \pi^{-1}\Omega^1_{\mathbb{U}_0} \wedge \Omega^n_{\mathbb{U}_1}} \cong \frac{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}}{df \wedge \Omega^n_{\mathbb{U}_1/\mathbb{U}_0} + f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}}.$$

Looking in this way, we have the well defined differential map

$$d : \mathsf{H}' \to \mathsf{W}_f.$$

Let Δ be the discriminant of f. We define the Gauss-Manin connection on H' as follows:

$$\nabla : \mathsf{H}' \to \Omega^1_T \otimes_{\mathsf{R}} \mathsf{H}'$$

$$\nabla\omega = \frac{1}{\Delta}\sum_i \alpha_i \otimes \beta_i,$$

where

$$\Delta d\omega - \sum_i \alpha_i \wedge \beta_i \in f\Omega^{n+1}_{\mathbb{U}_1} + df \wedge \Omega^n_{\mathbb{U}_1},\ \alpha_i \in \Omega^1_{\mathbb{U}_0},\ \beta_i \in \Omega^n_{\mathbb{U}_1},$$

and Ω^1_T is the localization of $\Omega^1_{\mathbb{U}_0}$ on the multiplicative group $\{1, \Delta, \Delta^2, \ldots\}$. From scheme theory point of view this is the set of differential forms defined in

$$T := \mathrm{Spec}(\mathsf{R}_\Delta) = \mathbb{U}_0 \setminus \{\Delta = 0\}.$$

To define the Gauss-Manin connection on H'' we use the fact that $\frac{\mathsf{H}''}{\mathsf{H}'} = \mathsf{W}_f$ and define

$$\nabla : \mathsf{H}'' \to \Omega^1_T \otimes_{\mathsf{R}} \mathsf{H}'',$$

$$\nabla(\omega) = \nabla(\frac{\Delta\cdot\omega}{\Delta}) = \frac{\nabla(\Delta\cdot\omega) - d\Delta \otimes \omega}{\Delta}, \tag{3.26}$$

where $\Delta\cdot\omega = df \wedge \eta,\ \eta \in \mathsf{H}'$.

The operator ∇ satisfies the Leibniz rule, i.e.

$$\nabla(p\cdot\omega) = p\cdot\nabla(\omega) + dp \otimes \omega,\ p \in \mathsf{R},\ \omega \in \mathsf{H}$$

and so it is a connection on the module H. It defines the operators

$$\nabla_i = \nabla : \Omega^i_T \otimes_{\mathsf{R}} \mathsf{H} \to \Omega^{i+1}_T \otimes_{\mathsf{R}} \mathsf{H}.$$

by the rules

$$\nabla_i(\alpha\otimes\omega) = d\alpha \otimes \omega + (-1)^i \alpha \wedge \nabla\omega,\ \alpha \in \Omega^i_T,\ \omega \in \mathsf{H}.$$

If there is no danger of confusion we will use the symbol ∇ for these operators too.

Proposition 3.11. *The connection ∇ is an integrable connection, i.e. $\nabla\circ\nabla = 0$.*

Proof. We have

$$d\omega = \sum_i \alpha_i \wedge \beta_i,\ \alpha_i \in \Omega^1_T,\ \beta_i \in \Omega^n_{\mathbb{U}_1}$$

modulo $f\Omega^{n+1}_{\mathbb{U}_1} + df \wedge \Omega^n_{\mathbb{U}_1}$. We take the differential of this equality and so we have

$$\sum_i d\alpha_i \wedge \beta_i - \alpha_i \wedge d\beta_i = 0$$

again modulo $f\Omega^{n+1}_{\mathbb{U}_1} + df \wedge \Omega^n_{\mathbb{U}_1}$. This implies that

$$\nabla\circ\nabla(\omega) = \nabla(\sum_i \alpha_i \otimes \beta_i) = \sum_i d\alpha_i \otimes \beta_i - \alpha_i \wedge \nabla\beta_i = 0.$$

□

3.12 Gauss-Manin connection along a vector field and Picard-Fuchs equations

It is is useful to look at the Guass-Manin connection in the following way: We have the operator

$$\mathcal{D}_{\mathbb{U}_0} \to \mathrm{Lei}(\mathsf{H}_\Delta),\ v \mapsto \nabla_v,$$

where $\mathcal{D}_{\mathbb{U}_0}$ is the set of vector fields in $\mathbb{U}_0$, ∇_v is the composition

$$\mathsf{H}_\Delta \xrightarrow{\nabla} \Omega^1_T \otimes_{\mathsf{R}_\Delta} \mathsf{H}_\Delta \xrightarrow{v\otimes 1} \mathsf{R}_\Delta \otimes_{\mathsf{R}_\Delta} \mathsf{H}_\Delta \cong \mathsf{H}_\Delta,$$

and $\mathrm{Lei}(\mathsf{H}_\Delta)$ is the set of additive maps ∇_v from H_Δ to itself which satisfy

$$\nabla_v(r\omega) = r\nabla_v(\omega) + dr(v)\cdot\omega,\ v \in \mathcal{D}_{\mathbb{U}_0},\ \omega \in \mathsf{H}_\Delta,\ r \in \mathsf{R}_\Delta.$$

In this way H_Δ is a (left) $\mathcal{D}$-module (differential module):

$$v\cdot\omega := \nabla_v(\omega),\ v \in \mathcal{D},\ \omega \in \mathsf{H}_\Delta.$$

Note that we can now iterate ∇_v, i.e. $\nabla_v^s = \nabla_v \circ \nabla_v \circ \cdots \circ \nabla_v$ s-times, and this is different from $\nabla \circ \nabla$ introduced before.

For a given vector field $v \in \mathcal{D}_{\mathbb{U}_0}$ and $\omega \in \mathsf{H}$ consider

$$\omega, \nabla_v(\omega), \nabla_v^2(\omega), \dots \in \mathsf{H} \otimes_{\mathsf{R}} \mathsf{k}.$$

Since the k-vector space $\mathsf{H} \otimes_{\mathsf{R}} \mathsf{k}$ is of dimension μ, there exists a positive integer $m \leq \mu$ and $p_i \in \mathsf{R}$, $i = 0, 1, 2, \dots, m$ such that

$$p_0\omega + p_1\nabla_v(\omega) + p_2\nabla_v^2(\omega) + \cdots + p_m\nabla_v^m(\omega) = 0 \tag{3.27}$$

This is called the Picard-Fuchs equation of ω along the vector field v. Since R is a unique factorization domain, we assume that there is no common factor between p_i.

3.13 Gauss-Manin connection matrix

Let $\omega_1, \omega_2, \dots, \omega_\mu$ be a basis of H and define $\omega = [\omega_1, \omega_2, \dots, \omega_\mu]^{\mathsf{t}}$. The Gauss-Manin connection in this basis can be written in the following way:

$$\nabla\omega = A \otimes \omega,\ A \in \frac{1}{\Delta}\mathrm{Mat}^{\mu\times\mu}(\Omega^1_T) \tag{3.28}$$

The integrability condition translates into $dA = A \wedge A$.

3.14 Calculating Gauss-Manin connection

Let

$$\tilde{d} : \Omega^\bullet_{\mathbb{U}_1} \to \Omega^{\bullet+1}_{\mathbb{U}_1}$$

be the differential map with respect to variable x, i.e. $\tilde{d}r = 0$ for all $r \in \mathsf{R}$, and

$$\check{d} : \Omega^\bullet_{\mathbb{U}_1} \to \Omega^{\bullet+1}_{\mathbb{U}_1}$$

be the differential map with respect to the elements of R. It is the pull-back of the differential in $\mathbb{U}_0$. We have

$$d = \tilde{d} + \check{d},$$

where d is the total differential mapping. Let s be a new parameter and $S(s)$ be the discriminant of $f - s$. We have

$$S(f) = \sum_{i=1}^{n+1} p_i \frac{\partial f}{\partial x_i},\ p_i \in \mathsf{k}[x]$$

or equivalently

$$S(f)dx = df \wedge \eta_f,\ \eta_f = \sum_{i=1}^{n+1} (-1)^{i-1} p_i \widehat{dx_i}. \tag{3.29}$$

To calculate ∇ of

$$\omega = \sum_{i=1}^{n+1} P_i \widehat{dx_i} \in \mathsf{H}'$$

we assume that ω has no dr, $r \in \mathsf{R}$, but the ingredient polynomials of ω may have coeficients in R. Let $\Delta = S(0)$ and

$$\tilde{d}\omega = P \cdot dx.$$

We have

$$S(f)d\omega = S(f)\tilde{d}\omega + S(f)\sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i} = \tilde{d}f \wedge (P \cdot \eta_f) + S(f)\sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i} \Rightarrow$$

$\Delta d\omega =$

$$\begin{aligned}
&= (\Delta - S(f))(d\omega - \sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i}) + df \wedge (P \cdot \eta_f) + (\Delta \sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i}) - \check{d}f \wedge (P \cdot \eta_f) \\
&= (\Delta \sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i}) - \check{d}f \wedge (P \cdot \eta_f) \text{ in } \Omega^1_{\mathbb{U}_0} \otimes \mathsf{H}' \\
&= \sum_j dt_j \wedge \left(\Delta(\sum_{i=1}^{n+1} \frac{\partial P_i}{\partial t_j} \widehat{dx_i}) - \frac{\partial f}{\partial t_j} \cdot P \cdot \eta_f \right), \text{ in } \Omega^1_{\mathbb{U}_0} \otimes \mathsf{H}'.
\end{aligned}$$

We conclude that

$$\nabla(\omega) = \frac{1}{\Delta} \left(\sum_j dt_j \otimes \left(\sum_{i=1}^{n+1} \left(\Delta \frac{\partial P_i}{\partial t_j} - (-1)^{i-1} \frac{\partial f}{\partial t_j} \cdot P \cdot p_i \right) \widehat{dx_i} \right) \right), \tag{3.30}$$

where

$$P = \sum_{i=1}^{n+1} (-1)^{i-1} \frac{\partial P_i}{\partial x_i}.$$

It is useful to define

$$\frac{\partial \omega}{\partial t_j} = \sum_{i=1}^{n+1} \frac{\partial P_i}{\partial t_j} \widehat{dx_i}.$$

Then

$$\nabla(\omega)=\frac{1}{\Delta}\left(\sum_j dt_j\otimes\left(\Delta\frac{\partial\omega}{\partial t_j}-\frac{\partial f}{\partial t_j}\cdot P\cdot\eta_f\right)\right). \tag{3.31}$$

The calculation of ∇ in H'' can be done using

$$\nabla(P\cdot dx)=\frac{df\wedge\nabla(P\eta_f)-d\Delta\otimes Pdx}{\Delta},\ Pdx\in\mathsf{H}''$$

which is derived from (3.26). Note that we calculate $\nabla(P\cdot\eta_f)$ from (3.30). We lead to the following explicit formula

$$\nabla(P\cdot dx)=\frac{1}{\Delta}\left(\sum_j dt_j\otimes\left(\tilde{d}f\wedge\frac{\partial(P\eta_f)}{\partial t_j}-\frac{\partial f}{\partial t_j}Q_P-\frac{\partial\Delta}{\partial t_j}P\right)\right), \tag{3.32}$$

where

$$Q_P=\sum_{i=1}^{n+1}(\frac{\partial P}{\partial x_i}p_i+P\frac{\partial p_i}{\partial x_i}).$$

To be able to calculate the iterations of the Gauss-Manin connection along a vector field v in $\mathbb{U}_0$, it is useful to introduce the operators:

$$\nabla_{v,k}:\mathsf{H}\to\mathsf{H},\ k=0,1,2,\dots$$

$$\nabla_{v,k}(\omega)=\nabla_v(\frac{\omega}{\Delta^k})\Delta^{k+1}=\Delta\cdot\nabla_v(\omega)-k\cdot d\Delta(v)\cdot\omega.$$

It is easy to show by induction on k that

$$\nabla_v^k=\frac{\nabla_{v,k-1}\circ\nabla_{v,k-2}\circ\cdots\circ\nabla_{v,0}}{\Delta^k}. \tag{3.33}$$

Remark 3.9. The formulas (3.32) and (3.31) for the Gauss-Manin connection usually produce polynomials of huge size, even for simple examples. Specially when we want to iterate the Gauss-Manin connection along a vector field, the size of polynomials is so huge that even with a computer (of the time of writing this text) we get the lack of memory problem. However, if we write the result of the Gauss-Manin connection, in the canonical basis of the R-module H, and hence reduce it modulo to those differential forms which are zero in H, we get polynomials of reasonable size.

3.15 $\mathsf{R}[\theta]$ structure of H''

In this section we consider the $\mathsf{R}[s]$-modules H'' and H', where $s\omega:=f\omega$. We have the following well-defined map:

$$\theta:H''\to H',\ \theta\omega=\eta,\ \text{where}\ \ \omega=d\eta.$$

We have used the fact that $H^n_{\mathrm{dR}}(\mathbb{U}_1/\mathbb{U}_0)=0$ (see Proposition 3.3). It is well-defined because:

$$df\wedge d\eta_1=d\eta_2\Rightarrow\eta_2=df\wedge\eta_1+d\eta_3,\ \text{for some}\ \eta_3\in\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

Using the inclusion $H'\to H''$, $\omega\mapsto df\wedge\omega$, both H' and H'' are now $\mathsf{R}[s,\theta]$-modules. The relation between $\mathsf{R}[s]$ and $\mathsf{R}[\theta]$ structures is given by:

Proposition 3.12. *We have:*

$$\theta \cdot s = s \cdot \theta - \theta \cdot \theta$$

and for $n \in \mathbb{N}$

$$\theta^n s = s\theta^n - n\theta^{n+1}.$$

Proof. The map $d : H' \to H''$ satisfies

$$d \cdot s = s \cdot d + df,$$

where s stands for the mapping $\omega \mapsto s\omega$ and df stands for the mapping $\omega \mapsto df \wedge \omega,\ \omega \in H'$. Composing the both sides of the above equality by θ we get the first statement. The second statement is proved by induction. □

For a homogeneous polynomial with an isolated singularity at the origin we have $d\eta_\beta = A_\beta \omega_\beta$ and so

$$\theta\omega_\beta = \frac{s}{A_\beta}\omega_\beta.$$

Remark 3.10. The action of θ on H'' is inverse to to the action of the Gauss-Manin connection with respect to the parameter s in $f - s = 0$ (we have composed the Gauss-Manin connection with $\frac{\partial}{\partial s}$). This arises the following question: Is it possible to construct similar structures for H' and H''?

Chapter 4

Gauss-Manin system and Hodge filtration

In this chapter we keep using the algebraic notations of the previous chapter. We define the Gauss-Manin system M associated to f which plays the same role as H and it has the advantage that the Hodge and weight filtrations in M are defined explicitly. The main role of the Hodge and weight filtrations in the present text is to distinguish between differential forms and hence the corresponding modular foliations. We state the Griffiths transversality theorem which is a direct consequence of our definitions. The transversality theorem poses restrictions on the codimension of modular foliations.

4.1 Gauss-Manin system

In this section we define the Gauss-Manin system associated to a tame polynomial. Our approach is by looking at differential forms with poles along $\{f = 0\}$ in $\mathbb{U}_1/\mathbb{U}_0$ which is a proper way when one deals with the tame polynomials in the sense of present text. We will later use the material of this section for residue of such differential forms along the pole $\{f = 0\}$.

The Gauss-Manin system for a tame polynomial f is defined to be:

$$\mathsf{M}_f = \mathsf{M} := \frac{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}[\frac{1}{f}]}{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0} + d(\Omega^{n}_{\mathbb{U}_1/\mathbb{U}_0}[\frac{1}{f}])} \cong \frac{\Omega^{n+1}_{\mathbb{U}_1}[\frac{1}{f}]}{\Omega^{n+1}_{\mathbb{U}_1} + d(\Omega^{n}_{\mathbb{U}_1}[\frac{1}{f}]) + \pi^{-1}\Omega^{1}_{\mathbb{U}_0} \wedge \Omega^{n}_{\mathbb{U}_1}[\frac{1}{f}]},$$

where $\Omega^{i}_{\mathbb{U}_1}[\frac{1}{f}]$ is the set of polynomials in $\frac{1}{f}$ with coefficients in $\Omega^{i}_{\mathbb{U}_1}$ and etc.. It has a natural filtration given by the pole order along $\{f = 0\}$, namely

$$\mathsf{M}_i := \{[\frac{\omega}{f^i}] \in \mathsf{M} \mid \omega \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}\},\ \mathsf{M}_1 \subset \mathsf{M}_2 \subset \cdots \subset \mathsf{M}_i \subset \cdots \subset \mathsf{M}_\infty := \mathsf{M}.$$

It is useful to identify H' by its image under $df \wedge \cdot$ in H'' and define $\mathsf{M}_0 := \mathsf{H}'$. Note that in M we have

$$[\frac{d\omega}{f^{i-1}}] = [\frac{(i-1)df \wedge \omega}{f^i}],\ \omega \in \Omega^{n}_{\mathbb{U}_1/\mathbb{U}_0}, i = 2, 3, \ldots \tag{4.1}$$

$$[\frac{df \wedge d\omega}{f^i}] = [d(\frac{df \wedge \omega}{f^i})] = 0,\ \omega \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}, i = 1, 2, \ldots \tag{4.2}$$

Proposition 4.1. *If the discriminant of the tame polynomial f is not zero then the differential form $\frac{\omega}{f^i}$, $i\in\mathbb{N}$, $\omega\in\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ is zero in M if and only if ω is of the form*

$$fd\omega_1-(i-1)df\wedge\omega_1+df\wedge d\omega_2+f^i\omega_3,\ \omega_1\in\Omega^n_{\mathbb{U}_1/\mathbb{U}_0},\ \omega_2\in\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},\ \omega_3\in\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

Proof. Let

$$\frac{\omega}{f^i}=d(\frac{\omega_1}{f^s})\mod \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}. \tag{4.3}$$

If $s=i-1$ then ω has the desired form. If $s\geq i$ then $df\wedge\omega_1\in f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ and so by Proposition 3.8 we have $\omega_1=f\omega_3+df\wedge\omega_2$ and so

$$\frac{\omega}{f^i}=d(\frac{f\omega_3+df\wedge\omega_2}{f^s}),\quad \mod \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}. \tag{4.4}$$

If $s=i$ then we obtain the desired form for ω. If $s>i$ we get $df\wedge d\omega_2+(s-1)df\wedge\omega_3\in f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ and so again by Proposition 3.8 we have $d\omega_2+(s-1)\omega_3=f\omega_4+df\wedge\omega_5$. We calculate ω_3 from this equality and substitute it in (4.4) and obtain

$$\frac{\omega}{f^i}=\frac{1}{s-1}d(\frac{f\omega_4+df\wedge\omega_5}{f^{s-1}})\mod \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

We repeat this until getting the situation $s=i$. □

The structure of M and its relation with H is described in the following proposition.

Proposition 4.2. *We have the well-defined canonical maps*

$$\mathsf{H}''\to\mathsf{M}_1,\omega\mapsto[\frac{\omega}{f}],$$

$$\mathsf{W}\to\mathsf{M}_i/\mathsf{M}_{i-1},\ \omega\mapsto[\frac{\omega}{f^i}],\ i=1,2,\ldots.$$

If the discriminant of the tame polynomial f is not zero then they are isomorphims of R*-modules.*

Proof. The fact that they are well-defined follows from the equalities (4.1) and (4.2). The non-trivial part of the second part is that they are injective. This follows from Proposition 4.1. □

4.2 The connection

The Gauss-Manin connection on M is the map

$$\nabla:\mathsf{M}\to\Omega^1_{\mathbb{U}_0}\otimes_{\mathsf{R}}\mathsf{M}$$

which is obtained by derivation with respect to the elements of R (the derivation of x_i is zero). By definition it maps M_i to $\Omega^1_{\mathbb{U}_0}\otimes_{\mathsf{R}}\mathsf{M}_{i+1}$ For any vector field in $\mathbb{U}_0$, ∇_v is given by

$$\nabla_v:\mathsf{M}\to\mathsf{M},\ \nabla_v([\frac{Pdx}{f^i}])=[\frac{v(P)\cdot f-iP\cdot v(f)}{f^{i+1}}dx],\ P\in\mathsf{R}[x]. \tag{4.5}$$

where $v(P)$ is the differential of P with respect to elements in R and along the vector field v $(v(\cdot):\mathsf{R}\to\mathsf{R},\ p\mapsto dp(v))$. In the case $i=0$ it is given by

$$\nabla_v([\frac{df\wedge\omega}{f}])=[\frac{f\cdot v(df\wedge\omega)-v(f)\cdot df\wedge\omega}{f^2}]=[\frac{v(df\wedge\omega)+d(v(f)\cdot\omega)}{f}]$$

and so ∇_v maps M_0 to M_1. The operator ∇_v is also called the Gauss-Manin connection along the vector field v. To see the relation of the Gauss-Manin connection of this section with the Gauss-Manin connection of §3.11 we need the following proposition:

Proposition 4.3. *Suppose that the discriminant Δ of the tame polynomial f is not zero. Then the multiplication by Δ in M maps M_i to M_{i-1} for all $i\in\mathbb{N}$.*

Proof. The multiplication by Δ in W is zero and so for a given $\frac{\omega}{f^i}$ we can write

$$\Delta\frac{\omega}{f^i}=\frac{f\omega_1+df\wedge\omega_2}{f^i}=\frac{\omega_1}{f^{i-1}}+\frac{1}{i-1}(\frac{d\omega_2}{f^{i-1}}-d(\frac{\omega_2}{f^{i-1}}))$$

which is equal to $\frac{\omega_1+\frac{1}{i-1}d\omega_2}{f^{i-1}}$ in M. □

Now, it is easy to see that $\Delta\cdot\nabla_v:\mathsf{H}\to\mathsf{H},\ \mathsf{H}=\mathsf{H}',\mathsf{H}''$ of this section and §3.11 coincide. Recall that for a R-module M and $a\in M$, M_a denotes the localization of M over the multiplicative set $\{1,a,a^2,\cdots\}$. As a corollary of Proposition 4.3 we have:

Proposition 4.4. *The inclusion $\mathsf{H}\to\mathsf{M}$ induces an isomorphism of R-modules $\mathsf{M}_\Delta\cong\mathsf{H}_\Delta$.*

4.3 Mixed Hodge structure of M

Recall that $\{x^\beta\mid\beta\in I\}$ is a monomial basis of the R-module V_g and $\omega_\beta,\ \beta\in I$ is a basis of the R-module H''.

Definition 4.1. We define the degree of $\frac{\omega}{f^k},\ k\in\mathbb{N},\omega\in\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ to be $\deg_x(\omega)-\deg_x(f^k)$. By definition we have $\deg(\frac{\omega_\beta}{f^k})=d(A_\beta-k)$. The degree of $\alpha\in\mathsf{M}$ is defined to be the minimum of the degrees of $\frac{\omega}{f^k}\in\alpha$.

In order to define the mixed Hodge structure of M we need the following proposition.

Proposition 4.5. *Every element of degree s of M can be written as an R-linear sum of the elements*

$$\frac{\omega_\beta}{f^k},\ \beta\in I,\ 1\le k,\ A_\beta\le k, \tag{4.6}$$

$$\deg(\frac{\omega_\beta}{f^k})\le s.$$

Proof. Let us be given an element $\frac{\omega}{f^k}$ of degree s in M. According to Corollary 3.1, we write $\omega=\sum_{\beta\in I}a_\beta\omega_\beta+df\wedge d\omega_2+f\omega_1$ and so

$$\frac{\omega}{f^k}=\sum_{\beta\in I}a_\beta\frac{\omega_\beta}{f^k}+\frac{\omega_1}{f^{k-1}}\text{ in }\mathsf{M}.$$

We repeat this argument for ω_1. At the end we get $\frac{\omega}{f^k}$ as a R-linear combination of $\frac{\omega_\beta}{f^i}$, $\beta \in I$, $k \in \mathbb{N}$. An alternative way is to say that ω can be written as an $\mathsf{R}[f]$-linear combinations of ω_β, $\beta \in I$ modulo $df \wedge d\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}$ (see Theorem 3.1). The degree conditions (3.15) implies that the we have used only $\frac{\omega_\beta}{f^i}$ with $\deg(\frac{\omega_\beta}{f^i}) \leq \deg(\frac{\omega}{f^k})$.

Now, we have to get rid of elements of type $\frac{\omega_\beta}{f^k}$, $A_\beta > k$. Given such an element, in M we have:

$$\frac{\omega_\beta}{f^k} = \frac{1}{A_\beta}\frac{d\eta_\beta}{f^k} = \frac{k}{A_\beta}\frac{df \wedge \eta_\beta}{f^{k+1}} = \frac{k}{A_\beta}\frac{f\omega_\beta + (g-f)\omega_\beta + d(f-g)\wedge\eta_\beta}{f^{k+1}}$$

and so

$$\frac{\omega_\beta}{f^k} = \frac{k}{A_\beta - k}\frac{(g-f)\omega_\beta + d(f-g)\wedge\eta_\beta}{f^{k+1}}. \tag{4.7}$$

The degree of the right hand side of (4.7) is less than $d(A_\beta - k)$, which is the degree of the left hand side. We write the right hand side in terms of $\frac{\omega_{\beta'}}{f^s}$, $\beta' \in I, s \in \mathbb{N}$ and repeat (4.7) for these new terms. Since each time the degree of the new elements $\frac{\omega'_\beta}{f^s}$ decrease, at some pont we get the desired form for $\frac{\omega_\beta}{f^k}$. □

By definition of ∇_v in (4.5) and Proposition 4.6, we have

$$\deg(\nabla_v(\alpha)) \leq \deg(\alpha),\ \alpha \in \mathsf{M}. \tag{4.8}$$

Now, we can define two natural filtration on M.

Definition 4.2. We define $\mathsf{W}_n = \mathsf{W}_n\mathsf{M}_\Delta$ to be the R_Δ-submodule of M_Δ generated by

$$\frac{\omega_\beta}{f^k},\ \beta \in I,\ A_\beta < k$$

and call

$$0 =: \mathsf{W}_{n-1} \subset \mathsf{W}_n \subset \mathsf{W}_{n+1} := \mathsf{M}_\Delta$$

the weight filtration of M_Δ. We also define $\mathsf{F}^i = \mathsf{F}^i\mathsf{M}_\Delta$ to be the R_Δ-submodule of of M_Δ generated by

$$\frac{\omega_\beta}{f^k},\ \beta \in I,\ A_\beta \leq k \leq n+1-i \tag{4.9}$$

and call

$$0 = \mathsf{F}^{n+1} \subset \mathsf{F}^n \subset \mathsf{F}^{n-1} \subset \cdots \subset \mathsf{F}^0$$

the Hodge filtration of M_Δ. The pair $(\mathsf{F}^\bullet, \mathsf{W}^\bullet)$ is called the mixed Hodge structure of M_Δ.

Since for $j = 0, 1, 2, \ldots, \infty$ we have the the inclusion $\mathsf{H} := \mathsf{M}_j \subset \mathsf{M}_\Delta$, we define the mixed Hodge structure of H to be the intersection of the (pieces) of the mixed Hodge structure of M_Δ with H:

$$\mathsf{W}_i\mathsf{H} := \mathsf{W}_i\mathsf{M}_\Delta \cap \mathsf{H},\ \mathsf{F}^j\mathsf{H} := \mathsf{F}^j\mathsf{M}_\Delta \cap \mathsf{H},$$

$$i = n-1, n, n+1,\ j = 0, 1, 2, \ldots, n+1.$$

The Hodge filtration induces a filtration on $\mathrm{Gr}^{\mathsf{W}}_i := \mathsf{W}_i/\mathsf{W}_{i-1},\ i = n, n+1$ and we set

$$\mathrm{Gr}^j_{\mathsf{F}}\mathrm{Gr}^{\mathsf{W}}_i := \mathsf{F}^j\mathrm{Gr}^{\mathsf{W}}_i/\mathsf{F}^{j+1}\mathrm{Gr}^{\mathsf{W}}_i = \frac{(\mathsf{F}^j \cap \mathsf{W}_i) + \mathsf{W}_{i-1}}{(\mathsf{F}^{j+1} \cap \mathsf{W}_i) + \mathsf{W}_{i-1}},\ j = 0, 1, 2, \ldots, n+1. \tag{4.10}$$

For the original definition of the mixed Hodge structure in the complex context $\mathsf{R}=\mathbb{C}$ or $\mathbb{C}(t)$(the field of rational functions in $t=(t_1,t_2,\ldots,t_s)$) see [79, 80]. In fact we have used Griffiths-Steenbrink theorem (see [77]) in order to formulate the above definition. In particular, in this context we have $\mathsf{F}^0\mathsf{H}=\mathsf{H}$.

Remark 4.1. Since R is a principal ideal domain and $\mathsf{H}:=\mathsf{H}',\ \mathsf{H}''$ is a free R-module (Corollary 3.1), any R-sub-module of H is also free and in particular the pieces of mixed Hodge structure of H are free R-modules.

