De Rham cohomologies associated to iterated integrals [1]

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Abstract

For a punctured Riemann surface we construct the de Rham cohomology type spaces which are dual to the quotients of the lower central series of the homotopy group of the Riemann surface. We also construct a basis of such de Rham cohomologies in terms of P. Hall's basic commutators.

The present text has arisen from many conversations of the author with I. Nakai and L. Gavrilov. It contains some results of the papers [5, 2]. The reader is referred to these texts for a detailed account of the subject.

1 Homotopy groups of punctured Riemann surfaces

Let $\bar{U}$ be compact Riemann surface, U be the complement of a finite non-empty set of points of $\bar{U}$ and $p \in U$. The fact that $\bar{U}\backslash U$ is not empty plays an important role in the present text. Its first consequence is that the homotopy group $F := \pi_1(U, p)$ is freely generated by $m := g(\bar{U}) + \#(\bar{U}\backslash U) - 1$ elements. We denote by 1 the identity element of F. For $\delta_1, \delta_2 \in F$ we denote by $(\delta_1, \delta_2) = \delta_1\delta_2\delta_1^{-1}\delta_2^{-1}$ the commutator of δ_1 and δ_2 and for two sets $A, B \subset F$ by (A, B) we mean the group generated by $(a, b),\ a \in A, b \in B$. Let

$$F_r := (F_{r-1}, F),\ r = 1, 2, 3, \dots,\ F_1 := F.$$

Each quotient

$$H_{1,r}(U, \mathbb{Z}) := F_r/F_{r+1}$$

is a free $\mathbb{Z}$-module of rank

$$M_m(r) := \frac{1}{r}\sum_{d|r} \mu(d) m^{\frac{r}{d}},$$

where $\mu(d)$ is the möbius function: $\mu(1) = 1$, $\mu(p_1p_2\cdots p_s) = (-1)^s$ for distinct primes p_i's, and $\mu(n) = 0$ otherwise. Note that for r prime we have $M_m(r) = \frac{m^r - m}{r}$. A basis of $H_{1,r}(U, \mathbb{Z})$ is given by basic commutators of weight r (see §5).

The $\mathbb{Z}$-module $H_{1,1}(U, \mathbb{Z})$ is the classical 1-th homology group $H_1(U, \mathbb{Z})$ of U with integer coefficients. Its dual $H^1(U, \mathbb{Z}) := \{a : H_1(U, \mathbb{Z}) \to \mathbb{Z},\ \mathbb{Z} - \text{linear}\}$ is the 1-th cohomology group of U. It can be constructed either by Cech cohomology or de Rham cohomology. In the second case we have

$$H^1_{\mathrm{d}R}(U) := \frac{\Omega^1_U}{d\Omega^0_U} \cong H^1(U, \mathbb{C}) := H^1(U, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C},$$

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where Ω^i_U is the set of meromorphic differential forms in $\bar{U}$ with poles in $\bar{U}\backslash U$. Here we have again used the fact that $\bar{U}\backslash U$ is not empty. An element $\omega \in H^1_{\mathrm{d}R}(U)$ maps to

$$H_1(U,\mathbb{Z}) \to \mathbb{C},\ \delta \mapsto \int_\delta \omega.$$

under the above isomorphism. In this text we are going to construct similar de Rham type cohomologies for the $\mathbb{Z}$-modules $H_{1,r}(U,\mathbb{Z})$.

2 Iterated integrals

Let $p_i \in U,\ i = 0, 1$ and

$$\Omega^{\bullet,r}_U = \mathbb{C} + \cdot\Omega^\bullet_U + \cdot\Omega^\bullet_U\Omega^\bullet_U + \cdots + \cdot\underbrace{\Omega^\bullet_U\Omega^\bullet_U\cdots\Omega^\bullet_U}_{r\text{ times}}.$$

