Iterated integrals in holomorphic foliations [1]

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Abstract

In this article we study the iterated integrals in holomorphic foliations. We define the corresponding Petrov/Brieskorn type modules, give a formula for the Gauss-Manin connection of iterated integrals and calculate the Melnikov functions for certain topological cycles in terms of iterated integrals. As an application we show that after a deformation of a holomorphic foliation with a generic first integral in the complex plane, one cannot get two commuting holonomies.

1 Introduction

In a deformation of an integrable foliation one obtains the first Melnikov function as an Abelian integral whose zeros give rise to limit cycles in the deformed foliation. In the case in which the Abelian integral is identically zero such limit cycles are controlled by higher order Melnikov functions and L. Gavrilov in [4] has shown that they can be expressed in terms of iterated integrals and so they satisfy certain Picard-Fuchs equations. In a different context, the second named author and K. Yanai in [19, 18] have used iterated integrals to investigate the existence of relations between formal diffeomorphisms. Basic properties of iterated integrals are established by A. N. Parsin in 1969 and a systematic approach for de Rham cohomology type theorems for iterated integrals was made by K.-T. Chen around 1977. In the articles [15, 16] the application of Abelian integrals in holomorphic foliations are given. In this article we give a survey of iterated integrals in holomorphic foliations and as an application we investigate the existence of relations between deformed holonomies.

Let us consider a polynomial $f(x,y) \in \mathbb{C}[x,y], \deg(f) \geq 3$ in two variables and perform a perturbation

(1) $$\mathcal{F}_\epsilon : df + \epsilon\omega,\ \epsilon \in (\mathbb{C}, 0),\ \deg(\omega) \leq \deg(f) - 1,$$

where $\omega = Pdx + Qdy$ is a polynomial differential form and $\deg(\omega)$ is the maximum of $\deg(P)$ and $\deg(Q)$. We take a path $\delta \in \pi_1(f^{-1}(b), p)$, where b is a regular value of f, and ask for the conditions on ω such that the deformed holonomy $h_\epsilon : \Sigma \to \Sigma$, where Σ is a transversal section to $\mathcal{F}_0$ at p, is identity. The first result in this direction is due to Yu. Ilyashenko (see [10]): Consider a polynomial f whose fibers intersect the line at

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infinity transversally and it has only Morse type singularities with distinct images (these are generic conditions on f). If h_ϵ is the identity map for all $\epsilon \in (\mathbb{C}, 0)$ and the homology class of δ is a vanishing cycle then ω is an exact form and so $\mathcal{F}_\epsilon$ is again Hamiltonian. The generalization of this result for pencils of type $\frac{F^p}{G^q}$ in $\mathbb{P}^2$, pencils in arbitrary projective manifolds and logarithmic foliations is done in the articles [14, 15, 16]. The theory of iterated integrals gives us further generalizations of the above theorem for cycles with zero homology classes. For a, b in a group G let $(a, b) := aba^{-1}b^{-1}$ be the commutator of a and b.

Theorem 1. *For a generic polynomial f as before, let us assume that h_ϵ is the identity map for all $\epsilon \in (\mathbb{C}, 0)$, $\delta = (\delta_1, \delta_2)$ and the homology classes of δ_1 and δ_2 vanish along two paths which do not intersect each other except at b. Further assume that the homology classes of δ_1 and δ_2 have non-zero intersection. Then ω is an exact form and so $\mathcal{F}_\epsilon$ is again Hamiltonian.*

The above theorem is a special case of Theorem 2 in §4 stated for strongly tame functions. Another special case of Theorem 2 is the following: Let M be a projective compact manifold of dimension two and $\mathcal{F}(\omega_0)$ be a holomorphic foliation in M obtained by a generic non rational Lefschetz pencil (see [11] and §4). Here ω_0 is a global holomorphic section of $\Omega^1_M \otimes L$ such that the zero locus of ω_0 is a finite set in M, where L is a line bundle on M and Ω^1_M is the sheaf of holomorphic 1-forms in M. Let

$$\mathcal{F}_\epsilon = \mathcal{F}(\omega_0 + \epsilon\omega_1), \epsilon \in (\mathbb{C}, 0),\ \omega_1 \in H^0(M, \Omega^1_M \otimes L)$$

be a linear deformation of $\mathcal{F}(\omega_0)$. If δ_1, δ_2 are two vanishing cycles with the same properties as in Theorem 1, $H_1(M, \mathbb{Q}) = 0$ and the holonomies associated to δ_1 and δ_2 commute then $\mathcal{F}_\epsilon$ has a first integral.

A typical example of the situation of Theorem 1 is the following: Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ has two non-degenerated critical points: p_1 a center singularity and p_2 a saddle singularity. Assume that there is no more critical value of f between $f(p_1)$ and $f(p_2)$. The real vanishing cycle around p_1 and the complex vanishing cycle around p_2 satisfy the hypothesis of Theorem 1. For a more explicit example take f the product of d degree 1 real polynomials which are in general position and deform it in order to obtain a generic polynomial required by Theorem 1. For a precise description of the generic properties we have posed on f see §4.

The zeros of iterated integrals have a big impact on the topology of the leaves of holomorphic foliations obtained by deformation of pencils in complex manifolds. This is similar to the real case in which the zeros of iterated integrals controls the birth of limit cycles. However, it is not proved or disproved whether there are limit cycles, real or complex, beyond the zeros of iterated integrals. This point of view may give some light into the infinitesimal version of the minimal set question investigated by C. Camacho, A. Lins Neto and P. Sad in [2].

The paper is organized as follows: In §2 we recall some basic terminology related to iterated integrals. In §3 we consider iterated integrals depending on a parameter and gives the formula (7) for the Gauss-Manin connection of iterated integrals which has the expected property stated in Proposition 2. We find also a formula for the Melnikov function of certain cycles. In §4 we consider a strongly tame function, which is a generalization of a generic polynomial in $\mathbb{C}^2$, in an affine variety and prove Theorem 1 for such functions. Finally in Appendix A we have listed some basic properties of iterated integrals.

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2 Iterated integrals and homotopy groups

In this section we collect the necessary machinery for dealing with iterated integrals. Our approach to iterated integrals is the homology type $\mathbb{Z}$-modules $H_{1,r}(U,\mathbb{Z}),\ r=1,2,\ldots$ (see §2.2) and the construction of their duals in terms of differential forms (see §2.3). This approach, which is more convenient in holomorphic foliations, is not the classical approach in the literature and this is the main reason why we have reproduced some well-known materials in this section.