Definition 4.3. *A set $B=\cup_{k=0}^n B_n^k\cup\cup_{k=1}^n B_{n+1}^k\subset\mathsf{H}$ is a basis of H compatible with the mixed Hodge structure if it is a basis of the R-module H and moreover each B_m^k form a basis of $\mathrm{Gr}_{\mathsf{F}}^k\mathrm{Gr}_m^{\mathsf{W}}\mathsf{H}$.*

4.4 Homogeneous tame polynomials

Bellow for simplicity we use d to denote the differential operator with respect to the variables $x_1,x_2,\ldots,x_{n+1}$. Let us consider a homogeneous polynomial g in the graded ring $\mathsf{R}[x],\ \deg(x_i)=\alpha_i$. We have the equalities

$$g=\sum_{i=1}^{n+1}w_ix_i\frac{\partial g}{\partial x_i}\ \text{ equivalently } gdx=dg\wedge\eta,$$

$$g\omega_\beta=dg\wedge\eta_\beta,\ d\eta=(w\cdot 1)dx,\ d\eta_\beta=A_\beta\omega_\beta. \tag{4.11}$$

The discriminant of the polynomial g is zero. We define $f:=g-t\in\mathsf{R}[t][x]$ which is tame and its discriminant is $(-t)^\mu$. The above qualities imply that

$$\nabla_{\frac{\partial}{\partial t}}\eta_\beta=\frac{A_\beta}{t}\eta_\beta,\ \nabla_{\frac{\partial}{\partial t}}(\omega_\beta)=\frac{(A_\beta-1)}{t}\omega_\beta.$$

We have
$$\frac{t\omega_\beta}{f^k}=\frac{-f\omega_\beta+dg\wedge\eta_\beta}{f^k}=(-1+\frac{A_\beta}{k-1})\frac{\omega_\beta}{f^{k-1}}\text{ in }\mathsf{M}$$
and so

$$\frac{\omega_\beta}{f^k}=\frac{1}{t^{k-1}}(-1+\frac{A_\beta}{k-1})(-1+\frac{A_\beta}{k-2})\cdots(-1+\frac{A_\beta}{1})\frac{\omega_\beta}{f}\text{ in }\mathsf{M}_t. \tag{4.12}$$

Note that under the canonical inclusion $\mathsf{H}'\subset\mathsf{H}''$ of the Brieskorn modules of f we have

$$t\omega_\beta=\eta_\beta.$$

Theorem 4.1. *For a weighted homogeneous polynomial $g\in\mathsf{R}[x]$ with an isolated singularity at the origin, the set*
$$B=\cup_{k=1}^n B_{n+1}^k\cup\cup_{k=0}^n B_n^k$$
with
$$B_{n+1}^k=\{\eta_\beta\mid A_\beta=n-k+1\},B_n^k=\{\eta_\beta\mid n-k<A_\beta<n-k+1\},$$
is a basis of the R-module H' associated to $g-t\in\mathsf{R}[t][x]$ compatible with the mixed Hodge structure. The same is true for H'' replacing η_β with ω_β.

Proof. This theorem with the classical definition of the mixed Hodge structures is proved by Steenbrink in [77]. In our context it is a direct consequence of Definition 4.2 and the equality (4.12). □

4.5 Griffiths transversality

In the free module H we have introduced the mixed Hodge structure and the Gauss-Manin connection. It is natural to ask whether there is any relation between these two concepts or not. The answer is given by the next theorem. First, we give a definition

Definition 4.4. A vector field v in $\mathbb{U}_0$ is called a basic vector field if for any $p \in \mathsf{R}$ there is $k \in \mathbb{N}$ such that $v^k(p) = 0$, where v^k is the k-th iteration of $v(\cdot) : \mathsf{R} \to \mathsf{R},\ p \mapsto dp(v)$.

For $\mathsf{R} = \mathbb{C}[t_1, t_2, \dots, t_s]$, the vector fields $\frac{\partial}{\partial t_i},\ i = 1, 2, \dots, s$ are basic.

Theorem 4.2. *Let* $(\mathsf{W}_\bullet, \mathsf{F}^\bullet)$ *be the mixed Hodge structure of* H*. The Gauss-Manin connection on* H *satisfies:*

1. *Griffiths transversality:*
$$\nabla(\mathsf{F}^i) \subset \Omega^1_T \otimes_{\mathsf{R}} \mathsf{F}^{i-1},\ i = 1, 2, \dots, n.$$
2. *No residue at infinity: We have*
$$\nabla(\mathsf{W}_n) \subset \Omega^1_T \otimes_{\mathsf{R}} \mathsf{W}_n.$$
3. *Residue killer: For a tame polynomial* f *of degree* d*,* $\omega \in \mathsf{H}$ *and a basic vector field* $v \in \mathcal{D}_{\mathbb{U}_0}$ *such that* $\deg(v(f)) < d$ *there exists* $k \in \mathbb{N}$ *such that* $\nabla_v^k \omega \in \mathsf{W}_n\mathsf{M}_\Delta$*.*

Griffiths transversality has been proved in [25, 24] for Hodge structures. For a recent text see also [79]. The proof for mixed Hodge structures is similar and can be found in [81, 82].

Proof. It is enough to prove the theorem for the Gauss-Manin connection ∇_v along a vector field $v \in \mathcal{D}_{\mathbb{U}_0}$ and the mixed Hodge of M_Δ.

For the Griffiths transversality, we have to prove that ∇_v maps $\mathsf{F}^i\mathsf{M}_\Delta$ to $\mathsf{F}^{i-1}\mathsf{M}_\Delta$. By Leibniz rule, it is enough to take an element $\omega = \frac{\omega_\beta}{f^k},\ A_\beta \leq k$ in the set (4.9) and prove that $\nabla_v\omega$ is in $\mathsf{F}^{i-1}\mathsf{M}$. This follows from (4.5) and:

$$\nabla_v \frac{\omega_\beta}{f^k} = \frac{v(f)\omega_\beta}{f^{k+1}},\ \deg(\frac{v(f)\omega_\beta}{f^{k+1}}) \leq \deg(\frac{\omega_\beta}{f^k}) = d(A_\beta - k) \leq 0$$

For the second part of the theorem we have to prove that ∇_v maps $\mathsf{W}_n\mathsf{M}_\Delta$ to $\mathsf{W}_n\mathsf{M}_\Delta$. This follows follows form (4.8) and the fact that $\frac{\omega_\beta}{f^k},\ A_\beta < k$ generate $\mathsf{W}_n\mathsf{M}_\Delta$

For the third part of the theorem we proceed as follows: For $\omega \in \mathsf{M}$ we use Proposition 4.6 and write ω as a R-linear combination of $\frac{\omega_\beta}{f^k},\ \beta \in I,\ A_\beta \leq k$. By the second part of the theorem, it is enough to prove that for $\frac{\omega_\beta}{f^k},\ \ A_\beta = k$ and $p \in \mathsf{R}$, there exists some $s \in \mathbb{N}$ such that

$$\nabla_v^s(p[\frac{\omega_\beta}{f^k}]) \in \mathsf{W}_n\mathsf{M}_\Delta$$

Since $\deg(v(f)) < d$ we have

$$\deg \nabla_v[\frac{\omega_\beta}{f^k}] < \deg(\frac{\omega_\beta}{f^k}) = 0$$

and so $\nabla_v[\frac{\omega_\beta}{f^k}] \in \mathsf{W}_n\mathsf{M}_\Delta$. Now modulo $\mathsf{W}_n\mathsf{M}_\Delta$ we have

$$\nabla_v^k(p[\frac{\omega_\beta}{f^k}]) = v^k(p) \cdot [\frac{\omega_\beta}{f^k}]$$

and the affirmation follows from the fact that v is a basic vector field. $\square$

Definition 4.5. We say that a polynomial $g \in \mathsf{R}[x]$ does not depend on R (or parameters in R) if all the coefficients of g lies in the kernel of the map $d : \mathsf{R} \to \Omega^1_{\mathbb{U}_0}$. In other words, $v(g) = 0$ for all vector field $v \in \mathcal{D}_{\mathbb{U}_0}$.

In the case $\mathsf{R} := \mathbb{Q}[t_1, t_2, \dots, t_s]$ the above definition simply means that in g the parameters $t_i,\ i = 1, 2, \dots, s$ do not appear. In this case, for a tame polynomial f over R such that its last homogeneous piece g does not depend on R, all $v = \frac{\partial}{\partial t_i}$'s are basic and $\deg(v(f)) \leq \deg(f)$. In practice, we use this as an example for the third part of Theorem 4.2.

4.6 Foliations

In this section we introduce foliations in an algebraic context. It is left to the reader to justify the geometric interpretation of foliations and the associated notions.

Definition 4.6. A foliation $\mathcal{F}$ in the affine variety $\mathbb{U}_0$ is a submodule of the R-module $\Omega^1_{\mathbb{U}_0}$ such that for all $\omega \in \mathcal{F}$ we have

$$d\omega \in \mathcal{F} \wedge \Omega^1_{\mathbb{U}_0}. \tag{4.13}$$

In the geometric context of foliations (4.13) is the integrability condition for the distribution induced by $\mathcal{F}$ in the tangent space of $\mathbb{U}_0 = \mathbb{C}^s$. Recall that k is the quotient field of R.

Definition 4.7. The codimension of a foliation $\mathcal{F}$ is defined in the following way:

$$\operatorname{codim}(\mathcal{F}) := \dim_{\mathsf{k}}(\mathcal{F} \otimes_{\mathsf{R}} \mathsf{k}).$$

The word codimension refers to the codimension of the leaves of the foliation induced by $\mathcal{F}$ in the geometric context $\mathbb{U}_0 = \mathbb{C}^s$.

Definition 4.8. The first integral field of a R-submodule $\mathcal{F}$ of $\Omega^1_{\mathbb{U}_0}$ is defined to be the set

$$\{f \in \mathsf{k} \mid df \in \mathcal{F} \otimes_{\mathsf{R}} \mathsf{k}\}.$$

It is not hard to see that the above set is indeed a field.

4.7 Modular foliations

We have constructed the Gauss-Manin connection

$$\nabla : \mathsf{M} \to \Omega^1_{\mathbb{U}_0} \otimes_{\mathsf{R}} \mathsf{M}$$

which maps M_i to $\Omega^1_{\mathbb{U}_0} \otimes_{\mathsf{R}} \mathsf{M}_{i+1}$ and is integrable. Therefore, we can define the notion of a modular foliation associated to ∇ in an algebraic context.

Definition 4.9. *The modular foliation associated to $\omega \in \mathsf{H}$ is the R-module $\mathcal{F}_\omega$ generated by $\alpha_\beta \in \Omega^1_{\mathbb{U}_0}$, where*

$$\nabla(\omega) = \sum_{j=1}^{\mu} \alpha_j \otimes \omega_j.$$

and ω_j's form a basis of the R-module H.

One has to verify that $\mathcal{F}_\omega$ does not depend on the choice of the basis $\omega_j,\ j=1,2,\ldots,\mu$. The fact that $\mathcal{F}_\omega$ is a foliation, i.e. it satisfies (4.13) follows from $\nabla(\nabla(\omega))=0$.

Theorem 4.3. *The codimension of a modular foliation $\mathcal{F}_\omega,\ \omega\in \mathsf{F}^i\mathsf{H}$ is at most the rank of the (free) R-module F^{i-1}.*

Proof. Let us choose a basis $\omega_i,\ i=1,2,\ldots,s$ of the free R-module F^{i-1}. According to Griffiths transversality for $\omega\in\mathsf{F}^i$ we can write $\nabla\omega=\sum_{i=1}^s\eta_i\otimes\omega_i$ and so $\mathcal{F}_\omega$ is generated by $\eta_i,\ i=1,2,\ldots,s$. □

Let us assume that the last homogeneous piece g of a tame polynomial f does not depend on R. For $\omega\in\mathsf{M}$ we may find first integrals for $\mathcal{F}_\omega$ in the following way: By Proposition 4.5 we can write ω as

$$\omega=\sum_{A_\beta=k,k\in\mathbb{N},\beta\in I} a_{\beta,k}\frac{\omega_\beta}{f^k}+\sum_{A_\beta<k,k\in\mathbb{N},\beta\in I} b_{\beta,k}\frac{\omega_\beta}{f^k}\tag{4.14}$$

and so

$$\nabla\omega=\sum_{A_\beta=k,k\in\mathbb{N},\beta\in I} da_{\beta,k}\otimes\frac{\omega_\beta}{f^k}+\alpha,\ \ \alpha\in\Omega^1_{\mathbb{U}_0}\otimes\mathsf{W}_n\mathsf{M}.$$

We have used the second part of Theorem 4.2 and

$$\deg(\nabla(\frac{\omega_\beta}{f^k}))<0,\quad\text{for } A_\beta=k.$$

Since $\frac{\omega_\beta}{f^k},\ A_\beta=k$ are k-linearly independent in $\mathsf{M}\otimes_{\mathsf{R}}\mathsf{k}/(\mathsf{W}_n\mathsf{M}\otimes_{\mathsf{R}}\mathsf{k})$, we conclude that $da_{\beta,k}\in\mathcal{F}_\omega$ and so $a_{\beta,k}$ is in the first integral field of $\mathcal{F}_\omega$. Later in Chapter 6 we will see the geometric interpretation of $a_{\beta,k}$'s.

Chapter 5

Topology of tame polynomials

It was S. Lefschetz who for the first time studied systematically the topology of smooth projective varieties. Later, his theorems were translated into the language of modern Algebraic Geometry, using Hodge theory, sheaf theory and spectral sequences. "But none of these very elegant methods yields Lefschetz's full geometric insight, e.g. they do not show us the famous vanishing cycles"(K. Lamotke). A direction in which Lefschetz's topological ideas were developed was in the study of the topology of hypersurface singularities. The objective of this chapter is to study the topology of the fibers of tame polynomials following the local context [2] and the global context [43]. To make this chapter self sufficient, we have put many well-known materials from the mentioned references. We mainly use a tame polynomial $f \in \mathbb{C}[x]$ in the sense of Chapter 3. We denote by C the set of critical values of f and by μ the Milnor number of f.

5.1 Vanishing cycles and orientation

We consider in $\mathbb{C}$ the canonical orientation $\frac{1}{-2\sqrt{-1}} dx \wedge d\bar{x} = d(\mathrm{Re}(x)) \wedge d(\mathrm{Im}(x))$. This corresponds to the anti-clockwise direction in the complex plane. In this way, every complex manifold carries an orientation obtained by the orientation of $\mathbb{C}$, which we call it the canonical orientation. For a complex manifold of dimension n and an holomorphic nowhere vanishing differential n-form ω on it, the orientation obtained from $\frac{1}{(-2\sqrt{-1})^n} \omega \wedge \overline{\omega}$ differs from the canonical one by $(-1)^{\frac{n(n-1)}{2}}$ (as an exercise compare the orientation

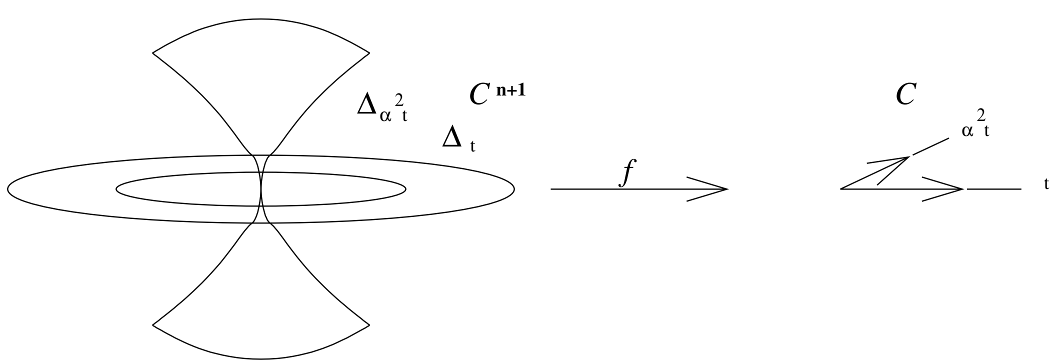

Figure 5.1: Intersection of thimbles

$\mathrm{Re}(\omega) \wedge \mathrm{Im}(\omega)$ with the canonical one. Assume that the complex manifold is $(\mathbb{C}^n, 0)$ and $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$). For instance for a tame polynomial f, the Gelfand-Leray form $\frac{dx}{df}$ in each regular fiber of f is such an n-form. Holomorphic maps between complex manifolds preserve the canonical orientation. For a zero dimensional manifold an orientation is just a map which associates ± 1 to each point of the manifold.

Let $f = x_1^2 + x_2^2 + \cdots + x_{n+1}^2$. For a real positive number t, the n-th homology of the complex manifold $L_t := \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \mid f(x) = t\}$ is generated by the so called vanishing cycle

$$\delta_t = \mathbb{S}_n(t) := L_t \cap \mathbb{R}^{n+1}.$$

It vanishes along the vanishing path γ which connects t to 0 in the real line. The (Lefschetz) thimble

$$\Delta_t := \cup_{0 \le s \le t} \delta_s = \{x \in \mathbb{R}^{n+1} \mid f(x) \le t\}$$

is a real $(n+1)$-dimensional manifold which generates the relative $(n+1)$-th homology $H_{n+1}(\mathbb{C}^{n+1}, L_t, \mathbb{Z})$. We consider for $\mathbb{S}_n(t)$ the orientation η such that $\eta \wedge \mathrm{Re}(df)$ is $\mathrm{Re}(dx_1) \wedge \mathrm{Re}(dx_2) \wedge \cdots \wedge \mathrm{Re}(dx_{n+1})$, which is an orientation for Δ_t. Let α be a complex number near to 1 with $\mathrm{Im}(\alpha) > 0, |\alpha| = 1$ and

$$h : L_t \to L_{\alpha^2 t}, \ x \mapsto \alpha \cdot x.$$

The oriented cycle $h_* \delta_t$ is obtained by the monodromy of δ_t along the shortest path which connects t to $\alpha^2 t$. Now the orientation of Δ_t wedge with the orientation of $h_* \Delta_t$ is:

$$\begin{aligned}
&= \mathrm{Re}(dx_1) \wedge \mathrm{Re}(dx_2) \wedge \cdots \wedge \mathrm{Re}(dx_{n+1}) \wedge \mathrm{Re}(\alpha^{-1} dx_1) \wedge \mathrm{Re}(\alpha^{-1} dx_2) \wedge \cdots \wedge \mathrm{Re}(\alpha^{-1} dx_{n+1}) \\
&= (-1)^{\frac{n^2+n}{2}} \mathrm{Im}(\alpha)^{n+1} \mathrm{Re}(dx_1) \wedge \mathrm{Im}(dx_1) \wedge \mathrm{Re}(dx_2) \wedge \mathrm{Im}(dx_2) \wedge \cdots \wedge \mathrm{Re}(dx_{n+1}) \wedge \mathrm{Im}(dx_{n+1}) \\
&= (-1)^{\frac{n^2+n}{2}} \text{ the canonical orientation of } \mathbb{C}^{n+1}.
\end{aligned}$$

This does not depend on the orientation η that we chose for δ_t. The assumption $\mathrm{Im}(\alpha) > 0$ is equivalent to the fact that $\mathrm{Re}(dt) \wedge h_* \mathrm{Re}(dt)$ is the canonical orientation of $\mathbb{C}$. We conclude that:

Proposition 5.1. *The orientation of* $(\mathbb{C}^{n+1}, 0)$ *obtained by the intersection of two thimbles is* $(-1)^{\frac{n(n+1)}{2}}$ *times the orientation of* $(\mathbb{C}, 0)$ *obtained by the intersection of their vanishing paths.*

See Figure 5.1.

5.2 Picard-Lefschetz theory of tame polynomials

Let us consider a tame polynomial f in the ring $\mathsf{R}[x]$, where R is a localization of $\mathbb{Q}[t]$, $t = (t_1, t_2, \ldots, t_s)$ a multi parameter, over a multiplicative group generated by $a_i \in \mathsf{R}, \ i = 1, 2, \ldots, r$. Let also

$$\mathbb{U}_0 := \mathbb{C}^s \backslash (\cup_{i=1}^r \{t \in \mathbb{C}^s \mid a_i(t) = 0\}), \ \mathbb{U}_1 := \{(x, t) \in \mathbb{C}^{n+1} \times \mathbb{U}_0 \mid f(x, t) = 0\},$$

$$T := \mathbb{U}_0 \backslash \{t \in \mathbb{U}_0 \mid \Delta(t) = 0\},$$

where Δ is the discriminant of f. We have a canonical projection $\pi : \mathbb{U}_1 \to \mathbb{U}_0$ and we define:

$$L_t := \pi^{-1}(t) = \{x \in \mathbb{C}^{n+1} \mid f_t(x) = 0\}.$$

where f_t is the poynomial obtained by fixing the value of t. Let g be the last homogeneous piece of f and $\mathbb{N}_{n+1} = \{1, 2, \ldots, n+1\}$, $S = \{i \in \mathbb{N}_{n+1} \mid \alpha_i = 1\}$ and $S^c = \mathbb{N}_{n+1} \backslash S$.

Definition 5.1. The homogeneous polynomial g has a strongly isolated singularity at the origin if g has an isolated singularity at the origin and for all $R \subset \{1, 2, 3 \ldots, n+1\}$ with $S \subset R$, g restricted to $\cap_{i \in R}\{x_i = 0\}$ has also an isolated singularity at the origin.

If $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 1$ then the condition 'strongly isolated' is the same as 'isolated'. The Picard-Lefschetz theory of tame polynomials is based on the following statement:

Theorem 5.1. *If the last homogeneous piece of a tame polynomial f is either independent of any parameter in R or it has a strongly isolated singularity at the origin then the projection $\pi : \mathbb{U}_1 \to \mathbb{U}_0$ is a locally trivial C^∞ fibration over T.*

Proof. We give only a sketch of the proof. First, assume that the last homogeneous piece of f, namely g, has a strongly isolated singularity at the origin. Let us add the new variable x_0 to $\mathsf{R}[x]$ and consider the homogenization $F(x_0, x) \in \mathsf{R}[x_0, x]$ of f. Let F_t be the specialization of F in $t \in T$. Define

$$\bar{\mathbb{U}}_1 := \{([x_0 : x], t) \in \mathbb{P}^{1,\alpha} \times T \mid F_t(x_0, x) = 0\},$$

where $\mathbb{P}^{1,\alpha}$ is the weighted projective space of type $(1, \alpha) = (1, \alpha_1, \alpha_2, \ldots, \alpha_{n+1})$. Let $\bar{\pi} : \bar{\mathbb{U}}_1 \to \mathbb{U}_0$ be the projection in $\mathbb{U}_0$. If all the weights α_i are equal to 1 then $D := \bar{\mathbb{U}}_1 \backslash \mathbb{U}_1$ is a smooth submanifold of $\bar{\mathbb{U}}_1$ and $\bar{\pi}$ and $\bar{\pi} \mid_D$ are proper regular (i.e. the derivative is surjective). For this case one can use directly Ehresmann's fibration theorem (see [15, 43]). For arbitrary weights we use the generalization of Ehresmann's theorem for stratified varieties. In $\mathbb{P}^{1,\alpha}$ we consider the following stratification

$$(\mathbb{P}^{1,\alpha} \backslash \mathbb{P}^{\alpha}) \cup (\mathbb{P}^{\alpha} \backslash \mathbb{P}^{S^c}) \cup \cup_{I \subset S^c} (\mathbb{P}^I \backslash \mathbb{P}^{<I}),$$

where for a subset I of $\mathbb{N}_{n+1}$, $\mathbb{P}^I$ denotes the sub projective space of the weighted projective space $\mathbb{P}^\alpha$ given by $\{x_i = 0 \mid i \in \mathbb{N}_{n+1} \backslash I\}$ and

$$\mathbb{P}^{<I} := \cup_{J \subset I,\ J \neq I} \mathbb{P}^J.$$

Now in T consider the one piece stratification and in $\mathbb{P}^{1,\alpha} \times T$ the product stratification. This gives us a stratification of $\bar{\mathbb{U}}_1$. The morphism $\bar{\pi}$ is proper and the fact that g has a strongly isolated singularity at the origin implies that $\bar{\pi}$ restricted to each strata is regular. We use Verdier Theorem ([78], Theorem 4.14, Remark 4.15) and obtain the local trivialization of π on a small neighborhood of $t \in T$ and compatible with the stratification of $\bar{\mathbb{U}}_1$. This yields to a local trivialization of π around t. If g is independent of any parameter in R then $\bar{\mathbb{U}}_1 \backslash \mathbb{U}_1 = G \times \mathbb{U}_0$, where G is the variety induced in $\{g = 0\}$ in $\mathbb{P}^\alpha$. We choose an arbitrary stratification in G and the product stratification in $G \times \mathbb{U}_0$ and apply again Verdier Theorem. □

The hypothesis of Theorem 5.1 is not the best one. For instance, the homogeneous polynomial $g = x^3 + tzy + tz^2$ in the ring $\mathsf{R}[x, y, z]$, $\mathsf{R} = \mathbb{C}[t, s, \frac{1}{t}]$, $\deg(x) = 2$, $\deg(y) = \deg(z) = 3$ depends on the parameter t and $g(x, y, 0)$ has not an isolated singularity at the origin. However, π is a C^∞ locally trivial fibration over T. I do not know any theorem describing explicitly the atypical values of the morphism π. Such theorems must be based either on a precise desingularization of $\bar{\mathbb{U}}_1$ and Ehresmann's theorem or various types of stratifications depending on the polynomial g. For more information in this direction the reader is referred to the works of J. Mather, R. Thom and J. L. Verdier around 1970 (see [47] and the references there). Theorem 5.1 (in the general context of morphism of algebraic varieties) is also known as the second theorem of isotopy (see [78] Remark 4.15).

5.3 Monodromy group

Let b_0 and b_1 two points in T and λ be a path in T connecting b_0 to b_1 and defined up to homotopy. Theorem 5.1 gives us a unique map

$$h_\lambda : L_{b_0} \to L_{b_1}$$

defined up to homotopy. In particular, for $b := b_0 = b_1$ we have the action of $\pi_1(\mathbb{U}_0, b)$ on the homology group $H_n(L_b, \mathbb{Z})$. The image of $\pi_1(\mathbb{U}_0, b)$ in $\mathrm{Aut}(H_n(L_b, \mathbb{Z}))$ is called the monodromy group.

Example 5.1. We consider the one variable tame polynomial $f = f_t = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0$ in $\mathsf{R}[x]$, where $\mathsf{R} = \mathbb{C}[t_0, t_1, \ldots, t_{d-1}]$. The homology $H_0(\{f_t = 0\}, \mathbb{Z})$ is the set of all finite sums $\sum_i r_i[x_i]$, where $r_i \in \mathbb{Z}, \sum_i r_i = 0$ and x_i's are the roots of f_t. The monodromy is defined by the continuation of the roots of f along a path in $\pi_1(T, b)$. To calculate the monodromy we proceed as follows:

The polynomial $f = (x-1)(x-2)\cdots(x-d)$ has $\mu := d-1$ distinct real critical values, namely $c_1, c_2, \ldots, c_\mu$. Let b the point in T corresponding to f. We consider f as a function from $\mathbb{C}$ to itself and take a distinguished set of paths $\lambda_i,\ i = 1, 2, \ldots, \mu$ in $\mathbb{C}$ which connects 0 to the critical values of f. This mean that the paths λ_i do not intersect each other except at 0 and the order $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ around 0 is anti-clockwise. The cycle $\delta_i = [i+1] - [i], i = 1, 2, \ldots, \mu$ vanishes along the path λ_i and $\delta = (\delta_1, \delta_2, \ldots, \delta_\mu)$ is called a distinguished set of vanishing cycles in $H_0(L_b, \mathbb{Z})$. Now, the monodromy around the critical value c_i is given by

$$\delta_j \mapsto \begin{cases} \delta_j & j \neq i-1, i, i+1 \\ -\delta_j & j = i \\ \delta_j + \delta_i & j = i-1, i+1 \end{cases}.$$

In $H_0(L_b, \mathbb{Z})$ we have the intersection form induced by

$$\langle x, y \rangle = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \quad x, y \in L_b.$$

By definition $\langle \cdot, \cdot \rangle$ is a symmetric form in $H_0(L_b, \mathbb{Z})$, i.e. for all $\delta_1, \delta_2 \in H_0(L_b, \mathbb{Z})$ we have $\langle \delta_1, \delta_2 \rangle = \langle \delta_2, \delta_1 \rangle$. Let Ψ_0 be the intersection matrix in the basis δ:

$$\Psi_0 := \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{5.1}$$

The monodromy group keeps the intersection form in $H_0(L_b, \mathbb{Z})$. In other words:

$$\Gamma_{\mathbb{Z}} \subset \{A \in \mathrm{GL}(\mu, \mathbb{Z}) \mid A\Psi_0 A^{\mathsf{t}} = \Psi_0\}. \tag{5.2}$$

Consider the case $d = 3$. We choose the basis $\delta_1 = [2] - [1],\ \delta_2 = [3] - [2]$ for $H_0(L_b, \mathbb{Z})$. In this basis the intersection matrix is given by

$$\Psi_0 := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

There are two critical points for f for which the monodromy is given by:

$$\delta_1 \mapsto -\delta_1,\ \delta_2 \mapsto \delta_2 + \delta_1,$$

$$\delta_2 \mapsto -\delta_2,\ \delta_1 \mapsto \delta_2 + \delta_1.$$

Let $g_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. The monodromy group satisfies the equalities:

$$\Gamma_{\mathbb{Z}} = \langle g_1, g_2 \mid g_1^2 = g_2^3 = I, g_1g_2g_1 = g_2g_1g_2 \rangle = \{I, g_1, g_2, g_1g_2g_1, g_2g_1, g_1g_2\} =$$

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\}.$$

For this example (5.2) turns out to be an equality (one obtains equations like $(a-b)^2 + a^2 + b^2 = 2$ for the entries of the matrix A and the calculation is explicit).

5.4 Distinguished set of vanishing cycles

First, let us recall some definitions from local theory of vanishing cycles. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at $0 \in \mathbb{C}^{n+1}$. We take convenient neighborhoods U of $0 \in \mathbb{C}^{n+1}$ and V of $0 \in \mathbb{C}$ such that $f : U \to V$ is a C^∞ fiber bundle over $V \backslash \{0\}$. Let $t_i \in V\backslash\{0\},\ i = 1, 2, \cdots, s$ (not necessarily distinct) and λ_i be a path which connects 0 to t_i in V. We assume that λ_i's do not intersect each other except at their start/end points and at 0 they intersect each other transversally. We also assume that the embedded oriented sphere $\delta_i \subset f^{-1}(t_i)$ vanishes along λ_i. The sphere δ_i is called a vanishing cycle and is defined up to homotopy.