An element of $\Omega^{\bullet,r}_U$ is called to be of length$\leq r$. By definition $\Omega^{1,r}_U \subset \Omega^{\bullet,r}_U$ contains only differential 1-forms and in each homogeneous piece of an element of $\Omega^{0,r}_U \subset \Omega^{\bullet,r}_U$ there exists exactly one differential 0-form. We have the differential map

$$d = d_U : \Omega^{0,\bullet}_U \to \Omega^{1,\bullet}_U$$

which is $\mathbb{C}$-linear and is given by the rules

$$d(g) = dg - g(p_1) + g(p_0) \tag{1}$$

$$d(g\omega_1\omega_2\cdots\omega_r) = dg\omega_1\omega_2\cdots\omega_r - (g\omega_1)\omega_2\cdots\omega_r + g(p_0)\omega_1\omega_2\cdots\omega_r$$

$$d(\omega_1\cdots\omega_{i-1}g\omega_{i+1}\cdots\omega_r) =$$

$$\omega_1\cdots\omega_{i-1}dg\omega_{i+1}\cdots\omega_r - \omega_1\cdots\omega_{i-1}(g\omega_{i+1})\cdots\omega_r + \omega_1\cdots(\omega_{i-1}g)\omega_{i+1}\cdots\omega_r$$

$$d(\omega_1\omega_2\cdots\omega_r g) = \omega_1\omega_2\cdots\omega_r dg - g(p_1)\omega_1\omega_2\cdots\omega_r + \omega_1\omega_2\cdots(\omega_r g).$$

Let

$$\Omega = \frac{\Omega^{1,\bullet}_U}{d\Omega^{0,\bullet}_U} \tag{2}$$

and

$$\mathbb{C} = \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega_r \subset \cdots \subset \Omega$$

be the filtration given by the length:

$$\Omega_r := \frac{\Omega^{1,\leq r}_U}{d\Omega^{0,\leq r}_U}.$$

The map $\epsilon : \Omega \to \mathbb{C}$ associate to each ω its constant term in $\Omega_0 = \mathbb{C}$. Take a basis $x_1, x_2, \ldots, x_m$ of the $\mathbb{C}$-vector space $H^1_{\mathrm{d}R}(U)$. For simplicity we take a basis dual to δ_i's i.e., $\int_{\delta_i}\omega_j = 1$ if $i = j$ and $= 0$ otherwise.

The $\mathbb{C}$-vector space Ω is freely generated by $x_{i_1}x_{i_2}\cdots x_{i_k},\ 1 \leq i_1, i_2, \ldots i_k \leq \mu,\ k \in \mathbb{N}_0$. The fact that these elements generate Ω follows from the definition of the differential d and various uses of the fact that every $\omega \in \Omega^1_U$ can be written as a $\mathbb{C}$-linear combination of x_i's plus some $dg,\ g \in \Omega^0_U$. We obtain an isomorphism between Ω and the abstract associative ring generated by x_i's. In this way Ω turns to be an associative, but non commutative, $\mathbb{C}$-Algebra. Note that the $\mathbb{C}$-algebra structure of Ω does depend on the choice of the basis and $p_0,\ p_1$. However, the isomorphism of $\mathbb{C}$-vector spaces obtained in the quotient $\Omega_r/\Omega_{r-1},\ r = 1, 2, \ldots$ does not depend on the basis and p_0, p_1.

Let $\delta : [0, 1] \to U$ be a path which connects p_0 to p_1 and $\omega_i \in \Omega^1_U,\ i = 1, 2, \ldots, r$. The iterated integral is defined by induction and according to the rule:

$$\int_\delta \omega_1\omega_2 \cdots \omega_r = \int_\delta \omega_1 (\int_{\delta_x} \omega_2 \cdots \omega_r),$$

where for $\delta(t_1) = x$ we have $\delta_x := \delta|_{[0,t_1]}$. By $\mathbb{C}$-linearity one extends the definition to $\Omega^{1,\bullet}_U$ and it is easy to verify that an iterated integral of the elements in $d\Omega^{0,\bullet}_U$ is zero ([3] Proposition 1.3) and hence $\int_\delta \omega, \omega \in \Omega$ is well-defined. It is homotopy functorial. This can be checked by induction on r. We have

$$\int_\delta \omega_1\omega_2 \cdots \omega_r = \int_\delta \omega_1 \cdots \omega_i (\int_{\delta_x} \omega_{i+1} \cdots \omega_r),\ i = 1, 2, \ldots, r-1.$$

3 The properties of iterated integrals

In this section we list properties of iterated integrals in the context of this paper. The following four statements can be considered as the axioms of iterated integrals:

I 1. By definition the iterated integral is $\mathbb{C}$-linear with respect to the elements of Ω and

$$\int_1 \omega := \epsilon(\omega), \omega \in \Omega,\ \int_\alpha 1 = 1,\ \alpha \in F.$$

We use the convention $\omega_1\omega_2 \cdots \omega_r = 1$ for $r = 0$.