2.1 Iterated integrals

Let $\bar{U}$ be compact Riemann surface, U be the complement of a finite non-empty set of points of $\bar{U}$ and $p_i \in U,\ i=0,1$. Let $\Omega_U^\bullet$ be the set of meromorphic differential forms in $\bar{U}$ with poles in $\bar{U}\backslash U$ and

$$\Omega_U^{\bullet,r} := \mathbb{C}+\Omega_U^\bullet+\Omega_U^\bullet\Omega_U^\bullet+\cdots+\underbrace{\Omega_U^\bullet\Omega_U^\bullet\cdots\Omega_U^\bullet}_{r \text{ times}}.$$

For simplicity, in the above definition $+$ denotes the direct sum and $\Omega_U^\bullet\Omega_U^\bullet$ denotes $\Omega_U^\bullet\otimes_{\mathbb{C}}\Omega_U^\bullet$. An element of $\Omega_U^{\bullet,r}$ is called to be of length$\leq r$. By definition $\Omega_U^{1,r}\subset\Omega_U^{\bullet,r}$ contains only differential 1-forms and in each homogeneous piece of an element of $\Omega_U^{0,r}\subset\Omega_U^{\bullet,r}$ there exists exactly one differential 0-form. We have the differential map

$$d=d_U:\Omega_U^{0,\bullet}\to\Omega_U^{1,\bullet}$$

which is $\mathbb{C}$-linear and is given by the rules

$$d(g)=dg-g(p_1)+g(p_0) \tag{2}$$

$$d(g\omega_1\omega_2\cdots\omega_r)=(dg)\omega_1\omega_2\cdots\omega_r-(g\omega_1)\omega_2\cdots\omega_r+g(p_0)\omega_1\omega_2\cdots\omega_r$$

$$d(\omega_1\cdots\omega_i g\omega_{i+1}\cdots\omega_r)=$$

$$\omega_1\cdots\omega_i(dg)\omega_{i+1}\cdots\omega_r-\omega_1\cdots\omega_i(g\omega_{i+1})\cdots\omega_r+\omega_1\cdots(\omega_i g)\omega_{i+1}\cdots\omega_r$$

$$d(\omega_1\omega_2\cdots\omega_r g)=\omega_1\omega_2\cdots\omega_r(dg)-g(p_1)\omega_1\omega_2\cdots\omega_r+\omega_1\omega_2\cdots(\omega_r g),$$

where $1\leq i\leq r-1$. Let

$$B=\frac{\Omega_U^{1,\bullet}}{d\Omega_U^{0,\bullet}} \tag{3}$$

and

$$\mathbb{C}=B_0\subset B_1\subset B_2\subset B_3\subset\cdots\subset B_r\subset\cdots\subset B$$

be the filtration given by the length:

$$B_r:=\frac{\Omega_U^{1,\leq r}}{d\Omega_U^{0,\leq r}}.$$

The map $\epsilon : B \to \mathbb{C}$ associates to each ω its constant term in $B_0 = \mathbb{C}$. Take a basis $\omega_1, \omega_2, \ldots, \omega_m$ of the $\mathbb{C}$-vector space

$$H^1(U, \mathbb{C}) \cong H^1_{\mathrm{d}R}(U) = \frac{\Omega^1_U}{d\Omega^0_U}.$$

Note that $\bar{U} \backslash U$ is not empty. The $\mathbb{C}$-vector space B is freely generated by $\omega_{i_1}\omega_{i_2}\cdots\omega_{i_k},\ 1 \leq i_1, i_2, \ldots i_k \leq m,\ k \in \mathbb{N}_0$. The fact that these elements generate B follows from the definition of the differential d and various applications of the fact that every $\omega \in \Omega^1_U$ can be written as a $\mathbb{C}$-linear combination of ω_i's plus some $dg,\ g \in \Omega^0_U$. We obtain an isomorphism between B and the abstract associative ring generated by ω_i's. In this way B turns to be an associative, but non commutative, $\mathbb{C}$-Algebra. Note that the $\mathbb{C}$-algebra structure of B does depend on the choice of the basis and $p_0,\ p_1$. However, the isomorphism of $\mathbb{C}$-vector spaces obtained in the quotient $B_r/B_{r-1},\ r = 1, 2, \ldots$ does not depend on the base p_0, p_1.

Let $\delta : [0, 1] \to U$ be a path which connects p_0 to p_1 and $\omega_i \in \Omega^1_U,\ i = 1, 2, \ldots, r$. The iterated integral is defined by induction and according to the rule:

$$\int_\delta \omega_1\omega_2\cdots\omega_r = \int_\delta \omega_1(\int_{\delta_x} \omega_2\cdots\omega_r),$$

where for $\delta(t_1) = x$ we have $\delta_x := \delta|_{[0,t_1]}$. By $\mathbb{C}$-linearity one extends the definition to $\Omega^{1,\bullet}_U$ and it is easy to verify that an iterated integral of the elements in $d\Omega^{0,\bullet}_U$ is zero ([7] Proposition 1.3) and hence $\int_\delta \omega, \omega \in B$ is well-defined. It is homotopy functorial. This can be checked by induction on r. We will frequently use the equality

$$\int_\delta \omega_1\omega_2\cdots\omega_r = \int_\delta \omega_1\cdots\omega_i(\int_{\delta_x} \omega_{i+1}\cdots\omega_r),\ i = 1, 2, \ldots, r-1.$$

2.2 Homotopy groups

From now on we take $p := p_0 = p_1$ and let

$$G := \pi_1(U, p),\ m = \text{ number of generators of } G.$$

We denote by 1 the identity element of G. For $\delta_1, \delta_2 \in G$ we denote by $(\delta_1, \delta_2) = \delta_1\delta_2\delta_1^{-1}\delta_2^{-1}$ the commutator of δ_1 and δ_2 and for two sets $A, B \subset G$ by (A, B) we mean the group generated by $(a, b),\ a \in A, b \in B$. Let

$$G_r := (G_{r-1}, G),\ r = 1, 2, 3, \ldots,\ G_1 := G.$$

Each quotient

$$H_{1,r}(U, \mathbb{Z}) := G_r/G_{r+1}$$

is a free $\mathbb{Z}$-module of rank

$$M_m(r) := \frac{1}{r}\sum_{d|r} \mu(d) m^{\frac{r}{d}},$$

where $\mu(d)$ is the möbius function: $\mu(1) = 1,\ \mu(p_1p_2\cdots p_s) = (-1)^s$ for distinct primes p_i's, and $\mu(n) = 0$ otherwise. Note that for r prime we have $M_m(r) = \frac{m^r - m}{r}$. A basis of $H_{1,r}(U, \mathbb{Z})$ is given by basic commutators of weight r (see [9] Chapter 11).

There is another way to study G by finite rank $\mathbb{Z}$-modules mainly used in Hodge theory (see [7]). Let $\mathbb{Z}[G]$ be the integral group ring of G, J be the kernel of $\mathbb{Z}[G] \to \mathbb{Z}$, $\sum_{i=1}^{k} a_i\alpha_i \mapsto \sum_{i=1}^{k} a_i$, $a_i \in \mathbb{Z}, \alpha_i \in G$. We have the canonical filtration of $\mathbb{Z}[G]$ by subideals:

$$\cdots \subset J^3 \subset J^2 \subset J^1 = J \subset \mathbb{Z}[G].$$

Each quotient $\mathbb{Z}[G]/J^r$ is a freely generate $\mathbb{Z}$-module of finite rank.