Definition 5.2. *The ordered set of vanishing cycles $\delta_1, \delta_2, \cdots, \delta_s$ is called distinguished if*

1. $(\lambda_1, \lambda_2, \cdots, \lambda_s)$ *near* 0 *is the clockwise direction;*
2. *for a versal deformation $\tilde{f}$ of f with μ distinguished critical values, where μ is the Milnor number of f, the deformed paths $\tilde{\lambda}_i$ do not intersect each other except possibly at their end points t_i's.*

Historically, one is interested to the full distinguished set of vanishing cycles, i.e. the one with μ elements and with $b := t_1 = t_2 = \cdots = t_\mu$. From now on by a distinguished set of vanishing cycles we mean the full one. It is well-known that a full distinguished set of vanishing cycles form a basis of $H_n(f^{-1}(b), \mathbb{Z})$ (see [2]).

Example 5.2. For $f := x^d$ the point $0 \in \mathbb{C}$ is the only critical value of f. Let $\lambda(s) = s,\ 0 \leq s \leq 1$. The set

$$\delta_i := [\zeta_d^{i+1}] - [\zeta_d^i],\ i = 0, \ldots, d-2$$

is a distinguished set of vanishing cycles for $H_0(\{f = 1\}, \mathbb{Z})$. The vanishing takes place along λ (see [2] Theorem 2.15).

Let $f \in \mathbb{C}[x]$ be a tame polynomial. We fix a regular value $b \in \mathbb{C}\backslash C$ of f and consider a system of paths $\lambda_i,\ i = 1, 2, \ldots, \mu$ connecting the points of C to the point b. Again, we assume that λ_i's do not intersect each other except at their start/end points and at the points of C they intersect each other transversally. We call λ_i's a distinguished set of paths. In a similar way as in Definition 5.2 we define a distinguished set of vanishing cycles $\delta_i \subset f^{-1}(b),\ i=1,2,\dots,\mu$ (defined up to homotopy). For each singularity p of f we use a separate versal deformation which is defined in a neighborhood of p. If the completion of f has a non zero double discriminant then we can deform f and obtain another tame polynomial $\tilde{f}$ with the same Milnor number in a such a way that f and $\tilde{f}$ have C^∞ isomorphic regular fibers and $\tilde{f}$ has distinct μ critical values. In this case we can use $\tilde{f}$ for the definition of a distinguished set of vanishing cycles.

Fix an embedded sphere in $f^{-1}(b)$ representing the vanishing cycle δ_i. For simplicity we denote it again by δ_i.

Theorem 5.2. *For a tame polynomial $f \in \mathbb{C}[x]$ and a regular value b of f, the complex manifold $f^{-1}(b)$ has the homotopy type of $\cup_{i=1}^{\mu}\delta_i$. In particular, a distinguished set of vanishing cycles generates $H_n(f^{-1}(b),\mathbb{Z})$.*

Proof. The proof of this theorem is a well-known argument in Picard-Lefschetz theory, see for instance [43] §5, [7] Theorem 1.2, [52] Theorem 2.2.1 and [14]. We have reproduced this argument in the proof of Theorem 5.4 □

In the literature the union $\cup_{i=1}^{\mu}\delta_i$ is known as the bouquet of μ spheres.

Theorem 5.3. *If the tame polynomial $f \in \mathbb{C}[x]$ has μ distinct critical values and the discriminant of its completion is irreducible then for two vanishing cycles δ_0, δ_1 in a regular fiber of f, there is a homotopy class $\gamma \in \pi_1(\mathbb{C}\backslash C, b)$ such that $h_\gamma(\delta_0) = \pm\delta_1$, where C is the set of critical values of f.*

Similar theorems are stated in [43] 7.3.5 for generic Lefschetz pencils, in [52] Theorem 2.3.2, Corollary 3.1.2 for generic pencils of type $\frac{F^p}{G^q}$ in $\mathbb{P}^n$ and in [2] Theorem 3.4 for a versal deformation of a singularity. Note that in the above theorem we are still talking about the homotopy classes of vanishing cycles. I believe that the discriminant of complete tame polynomials is always irreducible. This can be checked easily for $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 1$ and many particular cases of weights.

Proof. Let $F \in \mathsf{R}[x]$ be the completion of f, where R is some localization of $\mathbb{C}[t]$, and $\Delta_0 := \{t \in \mathbb{U}_0 \mid \Delta_F(t) = 0\}$. We consider $f - s,\ s \in \mathbb{C}$ as a line G_{c_0} in $\mathbb{U}_0$ which intersects Δ_0 transversally in μ points. If there is no confusion we denote by b the point in $\mathbb{U}_0$ corresponding to $f - b$. Let D be the locus of points $t \in \Delta_0$ such that the line G_t through b and t intersects Δ_0 at μ distinct points. Let also δ_0 and δ_1 vanish along the paths λ_0 and λ_1 which connect b to $c_0, c_1 \in G_{c_0} \cap \Delta_0$, respectively. Since the set D is a proper algebraic subset of Δ_0 and Δ_0 is an irreducible variety and $c_0, c_1 \in \Delta_0\backslash D$, there is a path w in $\Delta_0\backslash D$ from c_0 to c_1. After a blow up at the point b and using the Ehresmann's theorem, we conclude that: There is an isotopy

$$H : [0,1] \times G_{c_0} \to \cup_{t\in[0,1]} G_{w(t)}$$

such that

1. $H(0,\cdot)$ is the identity map;
2. for all $a \in [0,1]$, $H(a,.)$ is a C^∞ isomorphism between G_{c_0} and $G_{w(a)}$ which sends points of Δ_0 to Δ_0;
3. For all $a \in [0,1]$, $H(a,b) = b$ and $H(a,c_0) = w(a)$

Let $\lambda_a' = H(a, \lambda_0)$. In each line $G_{w(a)}$ the cycle δ_0 vanishes along the path λ_a' in the unique critical point of $\{F_{w(s)} = 0\}$. Therefore δ_0 vanishes along λ_1' in $c_1 = w(1)$. Consider λ_1 and λ_1' as the paths which start from b and end in a point b_1 near c_1 and put $\lambda = \lambda_1' - \lambda_1$. By uniqueness (up to sign) of the Lefschetz vanishing cycle along a fixed path we can see that the path λ is the desired path. $\square$

Let $f \in \mathbb{C}[x]$ be a tame polynomial and λ be a path in $\mathbb{C}$ which connects a regular value $b \in \mathbb{C}\backslash C$ to a point $c \in C$ and do not cross C except at the mentioned point c. To λ one can associate an element in $\tilde{\lambda} \in \pi_1(\mathbb{C}\backslash\mathbb{C}, b)$ as follows: The path $\tilde{\lambda}$ starts from b goes along λ until a point near c, turns around c anti clockwise and returns to b along λ. By the monodromy along the path and around c we mean the monodromy associated to $\tilde{\lambda}$. The associated monodromy is given by the Picard-Lefschetz formula/mapping:

$$a \mapsto a + \sum_{\delta} (-1)^{\frac{(n+1)(n+2)}{2}} \langle a, \delta \rangle \delta, \tag{5.3}$$

where δ runs through a basis of distinguished vanishing cycles which vanish in the critical points of the fiber $f^{-1}(c)$. The above mapping keeps the intersection form $\langle \cdot, \cdot \rangle$ invariant, i.e.

$$\left\langle a + (-1)^{\frac{(n+1)(n+2)}{2}} \langle a, \delta \rangle \delta, b + (-1)^{\frac{(n+1)(n+2)}{2}} \langle b, \delta \rangle \delta \right\rangle = \langle a, b \rangle, \ \forall a, b \in H_n(\{f = 0\}, \mathbb{Z}).$$

This follows from (5.5) and the fact that $\langle \cdot, \cdot \rangle$ is $(-1)^n$-symmetric. I do not know whether in general a $\langle \cdot, \cdot \rangle$-preserving map from $H_n(\{f = 0\}, \mathbb{Z})$ to itself is a composition of some Picard-Lefschetz mappings. The positive answer to this question may change our point of view on the moduli of polarized Hodge structures (see §10.16).

Definition 5.3. A cycle $\delta \in H_n(\{f = 0\}, \mathbb{Z})$, f a tame polynomial, is called the cycle at infinity if its intersection with all other cycles in $H_n(\{f = 0\}, \mathbb{Z})$ (including itself) is zero.

5.5 Join of topological spaces

We start with a definition.

Definition 5.4. The join $X * Y$ of two topological spaces X and Y is the quotient space of the direct product $X \times I \times Y$, where $I = [0, 1]$, by the equivalence relation:

$$(x, 0, y_1) \sim (x, 0, y_2) \ \forall \ y_1, y_2 \in Y, \ x \in X,$$

$$(x_1, 1, y) \sim (x_2, 1, y) \ \forall \ x_1, x_2 \in X, \ y \in Y.$$

Let X and Y be compact oriented real manifolds and $\pi : X*Y \to I$ be the projection on the second coordinate. The real manifold $X * Y\backslash\pi^{-1}(\{0, 1\})$ has a canonical orientation obtained by the wedge product of the orientations of X, I and Y. Does $X * Y$ have a structure of a real oriented manifold? It does not seem to me that the answer is positive for arbitrary X and Y. In the present text we only need the following proposition which gives partially a positive answer to our question. Let

$$\mathbb{S}_n := \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

be the n-dimensional sphere with the orientation $\frac{dx}{d(x_1^2+x_2^2+\cdots+x_{n+1}^2)}$.

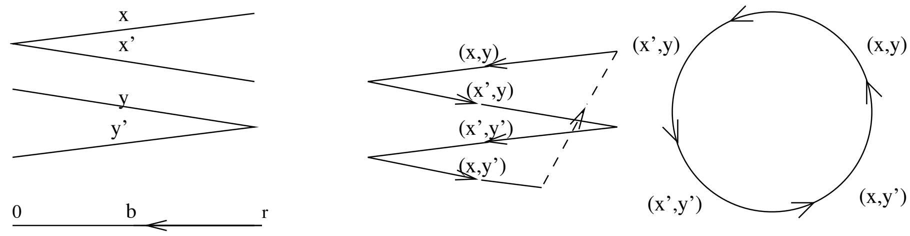

Figure 5.2: Join of zero dimensional cycles

Proposition 5.2. *We have*

$$\mathbb{S}_n * \mathbb{S}_m \overset{C^0}{\cong} \mathbb{S}_{n+m+1}, \ n, m \in \mathbb{N}_0,$$

which is an isomorphisim of oriented manifolds outside $\pi^{-1}(\{0,1\})$.

Proof. For the proof of the above diffeomorphism we write $\mathbb{S}_{n+m+1}$ as the set of all $(x, y) \in \mathbb{R}^{n+m+2}$ such that

$$x_1^2 + \cdots + x_{n+1}^2 = 1 - (y_1^2 + \cdots + y_{m+1}^2)$$

Now, let t be the above number and let it varies from 0 to 1. We have the following isomorphism of topological spaces:

$$\mathbb{S}_{n+m+1} \to \mathbb{S}_n * \mathbb{S}_m, \ (x, y) \mapsto \begin{cases} (\frac{x}{\sqrt{t}}, t, \frac{y}{\sqrt{1-t}}) & t \neq 0, 1 \\ (0, 0, y) & t = 0 \\ (x, 1, 0) & t = 1 \end{cases}$$

The Figure (5.2) shows a geometric construction of $\mathbb{S}_0 \times \mathbb{S}_0$. The proof of the statement about orientations is left to the reader. □

5.6 Direct sum of polynomials

Let $f \in \mathbb{C}[x]$ and $g \in \mathbb{C}[y]$ be two polynomials in variables $x := (x_1, x_2, \ldots, x_{n+1})$ and $y := (y_1, y_2 \ldots, y_{m+1})$ respectively. In this section we study the topology of the variety

$$X := \{(x, y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} \mid f(x) = g(y)\}$$

in terms of the topology of the fibrations $f : \mathbb{C}^{n+1} \to \mathbb{C}$ and $g : \mathbb{C}^{m+1} \to \mathbb{C}$. Let also C_1 (resp. C_2) denotes the set of critical values of f (resp. g). We assume that $C_1 \cap C_2 = \emptyset$, which implies that the variety X is smooth. Fix a regular value $b \in \mathbb{C}\backslash(C_1 \cup C_2)$ of both f and g. Let $\delta_{1b} \in H_n(f^{-1}(b), \mathbb{Z})$ and $\delta_{2b} \in H_m(g^{-1}(b), \mathbb{Z})$ be two vanishing cycles and $t_s, \ s \in [0, 1]$ be a path in $\mathbb{C}$ such that it starts from a point in C_1, crosses b and ends in a point of C_2 and never crosses $C_1 \cup C_2$ except at the mentioned cases. We assume that δ_{1b} vanishes along $t_{\cdot}^{-1}$ when s tends to 0 and δ_{2b} vanishes along $t_{\cdot}$ when s tends to 1. Now

$$\delta_{1b} * \delta_{2b} \cong \delta_{1b} *_{t_{\cdot}} \delta_{2b} := \cup_{s \in [0,1]} \delta_{1t_s} \times \delta_{2t_s} \in H_{n+m+1}(X, \mathbb{Z})$$

is an oriented cycle. Note that its orientation changes when we change the direction of the path $t_{\cdot}$. We call the triple $(t_s, \delta_1, \delta_2) = (t_s, \delta_{1t_{\cdot}}, \delta_{2t_{\cdot}})$ an admissible triple.

Let $b \in \mathbb{C}\backslash(C_1 \cup C_2)$. We take a system of distinguished paths λ_c $c \in C_1 \cup C_2$, where λ_c starts from b and ends at c. Let $\delta_1^1, \delta_1^2, \cdots, \delta_1^{\mu} \in H_n(f^{-1}(b), \mathbb{Z})$ and $\delta_2^1, \delta_2^2, \cdots, \delta_2^{\mu'} \in H_m(g^{-1}(b), \mathbb{Z})$ be the corresponding distinguished basis of vanishing cycles. Note that many vanishing cycles may vanish along a path in one singularity.

Theorem 5.4. *The $\mathbb{Z}$-module $H_{n+m+1}(X, \mathbb{Z})$ is freely generated by*

$$\gamma := \delta_1^i * \delta_2^j, \ i = 1, 2, \ldots, \mu, \ j = 1, 2, \cdots, \mu',$$

where we have taken the admissible triples $(\lambda_{c_j}\lambda_{c_i}^{-1}, \delta_1^i, \delta_2^j), \ c_i \in C_1, \ c_j \in C_2$.

Proof. The proof which we present for this theorem is similar to a well-known argument in Picard-Lefschetz theory, see for instance [43] or Theorem 2.2.1 of [52]. The homologies bellow are with $\mathbb{Z}$ coefficients.

The fibration $\pi : X \to \mathbb{C}, \ (x, y) \mapsto f(x) = f(y)$ is toplogically trivial over $\mathbb{C}\backslash(C_1 \cup C_2)$. Let $Y = f^{-1}(b) \times g^{-1}(b)$. We have
(5.4)

$$0 = H_{n+m+1}(Y) \to H_{n+m+1}(X) \to H_{n+m+1}(X, Y) \xrightarrow{\partial} H_{n+m}(Y) \to H_{n+m}(X) \to \cdots$$

We take small open disks D_c around each point $c \in C_1 \cup C_2$. Let b_c be a point near c in D_c and $X_c = \pi^{-1}(\lambda_c \cup D_c)$. We have

$$H_{n+m}(Y) \cong H_n(f^{-1}(b)) \otimes_{\mathbb{Z}} H_n(g^{-1}(b))$$

and

$$\begin{aligned} H_{n+m+1}(X, Y) &\cong \oplus_{c \in C_1 \cup C_2} H_{n+m+1}(X_c, Y) \\ &\cong \oplus_{c \in C_1 \cup C_2} H_{n+m+1}(X_c, Y_{b_c}) \\ &\cong \oplus_{c \in C_1} H_{n+1}(f^{-1}(D_c), f^{-1}(b_c)) \oplus \oplus_{c \in C_2} H_{m+1}(g^{-1}(D_c), g^{-1}(b_c)). \end{aligned}$$

We look $H_{n+m+1}(X)$ as the kernel of the boundary map ∂ in (5.4). Let us take two cycles δ_1 and δ_2 form the pieces of the last direct sum in the above equation and assume that $\partial\delta = 0$, where $\delta = \delta_1 - \delta_2$. If δ_1 and δ_2 belongs to different classes, according to $c \in C_1$ or $c \in C_2$, then δ is the join of two vanishing cycles. Otherwise, $\delta = 0$ in $H_{n+m+1}(X, \mathbb{Z})$. □

It is sometimes useful to take $g = b' - g'$, where b' is a fixed complex number and g' is a tame polynomial . The set of critical values of g' is denoted by C_2' and hence the set of critical values of g is $C_2 = b' - C_2'$. We define $t = F(x, y) := f(x) + g'(y)$ and so $X = F^{-1}(b')$. The set of critical values of F is $C_1 + C_2'$ and the assumption that $C_1 \cap (b' - C_2')$ is empty implies that b' is a regular value of F. Let $(t_s, \delta_{1b}, \delta_{2b})$ be an admissible triple and t_s starts from c_1 and ends in $b' - c_2'$.

Proposition 5.3. *The topological cycle $\delta_{1b} * \delta_{2b}$ is a vanishing cycle along the path $t_{.} + c_2$ with respect to the fibration $F = t$.*

Proof. See Figure 5.3. □

Remark 5.1. Let $b \in \mathbb{C}\backslash(C_1 \cup C_2)$. We take a system of distinguished paths λ_c $c \in C_1 \cup C_2$, where λ_c starts from b and ends at c (see Figure 5.3). If the points of the set C_1 (resp. C_2) are enough near (resp. far from) each other then the collection of translations given in Proposition 5.3 gives us a system of paths, which is distinguished after performing a proper homotopy, starting from the points of $C_1 + C_2'$ and ending in b'. This together with Theorem 5.2 gives an alternative proof to Theorem 5.4.

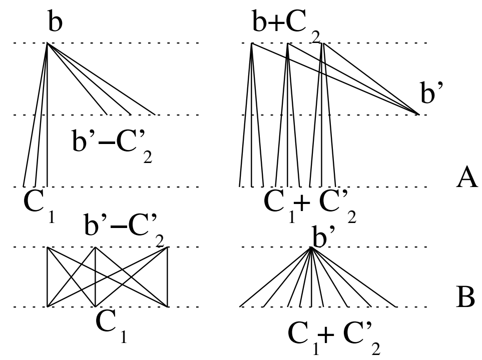

Figure 5.3: A system of distinguished paths

Example 5.3. Let us assume that all the critical values of f and $g' = b' - g$ are real. Moreover, assume that f (resp. g) has non-degenerated critical points with distinct images. For instance, in the case $n = m = 0$ take

$$f := (x-1)(x-2)\cdots(x-m_1),\ g' := (x-m_1-1)(x+2)\cdots(x-m_1-m_2).$$

Take $b' \in \mathbb{C}$ with $\mathrm{Im}(b') > 0$. We take direct segment of lines which connects the points of C_1 to the points of $b' - C_2'$. The set of joint cycles constructed in this way, is a basis of vanishing cycles associated the direct segment of paths which connect b' to the points of $C_1 + C_2'$ (see Figure 5.3, B).

Example 5.4. Using the machinery introduced in this section, we can find a distinguished basis of vanishing cycles for $H_n(\{g=1\},\mathbb{Z})$, where $g = x_1^{m_1}+x_2^{m_2}+\cdots+x_{n+1}^{m_{n+1}},\ 2 \le m_i \in \mathbb{N}$ is discussed in Example 3.3. Let

$$\Gamma := \{(t_1,t_2,\ldots,t_{n+1}) \in \mathbb{R}^{n+1} \mid t_i \ge 0, \sum_{i=1}^{n+1} t_i = 1\}.$$

For $i = 1,2,\ldots,n+1$ we take the distinguished set of vanishing cycles $\delta_{i,\beta_i},\ \beta_i = 0,1,\ldots,m_i-2$ given in Example 5.2 and define the joint cycles

$$\delta_\beta = \delta_{m_1,\beta_1} * \delta_{m_2,\beta_2} * \cdots * \delta_{m_{n+1},\beta_{n+1}} :=$$

$$\cup_{t\in\Gamma}\delta_{m_1,\beta_1,t_1} \times \delta_{m_2,\beta_2,t_2} \times \cdots \times \delta_{m_{n+1},\beta_{n+1},t_{n+1}} \in H_n(\{g=1\},\mathbb{Z}),\ \beta \in I,$$

where $I := \{(\beta_1,\ldots,\beta_{n+1}) \mid 0 \le \beta_i \le m_i - 2\}$. They are ordered lexicographically and form a distinguished set of vanishing cycles in $H_n(\{g=1\},\mathbb{Z})$. Another description of δ_β's is as follows: For $\beta \in I$ and $a_i = 0,1$, where $i = 1,2,\ldots,n+1$, let

$$\Gamma_{\beta,a} : \Gamma \to \{g=1\},\ \Gamma_{\beta,a}(t) = (t_1^{\frac{1}{m_1}}\zeta_{m_1}^{\beta_1+a_1}, t_2^{\frac{1}{m_2}}\zeta_{m_2}^{\beta_2+a_2},\ldots,t_{n+1}^{\frac{1}{m_{n+1}}}\zeta_{m_{n+1}}^{\beta_{n+1}+a_{n+1}}),$$

where for a positive number r and a natural number s, $r^{\frac{1}{s}}$ is the unique positive s-th root of r. We have

$$\delta_\beta := \sum_a (-1)^{\sum_{i=1}^{n+1}(1-a_i)}\Gamma_{\beta,a}.$$

$0 \times \delta_2'$ $\qquad$ $\delta_1' \times \delta_2'$

$0 \times \delta_2$ $\qquad$ $\delta_1 \times \delta_2$

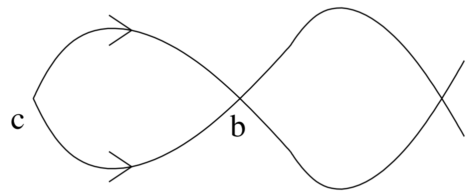

Figure 5.4: Two paths in $\mathbb{C}$

5.7 Calculation of the Intersection form

Let us consider two tame polynomials $f, g \in \mathbb{C}[x]$. A critical value c of f is called non-degenerated if the fiber $f^{-1}(c)$ contains only one singularity and the Milnor number of that singularity is one. Around such a singularity f can be written in the form $X_1^2 + X_2^2 + \cdots + X_{n+1}^2 + c$ for certain local coordinate functions X_i.

For two oriented paths $t_{.}, t'_{.}$ in $\mathbb{C}$ which intersect each other at b transversally the notation $t_{.} \times_b^+ t'_{.}$ means that $t_{.}$ intersects $t'_{.}$ in the positive direction, i.e. $dt_{.} \wedge dt'_{.}$ is the canonical orientation of $\mathbb{C}$. In a similar way we define $t_{.} \times_b^- t'_{.}$ (see Figure 5.4).

Theorem 5.5. *Let $(t_{.}, \delta_1, \delta_2)$ and $(t'_{.}, \delta'_1, \delta'_2)$ be two admissible triples. Assume that $t_{.}$ and $t'_{.}$ intersect each other transversally in their common points and the start/end critical points of $t_{.}$ and $t'_{.}$ are non-degenerated. Then*

$$\langle \delta_1 * \delta_2, \delta'_1 * \delta'_2 \rangle = (-1)^{nm+n+m} \sum_b \epsilon_1(b) \langle \delta_{1b}, \delta'_{1b} \rangle \langle \delta_{2b}, \delta'_{2b} \rangle$$

where b runs through all intersection points of $t_{.}$ and $t'_{.}$,

$$\epsilon_1(b) = \begin{cases} 1 & t_{.} \times_b^+ t'_{.} \text{ and } b \text{ is not a start/end point} \\ -1 & t_{.} \times_b^- t'_{.} \text{ and } b \text{ is not a start/end point} \\ (-1)^{\frac{n(n-1)}{2}} & t_{.} \times_b^+ t'_{.} \text{ and } b \text{ is a start point} \\ (-1)^{\frac{n(n+1)}{2}+1} & t_{.} \times_b^- t'_{.} \text{ and } b \text{ is a start point} \\ (-1)^{\frac{m(m-1)}{2}} & t_{.} \times_b^+ t'_{.} \text{ and } b \text{ is an end point} \\ (-1)^{\frac{m(m+1)}{2}+1} & t_{.} \times_b^- t'_{.} \text{ and } b \text{ is an end point} \end{cases},$$

and by $\langle 0, 0 \rangle$ we mean 1.

Proof. Let $t_{.}$ intersect $t'_{.}$ transversally at a point b. Let also a_1, a_2, a'_1, a'_2 be the orientation elements of the cycles $\delta_1, \delta_2, \delta'_1, \delta'_2$ and a and a' be the orientation element of $t_{.}$ and $t'_{.}$. We consider two cases:

1. b is not the end/start point of neither $t_{.}$ nor $t'_{.}$: Assume that the cycles δ_1 and δ'_1 (resp. δ_2 and δ'_2) intersect each other at p_1 (resp. p_2) transversally. The cycles $\gamma = \delta_1 * \delta_2$ and $\gamma' = \delta'_1 * \delta_2$ intersect each other transversally at (p_1, p_2). The orientation element of the whole space X obtained by the intersection of γ and γ' is:

$$a_1 \wedge a \wedge a_2 \wedge a'_1 \wedge a' \wedge a'_2 = (-1)^{nm+n+m} (a_1 \wedge a'_1) \wedge (a \wedge a') \wedge (a_2 \wedge a'_2)$$

This is $(-1)^{nm+n+m}$ times the canonical orientation of X.

2. $b = c$ is, for instance, the start point of both $t_.$ and $t'_.$ and δ_1, δ'_1 vanish in the point $p_1 \in \mathbb{C}^{n+1}$ when t tends to c. Assume that the cycles δ_2 and δ'_2 intersect each other transversally at p_2. By assumption, p_1 is a non-degenerated critical point of f and so both cycles γ, γ' are smooth around (p_1, p_2) and intersect each other transversally at (p_1, p_2). The orientation element of the whole space X obtained by the intersection of γ and γ' is:

$$(a_1 \wedge a) \wedge a_2 \wedge (a'_1 \wedge a') \wedge a'_2 = (-1)^{(n+1)m}(a_1 \wedge a) \wedge (a'_1 \wedge a') \wedge a_2 \wedge a'_2.$$

Note that $a_1 \wedge a$ has meaning and is the orientation of the thimble formed by the vanishing of δ_1 at p_1. According to Proposition 5.1, $(a_1 \wedge a) \wedge (a'_1 \wedge a')$ is the canonical orientation of $\mathbb{C}^{n+1}$ multiplied with ϵ, where $\epsilon = (-1)^{\frac{n(n+1)}{2}}$ if $t_. \times^+_b t'_.$ and $= (-1)^{\frac{n(n+1)}{2}+n+1}$ otherwise.

□

Remark 5.2. One can use Theorem 5.5 to calculate the intersection matrix of $H_n((f + g')^{-1}(b'), \mathbb{Z})$ in the basis given by Theorem 5.4. This calculation in the local case is done by A. M. Gabrielov (see [2] Theorem 2.11). To state Gabrielov's result in the context of this text take f and g two tame polynomials such that the set C_1 can be separated from C_2 by a real line in $\mathbb{C}$. Then take b a point in that line. The advantage of our calculation is that it works in the global context and the vanishing cycles are constructed explicitly.

Remark 5.3. In Theorem 5.5 we may discard the assumption on the critical points in the following way: In the case in which δ_1 and δ'_1 (resp. δ_2 and δ'_2) vanish on the same critical point, we assume that they are distinguished (see Definition 5.2). Note that if two vanishing cycles vanish along transversal paths in the same singularity then the corresponding thimbles are not necessarily transversal to each other, except when the singularity is non-degenerated.

Proposition 5.4. *The self intersection of a vanishing cycle of dimension n is given by*

$$(-1)^{\frac{n(n-1)}{2}}(1 + (-1)^n). \tag{5.5}$$

Proof. By Proposition 5.3 a joint cycle of two vanishing cycle is also a vanishing cycle. We apply Theorem 5.5 in the case $\delta_1 = \delta'_1$ and $\delta_2 = \delta'_2$ and conclude that the self intersection a_n of a vanishing cycle of dimension n satisfies

$$a_{n+m+1} = (-1)^{nm+n+m}((-1)^{\frac{n(n-1)}{2}} a_m + (-1)^{\frac{m(m+1)}{2}+1} a_n), \ a_0 = 2, \quad n, m \in \mathbb{N}_0.$$

It is easy to see that (5.5) is the only function with the above property. □

Example 5.5. (Stabilization) We take $g = y_1^2 + y_2^2 + \cdots + y_{m+1}^2$ and f an arbitrary tame polynomial. Let $\delta_1, \delta_2, \cdots, \delta_\mu$ be a distinguished set of vanishing cycles in $H_n(f^{-1}(0), \mathbb{Z})$ and δ be the vanishing cycle in $H_n(f^{-1}(0), \mathbb{Z})$ (up to multiplication by ± 1 it is unique). The intersection form in the basis $\tilde{\delta}_i = \delta_i * \delta$ is given by

$$\langle \tilde{\delta}_i, \tilde{\delta}_j \rangle = (-1)^{nm+n+m+\frac{m(m-1)}{2}} \langle \delta_i, \delta_j \rangle, \ i > j,$$

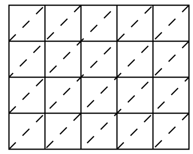

Figure 5.5: Dynkin diagram of $x^5 + y^4$

$$\langle \tilde{\delta}_i, \tilde{\delta}_j \rangle = (-1)^{nm+n+m+\frac{m(m+1)}{2}+1} \langle \delta_i, \delta_j \rangle, \ i < j,$$

(see [2] Theorem 2.14). Now let us assume that $m = 0$ and $n = 1$. Choose δ_i's as in Example 5.2. In this basis the intersection matrix is:

$$\Psi_0 = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

The intersection matrix in the basis $\tilde{\delta}_i, \ i = 1, 2, \cdots, \mu$ is of the form:

$$\tilde{\Psi}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

As an exercise, construct a symplectic basis of the Riemann surface X using the basis $\tilde{\delta}_i, \ i = 1, 2, \ldots, \mu$ and its intersection matrix.