I 2. For $\alpha, \beta \in F$ and $\omega_1, \omega_2, \ldots \omega_r \in \Omega^1_U$

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{i=0}^{r} \int_\alpha \omega_1 \cdots \omega_i \int_\beta \omega_{i+1} \cdots \omega_r$$

([3], Proposition 2.9).

I 3. For $\alpha \in F$ and $\omega_1, \omega_2, \ldots \omega_r \in \Omega^1_U$

$$\int_{\alpha^{-1}} \omega_1\omega_2 \cdots \omega_r = (-1)^r \int_\alpha \omega_r \cdots \omega_1.$$

([3], Proposition 2.12).

I 4. For $\alpha \in F$ and $\omega_1, \omega_2, \ldots \omega_{r+s} \in \Omega^1_U$ we have the shuffle relations

$$\int_\alpha \omega_{i_1} \cdots \omega_{i_r} \int_\alpha \omega_{j_1} \cdots \omega_{j_s} = \sum_{(k_1,k_2,\ldots,k_{r+s})} \int_\alpha \omega_{k_1}\omega_{k_2} \cdots \omega_{k_{r+s}}, \tag{3}$$

where $(k_1, k_2, \ldots, k_{r+s})$ runs through all shuffles of $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_s)$ ([3], Lemma 2.11). This means that there is a partition of $\{1, 2, \ldots, r + s\}$ into two disjoint sets I, J such that $(k_i,\ i \in I)$ (resp. $(k_i,\ i \in J)$) ordered as I(resp. J) is equal to $(i_1, \ldots, i_r)$ (resp. $= (j_1, \ldots, j_s)$).

Note that I1, I2 and I3 imply that every iterated integral can be written as a polynomial in $\int_\delta \omega_1\omega_2\cdots\omega_r$, where δ runs through a set which generated F freely and ω_i runs through a fixed basis of $H^1_{\mathrm{d}R}(U)$. However by I4 this way of writing is not unique.

Let $\mathbb{Z}[F]$ be the integral group ring of F, J be the kernel of $\mathbb{Z}[F]\to\mathbb{Z},\ \sum_{i=1}^k a_i\alpha_i \mapsto \sum_{i=1}^k a_i,\ a_i\in\mathbb{Z}, \alpha_i\in F$. We have the canonical filtration of $\mathbb{Z}[F]$ by subideals:

$$\cdots\subset J^3\subset J^2\subset J^1=J\subset\mathbb{Z}[F].$$

By definition an iterated integral over $\mathbb{Z}[F]$ is $\mathbb{Z}$-linear. All the well-known properties of iterated integrals in the literature can be deduced form I1,I2,I3 and I4.

I 5. For $\alpha,\beta\in J$ and $\omega_1,\omega_2,\ldots\omega_r\in\Omega^1_U,\ r\geq 1$

$$\int_{\alpha\beta}\omega_1\cdots\omega_r=\sum_{i=1}^{r-1}\int_\alpha\omega_1\cdots\omega_i\int_\beta\omega_{i+1}\cdots\omega_r.$$

In particular, $\int_{\alpha\beta}\omega_1=0$.This statement follows from I1 and I2.

I 6. We have

$$\int_{J^s}\Omega_r=0,\ \text{for } 0\leq r<s.$$

This follows by induction on r from I5.

I 7. For $\alpha_1,\alpha_2,\cdots,\alpha_r\in F$ and $\omega_1,\omega_2,\ldots,\omega_r\in\Omega^1_U$

$$\int_{(\alpha_1-1)(\alpha_2-1)\cdots(\alpha_r-1)}\omega_1\cdots\omega_r=\prod_{i=1}^r\int_{\alpha_i}\omega_i.$$

This follows by induction on r from I5, I6 and I1.

We conclude that $\int_\alpha\omega,\ \omega\in\Omega_r/\Omega_{r-1},\ \alpha\in J^r/J^{r+1}$ is well-defined. Now we list some properties related to F_r's.

I 8. For $r<s$ and $\omega_1,\omega_2,\ldots,\omega_r\in\Omega^1_U$ we have

$$\int_{\beta_s}\omega_1\omega_2\cdots\omega_r=0,\ \beta_s=(\alpha_1,\alpha_2,\cdots,\alpha_s)\text{ or its inverse,}$$

where $(\alpha_1,\alpha_2,\ldots,\alpha_r)=((\cdots((\alpha_1,\alpha_2),\alpha_3)\cdots),\alpha_r)$.