2.3 The dual of $H_{1,r}(U,\mathbb{C})$

Using the properties of iterated integrals it is easy to see that

$$\int_\delta \omega, \delta \in H_{1,r}(U,\mathbb{Z}),\ \omega \in B_r/B_{r-1}$$

is well-defined (see Appendix A). Knowing the fact that

$$\dim_{\mathbb{C}}(B_r/B_{r-1}) = m^r \geq \mathrm{rank}_{\mathbb{Z}}(G_r/G_{r-1}) = M_m(r)$$

we expect that

$$V_r := \{\omega \in B_r/B_{r-1} \mid \int_{H_{1,r}(U,\mathbb{Z})} \omega = 0\}$$

has non zero dimension. In fact by shuffle formula (see Appendix A), we know that in general $V_r \neq 0$. It has been recently proved in [5] that V_r is generated by the shuffle relations. By the extension of Atyiah-Hodge-Grothendieck theorem to iterated integrals, see [8] commentary after Theorem 13.5 and Corollary 7.3, we know that for every $\delta \in H_{1,r}(U,\mathbb{Z})$ there is a $\omega \in B_r/B_{r-1}$ such that $\int_\delta \omega \neq 0$. Therefore,

$$H^{1,r}_{\mathrm{d}R}(U) := B_r/(B_{r-1}+V_r) \cong \check{H}_{1,r}(U,\mathbb{C}),$$

where $\check{}$ means dual and $H_{1,r}(U,\mathbb{C}) = H_{1,r}(U,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. One may be interested to find a basis of $H^{1,r}_{\mathrm{d}R}(U)$ similar to basic commutators (see [9]). For instance one can construct $M_m(r)$ elements of $H^{1,r}_{\mathrm{d}R}(U)$ in the following way: In the construction of basic commutators we replace the set which generates G freely with a basis of $H^1_{\mathrm{d}R}(U)$ and $(\cdot,\cdot)$ with $[\cdot,\cdot]$. By definition $[u,v] = uv - vu$ for $u, v \in B$. The basic commutators of weight r obtained in this way form a basis of $H^{1,r}_{\mathrm{d}R}(U)$. This is easy to see for $r = 1, 2$ and the complete proof is given in [5].

Remark 1. The authors of [19] have used another construction of $H^{1,r}(U,\mathbb{Z})$ by means of iterated integrals over Feynman diagrams. Such a construction is associated to a basis of the freely generated group G and has the advantage that it does not require to perform a tensor product of the $\mathbb{Z}$-module $H^{1,r}(U,\mathbb{Z})$ with $\mathbb{C}$.

3 Iterated integrals depending on a parameter

In this section we consider a two dimensional complex manifold M, a one dimensional submanifold D of M (possibly not connected) and a regular proper holomorphic map $f : M \to V$ such that $f\mid_D$ is also regular, where V is some small open disk in $\mathbb{C}$. By Ehresmann's theorem $f : (M,D) \to V$ is topologically trivial over V. We are going to work with iterated integrals in $U_t := f^{-1}(t)\backslash D,\ t \in V$. In other words, the Riemann surface of the previous section depends on the parameter t. Instead of two points p_0, p_1 we use two transversal section Σ_0, Σ_1 to the fibers of f at points $p_0, p_1 \in U_{t_0}$ for some $t_0 \in V$. We assume that $\Sigma_i, i = 0, 1$ are parameterized by the the image $t \in V$ of $f : \Sigma_i \to V$.

3.1 Gauss-Manin connection of iterated integrals

Let $U := M\backslash D$, Ω^1_U be the set of of meromorphic differential 1-forms in M with poles along D, Ω^1_V be the set of holomorphic differential 1-forms in V and $\Omega^1_{U/V} = \frac{\Omega^1_U}{f^*\Omega^1_V}$ be the set of relative differentials. The set $\Omega^1_{U/V}$ is a $\mathcal{O}(V)$-module in a canonical way. We redefine the set B in (3) using

$$\Omega^{\bullet,r}_{U/V} = \mathcal{O}(V) + \Omega^\bullet_{U/V} + \Omega^\bullet_{U/V}\Omega^\bullet_{U/V} + \cdots + \underbrace{\Omega^\bullet_{U/V}\Omega^\bullet_{U/V}\cdots\Omega^\bullet_{U/V}}_{r \text{ times}}.$$

The differential $d = d_{U/V}$ is $\mathcal{O}_V$-linear and is defined by the equalities in (2). Here by $f(p_i),\ i = 0,1$ we mean $f\mid_{\Sigma_i}$ as a function in t (one has to verify that d is well-defined).

Let δ be a path in U_{t_0} which connects p_0 to p_1. We denote by (M,δ) a small neighborhood of δ in M which can be homotopically contracted to δ. By a holomorphic object (function, differential form etc.) along δ we mean a holomorphic object defined in a universal covering of (M,δ). Therefore it can be viewed as a holomorphic object in a neighborhood of δ in M which may be multi-valued in the self intersection points of δ.

Let ω be a holomorphic 1-form defined along the path δ. Let $x_0 \in \Sigma_0$ and δ_{x,x_0} be a path which connects x_0 to $x \in M$ in $f^{-1}(f(x_0))$ along the path δ. For simplicity, we use $\int_{x_0}^x \omega = \int_{\delta_{x,x_0}} \omega$ and consider it as a holomorphic function along δ. The Gelfand-Leray form $\frac{d\omega}{df}$ restricted to U_t's is well-defined. For $\omega \in \Omega^1_{U/V}$, the map $\omega \mapsto \frac{d\omega}{df}$ is also called the Gauss-Manin connection with respect to the parameter t. The reader is referred to [1, 15] for more details.

We denote by $\tilde{\omega}$ (resp. $\bar{\omega}$) the pullback of $\omega\mid_{\Sigma_i}$ by the the holonomy map $(M,\delta) \to \Sigma_i$ with $i = 0$ (resp. $i = 1$). The form $\tilde{\omega}$ is of the form $a(f)df,\ a_2 \in \mathcal{O}(V)$ and so we define

$$\frac{\tilde{\omega}_1}{df} := a \in \mathcal{O}(V).$$

If there is no confusion we will also use $\frac{\tilde{\omega}_1}{df}$ to denote $a(f)$. In a similar way we define $\frac{\bar{\omega}}{df}$.

Proposition 1. We have

$$d(\int_{x_0}^x \omega) = (\int_{x_0}^x \frac{d\omega}{df})df + \omega - \tilde{\omega}. \tag{4}$$

This is [4] Lemma 1. For the convenience of the reader we prove it here.