Example 5.6. We consider f and $g = b' - g'$, where f and g' are two homogeneous tame polynomials. The point $0 \in \mathbb{C}$ (resp. $b' \in \mathbb{C}$) is the only critical value of f (resp. g) and so, up to homotopy, there is only one path connecting 0 to b'. We choose the straight piece of line $t_s = sb', \ 0 \le s \le 1$ as the path for our admissible triples. For a point b between 0 and b' in t. we choose a distinguished set of vanishing cycles $\delta_i, \ i = 1, 2 \ldots, \mu_1$ (resp. $\gamma_j, \ j = 1, 2 \ldots, \mu_2$) of f (resp. g) in the fiber $f^{-1}(b)$ (resp. $g^{-1}(b)$). By Theorem 5.4, the cycles

$$\delta_i * \gamma_j, \ i = 1, 2, \ldots, \mu_1, \ j = 2, \ldots, \mu_2$$

generate $H_1(\{f + g = b'\}, \mathbb{Z})$. The intersection matrix in this basis is given by

$$\langle \delta_i * \gamma_j, \delta_{i'} * \gamma_{j'} \rangle = \begin{cases} \operatorname{sgn}(j' - j)^{n+1}(-1)^{(n+1)(m+1)+\frac{n(n+1)}{2}} \langle \gamma_j, \gamma_{j'} \rangle & \text{if } i' = i \ \& \ j' \neq j \\ \operatorname{sgn}(i' - i)^{m+1}(-1)^{(n+1)(m+1)+\frac{m(m+1)}{2}} \langle \delta_i, \delta_{i'} \rangle & \text{if } j' = j \ \& \ i' \neq i \\ \operatorname{sgn}(i' - i)(-1)^{(n+1)(m+1)} \langle \delta_i, \delta_{i'} \rangle \langle \gamma_j, \gamma_{j'} \rangle & \text{if } (i' - i)(j' - j) > 0 \\ 0 & \text{if } (i' - i)(j' - j) < 0 \end{cases}$$

Example 5.7. In the case $f := x^{m_1}$ and $g := b' - y^{m_2}$,

$$\delta_i := [\zeta_{m_1}^{i+1} b^{\frac{1}{m_1}}] - [\zeta_{m_1}^{i} b^{\frac{1}{m_1}}],\ i = 0, \dots, m_1 - 2$$

(resp.

$$\gamma_j := [\zeta_{m_2}^{j+1} (b' - b)^{\frac{1}{m_2}}] - [\zeta_{m_1}^{j} (b' - b)^{\frac{1}{m_2}}],\ j = 0, \dots, m_2 - 2)$$

is a distinguished set of vanishing cycles for $H_0(\{f = b\}, \mathbb{Z})$ (resp. $H_0(\{g = b\}, \mathbb{Z})$), where we have fixed a value of $b^{\frac{1}{m_1}}$ and $b^{\frac{1}{m_2}}$. See Figure (5.2) for a tentative picture of the join cycle $\delta_i * \gamma_j$ with $\delta_i = x - y$ and $\gamma_j = x' - y'$. The upper triangle of intersection matrix in this basis is given by:

$$\langle \delta_i * \gamma_j, \delta_{i'} * \gamma_{j'} \rangle = \begin{cases} 1 & \text{if } (i' = i \ \&\ j' = j + 1) \vee (i' = i + 1 \ \&\ j' = j) \\ -1 & \text{if } (i' = i \ \&\ j' = j - 1) \vee (i' = i + 1 \ \&\ j' = j + 1) \\ 0 & \text{otherwise} \end{cases}.$$

This shows that Figure 5.5 is the associated Dynkin diagram.

Example 5.8. The calculation of the Dynkin diagram of tame polynomials of the type $g = x^{m_1} + x^{m_2} + \cdots + x^{m_{n+1}}$ is done first by F. Pham (see [2] p. 66). It follows from Example 5.6 and by induction on n that the intersection map in the basis $\delta_\beta,\ \beta \in I$ of Example 5.4 is given by:

$$(5.6)\quad \langle \delta_\beta, \delta_{\beta'} \rangle = (-1)^{\frac{n(n+1)}{2}} (-1)^{\Sigma_{k=1}^{n+1} \beta'_k - \beta_k},\ \beta = (\beta_1, \beta_2, \dots, \beta_{n+1}), \beta' = (\beta'_1, \beta'_2, \dots, \beta'_{n+1})$$

for $\beta_k \leq \beta'_k \leq \beta_k + 1,\ k = 1, 2, \dots, n + 1, \beta \neq \beta'$, and

$$\langle \delta_\beta, \delta_\beta \rangle = (-1)^{\frac{n(n-1)}{2}} (1 + (-1)^n),\ \beta \in I.$$

In the remaining cases, except those arising from the previous ones by a permutation, we have $\langle \delta_\beta, \delta_{\beta'} \rangle = 0$.

Chapter 6

Integrals

In this chapter we unify the material of Chapters 3, 5 and 4 in order to study integrals of algebraic differential forms over topological cycles. We will also discuss some methods for reducing higher dimensional integrals to lower dimensional ones.

6.1 Notations

We fix a finite number of elements $a_i,\ i = 1, 2, \ldots, r$ of the polynomial ring $\mathbb{Q}[t]$, $t = (t_1, t_2, \ldots, t_s)$ a multi parameter, and we assume that R is the localization of $\mathbb{Q}[t]$ over the multiplicative group generated by a_i's. As before, f is a tame polynomial in $\mathsf{R}[x]$ and we will freely use the notations related to f introduced in §3.2. We have

$$\mathbb{U}_0 := \mathbb{C}^s \backslash (\cup_{i=1}^r \{t \in \mathbb{C}^s \mid a_i(t) = 0\})$$

and

$$T := \mathbb{U}_0 \backslash \{t \in \mathbb{U}_0 \mid \Delta(t) = 0\},$$

where Δ is the discriminant of f. In particular, Ω_T^i is the set of algebraic i-forms in T.

For a fixed value $c \in \mathbb{U}_0$ of t, we denote by f_c the polynomial obtained by replacing c instead of t in f. By a topological cycle $\delta \in H_n(\{f = 0\}, \mathbb{Z})$ we mean a continuous family of cycles $\{\delta_t\}_{t \in U},\ \delta_t \in H_n(\{f_t = 0\}, \mathbb{Z})$, where U is a small neighborhood in T.

The integral

$$\int_\delta \omega := \int_{\delta_t} \left(\omega \mid_{\{f_t=0\}}\right),\ \omega \in \mathsf{H}',\ \delta \in H_n(\{f = 0\}, \mathbb{Z})$$

is well-defined, i.e. it does not depend on the choice of the differential form (resp. cycle) in the class ω (resp. in the homology class δ). In the case $\omega \in \mathsf{H}''$ by $\int_\delta \omega$ we mean $\int_\delta \frac{\omega}{df}$, where the Gelfand-Leray form $\frac{\omega}{df}$ is defined in §3.3. The integral $\int_\delta \omega$ is a holomorphic function in U and it can be extended to a multi-valued holomorphic function in T.

In the zero dimensional case $n = 0$, recall that $H_0(\{f = 0\}, \mathbb{Z})$ is the set of all finite sums $\sum_i r_i[x_i]$, where $r_i \in \mathbb{Z}, \sum_i r_i = 0$ and x_i's are the roots of f. We define

$$\int_\delta \omega := \sum_i r_i \omega(x_i),$$

where

$$\omega \in \mathsf{H}',\ \delta = \sum_i r_i x_i \in H_0(\{f = 0\}, \mathbb{Z}),$$

and call them (zero dimensional Abelian) integrals/periods.

6.2 Integrals and Gauss-Manin connections

The following proposition gives us the most important property of the Gauss-Manin connection related to integrals.

Proposition 6.1. *Let U be a small open set in T and $\{\delta_t\}_{t\in U}, \delta_t \in H_n(\{f_t = 0\}, \mathbb{Z})$ be a continuous family of topological n-dimensional cycles. Then*

$$d(\int_{\delta_t} \omega) = \sum_{i=1}^{\mu} \alpha_i \int_{\delta_t} \omega_i, \ \omega \in \mathsf{H}, \tag{6.1}$$

where

$$\nabla\omega = \sum_{i=1}^{\mu} \alpha_i \otimes \omega_i, \ \alpha_i \in \Omega_T^1, \ \omega_i \in \mathsf{H},$$

and ω_i's form a R-basis of H.

See [2] for similar statements in the local context and their proof.

Proof. By Theorem 5.2 a distinguished set of vanishing cycles generate the n-th cohomology of $\{f = 0\}$ and so we assume that δ_t is a vanishing cycle in a smooth point c of the variety $\{\Delta = 0\}$. Therefore, there exists an $n+1$-dimensional real thimble

$$D_t = \cup_{s\in[0,1]}\delta_{\gamma_t(s)} \times \{\gamma_t(s)\} \subset \mathbb{C}^{n+1} \times \mathbb{C}^s$$

such that γ_t is a path in $\mathbb{U}_0$ connecting t to c and $\delta_{\gamma_t(s)}$ is the trace of δ_t when it vanishes along γ_t. In order to define the Gauss-Manin connection of $\omega \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ we wrote

$$d\omega - \sum_i \alpha_i \wedge \omega_i \in f\Omega^{n+1}_{\mathbb{U}_1} + df \wedge \Omega^n_{\mathbb{U}_1}, \ \alpha_i \in \Omega^1_T, \ \omega_i \in \Omega^n_{\mathbb{U}_1}.$$

Since $f \mid_{D_t} = 0$, the integral of the elements of $f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0} + df \wedge \Omega^n_{\mathbb{U}_1/\mathbb{U}_0}$ on D_t is zero and we have

$$\begin{aligned} \int_{\delta_t} \omega &= \int_{D_t} d\omega \\ &= \sum_i \int_{D_t} \alpha_i \wedge \omega_i \\ &= \int_{\gamma_t(s)} \left(\sum_i \alpha_i \int_{\delta_{\gamma_t(s)}} \omega_i \right). \end{aligned}$$

In the first equality we have used Stokes Lemma and in the last equality we have used integration by parts. Taking the differential of the above equality we get the desired equality.

□

Remark 6.1. From (6.1) it follows that

$$v(\int_{\delta_t} \omega) = \int_{\delta_t} \nabla_v \omega, \ \forall \omega \in \mathsf{H}, \ v \in \mathcal{D}_{\mathbb{U}_0} \tag{6.2}$$

for any continuous family of cycles δ_t in a small neighborhood in T. For a fixed v, the operator $\nabla_v : \mathsf{H} \to \mathsf{H}_\Delta$ with the above property is unique. This follows from the fact that if $\omega \in \mathsf{H}$ restricted to all regular fibers of f is exact then ω is zero in H (a consequence of Corollary 3.1). If we want to prove an equality for the Gauss-Manin connection of a tame polynomial f over the function field introduced at the beginning of this chapter then we may use (6.2). The proof of the same equality for an arbitrary R of Chapter 3 demands only algebraic methods.

6.3 Period matrix

Let $\omega = (\omega_1, \omega_2, \dots, \omega_\mu)^{\mathsf{t}}$ be a basis of of the free R-module H. In this basis we can write the matrix of the Gauss-Manin connection ∇:

$$\nabla\omega = A \otimes \omega,\ A \in \mathrm{Mat}^{\mu\times\mu}(\Omega_T^1).$$

A fundamental matrix of solutions for the linear differential equation

$$dY = A \cdot Y \tag{6.3}$$

(with Y a $\mu \times 1$ unknown matrix function defined in an small open neighborhood in $\mathbb{U}_0\backslash\{\Delta = 0\}$) is given by $Y = \mathsf{pm}^{\mathsf{t}}$, where

$$\mathsf{pm}(t) = [\int_\delta \omega^{\mathsf{t}}] = \begin{pmatrix} \int_{\delta_1}\omega_1 & \int_{\delta_1}\omega_2 & \cdots & \int_{\delta_1}\omega_\mu \\ \int_{\delta_2}\omega_1 & \int_{\delta_2}\omega_2 & \cdots & \int_{\delta_2}\omega_\mu \\ \vdots & \vdots & \vdots & \vdots \\ \int_{\delta_\mu}\omega_1 & \int_{\delta_\mu}\omega_2 & \cdots & \int_{\delta_\mu}\omega_\mu \end{pmatrix}, \tag{6.4}$$

and $\delta = (\delta_1, \delta_2, \dots, \delta_\mu)^{\mathsf{t}}$ is a basis of the $\mathbb{Z}$-module $H_0(\{f = 0\}, \mathbb{Z})$. This follows from Proposition 6.1. The matrix pm is called the period matrix of f (in the basis δ and ω). Looking pm as a function matrix in t, it is also called the period map. By Theorem 5.2 we know that δ can be chosen as a distinguished set of vanishing cycles.

Proposition 6.2. *Let $\Delta_i,\ i = 1, 2, \dots, m$ be the irreducible components of the discriminant of a tame polynomial in $\mathsf{R}[x]$. We have*

$$\det(\mathsf{pm})^2 = c \cdot \Delta_1^{k_1}\Delta_2^{k_2}\cdots\Delta_m^{k_m}$$

for some non-zero constant c and $k_1, k_2, \dots, k_m \in \mathbb{Z}$.

Proof. First, we prove that $\det(\mathsf{pm})^2$ is a one-valued function in T. If δ' is another basis of $H_n(\{f = 0\}, \mathbb{Z})$ obtained by the monodromy of δ then

$$\delta' = A\delta,\ A\Psi_0 A^{\mathsf{t}} = \Psi_0,$$

where Ψ_0 is the intersection matrix of $H_n(\{f = 0\}, \mathbb{Z})$ in the basis δ. This implies that $\det(A)^2 = 1$ and so $\det(\mathsf{pm})^2$ is a one-valued function in T. Since our integrals have a finite growth at infinity and $\{\Delta = 0\}$ we conclude that $\det(\mathsf{pm})^2$ is rational function in $\mathbb{U}_0$ with poles along $\{\Delta = 0\}$. It does not have zeros outside $\{\Delta = 0\}$ and so it must be of the desired form. $\square$

6.4 Picard-Fuchs equation

We saw in §3.12 that for $\omega \in \mathsf{H}$ and $v \in \mathcal{D}_{\mathbb{U}_0}$ a vector field in $\mathbb{U}_0$, there exists $m \leq \mu$ and $p_i \in \mathsf{R},\ i = 0, 1, 2, \dots, m$ such that we have the Picard-Fuchs equation of ω along v:

$$p_0\omega + p_1\nabla_v(\omega) + p_2\nabla_v^2(\omega) + \cdots + p_m\nabla_v^m(\omega) = 0 \tag{6.5}$$

For $\delta \in H_n(\{f = 0\}, \mathbb{Z})$ we take $\int_\delta$ of the above equality, we use the equality $v(\int_\delta \cdot) = \int_\delta \nabla_v(\cdot)$ and finally we conclude that the analytic functions

$$\int_\delta \omega,\ \delta \in H_n(\{f = 0\}, \mathbb{Z}) \tag{6.6}$$

satisfy the linear differential equation:

$$p_0(t)y + p_1(t)y' + p_2(t)y'' + \cdots + p_m(t)y^{(m)} = 0,\ y' := dy(v) \tag{6.7}$$

In fact, they span the μ-dimensional vector space of the solutions of (6.7). This follows from the fact that the period matrix (6.4) is a fundamental system for the linear differential equation (6.3). The number m is called the order of the differential equation (6.7). If $m = \mu$ then the integrals (6.6) form a basis of the solution space of (6.7).

Remark 6.2. Note that if $v = \frac{\partial}{\partial t_i},\ i = 1, 2, \dots, s$ then y' means the derivation with respect to the parameter t_i. Almost all the examples of Picard-Fuchs equations in the literature are obtained by such vector fields.

6.5 Modular foliations and integrals

As a corollary of Proposition 6.1 we have:

Proposition 6.3. *The leaves of the modular foliation $\mathcal{F}_\omega,\ \omega \in \mathsf{H}$ are the loci of parameters in which the integrals $\int_\delta \omega,\ \delta \in H_n(\{f = 0\}, \mathbb{Z})$ are constant.*

Proof. Let $\Delta = (\delta_1, \delta_2, \dots, \delta_\mu)^{\mathsf{t}}$ be a $\mathbb{Z}$-basis of $H_n(\{f = 0\}, \mathbb{Z})$ and $\omega = (\omega_1, \omega_2, \dots, \omega_\mu)^{\mathsf{t}}$ be a basis of the free R-module H. Let also pm be the corresponding period matrix. By Proposition 6.1 we have

$$[d(\int_{\delta_1} \omega), d(\int_{\delta_2} \omega), \cdots, d(\int_{\delta_\mu} \omega)]^{\mathsf{t}} = \mathsf{pm} \cdot [\alpha_1, \alpha_2, \dots, \alpha_\mu]^{\mathsf{t}},$$

where $\nabla\omega = \sum_{i=1}^{\mu} \alpha_i \otimes \omega_i,\ \alpha_i \in \Omega_T^1,\ \omega_i \in \mathsf{H}$. By Proposition 6.2, the period matrix has a non-zero determinant outside of $\{\Delta = 0\}$ and the foliation induced by $d(\int_{\delta_i} \omega),\ i = 1, 2, \dots, \mu$ and $\alpha_i,\ i = 1, 2, \dots, \mu$ are the same. □

6.6 Homogeneous polynomials

For a homogeneous polynomial $g(x) = g(x_1, x_2, \dots, x_{n+1})$ let us define:

$$\mathsf{p}(\beta, \delta) := \int_\delta \omega_\beta \in \mathbb{C}, \tag{6.8}$$

where $\omega_\beta := x^\beta dx$, x^β a monomial in x, and $\delta \in H_n(\{f=1\},\mathbb{Z})$. We define $f := g - t \in \mathsf{R}[t][x]$ which is tame and its discriminant is $(-t)^\mu$. We have

$$\nabla_{\frac{\partial}{\partial t}}(\omega_\beta) = \frac{(A_\beta - 1)}{t}\omega_\beta$$

and so

$$\frac{\partial}{\partial t}\int_{\delta_t}\omega_\beta = \frac{A_\beta - 1}{t}\int_{\delta_t}\omega_\beta.$$

Therefore

$$\int_{\delta_t}\omega_\beta = \mathsf{p}(\beta,\delta)t^{A_\beta - 1}. \tag{6.9}$$

Here we have chosen a branch of t^{A_β} whose evaluation on 1 is 1. Using $\eta_\beta = t\omega_\beta$ in H'' of $f - t$ we can obtain similar formulas for η_β.

6.7 Integration over joint cycles

The objective of this and the next section is to introduce techniques for simplifying integrals and in the best case to calculate them. For simplicity, we take the tame polynomials over $\mathbb{C}$ but the whole discussion is valid for the tame polynomials depending on parameters as it is explained at the beginning of the present chapter.

Let $f \in \mathbb{C}[x]$ and $g \in \mathbb{C}[y]$ be two tame polynomials in $n+1$, respectively $m+1$, variables. Recall the definition of an admissible triple from §5.6.

Proposition 6.4. *Let ω_1 (resp. ω_2) be an $(n+1)$-form (resp. $(m+1)$-form) in $\mathbb{C}^{n+1}$ (resp. $\mathbb{C}^{m+1}$). Let also $(t_s,\ s \in [0,1], \delta_{1b}, \delta_{2b})$ be an admissible triple and*

$$I_1(t_s) = \int_{\delta_{1,t_s}} \frac{\omega_1}{df},\ I_2(t_s) = \int_{\delta_{2,t_s}} \frac{\omega_2}{dg}.$$

Then

$$\int_{\delta_{1b} *_{t.} \delta_{2b}} \frac{\omega_1 \wedge \omega_2}{d(f-g)} = \int_{t_s, s\in[0,1]} I_1(t_s) I_2(t_s) dt_s$$

Proof. We have

$$\omega_1 \wedge \omega_2 = df \wedge \frac{\omega_1}{df} \wedge dg \wedge \frac{\omega_2}{dg} = d(f-g) \wedge \frac{\omega_1}{df} \wedge dg \wedge \frac{\omega_2}{dg}$$

and so restricted to the variety $X := \{(x,y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} \mid f(x) - g(y) = 0\}$ we have

$$\frac{\omega_1 \wedge \omega_2}{d(f-g)} = \frac{\omega_1}{df} \wedge dt \wedge \frac{\omega_2}{dg},$$

where t is the holomorphic function on X defined by $t(x,y) := f(x) = g(y)$. Now, the proposition follows by integration in parts. $\square$

Recall the B-function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 s^{a-1}(1-s)^{b-1} ds,\ a, b, \in \mathbb{Q}.$$

and its multi parameter form:

$$B(a_1, a_2, \cdots, a_r) = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_r)}{\Gamma(a_1 + a_2 + \cdots + a_r)}.$$

Proposition 6.5. *Let $f(x) = f(x_1, x_2, \dots, x_{n+1})$ and $g(y) = g(y_1, y_2, \dots, y_{m+1})$ be two tame homogeneous polynomials. Let also $(t_s,\ s \in [0,1], \delta_{1b}, \delta_{2b})$ be an admissible triple, x^{β_1} be a monomial in x and y^{β_2} be a monomial in y. We have*

$$\mathsf{p}(\{f+g=1\}, (\beta_1, \beta_2), \delta_1 *_{t.} \delta_2) = \mathsf{p}(\{f=1\}, \beta_1, \delta_1)\mathsf{p}(\{g=1\}, \beta_2, \delta_2)B(A_{\beta_1}, A_{\beta_2})$$

Proof. In Proposition 6.4 let us replace g with $-g+1$. We use (6.9) and we have

$$\begin{aligned}\int_{\delta_1 *_{t.} \delta_2} \frac{x^{\beta_1} y^{\beta_2} dx \wedge dy}{d(f+g)} &= \mathsf{p}(\{f=1\},\ \beta_1, \delta_1)\mathsf{p}(\{g=1\}, \beta_2, \delta_2) \int_0^1 s^{A_{\beta_1}-1}(1-s)^{A_{\beta_2}-1} ds \\ &= \mathsf{p}(\{f=1\},\ \beta_1, \delta_1)\mathsf{p}(\{g=1\}, \beta_2, \delta_2)B(A_{\beta_1}, A_{\beta_2}).\end{aligned}$$

□

As a corollary of the above proposition we have:

Proposition 6.6. *For zero dimensional cycles $\delta_i = [a_i] - [b_i] \in H_0(\{x_i^{m_i} - 1\}, \mathbb{Z})$ we have*

$$\mathsf{p}(\{x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} = 1\}, (\beta_1, \beta_2, \dots, \beta_{n+1}), \delta_1 * \delta_2 * \cdots * \delta_{n+1}) =$$

$$(\int_{\delta_1} \frac{x_1^{\beta_1} dx_1}{dx_1^{m_1}})(\int_{\delta_2} \frac{x_2^{\beta_2} dx_2}{dx_2^{m_2}}) \cdots (\int_{\delta_{n+1}} \frac{x_{n+1}^{\beta_{n+1}} dx_{n+1}}{dx_{n+1}^{m_{n+1}}}) B(\frac{\beta_1+1}{m_1}, \frac{\beta_2+1}{m_2}, \cdots, \frac{\beta_{n+1}+1}{m_{n+1}}) =$$

$$\frac{1}{m_1 m_2 \cdots m_{n+1}} (a_1^{\beta_1+1} - b_1^{\beta_1+1})(a_2^{\beta_2+1} - b_2^{\beta_2+1}) \cdots (a_{n+1}^{\beta_{n+1}+1} - b_{n+1}^{\beta_{n+1}+1}) \cdot$$

$$B(\frac{\beta_1+1}{m_1}, \frac{\beta_2+1}{m_2}, \cdots, \frac{\beta_{n+1}+1}{m_{n+1}})$$

Proof. Successive uses of Proposition 6.5 will give us the desired equality of the proposition. □

6.8 Reduction of integrals

In this section we describe some simple rules for reducing a higher dimensional integral to a lower dimensional one.

Proposition 6.7. *Let $f(x) = f(x_1, x_2, \dots, x_{n+1})$ be a tame polynomial and $g(y) = g(y_1, y_2, \dots, y_{m+1})$ be a homogeneous tame polynomial. Let $\delta_1 \in H_n(\{f=1\}, \mathbb{Z})$, $\delta_2 \in H_m(\{g=1\}, \mathbb{Z})$, x^{β_1} be a monomial in x and y^{β_2} be a monomial in y. Let also $t_s,\ s \in [0,1]$ is a path in the $\mathbb{C}$-plane which connects a critical value of f to 0 (the unique critical value of g). We assume that δ_1 vanishes along $t_.^{-1}$ and δ_2 vanishes along $t_.$. Then we have*

$$\int_{\delta_1 *_{t.} \delta_2} \frac{x^{\beta_1} y^{\beta_2} dx \wedge dy}{d(f-g)} = \begin{cases} \frac{\mathsf{p}(\beta_2, \delta_2)}{\mathsf{p}(\beta_3, \delta_3)} \int_{\delta_1 *_{t.} \delta_3} \frac{x^{\beta_1} z^{\beta_3} dx \wedge dz}{d(f - z^q)} & A_{\beta_2} \notin \mathbb{N} \\ \\ \mathsf{p}(\beta_2, \delta_2) \int_{\tilde{\delta}_1} \theta(\frac{f^{A_{\beta_2}-1} x^{\beta_1} dx}{df}) & A_{\beta_2} \in \mathbb{N} \end{cases}$$

In the first case q and β_3 are given by the equality $A_{\beta_2} = \frac{\beta_3+1}{q}$ and δ_3 is any cycle in $H_0(\{z^q = 1\}, \mathbb{Z})$ with $p(\beta_3, \delta_3) \neq 0$. In the second case, $\tilde{\delta}_1 \in H_n(\{f=0\}, \mathbb{Z})$ is the monodromy of δ_1 along the path $t_s,\ s \in [0,1]$ and θ is the operator in §3.15.

Proof. Using Proposition 6.4 we have:

$$\begin{aligned}\int_{\delta_1*\delta_2}\frac{x^{\beta_1}y^{\beta_2}dx\wedge dy}{d(f-g)} &= \mathsf{p}(\beta_2,\delta_2)\int_{t_s,\ s\in[0,1]} t^{A_{\beta_2}-1}I_1(t_s)dt_s \\ &= \mathsf{p}(\beta_2,\delta_2)\int_{t_s,\ s\in[0,1]} t^{\frac{\beta_3+1}{q}-1}I_1(t_s)dt_s,\end{aligned}$$

where $I_1(t_s):=\int_{\delta_{1,t_s}}\frac{x^{\beta_1}dx}{df}$. We consider two cases: If $A_{\beta_2}\not\in\mathbb{N}$ then we can choose a cycle $\delta_3\in H_0(\{z^q=1\},\mathbb{Z})$ such that $\mathsf{p}(\beta_3,\delta_3)\neq 0$ and so

$$t^{\frac{\beta_3+1}{q}}=\frac{1}{\mathsf{p}(\beta_3,\delta_3)}I_3(t),\ \ I_3(t):=\int_{\delta_{3,t.}}\frac{z^{\beta_3}dz}{dz^q}.$$

We again use Proposition 6.4 and get the desired equality.

If $A_{\beta_2}\in\mathbb{N}$ then z^{β_3} is zero in H'' of the tame one variable polynomial z^q-t and we cannot repeat the argument of the first part. In this case we have

$$\begin{aligned} &= \mathsf{p}(\beta_2,\delta_2)\int_{t_s,\ s\in[0,1]}\left(\int_{\delta_{1,t_s}}\frac{f^{A_{\beta_2}-1}x^{\beta_1}dx}{df}\right)dt_s \\ &= \mathsf{p}(\beta_2,\delta_2)\int_{\Delta} f^{A_{\beta_2}-1}x^{\beta_1}dx \\ &= \mathsf{p}(\beta_2,\delta_2)\int_{\tilde\delta_1}\theta\left(\frac{f^{A_{\beta_2}-1}x^{\beta_1}dx}{df}\right),\end{aligned}$$

where

$$\Delta:=\cup_{s\in[0,1]}\delta_{1,t_s}\in H_{n+1}(\mathbb{C}^{n+1},f^{-1}(0),\mathbb{Z})$$

is the Lefschetz thimble with boundary $\tilde\delta_1$. □

Proposition 6.8. *With the notations of Proposition 6.7*

$$\int_{\delta_1*_{t.}\delta_2}\frac{x^{\beta_1}y^{\beta_2}dx\wedge dy}{(f-g)^k}=\begin{cases}\frac{\mathsf{p}(\beta_2,\delta_2)}{\mathsf{p}(\beta_3,\delta_3)}\int_{\delta_1*_{t.}\delta_3}\frac{x^{\beta_1}z^{\beta_3}dx\wedge dz}{(f-z^q)^k} & A_{\beta_2}\not\in\mathbb{N}\\ \\ \mathsf{p}(\beta_2,\delta_2)\int_{\tilde\delta_1}\frac{f^{A_{\beta_2}-1}x^{\beta_1}dx}{f^{k-1}} & A_{\beta_2}\in\mathbb{N}\end{cases}$$

for $k\geq 2$.