It is enough to prove the statement for β_s. For β_s^{-1} it follows from I2 applied on $\beta_s\beta_s^{-1}=1$. The proof for $\beta_s=(\beta_{s-1},\alpha_s)$ is by induction on s. For $s=1$ it is trivially true. Suppose that the statement is true for s and let us prove it for $s+1$. After various applications of I2 and the induction hypothesis we have

$$\int_{\beta_{s+1}}\omega_1\omega_2\cdots\omega_r=\int_{\beta_s}\omega_1\omega_2\cdots\omega_r+\int_{\beta_s^{-1}}\omega_1\omega_2\cdots\omega_r$$

Now we apply I2 for $\beta_s\beta_s^{-1}=1$ and we conclude that the right hand side of the above equality is zero.

I 9. For $\omega_1, \omega_2, \ldots, \omega_r \in \Omega^1_U$ we have

$$\int_\alpha \omega_1\omega_2\cdots\omega_r = 0,\ \alpha \in F_s,\ r < s,$$

$$\int_{\alpha\beta} \omega_1\cdots\omega_r = \int_\alpha \omega_1\cdots\omega_r + \int_\beta \omega_1\cdots\omega_r,\ \alpha,\beta \in F_r,$$

$$\int_{\alpha^{-1}} \omega_1\cdots\omega_r = -\int_\alpha \omega_1\cdots\omega_r,\ \alpha \in F_r,$$

$$\int_\alpha (\omega_1\omega_2\cdots\omega_r + (-1)^r\omega_r\cdots\omega_1) = 0,\ \alpha \in F_r.$$

I9 implies that $\int_\alpha \omega,\ \alpha \in F_r/F_{r+1},\ \omega \in \Omega_r/\Omega_{r-1}$ is well-defined.

I 10. For $\alpha \in F_r$ and $\beta \in F_s$

$$\int_{(\alpha,\beta)} \omega_1\omega_2\cdots\omega_{r+s} = \int_\alpha \omega_1\cdots\omega_r \int_\beta \omega_{r+1}\cdots\omega_{r+s} - \int_\beta \omega_1\cdots\omega_s \int_\alpha \omega_{s+1}\cdots\omega_{r+s}$$

In particular

$$\int_{(\alpha,\beta)} \omega_1\omega_2 = \det\begin{pmatrix} \int_\alpha \omega_1 & \int_\beta \omega_1 \\ \int_\alpha \omega_2 & \int_\alpha \omega_2 \end{pmatrix}, \alpha,\beta \in F,\ \omega_1,\omega_2 \in \Omega^1_U. \tag{4}$$

The above statement follows by several application of I2,I9 (see also [1] Lemma 3).

I 11. For $\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_r, \beta_r \in F$ and $\omega_1, \omega_2 \in \Omega^1_U$

$$\int_{\prod_{i=1}^s (\alpha_i,\beta_i)} \omega_1\omega_2 = \sum_{i=1}^s \det\begin{pmatrix} \int_{\alpha_i} \omega_1 & \int_{\beta_i} \omega_1 \\ \int_{\alpha_i} \omega_2 & \int_{\beta_i} \omega_2 \end{pmatrix}.$$

The above statement follows by induction on s.

4 Free $\mathbb{Z}$-Lie algebras

One can associate to a free group F the $\mathbb{Z}$-Lie algebra

$$L_F := \oplus_{i=1}^\infty F_i/F_{i+1},\ [xF_i, yF_j] = (x,y)F_{i+j}.$$

It is in fact freely generated by $\delta_1, \delta_2, \ldots, \delta_m$. Another way to construct L_F is as follows: The remarks after I7 and I9 suggest that there may be an isomorphism between F_r/F_{r+1} and J^r/J^{r+1}. In fact the maps $F_r/F_{r+1} \to J^r/J^{r+1}$ induced by $x \mapsto x-1$ are well-defined and gives us an isomorphism of $\mathbb{Z}$-Lie algebras:

$$L_F \to \oplus_{r=1}^\infty J^r/J^{r+1}$$

This is proved in [6]. There is also a third way to define a free Lie algebra: Let Ω be the free non-commutative ring generated by $x_1, x_2, \ldots, x_m$. We denote by Ω_n the subset of Ω containing homogeneous polynomials of degree n, $\Omega = \oplus_{i=0}^\infty \Omega_i$. In Ω we define the bracket

$$[\alpha, \beta] = \alpha\beta - \beta\alpha,\ \alpha, \beta \in \Omega.$$

In this way Ω turns out to be a Lie algebra and we consider the smallest sub Lie algebra Ω^l of Ω generated by $x_i,\ i=1,2,\ldots,m$. Note that only $+$ and $[\cdot,\cdot]$ is allowed. A element of Ω^l is called a Lie element. We have an isomorphism of $\mathbb{Z}$-Lie algebras:

$$\mathsf{A}:\Omega^l \to L_F, \ \text{ induced by } x_i \mapsto \delta_i - 1,\ i=1,2,\ldots,m.$$