Proof. First, we remark that if the equality (4) is true for ω then it is also true for $\omega + gdf$, where g is a holomorphic function along δ. By analytic continuation argument, it is enough to prove the proposition in a small neighborhood of p_0. We take coordinates $(z_1, z_2) : V_{p_0} \to (\mathbb{C}^2, 0)$ around p_0 such that

$$p_0 = (0,0), x = (z_1,z_2),\ x_0 = (0,z_2),\ f = z_2,\ \Sigma_0 = \{0\} \times (\mathbb{C},0).$$

Based on the first remark we can assume that $\omega = a(z_1,z_2)dz_1,\ a \in \mathcal{O}_{(\mathbb{C}^2,0)}$. We have

$$\begin{aligned} d(\int_{x_0}^x \omega) &= d(\int_0^{z_1} a(\tilde{z}_1, z_2)d\tilde{z}_1) \\ &= a(z_1,z_2)dz_1 + (\int_0^{z_1} \frac{\partial a(\tilde{z}_1,z_2)}{\partial z_2} d\tilde{z}_1)dz_2 \\ &= \omega - \tilde{\omega} + (\int_{x_0}^x \frac{d\omega}{df})df. \end{aligned}$$

Here $\int_0^{z_1}$ is the integration on the straight line which connects 0 to z_1. □

Let $\omega_i,\ i=1,2,\dots t$ be holomorphic differential 1-forms along δ, $p_1=\int_{x_0}^x \omega_1$ and $\omega=\omega_2\cdots\omega_r$. We have

$$\int_{x_0}^{x_1}\omega\frac{d(\omega_2 p_1)}{df}=\int_{x_0}^{x_1}\omega\left(-\frac{\omega_2\wedge\omega_1}{df}-\frac{\tilde{\omega}_1}{df}\omega_2+\omega_2\frac{d\omega_1}{df}+\frac{d\omega_2}{df}\omega_1\right)\tag{5}$$

because

$$\begin{aligned}\int_{x_0}^{x_1}\omega\frac{d(\omega_2 p_1)}{df} &= \int_{x_0}^{x_1}\omega(\frac{dp_1\wedge\omega_2+p_1 d\omega_2}{df})\\ &= \int_{x_0}^{x_1}\omega\left(-\frac{\omega_2\wedge\omega_1}{df}-\frac{\tilde{\omega}_1}{df}\omega_2+\omega_2(\int_{x_0}^{x}\frac{d\omega_1}{df})+\frac{d\omega_2}{df}(\int_{x_0}^{x}\omega_1)\right)\\ &= L.H.S.\end{aligned}$$

Now we define the Gauss-Manin connection:

$$\Omega^{1,\bullet}_{U/V}\to\Omega^{1,\bullet}_{U/V},\ \omega\mapsto\omega'.$$

By definition it is a $\mathbb{C}$-linear map, it is zero on $\Omega_U^{1,0}$ and for $\omega\in\Omega^1_{U/V}$ we have:

$$\omega'=\frac{d\omega}{df}+\frac{\bar{\omega}}{df}-\frac{\tilde{\omega}}{df}\tag{6}$$

For $r\geq 2$ and $\omega_1,\omega_2,\cdots,\omega_r\in\Omega^1_{U/V}$

$$(\omega_1\omega_2\cdots\omega_r)':=\tag{7}$$

$$\sum_{i=1}^{r}\omega_1\omega_2\cdots\omega_{i-1}\frac{d\omega_i}{df}\omega_{i+1}\cdots\omega_r-\sum_{i=1}^{r-1}\omega_1\cdots\omega_{i-1}\frac{\omega_i\wedge\omega_{i+1}}{df}\omega_{i+2}\cdots\omega_r+$$

$$\frac{\bar{\omega}_1}{df}\omega_2\cdots\omega_r-\omega_1\cdots\omega_{r-1}\frac{\tilde{\omega}_r}{df}.$$

(For $r=1$ this is (6)). We have to show that this definition is well-defined and does not depend on the choice of ω_i in its class in $\Omega^1_{U/V}$. Since (7) is linear in ω_i, it is enough to prove that $(\omega_1\cdots\omega_r)'=0$ if for some i we have $\omega_i\in f^*\Omega^1_V$. This can be easily checked using the facts $\frac{d\omega_i}{df}=0$, $\frac{\omega_i\wedge\omega_j}{df}=\frac{\omega_i}{df}\omega_j$.

Note that the definition (7) does depend on the choice of the transversal sections. The idea behind the definition (7) lies in the proof of the following proposition:

Proposition 2. *For continuous family of paths δ_t connecting $x_0\in\Sigma_0$ to $x_1\in\Sigma_1$ in $U_t,\ t\in V$ and along the path δ, we have*

$$\frac{\partial}{\partial t}\int_{\delta_t}\omega=\int_{\delta_t}\omega',\ \omega\in\Omega_U^{1,\bullet}\tag{8}$$

Proof. Let $\omega=\omega_1\omega_2\cdots\omega_r$. For $r=0$ the equality (8) is true by definition. For $r=1$ it follows from Proposition 1. Let us assume that $r\geq 2$. Define

$$p_i(x):=\int_{x_0}^{x}\omega_i\cdots\omega_r=\int_{x_0}^{x}\omega_i p_{i+1},\ i=1,2,\cdots,r,\ p_{r+1}:=1.$$

Let $P_i,\ i=1,2,\ldots,r$ be the restriction of p_i to Σ_1. We consider P_i as a function in t. We have:

$$
\begin{aligned}
\frac{\partial P_1}{\partial t} &\overset{(4)}{=} \left(\int_{x_0}^{x_1}\frac{d(\omega_1 p_2)}{df}\right)+\frac{\overline{\omega_1 p_2}}{df}-\frac{\widetilde{\omega_1 p_2}}{df}=\left(\int_{x_0}^{x_1}\frac{d(\omega_1 p_2)}{df}\right)+\frac{\bar{\omega}_1}{df}P_2\\
&\overset{(5)}{=} \int_{x_0}^{x_1}\left(-\frac{\omega_1\wedge\omega_2}{df}\omega_3\cdots\omega_r-\frac{\widetilde{\omega_2 p_3}}{df}\omega_1+\omega_1\frac{d(\omega_2 p_3)}{df}+\frac{d\omega_1}{df}\omega_2\omega_3\cdots\omega_r+\frac{\bar{\omega}_1}{df}\omega_2\cdots\omega_r\right)\\
&\ \vdots\quad \vdots\\
&= \int_{x_0}^{x_1}\omega'
\end{aligned}
$$

In the $(i-1)$-th line, $2\le i\le r$, we have used the fact that $p_i\mid_{\Sigma_0}=0$ and so $\widetilde{\omega_{i-1}p_i}=0$. □

Similar to the previous section we define $V_r=\{\omega\in B_r/B_{r-1}\mid \int_{H_{1,r}(U_t,\mathbb{Z})}\omega=0,\ \forall t\in V\}$ and $H^{1,r}_{\mathrm{d}R}(U/V)=B_r/(B_{r-1}+V_r)$ for the case $\Sigma_0=\Sigma_1$. The Gauss-Manin connection does not necessarily maps $d\Omega^{0,r}_{U/V}, r\ge 2$ to itself (for instance check it for $r=2$) and so it may not induce a well-defined operator from B to itself. However, we have:

Proposition 3. *If $\Sigma_0=\Sigma_1$ then the Gauss-Manin connection induces a well-defined map*

$$
H^{1,r}_{\mathrm{d}R}(U/V)\to H^{1,r}_{\mathrm{d}R}(U/V),\ \omega_1\omega_2\cdots\omega_r\to\sum_{i=1}^{r}\omega_1\omega_2\cdots\omega_{i-1}\frac{d\omega_i}{df}\omega_{i+1}\cdots\omega_r \tag{9}
$$

which is independent of the choice of the transversal section Σ_0.