Proof. We assume that f is of the form $\tilde f-a$ and use the equality

$$\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial a^{k-1}}\int_{\delta_1*_{t.}\delta_2}\frac{x^{\beta_1}y^{\beta_2}dx\wedge dy}{(f-g)}=\int_{\delta_1*_{t.}\delta_2}\frac{x^{\beta_1}y^{\beta_2}dx\wedge dy}{(f-g)^k}$$

and Proposition 6.7. □

6.9 Residue map

Let us be given a closed submanifold N of real codimension c in a manifold M. The Leray (or Thom-Gysin) isomorphism is

$$\tau: H_{k-c}(N,\mathbb{Z})\tilde\to H_k(M,M-N,\mathbb{Z})$$

holding for any k, with the convention that $H_s(N)=0$ for $s<0$. Roughly speaking, given $\delta\in H_{k-c}(N)$, its image by this isomorphism is obtained by thickening a cycle representing δ, each point of it growing into a closed c-disk transverse to N in M (see for instance [10] p. 537). Let N be a connected codimension one submanifold of the complex manifold M of dimension n. Writing the long exact sequence of the pair $(M, M-N)$ and using τ we obtain:

$$\cdots\to H_{n+1}(M,\mathbb{Z})\to H_{n-1}(N,\mathbb{Z})\xrightarrow{\sigma} H_n(M-N,\mathbb{Z})\xrightarrow{i} H_n(M,\mathbb{Z})\to\cdots \tag{6.10}$$

where σ is the composition of the boundary operator with τ and i is induced by inclusion. Let $\omega\in H^n(M-N,\mathbb{C}):=\check{H}_n(M-N,\mathbb{Z})\otimes\mathbb{C}$, where $\check{H}_n(M-N,\mathbb{Z})$ is the dual of $H_n(M-N,\mathbb{Z})$. The composition $\omega\circ\sigma: H_{n-1}(N,\mathbb{Z})\to\mathbb{C}$ defines a linear map and its complexification is an element in $H^{n-1}(N,\mathbb{C})$. It is denoted by $\mathrm{Resi}_N(\omega)$ and called the residue of ω in N. We consider the case in which ω in the n-th de Rham cohomology of $M-N$ is represented by a meromorphic C^∞ differential form ω' in M with poles of order at most one along N. Let $f_\alpha=0$ be the defining equation of N in a neighborhood U_α of a point $p\in N$ in M and write $\omega'=\omega_\alpha\wedge\frac{df}{f}$. For two such neighborhoods U_α and U_β with non empty intersection we have $\omega_\alpha=\omega_\beta$ restricted to N. Therefore, we get a $(n-1)$-form on N which in the de Rham cohomology of N represents $\mathrm{Resi}_N\omega$ (see [26] for details). This is called the Poincaré residue.

Let us be given a tame polynomial f, $c\in T:=\mathbb{U}_0\backslash\{\Delta=0\}$ and $\omega\in\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$. We can associate to $\frac{\omega}{f^k}$, $k\in\mathbb{N}$ its residue in L_c which is going to be an element of $H^n(L_c,\mathbb{C})$ (we first substitute t with c in $\frac{\omega}{f}$ and then take the residue as it is explained in the previous paragraph). This gives us a global section $\mathrm{Resi}[\frac{\omega}{f^k}]$ of the n-th cohomology bundle of the fibration f over T. It is represented by the element $[\frac{\omega}{f^k}]\in\mathsf{M}$, where M is defined in §4.1. Having Proposition 4.3 in mind, we regard $\mathrm{Resi}(\frac{\omega}{f^k})$ as an element in the localization of H over $\{1,\Delta,\Delta^2,\ldots\}$. In the case $k=1$, $\mathrm{Resi}(\frac{\omega}{f})=[\omega]\in\mathsf{H}''$.

If v is a vector field in $\mathbb{U}_0$ then we have

$$v\int_\delta \mathrm{Resi}(\frac{\omega}{f^k})=v\int_{\sigma(\delta)}\frac{\omega}{f^k}=\int_{\sigma(\delta)}\nabla_v([\frac{\omega}{f^k}])=\int_\delta \mathrm{Resi}(\nabla_v([\frac{\omega}{f^k}]))$$

and so

$$\mathrm{Resi}(\nabla_v([\frac{\omega}{f^k}])=\nabla_v(\mathrm{Resi}([\frac{\omega}{f^k}]).$$

6.10 Geometric interpretation of Theorem 4.2

Let $\mathbb{P}^{(1,\alpha)}=\{[X_0:X_1:\cdots:X_{n+1}]\mid(X_0,X_1,\cdots,X_{n+1})\in\mathbb{C}^{n+2}\}$ be the projective space of weight $(1,\alpha)$, $\alpha=(\alpha_1,\ldots,\alpha_{n+1})$. One can consider $\mathbb{P}^{(1,\alpha)}$ as a compactification of $\mathbb{C}^{n+1}$ with coordinates $(x_1,x_2,\ldots,x_{n+1})$ by putting

$$x_i=\frac{X_i}{X_0^{\alpha_i}},\ i=1,2,\cdots,n+1. \tag{6.11}$$

The projective space at infinity $\mathbb{P}^\alpha_\infty=\mathbb{P}^{(1,\alpha)}-\mathbb{C}^{n+1}$ is of weight $\alpha:=(\alpha_1,\alpha_2,\ldots,\alpha_{n+1})$. Let $f\in\mathbb{C}[x]$ be a tame polynomial of degree d and g be its last quasi-homogeneous part. We take the homogenization $F=X_0^d f(\frac{X_1}{X_0^{\alpha_1}},\frac{X_2}{X_0^{\alpha_2}},\ldots,\frac{X_{n+1}}{X_0^{\alpha_{n+1}}})$ of f and so we can regard $\{f=0\}$ as an affine subvariety in $\{F=0\}\subset\mathbb{P}^{(1,\alpha)}$.

Proposition 6.9. *For a monomial x^β with $A_\beta = k \in \mathbb{N}$, the meromorphic form $\frac{x^\beta dx}{f^k}$ has a pole of order one at infinity and its Poincaré residue at infinity is $\frac{X^\beta \eta_\alpha}{g^k}$. If $A_\beta < k$ then it has no poles at infinity.*

Proof. Let us write the above form in the homogeneous coordinates (6.11). We use $d(\frac{X_i}{X_0^{\alpha_i}}) = X_0^{-\alpha_i} dX_i - \alpha_i X_i X_0^{-\alpha_i - 1} dX_0$ and

$$\begin{aligned} \frac{x^\beta dx}{f^k} &= \frac{(\frac{X_1}{X_0^{\alpha_1}})^{\beta_1} \cdots (\frac{X_{n+1}}{X_0^{\alpha_{n+1}}})^{\beta_{n+1}} d(\frac{X_1}{X_0^{\alpha_1}}) \wedge \cdots \wedge d(\frac{X_{n+1}}{X_0^{\alpha_{n+1}}})}{f(\frac{X_1}{X_0^{\alpha_1}}, \cdots, \frac{X_{n+1}}{X_0^{\alpha_{n+1}}})^k} \\ &= \frac{X^\beta \eta_{(1,\alpha)}}{X_0^{(\sum_{i=1}^{n+1} \beta_i \alpha_i) + (\sum_{i=1}^{n+1} \alpha_i) + 1 - kd} (X_0 \tilde{F} - g(X_1, X_2, \cdots, X_{n+1}))^k} \\ &= \frac{X^\beta \eta_{(1,\alpha)}}{X_0 (X_0 \tilde{F} - g(X_1, X_2, \cdots, X_{n+1}))^k} \\ &= \frac{dX_0}{X_0} \wedge \frac{X^\beta \eta_\alpha}{(X_0 \tilde{F} - g)^k} \end{aligned}$$

The last equality is up to forms without pole at $X_0 = 0$. The restriction of $\frac{X^\beta \eta_\alpha}{(X_0 \tilde{F} - g)^k}$ to $X_0 = 0$ gives us the desired form.

If $A_\beta < 0$ then the second equality above tells us that $\frac{\omega_\beta}{f^k}$ has no poles at infinity $X_0 = 0$. □

Proposition 6.9 shows that for a cycle $\delta \in H_n(\{f = 0\}, \mathbb{Z})$ at infinity we have

$$\int_\delta \frac{x^\beta dx}{f^k} = 0, \text{ if } A_\beta < k$$

and

$$\int_\delta \frac{x^\beta dx}{f^k} = \int_{\delta'} \frac{X^\beta \eta_\alpha}{g^k}, \text{ if } A_\beta = k$$

for some cycle $\delta' \in H_n(\mathbb{P}^\alpha \backslash \{g = 0\}, \mathbb{Z})$. In particular, if the last homogeneous part g of f does not depend on any parameter of R then for $A_\beta = k$ the integral $\int_\delta \frac{x^\beta dx}{f^k}$ is constant. Now it is evident that W_n is the set of differential forms which do not have any residue at infinity. This gives another proof of Theorem 4.2, part 2. The topological interpretation of part 3 is as follows: For simplicity we take $\mathsf{R} = \mathbb{Q}[t_1, t_2, \ldots, t_s]$ and assume that g does not depend on the parameters in R. We write an $\omega \in \mathsf{M}$ in the form (4.14) and for a cycle at infinity δ we see that

$$\int_\delta \omega = \sum_{A_\beta = k, k \in \mathbb{N}, \beta \in I} a_{\beta,k} \int_\delta \frac{\omega_\beta}{f^k}$$

which is a polynomial in $t_1, t_2, \ldots, t_s$ (according to the previous discussion $\int_\delta \frac{\omega_\beta}{f^k}$ are constant numbers and so the above polynomial has complex coefficients). Therefore, the n-th iterative derivation of $\int_\delta \omega$ with respect to t_i must be zero for n bigger than the degree in t_i of $\int_\delta \omega$.

Using the equality (4.14) we see that the first integrals of $\mathcal{F}_\omega$ that we have discussed at the end of section (4.7) are the integration of ω over cycles at infinity (up to a constant).

Chapter 7

Loci of Lefschetz-Hodge cycles

The objective of the present chapter is to introduce the Lefschetz-Hodge loci which is invariant under certain modular foliations. We will also state some conjectures which are consequences of the Hodge conjecture. We mainly use the notations in §6.1

7.1 Hodge-Lefschetz cycles and cycles at infinity

Let f be a tame polynomial over $\mathbb{C}$ with $n+1$ variables. We further assume that f has a non-zero discriminant Δ_f and so $\{f=0\}$ is a smooth variety.

Definition 7.1. For an $\omega \in \mathsf{H}$, a cycle $\delta \in H_n(\{f=0\},\mathbb{Z})$ is called an ω-cycle if $\int_\delta \omega = 0$. It is called a cycle at infinity if

$$\int_\delta \omega = 0,\ \forall \omega \in \mathsf{W}_n,$$

where $(\mathsf{W}_\bullet, \mathsf{F}^\bullet)$ is the mixed Hodge structure of H. Let n be an even number. A cycle $\delta \in H_n(\{f=0\},\mathbb{Z})$ is called a Hodge cycle if

$$\int_\delta \omega = 0,\ \forall \omega \in \mathsf{F}^{\frac{n}{2}+1} \cap \mathsf{W}_n.$$

For $n=2$ we will also call δ the Lefschetz cycle. By definition the cycles at infinity are Hodge cycles. We say that two Hodge cycles δ_1, δ_2 are equivalent if $\delta_1 - \delta_2$ is a cycle at infinity. We denote by $[\delta_1]$ the equivalent class of the Hodge cycle δ_1.

Let M be a smooth compactification of $\{f=0\}$ and $i: H_n(\{f=0\},\mathbb{Z}) \to H_n(M,\mathbb{Z})$ be the map induced by the inclusion $\{f=0\} \subset M$. It is a classical fact that the kernel of i is the set of cycles at infinity and a cycle $\delta \in H_n(\{f=0\},\mathbb{Z})$ is a cycle at infinity if and only if $<\delta, \delta'>=0$ for all $\delta' \in H_n(\{f=0\},\mathbb{Z})$. For a Hodge cycle δ, the cycle $i(\delta)$ is Hodge in the classical sense (see [79]).

Let n be an even natural number and $Z = \sum_{i=1}^{s} r_i Z_i$, where $Z_i,\ i=1,2,\dots,s$ is a subvariety of M of complex dimension $\frac{n}{2}$ and $r_i \in \mathbb{Z}$. Using a resolution map $\tilde{Z}_i \to M$, where $\tilde{Z}_i$ is a complex manifold, one can define an element $\sum_{i=1}^{s} r_i[Z_i] \in H_n(M,\mathbb{Z})$ which is called an algebraic cycle (see [5]). The assertion of the Hodge conjecture is that if we consider the rational homologies then a Hodge cycle $\delta \in H_n(\{f=0\},\mathbb{Q})$ is an algebraic cycle, i.e. there exist subvarieties $Z_i \subset M$ of dimension $\frac{n}{2}$ and rational numbers r_i such that $i(\delta) = \sum r_i[Z_i]$. The difficulty of this conjecture lies in constructing varieties just with their homological information.

By our definition of a Hodge cycle we do not lose anything as it is explained in the following remark.

Remark 7.1. Let M be a hypersurface of even dimension n in the projective space $\mathbb{P}^{n+1}$. By first Lefschetz theorem $H_m(M,\mathbb{Z}) \cong H_m(\mathbb{P}^{n+1},\mathbb{Z})$, $m < n$ and so the only interesting Hodge cycles are in $H_n(M,\mathbb{Z})$. Let P^n be a hyperplane in general position with respect to M. The intersection $N := P^n \cap M$ is a submanifold of M and is a smooth hypersurface in P^n. Let $\delta \in H_n(M,\mathbb{Z})$. There is an algebraic cycle $[Z] \in H_n(M,\mathbb{Z})$ and integer numbers a, b such that $<\delta - \frac{a}{b}[Z],[N]>= 0$ and so $b\delta - a[Z]$ is in the image of i. The proof of this fact goes as follows: Let $P^{\frac{n}{2}+1}$ be a sub-projective space of $\mathbb{P}^{n+1}$ such that $P^{\frac{n}{2}+1}$ and $P^{\frac{n}{2}} := P^{\frac{n}{2}+1} \cap P^n$ are in general position with respect to M. Put $Z = P^{\frac{n}{2}+1} \cap M$. By Lefschetz first theorem $H_{n-2}(M) \cong H_{n-2}(\mathbb{P}^{n+1}) \cong \mathbb{Z}$. If $a :=< \delta, [N] >$ and $b :=< [Z], [N] >$ then $<\delta - \frac{a}{b}[Z],[N]>= 0$ (b is the degree of $M \cap P^{\frac{n}{2}+1} \cap P^n$ in $P^{\frac{n}{2}+1} \cap P^n$ and so it is not zero).

It is remarkable to mention that:

Proposition 7.1. *Let f be a tame polynomial over $\mathbb{C}$ with a non-zero discriminant and $\delta \in H_n(\{f = 0\},\mathbb{Z})$. If $\int_\delta \mathsf{W}_n \cap \mathsf{F}^{n-[\frac{n}{2}]} = 0$ then $\int_\delta \mathsf{W}_n = 0$ and so δ is the cycle at infinity.*

Proof. Let M be the compactification of $\{f = 0\}$. The elements of $\mathsf{W}_n \cap \mathsf{F}^{n-[\frac{n}{2}]}$ induce elements in $H^n_{\mathrm{dR}}(M)$ which are represented by C^∞ $(n-p,p)$-differential forms with $p = 0, 1, \dots, n - [\frac{n}{2}]$. Since $\int_\delta \bar{\omega} = \overline{\int_\delta \omega}$, we conclude that the integration of all elements of $H^n_{\mathrm{dR}}(M)$ over δ is zero. □

7.2 Some conjectures

In this section we state two consequences of the Hodge conjecture. For the fact that these conjectures are followed by the Hodge conjecture the reader is referred to the Deligne's lecture [13].

Conjecture 7.1. *Let f be a tame polynomial over $\bar{\mathbb{Q}}$ with a non-zero discriminant. For a Hodge cycle $\delta \in H_n(\{f = 0\},\mathbb{Q})$ and a differential form $\omega \in \mathsf{W}_n\mathsf{H}$ we have:*

$$\int_\delta \omega \in (2\pi i)^{\frac{n}{2}}\bar{\mathbb{Q}}.$$

Let f be as above and $\sigma : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}$ is a field homomorphisim ($\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$). Let f_σ be the polynomial obtained by replacing the coefficients of f with their images under σ. The polynomial f_σ is also tame and we have a well-defined map $\mathsf{H}_f \to \mathsf{H}_{f_\sigma}$, $\omega \to \omega_\sigma$, where ω_σ is obtained by replacing the coefficients of ω with their images under σ. To present the second conjecture, it is better to make the following definition.

Definition 7.2. *A cycle $\delta \in H_n(\{f = 0\},\mathbb{Z})$ is called absolute if for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ there is a cycle $\delta_\sigma \in H_n(\{f_\sigma = 0\},\mathbb{Z})$ such that*

$$\int_\delta \omega = \int_{\delta_\sigma} \omega_\sigma, \ \forall \omega \in \mathsf{W}_n\mathsf{H}_f.$$

If such a δ_σ exists then it is unique (up to cycles at infinity) and so the Galois group acts on the space of absolute Hodge cycles. If the Hodge conjecture is true then we have:

Conjecture 7.2. *Every Hodge cycle is absolute.*

We will use a variational version of the Conjecture 7.1 as follows: Recall the notations at the beginning of Chapter 6.

Conjecture 7.3. *Let f be a tame polynomial over a localization of $\bar{\mathbb{Q}}[t_1, t_2, \ldots, t_s]$ and assume that its discriminant is not zero. Let also $\delta_t \in H_n(\{f_t = 0\}, \mathbb{Z})$, $t \in U$ be a continuous family of cycles, where U is an small open set in T, and assume that for all $t \in U$, δ_t is a Hodge cycle. Then for a differential form $\omega \in \mathsf{W}_n\mathsf{H}$ we have:*

$$\int_\delta \omega \in (2\pi i)^{\frac{n}{2}} \overline{\mathbb{Q}(t_1, t_2, \ldots, t_s)}.$$

The Hodge conjecture for the case $n = 2$ follows from the Lefschetz $(1, 1)$ theorem (see [22, 79]) and so the conjectures 7.1, 7.2 and 7.3 are proved for the case $n = 2$.

7.3 Lefschetz-Hodge loci

We consider a tame polynomial f defined over R. We take a basis $\omega_1, \omega_2, \ldots, \omega_k$ of the freely generated R-module $\mathsf{F}^{\frac{n}{2}+1} \cap \mathsf{W}_n$. We also take a Hodge cycle $\delta_{t_0} \in H_n(\{f_{t_0} = 0\}, \mathbb{Z})$, where f_{t_0} is the specialization of f in $t = t_0$ and $t_0 \in T$ is a regular parameter. By definition we have

$$\int_{\delta_{t_0}} \omega_i = 0, \ i = 1, 2, \ldots, k.$$

For t near to t_0, denote by $\delta_t \in H_n(\{f_t = 0\}, \mathbb{Z})$ the cycle obtained by the monodromy of δ_{t_0}. The variety

$$X_{t_0} := \{t \in (\mathbb{U}_0, t_0) \mid \int_{\delta_t} \omega_i = 0, \ i = 1, 2, \ldots, k\}$$

is called the (local) loci of Hodge cycles. It is a germ of an analytic variety, possibly reducible, defined around a small neighborhood of t_0. For $n = 2$ we may also call X_{t_0} the loci of Lefschetz cycles.

Theorem 7.1. *(Cattani-Deligne-Kaplan) There exists an algebraic set $Y_{t_0} \subset \mathbb{U}_0$ such that Y_{t_0} in a small neighborhood of t_0 in $\mathbb{U}_0$ coincide with X_{t_0}.*

For a proof of the above theorem see [8]. Again, we will call Y_{t_0} the loci of Hodge cycles through t_0. The importance of the loci of Hodge cycles from the point of view of the present text is described in the following proposition:

Proposition 7.2. *The loci of Hodge cycles through t_0 is invariant by the foliation:*

$$\mathcal{F}_{\text{Hodge}} := \cap_{i=1}^k \mathcal{F}_{\omega_i},$$

where ω_i, $i = 1, 2, \ldots, k$ generate the R-module $\mathsf{F}^{\frac{n}{2}+1} \cap \mathsf{W}_n$.

Proof. Let $\delta_{t_0} \in H_n(L_{t_0}, \mathbb{Z})$ be a Hodge cycle. By definition, on a leaf L_i of $\mathcal{F}_{\omega_i}$, $i = 1, 2, \ldots, k$, the integral $\int_{\delta_t} \omega_i$ is constant and so $(L_i, t_0) \subset \{t \in (\mathbb{U}_0, t_0) \mid \int_{\delta_t} \omega_i = 0\}$. Taking intersection for all $i = 1, 2, \ldots, k$ we get the statement of the proposition. $\square$

A priory, the foliation $\mathcal{F}_{\mathrm{Hodge}}$ could be trivial, i.e its leaves are points. For instance, if the ring R has few parameters compared to μ (which is the dimension of the n-th cohomology of the regular fibers of f), then $\mathcal{F}_\omega$ is trivial.

Example 7.1. Let us consider $g = x_1^d + x_2^d + \cdots + x_{n+1}^d$ with $d > n+1$ and the tame polynomial $f = g - \sum_\alpha t_\alpha x^\alpha$ over the polynomial ring $\mathsf{R} = \mathbb{Q}[\cdots, t_\alpha, \cdots]$, where α runs through $\deg(x^\alpha) < d,\ x^\alpha \neq x_i^{d-1},\ i = 1, 2, \ldots, n+1$. If the differential forms

$$\frac{\partial}{\partial t_\alpha}\frac{dx}{f} = \frac{x^\alpha dx}{f^2}$$

are R-independent in M then the modular foliation $\mathcal{F}_\omega$ is trivial. Note that $\omega_\alpha := x^\alpha dx$'s are R-linearly independent in H''.

We do not have any general theorem classifying this situation.

Chapter 8

Fermat varieties

In this chapter we are going to discuss many concepts, such as Hodge numbers, joint cycles, Lefschetz-Hodge cycles and so on, for the affine variety

$$V = V(m_1, m_2, \dots, m_{n+1}) : x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} - 1 = 0.$$

We call V the Fermat variety because it is the generalization of the classical Fermat curve $x^d + y^d = 1$.

8.1 Hodge numbers

We are going to consider the weighted polynomial ring $\mathbb{C}[x]$ with $\deg(x_i) = \alpha_i \in \mathbb{N}$. For a given degree $d \in \mathbb{N}$, we would like to have at least one homogeneous polynomial $g \in \mathbb{C}[x]$ with an isolated singularity at the origin and of degree d. For instance for $\alpha_1, \alpha_2, \dots, \alpha_{n+1} \mid d$ we have the polynomial

$$g = x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}},\ m_i := \frac{d}{\alpha_i}.$$

For other d's we do not have yet a general method which produces a tame polynomial of degree d. The vector space $\mathsf{V}_g = \mathbb{C}[x]/\mathrm{jacob}(f)$ has the following basis of monomials

$$x^\beta,\ \beta \in I := \{\beta \in \mathbb{Z}^{n+1} \mid 0 \le \beta_i \le m_i - 2\}$$

and

$$\mu = \# I = \Pi_{i=1}^{n+1}(m_i - 1).$$

In this case

$$A_\beta = \sum_{i=1}^{n+1} \frac{(\beta_i + 1)}{m_i}.$$

Proposition 8.1. *For the affine variety*

$$V = V(m_1, \dots, m_{n+1}) := \{g = 1\} \subset \mathbb{C}^{n+1}$$

we have

$$h_0^{k-1,n-k} := \dim_{\mathbb{C}}(\mathrm{Gr}_F^{n+1-k}\mathrm{Gr}_{n+1}^W V) = \#\{\beta \in I \mid A_\beta = k\},$$

$$h_0^{k-1,n-k+1} := \dim(\mathrm{Gr}_F^{n+1-k}\mathrm{Gr}_n^W V) = \#\{\beta \in I \mid k - 1 < A_\beta < k\}.$$

The above proposition follows from Theorem 4.1. For $\beta \in I$ we have

$$A_\beta = A_{m-\beta-2},$$

where

$$m - \beta - 2 := (m_1 - \beta_1 - 2, m_2 - \beta_2 - 2, \cdots).$$

We have the symmetric sequence of numbers

$$(h_0^{k-1,n-k},\ k = 1,2,\ldots,n),\ (h_0^{k-1,n-k+1},\ k = 1,2,\ldots,n+1)$$

which correspond to the classical Hodge numbers of the primitive cohomologies of the weighted projective varieties:

$$V_\infty = V_\infty(m_1,\ldots,m_{n+1}) := \{g = 0\} \subset \mathbb{P}^{\alpha_1,\alpha_2,\ldots,\alpha_{n+1}},$$

$$\bar{V} = V \cup V_\infty \subset \mathbb{P}^{1,\alpha_1,\ldots,\alpha_{n+1}}$$

respectively. Here are some tables of Hodge numbers of weighted hypersurfaces obtained by Proposition 8.1.

$n = 2, \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 1$

d	1	2	3	4	5	6	7	8	9	10
$h_0^{0,2}$	0	0	0	1	4	10	20	35	56	84
$h_0^{1,1}$	0	1	6	19	44	85	146	231	344	489
$h_0^{2,0}$	0	0	0	1	4	10	20	35	56	84

$n = 3, \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$

d	1	2	3	4	5	6	7	8	9	10
$h_0^{0,3}$	0	0	0	0	1	5	15	35	70	126
$h_0^{1,2}$	0	0	5	30	101	255	540	1015	1750	2826
$h_0^{2,1}$	0	0	5	30	101	255	540	1015	1750	2826
$h_0^{3,0}$	0	0	0	0	1	5	15	35	70	126

$n = 4, \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 1$

d	1	2	3	4	5	6	7	8	9	10
$h_0^{0,4}$	0	0	0	0	0	1	6	21	56	126
$h_0^{1,3}$	0	0	1	21	120	426	1161	2667	5432	10116
$h_0^{2,2}$	0	1	20	141	580	1751	4332	9331	18152	32661
$h_0^{3,1}$	0	0	1	21	120	426	1161	2667	5432	10116
$h_0^{4,0}$	0	0	0	0	0	1	6	21	56	126

$n = 2, \alpha_0 = \alpha_1 = \alpha_2 = 1,\ \alpha_3 = 3$

d	3	6	9	12	15	18	21	24	27	30
$h_0^{0,2}$	0	1	11	39	94	185	321	511	764	1089
$h_0^{1,1}$	0	19	92	255	544	995	1644	2527	3680	5139
$h_0^{2,0}$	0	1	11	39	94	185	321	511	764	1089

Remark 8.1. For polynomials $f \in \mathsf{R}[x]$ satisfying the hypothesis of Theorem 5.1, the dimensions of the pieces of the mixed Hodge structure of a regular fiber of f (Hodge numbers) are constants depending only on f and not the parameter (see for instance [79] Proposition 9.20).

8.2 Riemann surfaces

Let us consider the case $n = 1$ and let $\alpha_i := \frac{[m_1,m_2]}{m_1}$, $i = 1, 2$

Proposition 8.2. *The variety V_∞ has*

$$\#\{(\beta_1,\beta_2) \in \mathbb{Z}^2 \mid \frac{\beta_1+1}{m_1} + \frac{\beta_2+1}{m_2} = 1,\ 0 \le \beta_i \le m_i - 2,\ i = 1,2\} + 1 = (m_1, m_2)$$

points and the genus of $\bar{V}$ is

$$g(\bar{V}) = \frac{(m_1-1)(m_2-1) - (m_1,m_2) + 1}{2}$$

Proof. We can assume that $g = x^{m_1} - y^{m_2}$. A point of V_∞ can be written in the form

$$[1 : \zeta_{m_2}^i] = [(\zeta_{\alpha_1}^j)^{\alpha_1} : \zeta_{\alpha_1}^{j\alpha_2}\zeta_{m_2}^i] = [1 : \zeta_{m_2}^{i+jm_1}].$$

Therefore, the number of points of V_∞ is $\#(\mathbb{Z}_{m_2}/m_1\mathbb{Z}_{m_2}) = (m_1, m_2)$. According to Proposition 8.1 the number of β's such that $A_\beta = 1$ is the dimension of the 0-th primitive cohomology of V_∞ which is the number of points of V_∞ mines one. $\square$

The only cases in which $\overline{V(m_1,m_2)}$ is an elliptic curve are

$$\{m_1, m_2\} = \{2,4\}, \{2,3\}, \{3,3\}.$$

The Riemann surface $\overline{V(d,2)}$ belongs to the family of Hypergeometric Riemann surfaces. Its genus is

$$g(V(d,2)) = \begin{cases} \frac{d-1}{2} & \text{if } d \text{ is odd} \\ \frac{d-2}{2} & \text{if } d \text{ is even} \end{cases}.$$

8.3 Hypersurfaces of type $1, h, 1$

We want to find all the hypersufaces $\bar{V}$ with the first Hodge number equal to 1. This means that we have to find all $m := (m_1, m_2, \cdots, m_{n+1})$ such that

$$(8.1) \qquad 1 - \frac{1}{m_{n+1}} \le \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_{n+1}} < 1,\ 2 \le m_1 \le m_2 \le \cdots \le m_{n+1}.$$

Note that the above conditions imply that $m_1 \le n+2 \le m_{n+1}$. We have

$$1 - \frac{1}{m_k} \le 1 - \frac{1}{m_{n+1}} \le \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_{n+1}} \le \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_{k-1}} + \frac{n+2-k}{m_k}$$

and so

$$m_k \le (n+3-k)(1 - \frac{1}{m_1} - \frac{1}{m_2} - \cdots - \frac{1}{m_{k-1}})^{-1} \le (n+3-k) \cdot p_k,$$

where p_k is defined by induction as follows: $p_1 = 1,\ p_2 = 2$ and p_{k+1} is the maximum of $(1 - \frac{1}{m_1} - \frac{1}{m_2} - \cdots - \frac{1}{m_{k-1}})^{-1}$ for $m_i \le (n+3-i)p_i,\ i = 1, 2, \ldots, k-1$ with $\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_{k-1}} < 1$. All these imply that for a fixed n the number of such m's is finite. By some computer calculations one expects that we have always $n \le 3$. For a complete list of all such m see the author's homepage [55].

Hodge numbers $1, 10, 1$: The regular fibers of the following homogeneous tame polynomials have the Hodge numbers $1, 10, 1$:

$$g = x^7 + y^3 + z^2,\ x^5 + y^4 + z^2, x^4 + y^3 + z^3$$

For these examples there is no cycle at infinity and so the intersection form is non-degenerate(see §7.1).