For $\omega \in \Omega_r$ the integration $\int_\delta \omega$ is well-defined for $\delta \in F_r/F_{r+1} \cong J^r/J^{r+1}$ and so we can talk about $\mathsf{A}^{-1}(\delta) \in \Omega^l$. In this way

$$\int_\delta \omega = \langle \omega, \mathsf{A}^{-1}\delta \rangle \tag{5}$$

where $\langle\cdot,\cdot\rangle$ is defined in $\Omega_i^l,\ i=1,2,\ldots$ by the rules:

$$\langle x_{i_1}x_{i_2}\cdots x_{i_n}, x_{j_1}x_{j_2}\cdots x_{j_n}\rangle = \begin{cases} 1 & i_1=j_1,\ldots,i_n=j_n \\ 0 & \text{otherwise} \end{cases}$$

5 Basic commutators

P. Hall in [4] Chapter 11 proves that a basis of $H_{1,r}(U,\mathbb{Z})$ is given by the so called basic commutators of weight r. In this section we explain the construction of such a basis. We have adapted the notations of [8], Chapter IV.

We choose a basis $\delta_i,\ i=,2,\cdots,m$ and put the order $\delta_i < \delta_j$ if $i<j$. The basic commutators of weight 1 are δ_i's. Having defined the basic commutators of weight less than r, the basic commutators of weight r are (c_i,c_j), where

1. c_i and c_j are basic and $w(c_i)+w(c_j)=r$;
2. $c_i < c_j$ and if $c_j=(c_s,c_t)$ then $c_s \le c_i$;

The commutators of weight r follows those of weight less than r and are ordered arbitrarily with respect to each other. In practice, one takes the lexicographical order in two elements c_i, c_j for the basic commutators (c_i,c_j) of weight r.

In Ω we define the commutator $[\cdot,\cdot]$ by $[u,v]:=uv-vu$ and in a similar way we define the basic commutators in Ω. In the construction of basic commutators we replace $\{\delta_i\}$ with with a basis $\{x_i\}$ of $H^1_{\mathrm{d}R}(U)$ and $(\cdot,\cdot)$ with $[\cdot,\cdot]$.

6 The dual of $H_{1,r}(U,\mathbb{C})$

Let Ω^s be the sub $\mathbb{Z}$-module of Ω generated by the shuffle elements: ω is a shuffle element if it is of the form

$$\sum_{(k_1,k_2,\ldots,k_r)} \omega_{k_1}\omega_{k_1}\cdots\omega_{k_r}$$

for some $r\in\mathbb{N}$, where for a fixed indices $i=(i_1,i_2,\ldots,i_a)$ and $j=(j_1,j_2,\ldots,j_b),\ a+b=r$ the above sum runs through all shuffles of i and j.

Theorem 1. *The subspaces Ω_r^l and Ω_r^s of $\Omega_r,\ r=2,3,\ldots$ are orthogonal to each other with respect to the bilinear map $\langle\cdot,\cdot\rangle$ and*

$$\Omega = \Omega_r^l \oplus \Omega_r^s$$

Proof. The first statement follows from I4 and (5) for $\omega \in \Omega_r^s$ and $\mathsf{A}^{-1}\delta \in \Omega_r^l$. The second statement follows form the first part and

$$\dim(\Omega_r^l) = M_m(r) = m^r - \dim \Omega_r^s$$

(see for instance [7] p. 218). $\square$

Let us define

$$H_{\mathrm{d}R}^{1,r}(U) := \Omega_r / (\Omega_{r-1} + \Omega_r^s).$$

Corollary 1. *The map*

$$\alpha : H_{\mathrm{d}R}^{1,r}(U) \to \check{H}_{1,r}(U, \mathbb{C}),\ \ \alpha(\omega)(\delta) = \int_\delta \omega.$$

is an isomorphism of $\mathbb{C}$*-vector spaces, where* ˇ *means dual and* $H_{1,r}(U,\mathbb{C}) = H_{1,r}(U,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. *A basis of* $H_{\mathrm{d}R}^{1,r}(U)$ *is given by basic commutators of weight* r.

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