Proof. First note that the Gauss-Manin connection induces a well-defined map in $B_r/B_{r-1}=(H^1_{\mathrm{d}R}(U/V))^r$ even if it is not well-defined in B_r. By Proposition 2 it maps V_r to itself and so it induces a well-defined map in $H^{1,r}_{\mathrm{d}R}(U/V)$. In the formula (7) the terms after the first sum have length less that r and so they are zero in $H^{1,r}_{\mathrm{d}R}(U/V)$. □

3.2 Melnikov functions as iterated integrals

Recall the notations of the previous section. Let

$$
\mathcal{F}_\epsilon:\omega_\epsilon=df+\epsilon\omega_1+\epsilon^2\omega_2+\cdots=0,\ \omega_i\in\Omega^1_U\ \ \epsilon\in(\mathbb{C},0) \tag{10}
$$

be a holomorphic deformation of $\mathcal{F}=\mathcal{F}_0$. Let $h_\epsilon(t):\Sigma_0\to\Sigma_1$ be the holonomy of $\mathcal{F}_\epsilon$ along the path δ. We write

$$
h_\epsilon(t)-t=M_1(t)\epsilon+M_2(t)\epsilon^2+\cdots+M_i(t)\epsilon^i+\cdots,\ i!.M_i(t)=\frac{\partial^i h_\epsilon}{\partial\epsilon^i}\mid_{\epsilon=0}.
$$

M_i is called the i-th Melnikov function of the deformation along the path δ. Let $M_1\equiv M_2\equiv\cdots\equiv M_{k-1}\equiv 0$ and $M_k\not\equiv 0$. It is a well known fact that the multiplicity of M_k at $t=0$ is the number of fixed points of the holonomy h_ϵ (as a function in t).

Proposition 4. *If* $M_1 \equiv M_2 \equiv \cdots \equiv M_k \equiv 0$ *then*

$$M_{k+1}(t) = -\int_{\delta_t} (\sum_{i=0}^{k} \omega_{k+1-i} p_i)$$

where p_i *and* g_i *are holomorphic functions along* δ *defined recursively by*

$$p_i df + dg_i = -\sum_{j=0}^{i-1} \omega_{i-j} p_j, \ \ i = 1, 2 \ldots, k, \ \ p_0 = 1. \tag{11}$$

Moreover, the restriction of p_i *(resp.* g_i*) to* Σ_0 *and* Σ_1 *coincide (as functions in* t*).*

For the proof of the above theorem see [21] Proposition 6 p. 73 or [15] Theorem 7.1. Now we want to write M_{k+1} as an iterated integral. This has been done in [4] Theorem 2, for the linear deformation of df. The idea of the proof is based on various usages of

$$p_i = \int_{x_0}^{x} \frac{d}{df} (\sum_{j=0}^{i-1} \omega_{i-j} p_j) \tag{12}$$

and the equality (5). Note that by Proposition 4

$$dp_i = (\int_{x_0}^{x} \frac{d^2}{df^2} (\sum_{j=0}^{i-1} \omega_{i-j} p_j)) df + \frac{d}{df} (\sum_{j=0}^{i-1} \omega_{i-j} p_j) - \frac{d}{df} (\widetilde{\sum_{j=0}^{i-1} \omega_{i-j} p_j}).$$

Since $\frac{d}{df}(\sum_{j=0}^{i-1} \omega_{i-j} p_j)$ is defined modulo relatively zero differential 1-forms, the term $\frac{d^2}{df^2}(\sum_{j=0}^{i-1} \omega_{i-j} p_j))$ is not uniquely defined even modulo relatively zero 1-forms. Note that we can add any holomorphic differential form η with $\int_{\delta_t} \eta \equiv 0$ to $\sum_{j=0}^{i-1} \omega_{i-j} p_j$ in the definition of p_i and so p_i and g_i's are not uniquely defined.

For simplicity we define $\omega^* = \frac{d\omega}{df}$ and define $(\omega_1 \omega_2 \cdots \omega_r)^* = \sum_{i=1}^{r} \omega_1 \cdots \omega_{i-1} \omega_i^* \omega_{i+1} \cdots \omega_r$. The first Melnikov functions are given by:

$$M_1 = -\int_{\delta_t} \omega_1.$$

$$M_2 = -\int_{\delta_t} \omega_2 + \omega_1 p_1 = -\int_{\delta_t} \omega_2 + \omega_1 \omega_1^*.$$

We have

$$p_2 = \int_{x_0}^{x} (\omega_2 + \omega_1 p_1)^* = \int_{x_0}^{x} \omega_2^* - \frac{\omega_1 \wedge \omega_1^*}{df} - \omega_1 \frac{\tilde{\omega}_1^*}{df} + \omega_1^* \omega_1^* + \omega_1 \omega_1^{**}$$

and so

$$M_3 = -\int_{\delta_t} \omega_3 + \omega_2 p_1 + \omega_1 p_2 = -\int_{\delta_t} \omega_3 + \omega_2 \omega_1^* + \omega_1 (\omega_2^* - \frac{\omega_1 \wedge \omega_1^*}{df} - \omega_1 \frac{\tilde{\omega}_1^*}{df} + \omega_1^* \omega_1^* + \omega_1 \omega_1^{**})$$

$$= -\int_{\delta_t} \omega_3 + \omega_2 \omega_1^* + \omega_1 (\omega_2^* - \frac{\omega_1 \wedge \omega_1^*}{df} + \omega_1^* \omega_1^* + \omega_1 \omega_1^{**}).$$

In the last equality we have used $\int_{\delta_t} \omega_1 \omega_1 \equiv 0$. In a similar way one calculates M_i's as iterated integrals.

Remark 2. In the process of writing M_k as an iterated integral, we do not use the fact that $M_i = 0, i < k$. However, we have used them in the proof of Proposition 4. They may simplify the formula for M_k as we have seen in M_3.

Proposition 5. *If $\Sigma_0 = \Sigma_1$ and $\delta \in G_k$ then $M_1 = M_2 = \cdots = M_{k-1} = 0$ and*

$$M_k = \int_{\delta_t} \underbrace{\omega_1(\omega_1(\cdots(\omega_1(}_{k-1 \ \textit{times} \ (}\omega_1)'\cdots)')'. \tag{13}$$

Proof. For $k = 1, 2$ we have already checked the equalities. In general the proof is as follows: For an arbitrary path δ connecting p_0 to p_1 we claim that M_k can be written as the iterated integral in (13) plus integrals, call it I_{k-1}, of differential forms of length strictly less than k. It is enough to prove that $p_i, \ i \le k-1$ is given by

$$p_i = \int_{x_0}^{x} \underbrace{(\omega_1(\cdots(\omega_1(}_{i \text{ times } (}\omega_1)^*\cdots)^*)^* + I_{i-1}$$

because if this claim is true then

$$M_k = -\int_{\delta_t} \omega_1 p_{k-1} + I_{k-1} = \int_{\delta_t} \underbrace{\omega_1(\omega_1(\cdots(\omega_1(}_{k-1 \text{ times } (}\omega_1)'\cdots)')' + I_{k-1}.$$