8.4 Versal deformation vs. tame polynomial

We may consider homogeneous polynomial g with an isolated singularity at the origin as a holomorphic map from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}, 0)$ and hence consider its versal deformation

$$f(x) = g + \sum_{\beta \in I} t_\beta x^\beta \in \mathsf{R}[x],\ \mathsf{R} := \mathbb{C}[t_\beta \mid \beta \in I],$$

where $\{x^\beta \mid \beta \in I\}$ form a monomial basis of the vector space $\mathbb{C}[x]/\mathrm{jacob}(g)$. In general, the degree of f is bigger than the degree of g and so f may not be a tame polynomial in our context. From topological point of view, the middle cohomology of a generic fiber of f has dimension bigger than the dimension of the middle cohomology of a regular fiber of g and some new singularities and hence vanishing cycles appear after deforming g in the above way. Let us analyze a versal deformation in a special case:

Let $g := x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$. In this case $I = \{\beta \in \mathbb{Z}^{n+1} \mid 0 \leq \beta_i \leq m_i - 2\}$. The Milnor number of f is $\mu := \#I = (m_1 - 1)(m_2 - 1) \cdots (m_{n+1} - 1)$ and $A_\beta = \sum_{i=1}^{n+1} \frac{\beta_i + 1}{m_i}$. The versal deformation of g has degree less than $d := [m_1, m_2, \cdots, m_{n+1}]$ if and only if for all $x^\beta \in I$

$$\sum_{i=1}^{n+1} (m_i - 2) \frac{[m_1, m_2, \cdots, m_{n+1}]}{m_i} \leq [m_1, m_2, \cdots, m_{n+1}].$$

$$\Leftrightarrow \sum_{i=1}^{n+1} \frac{1}{m_i} \geq \frac{n}{2} \tag{8.2}$$

The equality may happen only for $x_1^{m_1-2} x_2^{m_2-2} \cdots x_{n+1}^{m_{n+1}-2}$. We have

$$\frac{n}{2} \leq \sum_{i=1}^{n+1} \frac{1}{m_i} \leq A_\beta \leq (n+1) - \sum_{i=1}^{n+1} \frac{1}{m_i} \leq \frac{n}{2} + 1.$$

It is an easy exercise to verify that (8.2) happens if and only if $m := (m_1, m_2, \ldots, m_{n+1})$ belongs to:

$$(2, 2, \ldots, 2, 2, a),\ a \geq 2,\ (2, 2, \ldots, 2, 3, b), b = 3, 4, 5 \tag{8.3}$$

$$(2, 2, \ldots, 2, 3, 6),\ (2, 2, \ldots, 3, 3, 3), (2, 2, \ldots, 2, 4, 4). \tag{8.4}$$

or their permutation. The equality in (8.2) happens only in the cases (8.4). We conclude that:

Proposition 8.3. *The versal deformation of $g = x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$, $m_1 \leq m_2 \leq \cdots \leq m_{n+1}$ does not increase the degree of g if and only if m is in the list (8.3) or (8.4). In this case we have*

1. *For $n = 2k$ even, the list of Hodge numbers of the variety $\{g = 0\} \subset \mathbb{P}^{\alpha}$ (resp. $\overline{\{g = 1\}} \subset \mathbb{P}^{1,\alpha}$) is of the form $\cdots 0, 1, 1, 0 \cdots$ (resp. $\cdots 0, \mu - 2, 0 \cdots$) if m belongs to the list (8.4) and it is of the form $\cdots 0, 0, 0 \cdots$ (resp. $\cdots, 0, \mu, 0, \cdots$) otherwise,*

2. *For $n = 2k - 1$ odd, the list of Hodge numbers of the variety $\{g = 0\} \subset \mathbb{P}^{\alpha}$ (resp. $\overline{\{g = 1\}} \subset \mathbb{P}^{1,\alpha}$) is of the form $\cdots 0, \mu - 2h, 0 \cdots$ (resp. $\cdots 0, h, h, 0 \cdots$),*

where

$$\mu := (m_1 - 1) \cdots (m_{n+1} - 1),\ \mu - 2h := \{\beta \in \mathbb{Z}^{n+1} \mid 0 \leq \beta_i \leq m_i - 2,\ A_\beta = k\}.$$

8.5 The main example

We are going to analyze the Hodge cycles of $g = x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} - 1$ in more details. We use Theorem 4.1 and Proposition 6.6 and obtain an arithmetic interpretation of Hodge cycles which does not involve any topological argument.

For each natural number m let

$$I_m := \{0, 1, 2, \ldots, m-2\},\ \Delta_m := \{\delta_0, \delta_1, \delta_2, \ldots, \delta_{m-2}\},\ \Omega_m := \{\omega_0, \omega_1, \ldots, \omega_{m-2}\}$$

be three sets with $m - 1$ elements and define:

$$\int_{\delta_\beta} \omega_{\beta'} := \zeta_m^{(\beta+1)(\beta'+1)} - \zeta_m^{\beta(\beta'+1)},\ \beta, \beta' \in I_m$$

$$\mathsf{pm}_m(\omega_\beta) := [\int_{\delta_0} \omega_\beta, \int_{\delta_1} \omega_\beta, \ldots, \int_{\delta_{m-2}} \omega_\beta]^{\mathsf{t}} =$$
$$[\zeta_m^{\beta+1} - 1, \zeta_m^{2(\beta+1)} - \zeta_m^{(\beta+1)}, \cdots, \zeta_m^{(m-1)(\beta+1)} - \zeta_m^{(m-2)(\beta+1)}]^{\mathsf{t}},\ \beta \in I_m.$$

For a set M let $\mathbb{Z}[M]$ be the free $\mathbb{Z}$-module generated by the elements of M. For arbitrary $\beta \in \mathbb{Z}$ we define $\delta_i \in \mathbb{Z}[\Delta_m]$ using the rules:

$$\delta_i := \delta_{i \bmod m},\ \delta_{m-1} := -\sum_{i=0}^{m-2} \delta_i. \tag{8.5}$$

Equivalently

$$\delta_i + \delta_{i+1} + \cdots + \delta_{i+m-1} = 0,\ \forall i \in \mathbb{Z}.$$

Let $m = (m_1, m_2, \ldots, m_{n+1})$, $2 \leq m_i \in \mathbb{N}$ and

$$I_m := I_{m_1} \times I_{m_2} \times \cdots \times I_{m_{n+1}},\ \Delta_m := \Delta_{m_1} \times \Delta_{m_2} \times \cdots \times \Delta_{m_{n+1}},\ \Omega_m := \Omega_{m_1} \times \Omega_{m_2} \times \cdots \times \Omega_{m_{n+1}}.$$

We denote the elements of I_m by $\beta = (\beta_1, \beta_2, \cdots, \beta_{n+1})$. We also denote an element of Δ_m (resp. Ω_m) by δ_β (resp. ω_β) with $\beta \in I_m$. We define

$$\int_{\delta_\beta} \omega_{\beta'} := \prod_{i=1}^{n+1} \int_{\delta_{\beta_i}} \omega_{\beta'_i},\ \beta, \beta' \in I_m,$$

$$\mathsf{pm}_m(\omega_\beta) = \mathsf{pm}_{m_1}(\omega_{\beta_1}) * \mathsf{pm}_{m_2}(\omega_{\beta_2}) * \cdots * \mathsf{pm}_{m_{n+1}}(\omega_{\beta_{n+1}}),\ \beta = (\beta_1, \beta_2, \ldots, \beta_{n+1}) \in I_m.$$

Here for two matrices A and B by $A * B$ we mean the coordinate wise product of A and B ordered lexicographically.

By $\mathbb{Z}$-linearity we define

$$\int_\delta \omega,\ \ \delta \in \mathbb{Z}[\Delta_m],\ \omega \in \mathbb{Z}[\Omega_m].$$

The elements of Δ_m are called vanishing cycles and Δ_m is called to be a distinguished set of vanishing cycles.

Definition 8.1. For $\omega \in \mathbb{Z}[\Omega_m]$, an ω-cycle is an element $\delta \in \mathbb{Z}[\Delta_m]$ such that $\int_\delta \omega = 0$.

An ω-cycle written in the canonical basis of $\mathbb{Z}[\Delta_m]$ is a $1 \times \mu$ matrix δ with coefficients in $\mathbb{Z}$ such that $\delta \cdot \mathsf{pm}_m(\omega) = 0$. Recall that

$$A_\beta := \sum_{i=1}^{n+1} \frac{\beta_i + 1}{m_i},\ \ \beta \in I_m.$$

Definition 8.2. An element $\delta \in \mathbb{Z}[\Delta_m]$ which is an ω-cycle for all

$$\omega_\beta \in \Omega_m, A_\beta \notin \mathbb{Z},\ A_\beta < \frac{n}{2}$$

is called a Hodge cycle.

Proposition 8.4. *Let $\omega_\beta \in \Omega_m$.*

1. *For natural numbers $m_1, m_2, \ldots, m_{n+1}$ the condition*

   $$[\mathbb{Q}(\zeta_{m_1}, \zeta_{m_2}, \ldots, \zeta_{m_{n+1}}), \mathbb{Q}] = (m_1 - 1)(m_2 - 1) \cdots (m_{n+1} - 1) \tag{8.6}$$

   is satisfied if and only if all m_i's are prime numbers.

2. *If (8.6) is satisfied then there does not exist a non zero ω_β-cycle.*

3. *In particular, there does not exist a non zero Hodge cycle, and also, there does not exist a cycle at infinity and so*

   $$\forall \beta' \in I_m, A_{\beta'} \notin \mathbb{N}.$$

Proof. Let $k_i = \mathbb{Q}(\zeta_{m_1}, \zeta_{m_2}, \ldots, \zeta_{m_i}),\ i = 1, 2, \ldots, n+1$. Since

$$[k_{n+1}, \mathbb{Q}] = [k_{n+1} : k_n] \cdots [k_2 : k_1][k_1 : \mathbb{Q}],\ [k_i, k_{i-1}] \le m_i - 1,$$

the condition (8.6) implies that $[k_i, k_{i-1}] = m_i - 1$ and so m_i is a prime number. If all m_i's are prime the condition (8.6) is trivially true.

For the proof of the second statement of the theorem, we have to prove that the entries of $\mathsf{pm}(\omega_\beta)$ form a $\mathbb{Q}$-basis of $\mathbb{Q}(\zeta_{m_1}, \zeta_{m_2}, \ldots, \zeta_{m_{n+1}})$. This statement can be easily proved by induction on n (since m_i's are prime, we can assume that $\beta = (0, 0, \ldots, 0)$). $\square$

Definition 8.3. In the freely generated $\mathbb{Z}$-module $\mathbb{Z}[\Delta_m]$ we consider the bilinear form $\langle \cdot, \cdot \rangle$ which satisfies

$$\langle \delta_\beta, \delta_{\beta'} \rangle = (-1)^n \langle \delta_{\beta'}, \delta_\beta \rangle, \beta, \beta \in I_m,$$

$$\langle \delta_{(\beta_1,\beta_2,\ldots,\beta_{n+1})}, \delta_{(\beta'_1,\beta'_2,\ldots,\beta'_{n+1})} \rangle = (-1)^{\frac{n(n+1)}{2}} (-1)^{\Sigma_{k=1}^{n+1} \beta'_k - \beta_k}$$

for $\beta_k \leq \beta'_k \leq \beta_k + 1,\ k = 1, 2, \ldots, n+1,\ \beta \neq \beta'$, and

$$\langle \delta_\beta, \delta_\beta \rangle = (-1)^{\frac{n(n-1)}{2}} (1 + (-1)^n),\ \beta \in I_m.$$

In the remaining cases, except those arising from the previous ones by a permutation, we have $\langle \delta_\beta, \delta_{\beta'} \rangle = 0$.

The above bilinear map corresponds to the intersection map of $H_n(\{g = 1\}, \mathbb{Z})$, see Example 5.8.

Definition 8.4. An element $\delta \in \mathbb{Z}[\Delta_m]$ is called a cycle at infinity if δ is an ω_β-cycle for all $\beta \in I_m$ with $A_\beta \notin \mathbb{N}$, i.e.

$$\int_\delta \omega_\beta = 0,\ \forall (\beta \in I_m,\ A_\beta \notin \mathbb{N}).$$

Using the geometric interpretation of cycles at infinity, one can see that:

Proposition 8.5. *An element $\delta \in \mathbb{Z}[\Delta_m]$ is a cycle at infinity if and only if*

$$\langle \delta, \delta_\beta \rangle = 0,\ \forall \beta \in I_m.$$

Definition 8.5. To each vanishing cycle $\delta \in \Delta_m$ we associate the monodromy map

$$h_\delta : \mathbb{Z}[\Delta_m] \to \mathbb{Z}[\Delta_m],\ h_\delta(a) = a + (-1)^{\frac{(n+1)(n+2)}{2}} \langle a, \delta \rangle.$$

and call it the Picard-Lefschetz monodromy map. The full mondromy group M is the subgroup of the group of $\mathbb{Z}$-linear isomorphisms of $\mathbb{Z}[\Delta_m]$ generated by all $h_\delta,\ \delta \in \Delta_m$. We enlarge the class of vanishing cycles in the following way. A cycle $\delta' \in \mathbb{Z}[\Delta_m]$ is called a vanishing cycle if there is an element $h \in M$ and $\delta \in \Delta_m$ such that $h(\delta) = \pm\delta'$.

Using the geometric interpretation of vanishing cycles and Theorem 5.3 we have:

Proposition 8.6. *For any two vanishing cycle δ_1, δ_2 there is a monodromy $h \in M$ such that $h(\delta_1) = \pm\delta_2$.*

For a decomposition $\{1, 2, \ldots, n+1\} = A \cup B,\ A \cup B = \emptyset$, we have a canonical map

$$\Delta_{m_A} \times \Delta_{m_B} \to \Delta_m,\ (\delta_1, \delta_2) \mapsto \delta_1 * \delta_2$$

$$m_A := (m_i)_{i \in A},\ m_B := (m_i)_{i \in B}$$

which is obtained by shuffling $\delta_1 \in \Delta_{m_A}$ and $\delta_2 \in \Delta_{m_B}$ according to the mentioned decomposition. By $\mathbb{Z}$-linearity it extends to

$$\mathbb{Z}[\Delta_A] \times \mathbb{Z}[\Delta_B] \to \mathbb{Z}[\Delta_m].$$

Definition 8.6. A cycle $\delta \in \mathbb{Z}[\Delta_m]$ is called a joint cycle if it has the following property: There exists a decomposition $\{1, 2, \ldots, m\} = A \cup B$ into disjoint non empty sets such that $\delta = \delta_1 * \delta_2,\ \delta_1 \in \mathbb{Z}[\Delta_{m_A}],\ \delta_2 \in \mathbb{Z}[\Delta_{m_B}]$.

By the definition, if $\delta_1 \in \mathbb{Z}[\Delta_{m_A}]$ is a ω_{β_1} cycle then for all $\beta_2 \in I_B$ and $\delta_2 \in \mathbb{Z}[\Omega_B]$, $\delta_1 * \delta_2$ is a (β_1, β_2)-cycle.

Let m and m' be $(n+1)$-tuple as before and assume that $m'_i \mid m_i,\ i = 1, 2, \ldots, n+1$. We have a $\mathbb{Z}$-linear map

$$a^* : \mathbb{Z}[\Omega_{m'}] \to \mathbb{Z}[\Omega_m]$$

which is induced by

$$(\beta'_1, \beta'_2, \ldots, \beta'_{n+1}) \mapsto \frac{m_1}{m'_1}\frac{m_2}{m'_2} \cdots \frac{m_{n+1}}{m'_{n+1}}((\beta'_1+1)\frac{m_1}{m'_1}-1, (\beta'_2+1)\frac{m_2}{m'_2}-1, \ldots, (\beta'_{n+1}+1)\frac{m_{n+1}}{m'_{n+1}}-1)$$

We have also the map

$$a_* : \mathbb{Z}[\Delta_m] \to \mathbb{Z}[\Delta_{m'}]$$

$$\delta_{(\beta_1, \beta_2, \ldots, \beta_{n+1})} \mapsto \delta_{(\beta_1 \bmod m'_1, \beta_2 \bmod m'_2, \ldots, \beta_{n+1} \bmod m'_{n+1})}$$

where we have used the rules (8.5). Again, using the geometric interpretation:

Proposition 8.7. *We have*

$$\int_{a_*\delta} \omega = \int_{\delta} a^*\omega,\ \ \delta \in \mathbb{Z}[\Delta_m],\ \omega \in \mathbb{Z}[\Omega_m].$$

Chapter 9

Examples of modular foliations

Our objective in the present Chapter is to analyze examples of modular foliations associated to the Gauss-Manin connection of tame polynomials discussed in Chapter 3. The tame polynomials in this Chapter are defined over a functional ring as it is described in Chapter 6.

9.1 Weierstrass family of elliptic curves

The historical examples of Darboux-Halphen and Ramanujan equations are already discussed in Chapter 2. For the tame polynomial f_{R} (resp. f_{H}) in

$$\mathsf{R}[x,y],\ \deg(x)=2,\ \deg(y)=3,\ \mathsf{R}=\mathbb{Q}[t_1,t_2,t_3],$$

the canonical basis of H'' is given by the entries of $\omega=(dx\wedge dy, xdx\wedge dy)^{\mathsf{t}}$. Note that, modulo relatively exact forms we have $\frac{x^j dx\wedge dy}{df}=-\frac{1}{2}\frac{x^j dx}{y}$ and so by Proposition 2.4 the matrix A_{R} (resp. A_{H}) in Proposition 2.2 (resp. Proposition 2.1) is the connection matrix of the tame polynomials f_{R} (resp. f_{H}) in the basis ω.

9.2 Halphen equations arising from tame polynomials

In this section we are going to consider the Halphen differential equations which are modular associated to tame polynomials. Such tame polynomials have many automorphisms and hence the codimension of certain modular foliations corresponding to them is strictly less than the Milnor number.

Let $\overline{\mathbb{Q}[t]}$ be the ring of integers of $\overline{\mathbb{Q}(t)}$. Set theoretically, $\overline{\mathbb{Q}[t]}$ contains all $a\in\overline{\mathbb{Q}(t)}$ which satisfy a monic polynomial with coefficients in $\mathbb{Q}[t]$. For instance, if $a\in\mathbb{Q}[t]$ and $k\in\mathbb{Q}$ then $a^k\in\overline{\mathbb{Q}[t]}$.

Proposition 9.1. *Let $q,i\in\mathbb{N}$ and assume that $2\leq q, \frac{i+1}{q}\notin\mathbb{N}$. The modular foliation associated to the differential form*

$$((t_1-t_2)(t_2-t_3)(t_3-t_1))^{\frac{1}{2}-\frac{i+1}{q}}y^i x dx\wedge dy$$

in the Brieskorn module H'' of the tame polynomial

$$f:=y^q-(x-t_1)(x-t_2)(x-t_3)\in\mathsf{R}[x,y],\ \mathsf{R}:=\overline{\mathbb{Q}[t_1,t_2,t_3]} \tag{9.1}$$

is given by the Halphen equation

$$\mathrm{H}(\alpha):\begin{cases}\dot t_1=(1-\alpha)(t_1t_2+t_1t_3-t_2t_3)+\alpha t_1^2\\ \dot t_2=(1-\alpha)(t_2t_1+t_2t_3-t_1t_3)+\alpha t_2^2\\ \dot t_3=(1-\alpha)(t_3t_1+t_3t_2-t_1t_2)+\alpha t_3^2\end{cases}\tag{9.2}$$

where

$$\alpha=\frac{q-2(i+1)}{q-3(i+1)}.$$

Proof. The proof of the above proposition is again a pure calculation which can be done along the lines of Chapter 3. A general proof in which α can be an arbitrary non-integer complex number is done in [59] (see Propositions 1.4 and 1.3 in Chapter 1). Note that

$$\frac{y^ixdx\wedge dy}{df}=-\frac{1}{q}\frac{xdx}{y^{q-1-i}}=-\frac{1}{q}\frac{xdx}{((x-t_1)(x-t_2)(x-t_3))^{\frac{q-1-i}{q}}}.$$

We need $\frac{q-1-i}{q}\notin\mathbb{Z}$ because in this case the linear integrals can be written as integrals over the cycles of the fibers of f. Note also that $1-\alpha=\frac{a-1}{3a-2}$ and $a=1-\frac{i+1}{q}$. □

In order to use $\mathbb{Q}[t_1,t_2,t_3]$ in Proposition 9.1 instead of $\overline{\mathbb{Q}[t_1,t_2,t_3]}$ we may reformulate it in the following way: the modular foliation associated to the differential form $y^ixdx\wedge dy$ in the Brieskorn module H'' of $f\in\mathsf{R}[x,y],\ \mathsf{R}=\mathbb{Q}[t_1,t_2,t_3]$ is induced by

$$\begin{cases}\dot t_1=(a-1)t_2t_3+at_1t_3+at_1t_2\\ \dot t_2=at_2t_3+(a-1)t_1t_3+at_1t_2\\ \dot t_3=at_2t_3+at_1t_3+(a-1)t_1t_2\end{cases}$$

for $a\neq\frac{2}{3}$(see Remark 1.5).

By Proposition 9.1 we see that for the tame polynomials with many automorphisms some elements of the corresponding Brieskorn module have codimension strictly less than the Milnor number of the tame polynomial. The Milnor number of the polynomial f in Proposition 9.1 is $2(q-1)$ and if 3 does not divide q then we need one point to compactify $\{f_t=0\}$, $\Delta_f(t)\neq 0$. Therefore, a priori a modular foliation $\mathcal{F}_\omega,\ \omega\in\mathsf{H}$ associated to f with $q\geq 3$ is trivial, i.e. is of codimension 3. However, in Proposition 9.1 we have non-trivial modular foliations and the reason is as follows: We have the action of

$$G_q:=\{\zeta_q^i\mid i=0,1,2,\dots,q-1\},\ \zeta_q:=e^{\frac{2\pi i}{q}}$$

on $\{f=0\}$ given by

$$a,(x,y)\mapsto(x,ay),\ (x,y)\in\mathbb{C}^2,\ a\in G_q.$$

It induces an action of G_q on $V:=H_1(\{f=0\},\mathbb{Q})$ (resp. H) and so V turns out to be a $\mathbb{Q}(\zeta_q)$-vector space. It can be proved that V is a $\mathbb{Q}(\zeta_q)$-vector space of dimension two (see for instance [76]). For $a\in G_q$ and $\omega=[xy^idx\wedge dy]\in\mathsf{H}''$ we have $a^*\omega=a^{i+1}\omega$ and

$$\int_{a_*\delta}\omega=\int_\delta a^*\omega=a^{i+1}\int_\delta\omega,$$

where a_* (resp. a^*) is the action of a in V(resp. H). Therefore, the $\bar{\mathbb{Q}}$-vector space generated by $\int_\delta\omega,\ \delta\in H_1(\{f=0\},\mathbb{Z})$ is of dimension at most 2 (by Proposition 9.1 its dimension is exactly 2) and hence the modular foliation associated to ω is of codimension at most 2.

In Proposition 9.1 we have excluded the case $q \mid i+1$. In fact, in this case the differential form $xy^i dx \wedge dy$ is zero in the Brieskorn module H'' of f and hence the corresponding modular foliation is given by $\frac{\partial}{\partial t_1}$, $\frac{\partial}{\partial t_2}$ and $\frac{\partial}{\partial t_3}$.

The modular foliations are not usually given by simple expressions. For instance, we have calculated the following modular foliations.

$$\mathcal{F}_{\frac{x^2 dx\wedge dy}{d(y^2-(x-t_1)(x-t_2)(x-t_3))}}:$$

$$\begin{cases} \dot t_1 = t_1^2t_2^2 - 2t_1^2t_2t_3 + t_1^2t_3^2 + 2t_1t_2^2t_3 + 2t_1t_2t_3^2 - 3t_2^2t_3^2 \\ \dot t_2 = t_1^2t_2^2 + 2t_1^2t_2t_3 - 3t_1^2t_3^2 - 2t_1t_2^2t_3 + 2t_1t_2t_3^2 + t_2^2t_3^2 \\ \dot t_3 = -3t_1^2t_2^2 + 2t_1^2t_2t_3 + t_1^2t_3^2 + 2t_1t_2^2t_3 - 2t_1t_2t_3^2 + t_2^2t_3^2 \end{cases}$$

$$\mathcal{F}_{\frac{x^3 dx\wedge dy}{d(y^2-(x-t_1)(x-t_2)(x-t_3))}}:$$

$$\begin{cases} \dot t_1 = t_1^3t_2^3 - t_1^3t_2^2t_3 - t_1^3t_2t_3^2 + t_1^3t_3^3 + t_1^2t_2^3t_3 - 2t_1^2t_2^2t_3^2 + t_1^2t_2t_3^3 + 3t_1t_2^3t_3^2 + 3t_1t_2^2t_3^3 - 5t_2^3t_3^3 \\ \dot t_2 = t_1^3t_2^3 + t_1^3t_2^2t_3 + 3t_1^3t_2t_3^2 - 5t_1^3t_3^3 - t_1^2t_2^3t_3 - 2t_1^2t_2^2t_3^2 + 3t_1^2t_2t_3^3 - t_1t_2^3t_3^2 + t_1t_2^2t_3^3 + t_2^3t_3^3 \\ \dot t_3 = -5t_1^3t_2^3 + 3t_1^3t_2^2t_3 + t_1^3t_2t_3^2 + t_1^3t_3^3 + 3t_1^2t_2^3t_3 - 2t_1^2t_2^2t_3^2 - t_1^2t_2t_3^3 + t_1t_2^3t_3^2 - t_1t_2^2t_3^3 + t_2^3t_3^3 \end{cases}$$

9.3 Modular foliations associated to zero dimensional integrals

In this section we consider the case $n = 0$. The polynomial

$$f = x^d + t_{d-1}x^{d-1} + \cdots + t_1x + t_0$$

is tame in $\mathsf{R}[x]$, $\deg(x) = 1$ with $\mathsf{R} = \mathbb{Q}[t_0, t_1, \ldots, t_{d-1}]$. By definition of a zero dimensional integral in Chapter 6, it is easy to see that a leaf of the foliation $\mathcal{F}_x, x \in \mathsf{H}'$ is given by the coefficients of x^i's in $(x+s)^d + a_{d-2}(x+s)^{d-2} + \cdots + a_1(x+s) + a_0$, where a_i's are some constant complex numbers and s is a parameter. In fact, $\mathcal{F}_x$ is given by the solutions of the vector field:

$$t_1\frac{\partial}{\partial t_0} + 2t_2\frac{\partial}{\partial t_1} + 3t_3\frac{\partial}{\partial t_2} + \cdots + (d-1)t_{d-1}\frac{\partial}{\partial t_{d-2}} + d\frac{\partial}{\partial t_{d-1}}.$$

Example 9.1. Consider the degree three polynomial

$$f = x^3 + t_2x^2 + t_1x + t_0.$$

The Gauss-Manin connection in the basis $\omega = (x, x^2)^{\mathsf{t}}$ of H' is given by

$$\nabla\omega = \frac{1}{\Delta}(\sum_{i=0}^{2} A_i dt_i) \otimes \omega,$$

where $\Delta = 27t_0^2 - 18t_0t_1t_2 + 4t_0t_2^3 + 4t_1^3 - t_1^2t_2^2$ and

$$A_0 = \begin{pmatrix} 18t_0 - 2t_1t_2 & 6t_0t_2 + 4t_1^2 - 2t_1t_2^2 \\ -6t_1 + 2t_2^2 & 9t_0 - 7t_1t_2 + 2t_2^3 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} -12t_0t_2 + 4t_1^2 & -6t_0t_1 - 4t_0t_2^2 + 2t_1^2t_2 \\ 9t_0 - t_1t_2 & 3t_0t_2 + 2t_1^2 - t_1t_2^2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -6t_0t_1 + 8t_0t_2^2 - 2t_1^2t_2 & -18t_0^2 + 14t_0t_1t_2 - 4t_1^3 \\ -6t_0t_2 + 2t_1^2 & -3t_0t_1 - 2t_0t_2^2 + t_1^2t_2 \end{pmatrix}.$$

We have the modular foliation:

$$\mathcal{F}_{x^2} : (-t_1)\frac{\partial}{\partial t_2} + (-t_2t_1 + 3t_0)\frac{\partial}{\partial t_1} + (2t_2t_0 - t_1^2)\frac{\partial}{\partial t_0} \tag{9.3}$$

for which $\mathrm{Sing}(\mathcal{F}_{x^2}) = \{t \in \mathbb{C}^3 \mid t_1 = t_0 = 0\}$ and the locus of parameters in which $\int_\delta x^2 = 0$, for some vanishing cycle $\delta = [x_i]-[x_j] \in H_0(\{f = 0\}, \mathbb{Z})$, is given by $t_0 - t_2t_1 = 0$ which is $\mathcal{F}_{x^2}$-invariant.

Example 9.2. *For $x^4 + t_3x^3 + t_2x^2 + t_1x + t_0$ we have*

$$\mathcal{F}_{x^2} : (-2t_0t_2 + t_1^2)\frac{\partial}{\partial t_0} + (-3t_0t_3 + t_1t_2)\frac{\partial}{\partial t_1} + (-4t_0 + t_1t_3)\frac{\partial}{\partial t_2} + t_1\frac{\partial}{\partial t_3}. \tag{9.4}$$

The foliation $\mathcal{F}_{x^3}$ is given by

$$(-3t_0^2t_3 + 3t_0t_1t_2 - t_1^3)\frac{\partial}{\partial t_0} + (-4t_0^2 + t_0t_1t_3 + 2t_0t_2^2 - t_1^2t_2)\frac{\partial}{\partial t_1} + \tag{9.5}$$

$$(t_0t_1 + 2t_0t_2t_3 - t_1^2t_3)\frac{\partial}{\partial t_2} + (2t_0t_2 - t_1^2)\frac{\partial}{\partial t_3}.$$

Example 9.3. We can take an arbitrary polynomial in some function field and define a modular foliation. For example, let $\mathsf{R} = \mathbb{Q}[s_1, s_2, \cdots, s_{d-1}, t]$, $f = g - t, g \in \mathbb{Q}[x]$, $\deg(g) = d$ and $\omega = s_1x + s_2x_2 + \cdots + s_{d-1}x^{d-1}$. The foliation $\mathcal{F}_\omega$ is given by the vector field:

$$p_1\frac{\partial}{\partial s_1} + p_2\frac{\partial}{\partial s_2} + \cdots + p_{d-1}\frac{\partial}{\partial s_{d-1}} - \frac{\partial}{\partial t},$$

where $\nabla_{\frac{\partial}{\partial t}}\omega = p_1x + p_2x^2 + \cdots + p_{d-1}x^{d-1}$.

Since zero-dimensional integrals are algebraic functions we conclude that the leaves of any modular foliation associated to a tame polynomial in one variable are algebraic and in particular:

Proposition 9.2. *The leaves of (9.3), (9.4) and (9.5) are algebraic.*

9.4 A family of elliptic curves with two marked points

Let us consider the tame polynomial

$$f = y^2 - (x - t_1)(x - t_2)(x - t_3)(x - t_4),$$

in $\mathsf{R}[x, y]$, $\deg(x) = 1$, $\deg(y) = 2$, $\mathsf{R} = \mathbb{Q}[t_1, t_2, t_3, t_4]$. The affine curve $\{f_t = 0\}$, $\Delta_f(t) \neq 0$ is an elliptic curve E_t mines two points (at infinity). Therefore, a differential form in H may have a non-trivial residue at infinity.

Proposition 9.3. *Both foliations $\mathcal{F}_{\frac{x^j dx}{y}}$, $j = 0, 1$ are of codimension two and the leaves of $\mathcal{F}_{\frac{dx}{y}}$ are algebraic.*

Proof. Since $\frac{dx}{y}$ restricted to E_t is a differential form of the first kind and the residue of $\frac{xdx}{y}$ at infinity is constant (§6.10), $\nabla\frac{x^j dx}{y},\ j=0,1$ can be written in terms of two differential forms and so the modular foliation $\mathcal{F}_{\frac{x^j dx}{y}}$ is of codimension at most two. To confirm that such foliations are exactly of codimension 2 we have used the algorithms of Chapter 3 and we have calculated explicit expressions for $\mathcal{F}_{\frac{x^j dx}{y}}$ (see [55]). □

From modular forms theory point of view the modular foliation $\mathcal{F}_{\frac{x^2 dx}{y}}$ can be of interest. It is given by the vector field.

$$V_d: \tag{9.6}$$

$$(t_1t_2\cdots t_d)^2\sum_{i=1}^{d}\left(-3\frac{1}{t_i^2}+\sum_{1\leq j\leq d,\ j\neq i}\frac{1}{t_j^2}+2\frac{1}{t_i}(\sum_{1\leq j\leq d,\ j\neq i}\frac{1}{t_j})-2\sum_{1\leq j_1\neq j_2\leq d,\ j_1,j_2\neq i}\frac{1}{t_{j_1}t_{j_2}}\right)\frac{\partial}{\partial t_i}$$

for $d=4$. Later we will encounter again this vector field for $d=5$. Note that the vector field V_d is polynomial and for simplicity we have written in the above form.

We know that $\mathcal{F}_{\frac{x^2dx}{y}}$ has a non trivial polynomial first integral $\int_\delta \frac{x^2dx}{y}$ (see §6.10), where δ is a cycle at infinity. In fact it is easy to see that $d(t_1+t_2+t_3+t_4)(V_4)=0$ and so $t_1+t_2+t_3+t_4$ is a first integral of $\mathcal{F}_{\frac{x^2dx}{y}}$. Since the first integral is linear, we may discard one of the parameters. This is best seen using the tame polynomial

$$f_{\mathrm{R}}=y^2-x^4+t_1x^2+t_2x+t_3$$

A priori we expect that $\mathcal{F}_{\frac{x^2dx}{y}}$ for f to be trivial but in fact it is given by:

$$(8t_2t_3)\frac{\partial}{\partial t_1}+(8t_1^2t_3-2t_1t_2^2+32t_3^2)\frac{\partial}{\partial t_3}+(12t_1t_2t_3-3t_2^3)\frac{\partial}{\partial t_3}$$

9.5 The case of Hodge numbers $0, h, 0$

In section 9.3 we saw that all the leaves of modular foliations associated to zero dimensional integrals are algebraic. In this section we discuss other examples of modular foliations with only algebraic leaves. The Hodge numbers of the corresponding tame polynomial are of the form $\cdots,0,h,0,\cdots$ and we strongly use Conjecture 7.3 which is known for the $n=2$ case. We discuss one example and leave the description of other examples to the reader.

Let us consider the tame polynomial

$$f:=(x-t_1)(x-t_2)(x-t_3)-(y^2+z^5)$$

in $\mathsf{R}[x,y,z],\ \deg(x)=\deg(y)=15,\ \deg(z)=6$ with $\mathsf{R}=\mathbb{Q}[t_1,t_2,t_3]$. The Hodge numbers of a compactified regular fiber of f is $0,8,0$.

Proposition 9.4. *We have*

1. *For any $\omega\in\mathsf{H}$ all the leaves of the modular foliation $\mathcal{F}_\omega$ are algebraic.*
2. *For $\omega_\beta := xy^{\beta_2}z^{\beta_3}dx\wedge dy\wedge dz$ the modular foliations $\mathcal{F}_{\omega_\beta}$ is the Halphen equation*

$$\mathrm{H}(\alpha),\ \alpha:=\frac{1-2(\frac{\beta_2+1}{2}+\frac{\beta_3+1}{5})}{1-3(\frac{\beta_2+1}{2}+\frac{\beta_3+1}{5})}$$

and so all the leaves of $\mathrm{H}(\alpha)$ *are algebraic.*

Proof. Since the Hodge numbers of f are $0,8,0$, all the cycles in the fibers of f are Lefschetz cycles and so by Conjecture 7.3, which is true for $n=2$, the functions $\frac{1}{(2\pi i)^2}\int_\delta \omega$ are algebraic. Using the interpretation of modular foliation as the constant locus of integrals, Proposition 6.3, the modular foliation $\mathcal{F}_\omega$ has only algebraic leaves.

To prove the second part, we use Proposition 6.7 and we have

$$\int_{\delta_1*\delta_2} \frac{xy^{\beta_2}z^{\beta_3}dx\wedge dy\wedge dz}{df} = B(\frac{1}{2},\frac{1}{5})\frac{\mathsf{p}((2,7),(\beta_2,\beta_3),\delta_2)}{\mathsf{p}(12,\beta_3,\delta_3)}\int_{\delta_1*\delta_3}\frac{y_3^{\beta}xdx\wedge dy}{d(\tilde{f})},$$

where

$$\frac{\beta_3+1}{12} = \frac{\beta_2+1}{2}+\frac{\beta_3+1}{5},\ \ \delta_3 := [\zeta_{12}^2]-[\zeta_{12}]$$

Here $\mathsf{p}((m_1,m_2,\dots,m_{n+1}),(\beta_2,\beta_3),\delta)$ is $\mathsf{p}((\beta_2,\beta_3),\delta)$ associated to $x_1^{m_1}+x_2^{m_2}+\cdots+x_{n+1}^{m_{n+1}}-1$ and defined in §6.6. The assertion follows from Proposition 9.1. □

9.6 Modular foliations associated to genus two curves

We consider the following family of hyperelliptic genus two curves

$$f := y^2-x^5+t_1x^4+t_2x^3+t_3x^2+t_4x+t_5=0,\ t\in T:=\mathbb{C}^5\backslash\{\Delta=0\},$$

where Δ is the discriminant of f. We calculate the Gauss-Manin connection matrix in the basis

$$\omega = (dx\wedge dy, xdx\wedge dy, x^2dx\wedge dy, x^3dx\wedge dy)^{\mathsf{t}}$$

using the algorithms developed in Chapter 3 (note that $\frac{x^idx}{y} = -2\frac{x^idx\wedge dy}{df}$). But the ingredient polynomials have so big size that they do not fit to a mathematical paper. However, the vector field X_i tangent to the foliation $\mathcal{F}_{\frac{x^{i-1}dx}{y}}, i=1,2,3,4$ has not a huge size: For $\mathcal{F}_{\frac{dx}{y}}$ we have:

$$X_1 = -5\frac{\partial}{\partial t_1}+4t_1\frac{\partial}{\partial t_2}+3t_2\frac{\partial}{\partial t_3}+2t_3\frac{\partial}{\partial t_4}+t_4\frac{\partial}{\partial t_5}.$$

The solution of X_1 passing through $a\in\mathbb{C}^5$ is given by the coefficients of

$$y^2-(x+z)^5+a_1(x+z)^4+a_2(x+z)^3+a_3(x+z)^2+a_4(x+z)+a_5$$

and so all solutions of X_1 are algebraic. This is natural because $\frac{dx}{y}$ is invariant under $(x,y)\mapsto(x+b,y),\ b\in\mathbb{C}$. We have also

$$\mathcal{F}_{\frac{xdx}{y}}:$$

$$-3t_4\frac{\partial}{\partial t_1}+(2t_1t_4-10t_5)\frac{\partial}{\partial t_2}+(8t_1t_5+t_2t_4)\frac{\partial}{\partial t_3}+6t_2t_5\frac{\partial}{\partial t_4}+(4t_3t_5-t_4^2)\frac{\partial}{\partial t_5}$$

$$\mathcal{F}_{\frac{x^2dx}{y}}:$$

$$(-4t_3t_5+t_4^2)\frac{\partial}{\partial t_1}+(-12t_4t_5)\frac{\partial}{\partial t_2}+(8t_1t_4t_5-4t_2t_3t_5+t_2t_4^2-40t_5^2)\frac{\partial}{\partial t_3}+$$
$$(32t_1t_5^2+4t_2t_4t_5-8t_3^2t_5+2t_3t_4^2)\frac{\partial}{\partial t_4}+(24t_2t_5^2-12t_3t_4t_5+3t_4^3)\frac{\partial}{\partial t_5}$$

$$\mathcal{F}_{\frac{x^3dx}{y}}:$$

$$(8t_2t_5^2-4t_3t_4t_5+t_4^3)\frac{\partial}{\partial t_1}+(-16t_1t_2t_5^2+8t_1t_3t_4t_5-2t_1t_4^3-8t_3t_5^2+2t_4^2t_5)\frac{\partial}{\partial t_2}+$$
$$(-24t_2^2t_5^2+12t_2t_3t_4t_5-3t_2t_4^3-24t_4t_5^2)\frac{\partial}{\partial t_3}+$$
$$(16t_1t_4t_5^2-40t_2t_3t_5^2+2t_2t_4^2t_5+16t_3^2t_4t_5-4t_3t_4^3-80t_5^3)\frac{\partial}{\partial t_4}+$$
$$(64t_1t_5^3-32t_2t_4t_5^2-16t_3^2t_5^2+24t_3t_4^2t_5-5t_4^4)\frac{\partial}{\partial t_5}.$$

We may recalculate all above for the family $y^2-(x-t_1)(x-t_2)(x-t_3)(x-t_4)(x-t_5)$. The foliation $\mathcal{F}_{\frac{dx}{y}}$ is given by the vector field $\sum_{i=1}^5\frac{\partial}{\partial t_i}$. We have also

$$\mathcal{F}_{\frac{xdx}{y}}:t_1t_2t_3t_4t_5\sum_{i=1}^5(\frac{-2}{t_i}+\sum_{j=1}^5\frac{1}{t_j})\frac{\partial}{\partial t_i}$$

$$\mathcal{F}_{\frac{x^2dx}{y}}:V_5 \tag{9.7}$$

where V_5 is the vector field (9.6). The ingredient polynomials of the modular foliation $\mathcal{F}_{\frac{x^3dx}{y}}$ are huge:

$$(t_1^3t_2^3t_3^3t_4^3-t_1^3t_2^3t_3^3t_4^2t_5-t_1^3t_2^3t_3^3t_4t_5^2+t_1^3t_2^3t_3^3t_5^3-t_1^3t_2^3t_3^2t_4^3t_5+2t_1^3t_2^3t_3^2t_4^2t_5^2-t_1^3t_2^3t_3^2t_4t_5^3-$$
$$t_1^3t_2^3t_3t_4^3t_5^2-t_1^3t_2^3t_3t_4^2t_5^3+t_1^3t_2^3t_4^3t_5^3-t_1^3t_2^2t_3^3t_4^3t_5+2t_1^3t_2^2t_3^3t_4^2t_5^2-t_1^3t_2^2t_3^3t_4t_5^3+2t_1^3t_2^2t_3^2t_4^3t_5^2+$$
$$2t_1^3t_2^2t_3^2t_4^2t_5^3-t_1^3t_2^2t_3t_4^3t_5^3-t_1^3t_2t_3^3t_4^3t_5^2-t_1^3t_2t_3^3t_4^2t_5^3-t_1^3t_2t_3^2t_4^3t_5^3+t_1^3t_3^3t_4^3t_5^3+t_1^2t_2^3t_3^3t_4^3t_5-$$
$$2t_1^2t_2^3t_3^3t_4^2t_5^2+t_1^2t_2^3t_3^3t_4t_5^3-2t_1^2t_2^3t_3^2t_4^3t_5^2-2t_1^2t_2^3t_3^2t_4^2t_5^3+t_1^2t_2^3t_3t_4^3t_5^3-2t_1^2t_2^2t_3^3t_4^3t_5^2-2t_1^2t_2^2t_3^3t_4^2t_5^3-$$
$$2t_1^2t_2^2t_3^2t_4^3t_5^3+t_1^2t_2t_3^3t_4^3t_5^3+3t_1t_2^3t_3^3t_4^3t_5^2+3t_1t_2^3t_3^3t_4^2t_5^3+3t_1t_2^3t_3^2t_4^3t_5^3+3t_1t_2^2t_3^3t_4^3t_5^3-5t_2^3t_3^3t_4^3t_5^3)\frac{\partial}{\partial t_1}+\cdots$$

Note that the above vector field is symmetric in t_i's and so the coefficient of $\frac{\partial}{\partial t_i}$ is obtained by changing the role of t_1 with t_i in the coefficient of $\frac{\partial}{\partial t_1}$.

Chapter 10

Moduli of Polarized Hodge Structures

In this chapter we construct an analytic variety U and an action of an algebraic group G_0 on U from the right such that U/G_0 is the moduli space of polarized Hodge structures of a fixed type. The space U lives over the so called Griffiths domain (see [27]) and has the advantage that it carries certain modular foliations such that their pull-back by period maps are the geometric modular foliations constructed in Chapter 4. Our hope is that U has a canonical structure of an algebraic variety such that the action of G_0 is algebraic and the corresponding modular foliations are of geometric origin. If this is the case then one may look to the action of G_0 on U from the point of view of geometric invariant theory, see [64]. Since we know partial compactifications of U/G_0 (see [38]) in the analytic context, the algebraic version would be also of interest.

The objective of this chapter in the case of Hodge structures of type $h^{10} = h^{01} = 1$ is already realized in [57, 58]. In this case $U = \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P}$, where

$$\mathcal{P} := \{\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \mid \mathrm{Im}(x_1\overline{x_3}) > 0\}.$$

The algebraic group

$$G_0 = \{\begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \mid \ k' \in \mathbb{C},\ k \neq 0\}$$

acts from the right on U by the usual multiplication of matrices. The period map associated to f_{R} in §2.4 gives us a biholomorphism:

$$U \cong \{(t_1,t_2,t_3) \in \mathbb{C}^3 \mid 27t_3^2 - t_2^3 \neq 0\}.$$

Under the above biholomorphism the action of G_0 is given by

$$t \bullet g = (t_1k^{-2} + k'k^{-1}, t_2k^{-4}, t_3k_1^{-6}),$$

$$t = (t_1,t_2,t_3) \in \mathbb{C}^3,\ g = \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \in G_0.$$

The Ramanujan foliation $\mathcal{F}(\mathrm{R})$ is mapped by the period map to the foliation x_2=const. x_4 =const. in U. The complex manifold $\mathcal{P}$ contains copies of $\mathbb{C}^*$ and so it is not a Hermitian symmetric domain (see [35],[48]). This rules out the direct use of Baily-Borel theorem (see [3]) on $\mathcal{P}$.

10.1 The space of polarized lattices

We fix a $\mathbb{C}$-vector space V_0 of dimension h, a natural number $m \in \mathbb{N}$ and a $h \times h$ integer valued matrix Ψ_0 such that the associated bilinear form

$$\mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z},\ (a, b) \to a\Psi_0 b^t$$

is non-degenerate, symmetric if m is even and skew if m is odd. Note that in the case of $\mathbb{Z}$-modules by non-degenerate we mean that the associated morphism

$$\mathbb{Z}^h \to (\mathbb{Z}^h)^\vee,\ a \to (b \to a^t \Psi_0 b)$$

is a an isomorphism, where $\vee$ means the dual of a $\mathbb{Z}$-module.

A lattice $V_\mathbb{Z}$ in V_0 is a $\mathbb{Z}$-module generated by a basis of V_0. A polarized lattice $(V_\mathbb{Z}, \psi_\mathbb{Z})$ of type Ψ_0 is a lattice $V_\mathbb{Z}$ together with a bilinear map $\psi_\mathbb{Z} : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{Z}$ such that in a $\mathbb{Z}$-basis of $V_\mathbb{Z}$, $\psi_\mathbb{Z}$ has the form Ψ_0.

Let $\mathcal{L}$ be the space of polarized lattices of type Ψ_0 in V_0. Usually, we denote an element of $\mathcal{L}$ by $x, y, \ldots$ and the associated lattice (resp. bilinear form) by $V_\mathbb{Z}(x), V_\mathbb{Z}(y), \ldots$ (resp. $\psi_\mathbb{Z}(x), \psi_\mathbb{Z}(y), \ldots$). Let R be any subring of $\mathbb{C}$. For instance, R can be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}[\sqrt{2}]$ and etc.. We define

$$V_R(x) := V_\mathbb{Z}(x) \otimes_\mathbb{Z} R \text{ and } \psi_R(x) : V_R(x) \times V_R(x) \to R \text{ the induced map.}$$

Let

$$I := \{1, 2, \ldots, h\}.$$

Conjugation with respect to $x \in \mathcal{L}$ of an element $a = \sum_{i \in I} a_i \delta_i \in V_0$, where $V_\mathbb{Z}(x) = \sum_{i \in I} \mathbb{Z}\delta_i$, is defined by

$$\overline{a}^x = \sum_{i \in I} \overline{a}_i \delta_i,$$

where $\overline{s},\ s \in \mathbb{C}$ is the usual conjugation of complex numbers.

10.2 Polarized lattices and automorphism groups

We fix $x_0 \in \mathcal{L}$ and define Γ_R to be a subgroup of $P := \mathrm{Aut}(V_0)$ containing all $p \in P$ such that p induces an element in $\mathrm{Aut}(V_R(x_0), \psi_R(x_0))$, i.e. it induces an R-linear map $V_R(x_0) \to V_R(x_0)$ with

$$\psi_R(x_0)(pa, pb) = \psi_R(x_0)(a, b),\ \forall a, b \in V_R.$$

We will mainly make use of $\Gamma_\mathbb{Z}$. We define the action of P on $\mathcal{L}$ from the right. For $p \in P$ and $x \in \mathcal{L}$, xp is defined by:

$$V_\mathbb{Z}(xp) := p^{-1}(V_\mathbb{Z}(x)),\ \psi_\mathbb{Z}(xp)(v_1, v_2) := \psi_\mathbb{Z}(pv_1, pv_2),\ \forall v_1, v_2 \in V_\mathbb{Z}(xp).$$

By definition we have

$$\psi_\mathbb{C}(xp)(v_1, v_2) = \psi_\mathbb{C}(pv_1, pv_2),\ \forall v_1, v_2 \in V_0.$$

Note that the group P acts on V_0 from the left in a natural way:

$$pv = p(v),\ p \in P,\ v \in V_0.$$

Proposition 10.1. *For all $v \in V_0$, $x \in \mathcal{L}$ and $p \in P$, we have*

$$\overline{v}^{xp} = p^{-1}\overline{pv}^{x}.$$

Proof. Take a $\mathbb{Z}$-basis $\delta_i, i \in I$ of $V_{\mathbb{Z}}(x)$. Then $p^{-1}(\delta_i),\ i \in I$ is a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}(xp)$ and we write

$$v = \sum_{i \in I} a_i p^{-1}(\delta_i) \text{ or equivalently } p(v) = \sum_{i \in I} a_i \delta_i,\ a_i \in \mathbb{C}.$$

Now

$$\overline{v}^{xp} = \sum_{i \in I} \overline{a_i} p^{-1}(\delta_i) = p^{-1}(\sum_{i \in I} \overline{a_i}\delta_i) = p^{-1}\overline{pv}^{x}.$$

□

Proposition 10.2. *We have:*

1. *$\Gamma_{\mathbb{Z}}$ is a discrete subgroup of P;*
2. *The canonical map*

   $$\alpha : \Gamma_{\mathbb{Z}} \backslash P \to \mathcal{L},\ \alpha(p) = x_0 p \tag{10.1}$$

   is well-defined and is an isomorphism;
3. *$\mathcal{L}$ has a canonical structure of a complex manifold.*

Proof. 1. The set $V_{\mathbb{Z}}(x_0)$ is a discrete subset of V_0. 2. We have

$$\Gamma_{\mathbb{Z}} = \{p \in P \mid x_0 p = x_0\}.$$

3. The action of $\Gamma_{\mathbb{Z}}$ on P has no fixed points and no accumulation points. □

From now on for $p \in P$ we define

$$V_{\mathbb{Z}}(p) := V_{\mathbb{Z}}(\alpha(p)), \psi_{\mathbb{Z}}(p) := \psi_{\mathbb{Z}}(\alpha(p)),\ \overline{v}^{p} = \overline{v}^{\alpha(p)}$$

and so on.

10.3 Poincaré dual

In this section we explain the notion of Poincaré dual in the context of current chapter. Let $(V_{\mathbb{Z}}(x), \psi_{\mathbb{Z}}(x))$ be a polarized lattice and $\delta \in V_{\mathbb{Z}}(x)^{\vee}$, where $\vee$ means the dual of a $\mathbb{Z}$-module. We will use the notation

$$\int_{\delta} \omega := \tilde{\delta}(\omega),\ \forall \omega \in V_0,$$

where $\tilde{\delta} \in V_0^{\vee}$ is the complexification of δ. The Poincaré dual $\delta^{\mathsf{pd}} \in V_{\mathbb{Z}}(x)$ is the unique element with the property:

$$\int_{\delta} \omega = \psi_{\mathbb{Z}}(x)(\delta^{\mathsf{pd}}, \omega),\ \forall \omega \in V_{\mathbb{Z}}(x).$$

It exists and is unique because $\psi_{\mathbb{Z}}$ is non-degenerate. Using the Poincaré duality one defines the dual polarization:

$$\psi_{\mathbb{Z}}(x)^{\vee}(\delta_i, \delta_j) := \psi_{\mathbb{Z}}(x)(\delta_i^{\mathsf{pd}}, \delta_j^{\mathsf{pd}}),\ \delta_i, \delta_j \in V_{\mathbb{Z}}(x)^{\vee}.$$

Proposition 10.3. *We have:*

$$(A^\vee \delta)^{\mathsf{pd}} = A^{-1}\delta^{\mathsf{pd}},\ \forall A \in \Gamma_{\mathbb{Z}},\ \delta \in V_{\mathbb{Z}}(x_0)^\vee,$$

where $A^\vee : V_{\mathbb{Z}}(x_0)^\vee \to V_{\mathbb{Z}}(x_0)^\vee$ *is the induced dual map.*

Proof. The proposition follows from:

$$\int_{A^\vee \delta} \omega = \int_\delta A\omega = \psi_{\mathbb{Z}}(x_0)(\delta^{\mathsf{pd}}, A\omega) = \psi_{\mathbb{Z}}(x_0)(A^{-1}\delta^{\mathsf{pd}}, \omega)$$

for all $\omega \in V_0$. □

We define

$$\Gamma_{\mathbb{Z}}^\vee := \mathrm{Aut}(V_{\mathbb{Z}}(x_0)^\vee, \psi_{\mathbb{Z}}(x_0)^\vee).$$

It follows from the above proposition that

$$\Gamma_{\mathbb{Z}} \to \Gamma_{\mathbb{Z}}^\vee,\ A \mapsto A^\vee$$

is an isomorphism of groups.

10.4 Period matrix

Sometimes it is convenient to have explicit coordinate functions on P. In this section we explain such functions.

Let $\omega = (\omega_i)_{i\in I}$ be a $\mathbb{C}$-basis of V_0. In this chapter a basis of V_0 is written as a $h \times 1$ matrix of elements of V_0. We take a $\mathbb{Z}$-basis δ_{x_0} of $V_{\mathbb{Z}}(x_0)^\vee$ such that the matrix of $\psi_{\mathbb{Z}}(x_0)$ in the basis δ_{x_0} is Ψ_0. For an arbitrary lattice $V_{\mathbb{Z}}(x)$ with $p \in P, \alpha(p) = x$, where α is the map in (10.1), we obtain a $\mathbb{Z}$-basis $\delta = \delta_x := p^\vee(\delta_{x_0})$, where $p^\vee : V_{\mathbb{Z}}(x_0)^\vee \to V_{\mathbb{Z}}(x)^\vee$ is the map induced in the dual spaces. We define the period matrix in the following way:

$$\mathsf{pm} = \mathsf{pm}(x) = [\int_\delta \omega^{\mathsf{t}}]_{h\times h} := \begin{pmatrix} \int_{\delta_1}\omega_1 & \int_{\delta_1}\omega_2 & \cdots & \int_{\delta_1}\omega_h \\ \int_{\delta_2}\omega_1 & \int_{\delta_2}\omega_2 & \cdots & \int_{\delta_2}\omega_h \\ \vdots & \vdots & \vdots & \vdots \\ \int_{\delta_h}\omega_1 & \int_{\delta_h}\omega_2 & \cdots & \int_{\delta_h}\omega_h \end{pmatrix}.$$

We identify P with the space of the above matrices, which is $\mathrm{GL}(h, \mathbb{C})$. We write an element A of $\Gamma_{\mathbb{Z}}$ in the basis δ_{x_0}, and redefine $\Gamma_{\mathbb{Z}}$:

$$\Gamma_{\mathbb{Z}} := \{A \in \mathrm{GL}(h, \mathbb{Z}) \mid A\Psi_0 A^{\mathsf{t}} = \Psi_0\}.$$

The action of $\Gamma_{\mathbb{Z}}$ (resp. P) on P from the left(resp. right) is the usual multiplication of matrices.

Instead of the period matrix it is useful to use the matrix

$$\mathsf{q} = \mathsf{q}(x), \quad \text{where } \delta^{\mathsf{pd}} = \mathsf{q}\omega.$$

Then we have:

$$(\delta^{\mathsf{pd}})^{\mathsf{t}} = \omega^{\mathsf{t}}\mathsf{q}^{\mathsf{t}} \Longrightarrow \Psi_0 = \mathsf{pm}\cdot\mathsf{q}^{\mathsf{t}}.$$

Again, the action of $\Gamma_{\mathbb{Z}}$ (resp. P) on P from the left(resp. right) is the usual multiplication of matrices. If we identify V_0 with $\mathbb{C}^h$ through the basis ω then q is a matrix whose rows are the entries of δ.

10.5 A canonical connection on $\mathcal{L}$

Recall the terminology related to connections in Chapter 1. We consider the trivial bundle $\mathcal{H} = \mathcal{L} \times V_0$ on $\mathcal{L}$. On $\mathcal{H}$ we have a well-defined connection

$$\nabla : \mathcal{H} \to \Omega^1_{\mathcal{L}} \times \mathcal{H}$$

such that a flat section s of ∇ in an small open set $U \subset \mathcal{L}$ satisfies

$$s(x) \in \{x\} \times V_{\mathbb{Z}}(x) \subset \{x\} \times V_0,\ x \in U.$$

Let $\omega = (\omega_1, \omega_2, \dots, \omega_h)$ be a basis of V_0. We can consider ω_i as a global section of $\mathcal{H}$ and so we have

$$\nabla\omega = A \otimes \omega,\ A = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1h} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{h1} & \omega_{h2} & \cdots & \omega_{hh} \end{pmatrix},\ \omega_{ij} \in H^0(\mathcal{L}, \Omega^1_{\mathcal{L}}). \tag{10.2}$$

A is called the connection matrix of ∇ in the basis ω. The connection ∇ is integrable and so $dA = A \wedge A$:

$$d\omega_{ij} = \sum_{k=1}^{h} \omega_{ik} \wedge \omega_{kj},\ i, j = 1, 2, \dots, h. \tag{10.3}$$

Let δ be a basis of flat sections. Write $\delta = \mathsf{q}\omega$. We have

$$\omega = \mathsf{q}^{-1}\delta \Rightarrow \nabla(\omega) = d(\mathsf{q}^{-1})\mathsf{q}\omega \Rightarrow$$

$$A = d\mathsf{q}^{-1} \cdot \mathsf{q} = d(\mathsf{pm}^{\mathsf{t}} \cdot \Psi_0^{-\mathsf{t}}) \cdot (\Psi_0^{\mathsf{t}} \cdot \mathsf{pm}^{-\mathsf{t}}) = d(\mathsf{pm}^{\mathsf{t}}) \cdot \mathsf{pm}^{-\mathsf{t}}.$$

We have used the equality $\Psi_0 = \mathsf{pm} \cdot \mathsf{q}^{\mathsf{t}}$. Note that the entries of A are holomorphic 1-forms on $\mathcal{L}$ and a fundamental system for the linear differential equation $dY = A \cdot Y$ in $\mathcal{L}$ is given by $Y = \mathsf{pm}^{\mathsf{t}}$.