Our claim on p_i's follows by various applications of (5) in the formula of (12). Note that if in (5) ω_1 is an arbitrary homogeneous element in $\Omega_U^{1,\bullet}$ of length k then we have

$$\int_{x_0}^{x_1} \omega(\omega_2 p_1)^* = \int_{x_0}^{x_1} \omega(\omega_2\omega_1)^* + I_{k+r-2}.$$

□

4 Deformation of Holomorphic foliations

In this section we consider a smooth affine variety U and a polynomial function f in U and look at it as a morphism $f : U \to \mathbb{C}$ of algebraic varieties. There is a compact projective manifold $M = U \cup D$ and a rational morphism $\bar{f} : M \to \mathbb{P}^1$ which coincides with $f : U \to \mathbb{C}$ in U. Let $\mathcal{F} = \mathcal{F}(df)$ be the foliation in U whose leaves are connected component of the fibers of f. We are going to consider the holomorphic foliation

$$\mathcal{F}_\epsilon : df + \epsilon\omega, \ \epsilon \in (\mathbb{C}, 0), \ \omega \in \Omega_U^1. \tag{14}$$

We will apply the machinery of the previous section for $f : f^{-1}(V) \to V$, where V is a small open disk in $\mathbb{C}$.

4.1 Tame functions

Let us first define a tame function.

Definition 1. The morphism $f : U \to \mathbb{C}$ is tame if

1. The divisor at infinity $D := M \backslash U$ is smooth and $H_1(U, \mathbb{Q}) = 0$;

2. The foliation $\mathcal{F}(df)$ is not rational, i.e. the closure of a generic fiber of f is not isomorphic to $\mathbb{P}^1$.

3. f has non-degenerated singularities $p_i,\ i = 1, 2, 3 \ldots, \mu$ with distinct images $c_i := f(p_i)$;

4. A generic fiber of f is connected and its closure in M intersects D transversally.

Ehresmann's theorem implies that a tame morphism is topologically trivial over $\mathbb{C}\backslash C$, where $C := \{c_1, c_2, c_3, \ldots, c_\mu\}$. We have two main examples in mind. The first is a generic Lefschetz pencil (see [11, 17]) in a projective manifold $M \subset \mathbb{P}^m$. The first and second conditions become intrinsic properties of the pair $(M, \mathbb{P}^m)$. For this example, one can take D in such a way that $\bar{f}$ is also topologically trivial over ∞. The second example is mainly used in planar differential equations. Let f be a polynomial in two variables with $\deg(f) = d \geq 3$. We may compactify $\mathbb{C}^2$ inside $\mathbb{P}^2$, and look at $\mathcal{F}(df)$ as a foliation in $\mathbb{P}^2$. For a generic choice of the coefficients, the polynomial f is tame. For instance, to obtain the fourth condition one assumes that $\{[x; y] \in \mathbb{P}^1_\infty \mid f_d(x, y) = 0\}$ has d distinct points, where f_d is the last homogeneous piece of f. In this case $D \cong \mathbb{P}^1$ is not a regular fiber of f. Geometrically seen, d sheets of a regular fiber of f accumulate at D.

We take a distinguished system of paths $\gamma_i,\ i = 1, 2, \ldots, \mu$ in $\mathbb{C}$ (see [1]). The path γ_i connects a regular value b of f to c_i and has not an intersection point with other paths except at b. Let $\delta_i \in H_1(U_b, \mathbb{Z})$ be the vanishing cycle along γ_i. One calls $\delta_i,\ i = 1, 2, \ldots, \mu$ a distinguished basis of vanishing cycles. The Dynkin diagram of f is a graph whose vertices are vanishing cycles δ_i. The vertex δ_i is connected to δ_j if and only if $\langle \delta_i, \delta_j \rangle \neq 0$. The morphism f is called strongly tame if f is tame and its Dynkin diagram is connected. A generic Lefschetz pencil and a generic polynomial in two variables discussed above are strongly tame. For a proof see [11] 7.3.5 and [13] Theorem 2.3.2, 2. The polynomial case has been proved in [10]. It follows also from the following: If a tame polynomial f is obtained by a topologically trivial deformation of a morphism $g : U \to \mathbb{C}$ with only one singularity then the Dynkin diagram of f is connected and so it is strongly tame (see [12, 3, 6]). By a topologically trivial deformation we mean the one in which the topological structure of the smooth fiber does not change.

Proposition 6. *If f is a strongly tame morphism then*

1. *A distinguished basis of vanishing cycles generate $H_1(U_b, \mathbb{Q})$;*

2. *For a cycle $\delta \in H_1(U_b, \mathbb{Q})$ such that $H_1(U_b, \mathbb{Q}) \to \mathbb{Q},\ \delta' \mapsto \langle \delta, \delta' \rangle$ is not the zero map, the action of the monodromy on δ generates $H_1(U_b, \mathbb{Q})$. In particular, this is true for vanishing cycles.*

Proof. The first part can be proved by a slight modifications of the arguments of [11], §5. For a precise proof see [13] Theorem 2.2.1. The second part follows from the first part, the connectivity of the Dynkin diagram and Picard-Lefschetz formula. □

A peculiar property of a tame polynomial is that if $\int_{\delta_t} \omega = 0$ for a continuous family of vanishing cycles δ_t and $\omega \in \Omega^1_U$ then ω is relatively exact, i.e. $\int_\delta \omega = 0,\ \forall \delta \in H_1(U_t, \mathbb{Z}), t \in \mathbb{C}\backslash C$ and then it turns out that it is zero in the Brieskorn module

$$H = \frac{\Omega^1_U}{df \wedge \Omega^0_U + d\Omega^0_U},$$

where Ω^1_U is the set of (algebraic) differential 1-forms in U (see [15] Theorem 5.1). The module H, called also a Brieskorn module, is a $\mathbb{C}[t]$-module in a canonical way, $t[\omega] = [f\omega]$. The Gauss-Manin connection $\nabla_{\frac{\partial}{\partial t}} = \frac{d}{df}$ on H takes the form

$$\frac{d}{df} : H \to H_C,\ \omega \mapsto \omega' := \frac{d\omega}{df},$$

where H_C is the localization of H on the multiplicative group generated by $t - c_i,\ i = 1, 2, \ldots, \mu$ (see [16]). According to discussion in §2.3 it is natural to define $\tilde{H}^r := H \otimes_{\mathbb{C}[t]} H \otimes_{\mathbb{C}[t]} \cdots \otimes_{\mathbb{C}[t]} H$, $V_r := \{\omega \in \tilde{H}^r \mid \int_{H_r(U_b,\mathbb{Z})} \omega = 0,\ \forall b \in \mathbb{C}\backslash C\}$, $H^r = \tilde{H}^r / V_r$ and define the connection $H^r \to H^r_C, \omega \mapsto H^r_C$ as in (9). This connection is the algebraization of the canonical connection of the vector bundle $H^{1,r}(U_t, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C},\ t \in \mathbb{C}\backslash\mathbb{C}$, where the sections with images in $H^{1,r}(U_t, \mathbb{Z})$ are flat sections.

Remark 3. A systematic way for further generalizations of the main results of this paper would be a kind of Picard-Lefschetz theory for $H_{1,r}(U_t, \mathbb{Z})$. This requires the definition of a reasonable intersection theory in $H_{1,r}(U_t, \mathbb{Z})$, Picard-Lefschetz type formulas and so on. Such a theory does not seem to be worked out in the literature.

4.2 Two commuting holonomies

First, we state a Lemma.

Lemma 1. *Consider a strongly tame morphism f, a differential 1-form $\omega \in \Omega^1_U$ and a family of vanishing cycles $\delta = \delta_t$ such that $P(t) := \int_\delta \omega$ has the following property: In each $c \in C$, P can be written locally in the form $P(t) = (t-c)^\alpha \cdot p(t)$ for some $\alpha \in \mathbb{C}$ and a one valued holomorphic function p in $(\mathbb{C}, c)\backslash\{c\}$. Then ω is a relatively exact 1-form and so $\int_\delta \omega$ is identically zero.*

Proof. For another vanishing cycle δ' with the corresponding critical value c' of f and the vanishing path γ', if $\langle \delta, \delta' \rangle \neq 0$ then by Picard-Lefschtz formula along the path γ' and for the cycle δ we have:

$$\int_{\delta'} \omega = c_{\delta'} P(t), \tag{15}$$

where $c_{\delta'}$ is some constant depending on δ'. Since the Dynkin diagram of f is connected, the equality in (15) holds for all vanishing cycles δ'. Let δ be a vanishing cycle along the path γ in the critical value c. Since $\langle \delta, \delta \rangle = 0$, the value of the integral $\int_\delta \omega$ after the monodromy along the path γ and around c does not change and so the corresponding α must be integer.

We conclude that $\int_\delta \omega$ for any vanishing cycle δ, is a one valued function in $\mathbb{C}\backslash C$. Using the Picard-Lefschetz formula for two vanishing cycles δ_i, δ_j with non zero intersection number, we conclude that ω is a relatively exact 1-form. □

For $\omega \in \Omega^1_U$ define $\deg(\omega)$ to be the pole order of ω along D.

Theorem 2. *In the deformation (14) with $\deg(\omega) \leq \deg(df)$ assume that f is a strongly tame polynomial. Consider $\delta_1, \delta_2 \in \pi_1(U_b, b)$ such that the corresponding cycles in $H_1(U_b, \mathbb{Z})$ vanishes along the paths which do not intersect each other except at b. Also assume that*

$$\forall \delta \in H_1(U_b, \mathbb{Z}) \tag{16}$$

$$\langle\delta,\delta_1\rangle = 0 \ or \ \langle\delta,\delta_2\rangle = 0 \ or \ \exists\delta' \in H_1(U_b,\mathbb{Z}) \ s.t. \ \langle\delta_1,\delta\rangle\langle\delta_2,\delta'\rangle - \langle\delta_2,\delta\rangle\langle\delta_1,\delta'\rangle \neq 0.$$

If the deformed monodromies along δ_1 and δ_2 commute then $\mathcal{F}_\epsilon$ has a first integral.

Note that a generic fiber of f is not a punctured $\mathbb{P}^1$ and hence if $\langle\delta_1,\delta_2\rangle \neq 0$ then the condition in (16) is fulfilled. Theorem 1 is a special case of Theorem 2.

Proof. The first Melnikov function associated to the path (δ_1,δ_2) is trivially zero. By Proposition 5 and the equality (19) in Appendix A we have:

$$\frac{\int_{\delta_1}\omega'}{\int_{\delta_1}\omega} = \frac{\int_{\delta_2}\omega'}{\int_{\delta_2}\omega}. \tag{17}$$

If for a continuous family of vanishing cycles δ, we have $\int_\delta \omega = 0$ then by Proposition 6 the 1-form ω is relatively exact and so $\omega = Pdf + dQ$ for two meromorphic function in M with poles along D. The hypothesis $\deg(\omega) \leq \deg(df)$ implies that $P = 0$ and so $\omega = dQ$.

Assume that that $\int_\delta \omega$ is not identically zero. Then the multi-valued function (17) is well-defined. We denote it by P and claim that P is a rational function. Since integrals have finite growth at critical points and at infinity, it is enough to prove that P is one valued. By the hypothesis on the vanishing paths $\gamma_i,\ i = 1,2$ of δ_i, we can put γ_i inside a distinguished system of paths Γ. Let $c \in C$ and δ be the corresponding vanishing cycle along the path $\gamma \in \Gamma$. By Picard-Lefschetz formula along the path γ we have:

$$\frac{\int_{\delta_1}\omega' + r_1\int_\delta\omega'}{\int_{\delta_1}\omega + r_1\int_\delta\omega} = \frac{\int_{\delta_2}\omega' + r_2\int_\delta\omega'}{\int_{\delta_2}\omega + r_2\int_\delta\omega},$$

where $r_i := \langle\delta_i,\delta\rangle,\ i = 1,2$. This and (17) imply that either $P(t) = \frac{\int_\delta\omega'}{\int_\delta\omega}$ or $\int_{r_1\delta_2 - r_2\delta_1}\omega = 0$. If one of r_i's is zero then P is one valued along γ. If both are non-zero then the second case cannot happen because of Proposition 6 and the hypothesis in (16). In the first case we conclude that P is again one valued in a neighborhood of γ.

We have proved that P is a rational function. Now $\ln(\int_{\delta_1}\omega)' = P(t)$ and so

$$\int_\delta \omega = e^{\int P(t)dt} = Q\prod_{c\in K}(t-c)^{\alpha_c},\ Q \in \mathbb{C}(t),\ \alpha_c \in \mathbb{C},\ K \subset \mathbb{C},\ \#K < \infty.$$

Lemma 1 finishes the proof. □

Concerning the comments after Theorem 1 note that for a hyperplane section D of M we have the long exact sequence

$$\cdots \to H_2(M,\mathbb{Q}) \xrightarrow{s_1} H_2(M,U,\mathbb{Q}) \to H_1(U,\mathbb{Q}) \to H_1(M,\mathbb{Q}) \to \cdots$$

and the Leray-Thom-Gysin isomorphism $s_2 : H_2(M,U,\mathbb{Q}) \to H_0(D,\mathbb{Q}) \cong \mathbb{Q}$. The composition $s_2 \circ s_1$ is the intersection with D and so s_1 is not the zero map. This implies that if $H_1(M,\mathbb{Q})$ is zero then $H_1(U,\mathbb{Q}) = 0$.

A The properties of iterated integrals

In this appendix we list properties of iterated integrals in the context of this paper. The following four statements can be considered as the axioms of iterated integrals:

I 1. By definition the iterated integral is $\mathbb{C}$-linear (resp. $\mathbb{Z}$-linear) with respect to the elements of B (resp. $\mathbb{Z}[G]$) and

$$\int_1 \omega := \epsilon(\omega), \omega \in B, \ \int_\alpha 1 = 1, \ \alpha \in G.$$

We use the convention $\omega_1\omega_2\cdots\omega_r = 1$ for $r = 0$.