10.6 Some functions on $\mathcal{L}$

First of all recall that if δ and ω be two bases of V_0, $\delta = p\omega$ for some $p \in \mathrm{GL}(h, \mathbb{C})$ and a linear form on V_0 in the basis δ (resp. ω) has the matrix form A (resp. B) then $pBp^t = A$.

For two vectors $\omega_1, \omega_2 \in V_0$ one can define the following holomorphic function on $\mathcal{L}$

$$f_{\omega_1,\omega_2}(x) = \psi_{\mathbb{C}}(x)(\omega_1, \omega_2). \tag{10.4}$$

To obtain all such possible holomorphic functions we first choose a basis ω of V_0 and for $x \in \mathcal{L}$ we write $\delta_x = \mathsf{q} \cdot \omega$. Then

$$\mathsf{pm}^t \Psi_0^{-\mathsf{t}} \mathsf{pm} = (\mathsf{q}^{-1})^t \Psi_0 \mathsf{q}^{-1} = [f_{\omega_i,\omega_j}]_{i,j \in I} \tag{10.5}$$

(we have used the identity $\Psi_0 = \mathsf{q} \cdot \mathsf{pm}^{\mathsf{t}}$). Other functions as in (10.4) are $\mathbb{C}$-linear combination of the entries of the above matrix. It is remarkable that the matrix $F = [f_{\omega_i,\omega_j}]_{i,j \in I}$ satisfies the differential equation:

$$dF = A \cdot F + F \cdot A^{\mathsf{t}}, \tag{10.6}$$

where A is the connection matrix. It is easy to check that every solution of the above differential equation is of the form $\mathsf{pm}^t \cdot C \cdot \mathsf{pm}$ for some constant $h \times h$ matrix C with entries in $\mathbb{C}$ (if F is a solution of (10.6) then $F \cdot \mathsf{pm}^{-1}$ is a solution of $dY = A \cdot Y$).

We fix an isomorphism of $\mathbb{C}$-vector spaces $o : \wedge^h V_0 \cong \mathbb{C}$. It is called an orientation. Now, we have the determinant map

$$\det : (V_0)^h \to \mathbb{C},\ \det(\omega_1, \omega_2, \ldots, \omega_h) := o(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_h).$$

Using det, one can define:

$$\det{}^2 : \mathcal{L} \to \mathbb{C},\ \det{}^2(x) := \det(\delta_1, \delta_2, \cdots, \delta_h)^2 = \det(\mathsf{q})^2 = \frac{\det(\Psi_0)^2}{\det(\mathsf{pm})^2},$$

where $\delta := (\delta_i)_{i\in I}$ is a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}(x)$ for which the bilinear form $\psi_{\mathbb{Z}}(x)$ has the form Ψ_0. Taking another basis will contribute $\det(A),\ A \in \Gamma_{\mathbb{Z}}$, which is $+1$ or -1, to the det function and so $\det^2$ is a well-defined function. In case $\det(A) = 1$ for all $A \in \Gamma_{\mathbb{Z}}$, we can define the det function in $\mathcal{L}$.

We have a plenty of non holomorphic functions on $\mathcal{L}$. For two elements $\omega_1, \omega_2 \in V_0$ we define:

$$f_{\omega_1,\bar{\omega}_2} : \mathcal{L} \to \mathbb{C},\ f_{\omega_1,\bar{\omega}_2}(x) = \psi_{\mathbb{C}}(x)(\omega_1, \overline{\omega_2}^x).$$

Let $\omega_i, i = 1, 2, \ldots$ be as before. We write $\delta = \overline{\mathsf{q}} \cdot \overline{\omega}^x$ The entries of the bellow matrix gives us a set which spans the $\mathbb{R}$-vector space of real functions obtained in the above way:

$$\mathsf{pm}^t \Psi_0^{-\mathsf{t}} \overline{\mathsf{pm}} = (\mathsf{q}^{-1})^t \Psi_0 \overline{\mathsf{q}}^{-1} = [f_{\omega_i,\bar{\omega}_j}]_{i,j\in I} \tag{10.7}$$

The matrix $G = [f_{\omega_i,\bar{\omega}_j}]_{i,j\in I}$ satisfies the differential equation:

$$dG = A \cdot G + G \cdot \overline{A}^{\mathsf{t}}, \tag{10.8}$$

where A is the connection matrix.

For $\omega \in V_0,\ x \in \mathcal{L}$ and $\epsilon \in \{0, 1\}$ define $\overline{\overline{\omega}}^{x,\epsilon} = \omega$ if $\epsilon = 0$ and $= \overline{\omega}^x$ otherwise. Let $\epsilon : I \to \{0, 1\}$ be a function and $\omega = (\omega_i)_{i\in I}$ be h elements in V_0. We have the following complex valued analytic function on $\mathcal{L}$:

$$f_\omega^\epsilon : \mathcal{L} \to \mathbb{C},\ f_\omega^\epsilon(x) = \det(\overline{\overline{\omega}}^{x,\epsilon}).$$

10.7 Hodge filtrations

We fix Hodge numbers

$$h^{i,m-i} \in \mathbb{N} \cup \{0\},\ h^i := \sum_{j=i}^{m} h^{j,m-j},\ i = 0, 1, \ldots, m,\ h^0 = h$$

and a filtration

$$F_0^\bullet : \{0\} = F_0^{m+1} \subset F_0^m \subset \cdots \subset F_0^1 \subset F_0^0 = V_0,\ dim(F_0^i) = h^i \tag{10.9}$$

on V_0. We define

$$H^{i,m-i}(x) := F_0^i \cap \overline{F_0^{m-i}}^x$$

and the following properties for $x \in \mathcal{L}$:

1. $\psi_{\mathbb{C}}(x)(F_0^i, F_0^j) = 0, \ \forall i, j, \ i + j > m;$

2. $V_0 = \oplus_{i=0}^m H^{i,m-i}(x);$

3. $(-1)^{i+\frac{m}{2}} \psi_{\mathbb{C}}(x)(v, \overline{v}^x) > 0, \ \forall v \in H^{i,m-i}(x), \ v \neq 0.$

Throughout the text we call these properties P1, P2 and P3.

Proposition 10.4. *Fix a polarized lattice $x \in \mathcal{L}$.*

1. *P1 implies that*

$$\psi_{\mathbb{C}}(H^{i,m-i}(x), H^{j,m-j}(x)) = 0 \text{ except for } i + j = m.$$

2. *$\sum_i H^{i,m-i}(x) = \oplus_i H^{i,m-i}(x)$ if and only if*

$$F_0^i \cap \overline{F_0^j}^x = 0, \ \forall \ i + j > m. \tag{10.10}$$

Proof. 1. We have $\psi_{\mathbb{C}}(H^{i,m-i}(x), H^{j,m-j}(x)) = \psi_{\mathbb{C}}(F_0^i \cap \overline{F_0^{m-i}}^x, F_0^j \cap \overline{F_0^{m-j}}^x) = 0$. Because if $i + j > m$ then $\psi_{\mathbb{C}}(F_0^i, F_0^j) = 0$ and if $i + j < m$ then $\psi_{\mathbb{C}}(\overline{F_0^i}^x, \overline{F_0^j}^x) = 0$.

2. If $a_{m-k,k} + \cdots + a_{0,m} = 0, \ a_{i,m-i} \in H^{i,m-i}(x)$ for some $0 \leq k \leq m$ with $a_{m-k,k} \neq 0$, then

$$-a_{m-k,k} = a_{m-k-1,k+1} + \cdots + a_{0,m} \in F_0^{m-k} \cap \overline{F_0^{k+1}}^x \Rightarrow a_{k,m-k} = 0.$$

which is a contradiction. The proof in other direction is a consequence of

$$F_0^i \cap \overline{F_0^j}^x = H^{i,m-i}(x) \cap H^{m-j,j}(x), \ i + j > m.$$

□

10.8 The analytic variety U

Define

$$\mathcal{K} := \{x \in \mathcal{L} \mid x \text{ satisfies P1 }\},$$

$$U := \{x \in \mathcal{L} \mid x \text{ satisfies P1,P2, P3 }\}.$$

We also define

$$\tilde{\mathcal{K}} := \{x \in P \mid x \text{ satisfies P1 }\},$$

$$\mathcal{P} := \{x \in P \mid x \text{ satisfies P1,P2, P3 }\}.$$

Definition 10.1. *A basis $\omega_i, i = 1, 2, \ldots, h$ of V_0 is compatible with the filtration $F_0^\bullet$ if $\omega_i, \ i = 1, 2, \ldots, h^i$ is a basis of F_0^i for all i.*

Proposition 10.5. *The set $\mathcal{K}$ is an analytic subset of $\mathcal{L}$ and U is an open subset of $\mathcal{K}$.*

Proof. Take a basis $\omega_i,\ i = 1, 2, \ldots, h$ of V_0 compatible with the Hodge filtration. The property P1 is given by

$$f_{\omega_r,\omega_s}(x) = 0,\ r \leq h^i,\ s \leq h^j,\ i + j > m$$

and so $\mathcal{K}$ is an analytic subset of $\mathcal{L}$.

Now choose a basis $\delta = (\delta_i)_{i\in I}$ of $V_{\mathbb{Z}}(x)$ and write $\delta = p\omega$ as before. Using ω we may assume that $V_0 = \mathbb{C}^h$ and δ constitutes of the rows of p. We have

$$\omega = p^{-1}\delta \Longrightarrow \overline{\omega}^x = \overline{p}^{-1}\delta = \overline{p}^{-1}p\omega$$

Therefore, the rows of $\overline{p}^{-1}p$ are complex conjugate of the the entries of ω. Now it is easy to verify that if the property (10.10), $\dim(H^{i,m-i}(x)) = h^{i,m-i}$ and P3 are valid for one x then they are valid for all points in a small neighborhood of x (for P3 we may first restrict $\psi_{\mathbb{C}}$ to the sphere of radius 1 and center $0 \in \mathbb{C}^h$). □

10.9 Moduli of polarized Hodge structures

We fix

$$G_0 := \{p \in P \mid p(F_0^\bullet) = F_0^\bullet\}$$

and let G_0 to act from the right on P.

Proposition 10.6. *The properties P1, P2 and P3 are invariant under the action of G_0.*

Proof. Let $x \in \mathcal{L},\ g \in G_0$ and $\omega \in V_0$. We have

$$\begin{aligned} H^{i,m-i}(xg) &= F_0^i \cap \overline{F_0^{m-i}}^{xg} = F_0^i \cap g^{-1}\overline{g(F_0^{m-i})}^{x} = F_0^i \cap g^{-1}(\overline{F_0^{m-i}}^{x}) \\ &= g^{-1}(F_0^i \cap \overline{F_0^{m-i}}^{x}) = g^{-1}(H^{i,m-i}(x)) \end{aligned}$$

and

$$\psi_{\mathbb{C}}(xg)(\omega, \overline{\omega}^{xg}) = \psi_{\mathbb{C}}(x)(g\omega, gg^{-1}\overline{g\omega}^x) = \psi_{\mathbb{C}}(x)(g\omega, \overline{g\omega}^x).$$

These equalities prove the proposition. □

The space U/G_0 is called the moduli of polarized Hodge structures.

10.10 The classical approach to the moduli of polarized Hodge structures

In this section we give the classical approach to the moduli of polarized Hodge structures due to P. Griffiths. The reader is referred to [39, 38] for more developments in this direction.

Let us fix the $\mathbb{C}$-vector space V_0 and the Hodge numbers as in §10.7. Let also F be the space of filtrations (10.9) in V_0. In fact, F has a natural structure of a compact smooth projective variety. We fix again the polarized lattice $(V_{\mathbb{Z}}(x_0), \psi_{\mathbb{Z}}(x_0))$ and define the Griffiths domain:

$$D := \{F^\bullet \in \mathrm{F} \mid (V_{\mathbb{Z}}(x_0), \psi_{\mathbb{Z}}(x_0), F^\bullet) \text{ satisfies P1, P2 and P3 }\}.$$

The group $\Gamma_{\mathbb{Z}}$ acts on V_0 from the right in the usual way and this gives us an action of $\Gamma_{\mathbb{Z}}$ on D. The space $\Gamma_{\mathbb{Z}}\backslash D$ is the moduli of polarized Hodge structure in the Griffiths sense.

Proposition 10.7. *There is a canonical isomorphism*

$$\beta : U/G_0 \xrightarrow{\sim} \Gamma_{\mathbb{Z}}\backslash D.$$

Proof. We take $x \in U$ and an isomorphism $\gamma : (V_{\mathbb{Z}}(x), \psi_{\mathbb{Z}}(x)) \xrightarrow{\sim} (V_{\mathbb{Z}}(x_0), \psi_{\mathbb{Z}}(x_0))$. The Hodge filtration $F_0^\bullet$ under this isomorphism gives us a Hodge filtration on V_0 with respect to the lattice $V_{\mathbb{Z}}(x_0)$ and so it gives us a point $\beta(x) \in D$. Different choices of γ leads us to the action of $\Gamma_{\mathbb{Z}}$ on $\beta(x)$. Therefore, we have a well-defined map

$$\beta : U \to \Gamma_{\mathbb{Z}}\backslash D.$$

Since $G_0 = \mathrm{Aut}(V_0, F_0^\bullet)$, β induces the desired isomorphism. □

The Griffiths domain is the moduli of polarized Hodge structures of a fixed type and with a $\mathbb{Z}$-basis in which the polarization has a fixed matrix form. Our domain U is the moduli of polarized Hodge structures of a fixed type and with a $\mathbb{C}$-basis compatible with Hodge filtration. Since cohomology with integer coefficients is not defined in algebraic geometry over an arbitrary field but de Rham cohomology and its Hodge filtration is defined, the Griffiths domain does not seem to have an algebraic counterpart but U corresponds to the moduli of smooth projective varieties $X/\mathbb{C}$ with certain differential forms on X and certain topological invariants fixed. This arises the hope that U has a natural algebraic structure.

Remark 10.1. We may define the space

$$DU := \{(F^\bullet, x) \in \mathrm{F} \times \mathcal{L} \mid (F^\bullet, x) \text{ satisfies P1, P2 and P3 }\}.$$

The canonical projection $\pi_D : DU \to \mathrm{F}$ (resp.$\pi_U : DU \to \mathcal{L}$) is a holomorphic fiber bundle with fibers biholomorphic to U (resp. D). If DU has a canonical structure of an algebraic variety and $\pi_D : DU \to \mathrm{F}$ is a morphism of algebraic varieties then U has a canonical structure of an algebraic variety. Therefore, to find an algebraic structure for DU is as much difficult as for U.

10.11 On biholomorphism group of $\mathcal{P}$

We would like to investigate the group of biholomorphic mappings of $\mathcal{P}$. We have seen that G_0 acts from the right on $\mathcal{P}$.

Let $P_{\mathbb{R}}$ be a subgroup of $P := \mathrm{Aut}(V_0)$ containing all $p \in P$ such that p induces an isomorphism of $V_{\mathbb{R}}(x_0)$. For all $A \in P_{\mathbb{R}},\ \omega \in\ V_0,\ p \in P$ we have

$$\overline{A\omega}^{x_0} = A\overline{\omega}^{x_0} \tag{10.11}$$

and also

$$\overline{\omega}^p = p^{-1}(\overline{p\omega}^{x_0}) \tag{10.12}$$

because

$$\overline{\omega}^p = \overline{\omega}^{\alpha(p)} = \overline{\omega}^{\alpha(1)p} = p^{-1}(\overline{p\omega}^{\alpha(1)}) = p^{-1}(\overline{p\omega}^{x_0}).$$

In particular (10.11) and (10.12) imply that

$$\overline{\omega}^{Ap} = \overline{\omega}^p,\ A \in P_{\mathbb{R}},\ p \in P. \tag{10.13}$$

Proposition 10.8. *The properties P1, P2 and P3 are invariant under the action of $P_{\mathbb{R}}$ from the left on P.*

Proof. We use the equalities (10.11), (10.12) and (10.13) and we have

$$H^{i,m-i}(Ap) = F_0^i \cap \overline{F_0^{m-i}}^{Ap} = H^{i,m-i}(p)$$

and

$$\begin{aligned}\psi_{\mathbb{C}}(Ap)(\omega, \overline{\omega}^{Ap}) &= \psi_{\mathbb{C}}(\alpha(Ap))(\omega, p^{-1}\overline{p\omega}^{x_0}) \\ &= \psi_{\mathbb{C}}(\alpha(1))(Ap\omega, App^{-1}\overline{p\omega}^{x_0}) = \psi_{\mathbb{C}}(x_0)(Ap\omega, \overline{Ap\omega}^{x_0}).\end{aligned}$$

for $A \in P_{\mathbb{R}}$ and $p \in P$. These equalities prove the proposition. □

The Propositions 10.8 and 10.6 imply that the biholomorphism group of $\tilde{\mathcal{P}}$ contains the algebraic group G_0 and the real group $P_{\mathbb{R}}$.

10.12 Period map

Let $f : X \to S$ be a regular and proper map of two smooth varieties over $\mathbb{C}$, $H = \cup_{t\in S} H^m(X_t, \mathbb{C})$ be the cohomology bundle and V_f be the space of global sections of H. We assume that m is the dimension of the fibers of f and so we have a non-degenerate map $\langle \cdot, \cdot \rangle : H^m(X_t, \mathbb{Z}) \times H^m(X_t, \mathbb{Z}) \to \mathbb{Z}$, where $H^m(X, \mathbb{Z})$ is the image of the m-th cohomology with integer coefficients in $H^m(X_t, \mathbb{C})$ (therefore, we have killed the torsion elements). Each fiber $H^m(X_t, \mathbb{C})$ is equipped with the so called Hodge filtration $F^\bullet$ which together with the polarized lattice $(H^m(X_t, \mathbb{Z}), \langle \cdot, \cdot \rangle)$ satisfy the properties P1, P2 and P3. We assume that there are global sections $\tilde{\omega}_i,\ i = 1, 2, \ldots, h$ of H such that in each point $t \in T$ they form a basis of $H^m(X_t, \mathbb{C})$ compatible with the Hodge filtration $F^\bullet$. For examples of such algebraic families see Chapter 3 and [63].

We fix a basis $(\omega_i)_{i=1}^h$ of V_0 compatible with $F_0^\bullet$ and we identify $(H^m(X_t, \mathbb{C}), F^\bullet, (\tilde{\omega}_i)_{i=1}^h)$ with $(V_0, F_0^\bullet, (\omega_i)_{i=1}^h)$, sending $\tilde{\omega}_i$ to ω_i. Then under this identification, $(H^m(X_t, \mathbb{Z}), \langle \cdot, \cdot \rangle)$ is mapped to a polarized lattice in V_0 and so we have a point in U. The obtained map

$$\mathsf{pm} : S \to U$$

is called the period map. The period map satisfy the so called Griffiths transversality:

$$\mathsf{pm}^{-1}(\omega_{ij}) = 0,\ i \leq h^{m-x},\ j \geq h^{m-x-1},\ x = 0, 1, \ldots, m-1. \tag{10.14}$$

where $A = [\omega_{ij}]$ is the restriction to U of the connection matrix in 10.5. Note that we have a commutative diagram

$$\begin{array}{ccc} H & \to & \mathcal{H}\mid_U \\ \downarrow & & \downarrow \\ S & \to & U \end{array}$$

and the the pull-back of the connection ∇ constructed in §10.5 is the Gauss-Manin connection of the family $X \to S$.

10.13 The Siegel upper half plane

In this section we consider the case in which the weight m is 1 and the polarization matrix is:

$$\Psi_0 = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

where I_g is the $g \times g$ identity matrix. It satisfies $\Psi_0^{\mathsf{t}} = \Psi_0^{-1}$. In this case $g := h^{10} = h^{01}$ and $h = 2g$. We take a basis $\omega = (\omega_1^1, \omega_2^1, \ldots, \omega_g^1, \omega_1^2, \omega_2^2, \ldots, \omega_g^2)$ of V_0 compatible with $F_0^\bullet$, i.e. the first g elements form a basis of F_0^1. We fix an orientation $o : \wedge^h V_0 \cong \mathbb{C}$ and assume that such a basis is positively oriented, i.e. $o(\omega_1^1 \wedge \omega_2^1 \wedge \ldots \wedge \omega_g^1 \wedge \omega_1^2 \wedge \omega_2^2 \wedge \ldots \wedge \omega_g^2) = 1$. We take a basis δ of $V_{\mathbb{Z}}^\vee$ and write the associated period matrix in the form:

$$\mathsf{pm}(x) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix},$$

where $x_i, i = 1, \ldots, 4$ are $g \times g$ matrices. We have

$$\Gamma_{\mathbb{Z}} = \mathrm{S}p(2g, \mathbb{Z}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2g, \mathbb{Z}) \mid a^{\mathsf{t}} c = c^{\mathsf{t}} a,\ b^t d = d^t b,\ a^t d - c^t b = 1 \},$$

$$G_0 = \{ \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in \mathrm{GL}(2g, \mathbb{C}) \mid \det(k_1)\det(k_4) \neq 0 \}.$$

The matrices (10.5) and (10.7) have the form:

$$\begin{pmatrix} f_{\omega_1,\omega_1} & f_{\omega_1,\omega_2} \\ f_{\omega_2,\omega_1} & f_{\omega_2,\omega_2} \end{pmatrix} = \begin{pmatrix} x_1^{\mathsf{t}} & x_3^{\mathsf{t}} \\ x_2^{\mathsf{t}} & x_4^{\mathsf{t}} \end{pmatrix} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} -x_3^{\mathsf{t}} x_1 + x_1^{\mathsf{t}} x_3 & -x_3^{\mathsf{t}} x_2 + x_1^{\mathsf{t}} x_4 \\ -x_4^{\mathsf{t}} x_1 + x_2^{\mathsf{t}} x_3 & -x_4^{\mathsf{t}} x_2 + x_2^{\mathsf{t}} x_4 \end{pmatrix}$$

respectively

$$\begin{pmatrix} f_{\omega_1,\bar{\omega}_1} & f_{\omega_1,\bar{\omega}_2} \\ f_{\omega_2,\bar{\omega}_1} & f_{\omega_2,\bar{\omega}_2} \end{pmatrix} = \begin{pmatrix} x_1^{\mathsf{t}} & x_3^{\mathsf{t}} \\ x_2^{\mathsf{t}} & x_4^{\mathsf{t}} \end{pmatrix} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{x}_3 & \bar{x}_4 \end{pmatrix} = \begin{pmatrix} -x_3^{\mathsf{t}} \bar{x}_1 + x_1^{\mathsf{t}} \bar{x}_3 & -x_3^{\mathsf{t}} \bar{x}_2 + x_1^{\mathsf{t}} \bar{x}_4 \\ -x_4^{\mathsf{t}} \bar{x}_1 + x_2^{\mathsf{t}} \bar{x}_3 & -x_4^{\mathsf{t}} \bar{x}_2 + x_2^{\mathsf{t}} \bar{x}_4 \end{pmatrix}.$$

The properties P1 and P3 imply that $x_3^{\mathsf{t}} x_1 = x_1^{\mathsf{t}} x_3$ and $-\sqrt{-1}(-x_3^{\mathsf{t}} \bar{x}_1 + x_1^{\mathsf{t}} \bar{x}_3)$ is a positive matrix. The property P2 implies that x_1 and x_2 have non zero determinant and so $x := x_1 x_2^{-1}$ is well-defined invertible matrix which satisfies the famous Riemann relations:

$$x^{\mathsf{t}} = x,\ \mathrm{Im}(x) \text{ is a positive matrix.}$$

The set of matrices $x \in \mathrm{Mat}^{g \times g}(\mathbb{C})$ with the above properties is called the Siegel upper half plane and is denoted by $\mathbb{H}_g$. The arithmetic group $\Gamma_{\mathbb{Z}}$ acts on $\mathbb{H}_g$ by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = (ax + b)(cx + d)^{-1},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathbb{Z}},\ x \in \mathbb{H}_g.$$

The morphism

$$U/G_0 \to \Gamma_{\mathbb{Z}} \backslash \mathbb{H}_g,$$

is given by

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \to x_1 x_3^{-1}.$$

Now let us calculate the entries of the connection matrix $A = [\omega_{ij}]_{i,j\in I}$.

$$A = d(\mathsf{pm}^{\mathsf{t}}) \cdot \mathsf{pm}^{-\mathsf{t}} = \begin{pmatrix} dx_1^{\mathsf{t}} & dx_3^{\mathsf{t}} \\ dx_2^{\mathsf{t}} & dx_4^{\mathsf{t}} \end{pmatrix} \begin{pmatrix} x_1^{\mathsf{t}} & x_3^{\mathsf{t}} \\ x_2^{\mathsf{t}} & x_4^{\mathsf{t}} \end{pmatrix}^{-1}.$$

One may use the formula

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^{-1} = \begin{pmatrix} x_2^{-1}x_4(x_1x_2^{-1}x_4 - x_3)^{-1} & -(x_1x_2^{-1}x_4 - x_3)^{-1} \\ -x_1^{-1}x_3(x_4 - x_2x_1^{-1}x_3)^{-1} & (x_4 - x_2x_1^{-1}x_3)^{-1} \end{pmatrix}$$

and obtain explicit expression for A. For instance in the case $g = 1$ we have:

$$A = \frac{1}{\det(x)} \begin{pmatrix} x_4dx_1 - x_2dx_3 & x_1dx_3 - x_3dx_1 \\ x_4dx_2 - x_2dx_4 & x_1dx_4 - x_3dx_2 \end{pmatrix}$$

See the books [41, 17, 46] for more information on Siegel modular forms.

10.14 Modular foliations in U

In §10.5 we defined the connection ∇ on $\mathcal{L}$ and determined its matrix A. We restrict ∇, A and so on to U and, if there is no danger of confusion, we use the same notations for the the new ones. We have now modular foliations in U associated to a global section $\sum \omega_i p_i$ of $\mathcal{H}$ and the connection ∇, where p_i's are holomorphic functions on U. In particular, the modular foliation $\mathcal{F}_{\omega_i}$ is the locus of points $x \in U$ such the i-th column of the period matrix is constant.

10.15 Loci of Hodge cycles

In this section we assume that m is even. A cycle $\delta \in V_{\mathbb{Z}}(x_0)^\vee, x_0 \in U$ is called a Hodge cycle if

$$\int_\delta F_0^{\frac{m}{2}+1} = 0.$$

Fix a Hodge cycle $\delta_{x_0} \in V_{\mathbb{Z}}(x_0)^\vee$. The loci of Hodge cycles through x_0 is given by

$$\{x \in (U, x_0) \mid \int_{\delta_x} F_0^{\frac{m}{2}+1} = 0\},$$

where $\delta_x = p^{-1}(x_0)$ and p is in a small neighborhood of the identity automorphism in $\mathcal{P}$. It is an analytic subset of (U, x_0) and it may not be irreducible. It is too premature claim, to say that a loci of of Hodge cycles is a part of a global analytic subvariety of U (similar to Theorem 7.1). However, it seems to me that the following statement is true: Let $p : T \to U$ be an analytic map from an algebraic variety variety T to the period domain U which satisfies the Griffiths transversality (10.14). Then the pull-back of any local loci of Hodge cycles by p is a part of an algebraic subvariety of T. The possible proof must be reconstructed from the arguments in [8].

10.16 Vanishing cycles and Picard-Lefschetz formula

For a family of n-dimensional hypersurfaces studied in the previous chapters, we have the notion of a vanishing cycle and we know that not every cycle in the n-th homology is a vanishing cycle. Therefore it would be reasonable to translate this notion into the moduli of polarized Hodge structures. In this section we want to do this.

We take a finite set $\delta_{x_0} = \{\delta_{1,x_0}, \delta_{2,x_0}, \ldots, \delta_{\mu,x_0}\}$ which generates $V_{\mathbb{Z}}(x_0)^\vee$ as $\mathbb{Z}$-module. This corresponds to a distinguished set of vanishing cycles in the geometric context of Chapter 5. We define each element of δ_{x_0} to be a vanishing cycle. For an arbitrary lattice $V_{\mathbb{Z}}(x)$ with $p \in P, \alpha(p) = x$, we obtain a $\mathbb{Z}$-basis $\delta = \delta_x := p^\vee(\delta_{x_0})$, where $p^\vee : V_{\mathbb{Z}}(x_0)^\vee \to V_{\mathbb{Z}}(x)^\vee$ is the map induced in the dual spaces. Again, each entry of δ_x is defined to be a vanishing cycle. Following the property (5.5) of geometric vanishing cycles, it is natural to assume that the dual polarization $\langle \cdot, \cdot \rangle$ in $V_{\mathbb{Z}}(x_0)^\vee$ satisfies

$$\langle \delta_i, \delta_i \rangle = (-1)^{\frac{m(m-1)}{2}} (1 + (-1)^m), \; i = 1, 2, \ldots, h. \tag{10.15}$$

To each vanishing cycle $\delta_i \in V_{\mathbb{Z}}(x)$ we define the Picard-Lefschetz mapping:

$$\mathsf{pl}_i(x) : V_{\mathbb{Z}}(x_0)^\vee \to V_{\mathbb{Z}}(x_0)^\vee, \; a \mapsto a + (-1)^{\frac{(m+1)(m+2)}{2}} \langle a, \delta_i \rangle \delta_i.$$

Note that $\mathsf{pl}_i(x_0) \in \Gamma_{\mathbb{Z}}$. Let $\Gamma_{\mathbb{Z}}^{\mathsf{pl}}$ be the subgroup of $\Gamma_{\mathbb{Z}}$ generated by $\mathsf{pl}_i(x_0), \; i = 1, 2, \ldots, \mu$. If in the geometric context of Chapter 5, a $\langle \cdot, \cdot \rangle$-preserving map from $H_n(\{f = 0\}, \mathbb{Z})$ to itself, where f is a tame polynomial over $\mathbb{C}$ with a non-zero discriminant, is a composition of some Picard-Lefschetz mappings, then it would be reasonable to assume that $\Gamma_{\mathbb{Z}}^{\mathsf{pl}} = \Gamma_{\mathbb{Z}}$.

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