I 2. For $\alpha, \beta \in G$ and $\omega_1, \omega_2, \ldots \omega_r \in \Omega_U^1$

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{i=0}^{r} \int_\alpha \omega_1 \cdots \omega_i \int_\beta \omega_{i+1} \cdots \omega_r$$

([7], Proposition 2.9).

I 3. For $\alpha \in G$ and $\omega_1, \omega_2, \ldots \omega_r \in \Omega_U^1$

$$\int_{\alpha^{-1}} \omega_1\omega_2 \cdots \omega_r = (-1)^r \int_\alpha \omega_r \cdots \omega_1.$$

([7], Proposition 2.12).

I 4. For $\alpha \in G$ and $\omega_1, \omega_2, \ldots \omega_{r+s} \in \Omega_U^1$ we have the shuffle relations

$$\int_\alpha \omega_1 \cdots \omega_r \int_\alpha \omega_{r+1} \cdots \omega_{r+s} = \sum_\sigma \int_\alpha \omega_{\sigma(1)}\omega_{\sigma(2)} \cdots \omega_{\sigma(r+s)}, \tag{18}$$

where σ runs through all shuffles of type (r, s)([7], Lemma 2.11). Recall that a permutation σ of $\{1, 2, 3, \ldots, r+s\}$ is a shuffle of type (r, s) if

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(r) \text{ and } \sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+2) < \sigma^{-1}(r+s).$$

Note that I1, I2 and I3 imply that every iterated integral can be written as a polynomial in $\int_\delta \omega_1\omega_2\cdots\omega_r$, where δ runs through a set which generated G freely and ω_i runs through a fixed basis of $H^1_{\mathrm{d}R}(U)$. However by I4 this way of writing is not unique. By various applications of I4, we can get shuffle type formulas for the products of $s \geq 2$ integrals. All the well-known properties of iterated integrals in the literature can be deduced form I1,I2,I3 and I4.

I 5. For $\alpha, \beta \in J$ and $\omega_1, \omega_2, \ldots \omega_r \in \Omega_U^1, \ r \geq 1$

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{i=1}^{r-1} \int_\alpha \omega_1 \cdots \omega_i \int_\beta \omega_{i+1} \cdots \omega_r.$$

In particular, $\int_{\alpha\beta} \omega_1 = 0$.This statement follows from I1 and I2.

I 6. We have

$$\int_{J^s} B_r = 0, \text{ for } 0 \leq r < s.$$

This follows by induction on r from I5.

I 7. For $\alpha_1, \alpha_2, \cdots, \alpha_r \in G$ and $\omega_1, \omega_2, \ldots, \omega_r \in \Omega^1_U$

$$\int_{(\alpha_1-1)(\alpha_2-1)\cdots(\alpha_r-1)} \omega_1 \cdots \omega_r = \prod_{i=1}^{r} \int_{\alpha_i} \omega_i.$$

This follows by induction on r from I5, I6 and I1.

We conclude that $\int_\alpha \omega,\ \omega \in B_r/B_{r-1},\ \alpha \in J^r/J^{r+1}$ is well-defined. Now we list some properties related to G_r's.

I 8. For $r < s$ and $\omega_1, \omega_2, \ldots, \omega_r \in \Omega^1_U$ we have

$$\int_{\beta_s} \omega_1\omega_2 \cdots \omega_r = 0,\ \beta_s = (\alpha_1, \alpha_2, \cdots, \alpha_s) \text{ or its inverse,}$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_r) = ((\cdots((\alpha_1, \alpha_2), \alpha_3)\cdots), \alpha_r)$.

It is enough to prove the statement for β_s. For β_s^{-1} it follows from I2 applied on $\beta_s\beta_s^{-1} = 1$. The proof for $\beta_s = (\beta_{s-1}, \alpha_s)$ is by induction on s. For $s = 1$ it is trivially true. Suppose that the statement is true for s and let us prove it for $s+1$. After various applications of I2 and the induction hypothesis we have

$$\int_{\beta_{s+1}} \omega_1\omega_2 \cdots \omega_r = \int_{\beta_s} \omega_1\omega_2 \cdots \omega_r + \int_{\beta_s^{-1}} \omega_1\omega_2 \cdots \omega_r$$

Now we apply I2 for $\beta_s\beta_s^{-1} = 1$ and we conclude that the right hand side of the above equality is zero.

I 9. For $\omega_1, \omega_2, \ldots, \omega_r \in \Omega^1_U$ we have

$$\int_\alpha \omega_1\omega_2 \cdots \omega_r = 0,\ \alpha \in G_s,\ r < s,$$

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \int_\alpha \omega_1 \cdots \omega_r + \int_\beta \omega_1 \cdots \omega_r,\ \alpha, \beta \in G_r,$$

$$\int_{\alpha^{-1}} \omega_1 \cdots \omega_r = -\int_\alpha \omega_1 \cdots \omega_r,\ \alpha \in G_r,$$

$$\int_\alpha (\omega_1\omega_2 \cdots \omega_r + (-1)^r \omega_r \cdots \omega_1) = 0,\ \alpha \in G_r.$$

I9 implies that $\int_\alpha \omega,\ \alpha \in G_r/G_{r+1},\ \omega \in B_r/B_{r-1}$ is well-defined.

I 10. For $\alpha \in G_r$ and $\beta \in G_s$

$$\int_{(\alpha,\beta)} \omega_1\omega_2 \cdots \omega_{r+s} = \int_\alpha \omega_1 \cdots \omega_r \int_\beta \omega_{r+1} \cdots \omega_{r+s} - \int_\beta \omega_1 \cdots \omega_s \int_\alpha \omega_{s+1} \cdots \omega_{r+s}$$

In particular

$$\int_{(\alpha,\beta)} \omega_1\omega_2 = \det \begin{pmatrix} \int_\alpha \omega_1 & \int_\beta \omega_1 \\ \int_\alpha \omega_2 & \int_\alpha \omega_2 \end{pmatrix}, \alpha, \beta \in G,\ \omega_1, \omega_2 \in \Omega^1_U. \tag{19}$$

The above statement follows by several application of I2,I9 (see also [4] Lemma 3).

I 11. For $\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_r, \beta_r \in G$ and $\omega_1, \omega_2 \in \Omega^1_U$

$$\int_{\prod_{i=1}^{s}(\alpha_i,\beta_i)} \omega_1\omega_2 = \sum_{i=1}^{s} \det \begin{pmatrix} \int_{\alpha_i}\omega_1 & \int_{\beta_i}\omega_1 \\ \int_{\alpha_i}\omega_2 & \int_{\beta_i}\omega_2 \end{pmatrix}.$$

The above statement follows by induction on s.

The remarks after I7 and I9 suggest that there may be a relation between G_r/G_{r+1} and J^r/J^{r+1}. In fact the maps $G_r/G_{r+1} \to J^r/J^{r+1}$ induced by $x \mapsto x-1$ are well-defined and gives us a morphism of Lie algebras over $\mathbb{Z}$:

$$\oplus_{r=1}^{\infty} G_r/G_{r+1} \to \oplus_{r=1}^{\infty} J^r/J^{r+1}$$

For further information see [20].

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